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School of Applied Mathematical and Physical Sciences

Department of Mathematics

Framization of the Temperley–Lieb algebra and related link invariants

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PhD Thesis

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Extended abstract in Greek

Η τεχνική του framization είναι ένας μηχανισμός ο οποίος αναπτύχθηκε από τους Juyumaya και Λαμπροπούλου, και αποτελείται από μια γενίκευση μιας άλγεβρας κόμβων έτσι ώστε το αποτέλεσμα να σχετίζεται με τους πλαισιωμένους κόμβους (framed knots). Πιο συγκεκριμένα, ο μηχανισμός του framization μπορεί να περιγραφεί ως η διαδικασία πρόσθεσης των γεννητόρων του framing στο σύνολο των γεννητόρων της άλγεβρας κόμβων, ο ορισμός σχέσεων αλληλεπίδρασης μεταξύ των νέων γεννητόρων και των αρχικών γεννητόρων της άλγεβρας και η εμφάνιση του framing στις σχέσεις της άλγεβρας. Οι σχέσεις που προκύπτουν θα πρέπει φυσικά να είναι τοπολογικά συνεπείς. Επίσης, για τετριμμένο framing θα πρέπει να ανακτούμε τις κλασικές άλγεβρες. Το πιο δύσκολο πρόβλημα αυτής της διαδικασίας είναι η επιβολή του framing στις σχέσεις πολυωνυμικού τύπου. Ένα βασικό παράδειγμα framization είναι η άλγεβρα Yokonuma–Hecke, $\mathrm{Y}_{d,n}(u)$, που είναι το framization της άλγεβρας Iwahori–Hecke.

Σε αυτήν την διδακτορική διατριβή προτείνουμε τρία πιθανά framization της άλγεβρας Temperley–Lieb ως πηλίκα της άλγεβρας Yokonuma–Hecke πάνω από κατάλληλα αμφίπλευρα ιδεώδη. Οι πιθανές άλγεβρες πηλίκα που προκύπτουν είναι τρεις: Η άλγεβρα *Yokonuma–Temperley–Lieb* ($\mathrm{YTL}_{d,n}(u)$), η άλγεβρα *Framization of Temperley–Lieb* ($\mathrm{FTL}_{d,n}(u)$) και η άλγεβρα *Complex Reflection Temperley–Lieb* ($\mathrm{CTL}_{d,n}(u)$). Από αυτές ξεχωρίζουμε την άλγεβρα $\mathrm{FTL}_{d,n}(u)$ διότι, όπως θα γίνει σαφές στην πορεία, μας οδηγεί στο framization που ανταναγκλά την κατασκευή του "πολυωνύμου Jones για πλαισιωμένους κόμβους" με τον πιο φυσιολογικό τρόπο. Σ' αυτήν την περίληψη παρουσιάζονται συνοπτικά τα αποτελέσματα της παρούσης διατριβής και παρατείθενται οι σημαντικότερες αποδείξεις.

i. Η άλγεβρα Temperley–Lieb

Η άλγεβρα Temperley–Lieb ορίστηκε αρχικά από τους Temperley και Lieb [34] κατά τη διάρκεια έρευνάς τους πάνω στο μοντέλο για το λιώσιμο του πάγου. Είναι μια προσεταιριστική άλγεβρα πάνω από το $\mathbb{C}$, με μονάδα το 1 και παράγεται από το σύνολο γεννητόρων $\{f_1, \ldots, f_{n-1}\}$ το οποίο ικανοποιεί τις σχέσεις:

$$\begin{aligned} f_i^2 &= f_i \\ f_i f_j &= f_j f_i, \quad |i-j| > 1 \\ f_i f_j f_i &= \delta f_i \quad |i-j| = 1 \end{aligned}$$

όπου δ μια απροσδιόριστη. Σημειώνουμε ότι από την πρώτη σχέση της παράστασης της άλγεβρας, προκύπτει ότι οι γεννήτορες είναι ιδιοδύναμοι και επομένως μη–αντιστρέψιμοι.

Η εκ νέου ανακάλυψή της άλγεβρας Temperley–Lieb από τον Jones [15] ως πηλίκο της άλγεβρας Iwahori–Hecke και ο ορισμός από τον ίδιο μιας απεικόνισης ίχνους πάνω στη συγκεκριμένη άλγεβρα, οδήγησε σε απρόβλεπτες, μέχρι τότε, εφαρμογές στην Θεωρία Κόμβων (στο γνωστό *πολυώνυμο Jones* $V(u)$, μια από τις πιο σημαντικές αναλλοίωτες ισοτοπίας για κόμβους), ενώ έδωσε μια νέα ώθηση στην αλληλεπίδραση μεταξύ της Θεωρίας Κόμβων και της Θεωρίας Αναπαραστάσεων. Συγκεκριμένα, μέσω του μετασχηματισμού:

$$h_i := (u+1)f_i - 1$$

όπου το u ορίζεται μέσω της σχέσης $\delta^{-1} = 2 + u + u^{-1}$, παίρνουμε την ακόλουθη παράσταση για την άλγεβρα Temperley–Lieb $\mathrm{TL}_n(u)$ με σύνολο γεννητόρων το $\{h_1, \ldots, h_{n-1}\}$ και τις ακόλουθες σχέσεις μεταξύ αύτων:

$$h_ih_jh_i = h_jh_ih_j, \quad |i-j| = 1 \tag{1}$$
$$h_ih_j = h_jh_i, \quad |i-j| > 1 \tag{2}$$
$$h_i^2 = (u-1)h_i + u \tag{3}$$
$$h_ih_jh_i + h_jh_i + h_ih_j + h_i + h_j + 1 = 0, \quad |i-j| = 1. \tag{4}$$

Από την τετραγωνική σχέση (2) προκύπτει πως οι γεννήτορες h_i είναι αντιστρέψιμοι. Επιπλέον, οι σχέσεις (1)–(3) είναι ακριβώς οι σχέσεις που ορίζουν την άλγεβρα Iwahori–Hecke. Επομένως η άλγεβρα Temperley–Lieb, μπορεί να θεωρηθεί ως πηλίκο της άλγεβρας Iwahori–Hecke, $\mathrm{H}_n(u)$, πάνω από το αμφίπλευρο ιδεώδες J που ορίζουν οι σχέσεις (4). Θέτοντας:

$$h_{i,j} := h_ih_jh_i + h_jh_i + h_ih_j + h_i + h_j + 1$$

αποδεικνύεται πως το ιδεώδες J είναι κύριο και παράγεται από το στοιχείο $h_{1,2}$. Τα στοιχεία $h_{i,j}$ ονομάζονται στοιχεία Steinberg, ενώ οι σχέσεις (1.15) σχέσεις Steinberg.

Το πολυώνυμο Jones 2–μεταβλητών ή Homflypt, $P(u,\zeta)$, [13, 29], κατασκευάστηκε μέσω μιας απεικόνισης ίχνους Markov, με παράμετρο ζ, που ορίστηκε από τον Ocneanu πάνω στις άλγεβρες Iwahori–Hecke. Εφόσον, όπως απέδειξε ο Jones, η άλγεβρα Temperley–Lieb είναι ένα πηλίκο της άλγεβρας Iwahori–Hecke η απεικόνιση ίχνους του Ocneanu περνάει στην άλγεβρα πηλίκο όταν αυτή μηδενίζει τον γεννήτορα του ιδεώδους J. Αυτό συμβαίνει όταν η παράμετρος ζ παίρνει δύο συγκεκριμένες τιμές. Από τις αυτές μόνο μια έχει τοπολογικό ενδιαφέρον, η $\zeta = -\frac{1}{u+1}$. Για αυτή την τιμή του ζ το πολυώνυμο Jones 1–μεταβλητής προκύπτει από το πολυώνυμο Homflypt.

Οι άλγεβρες Iwahori–Hecke και Temperley–Lieb είναι τα πιο σημαντικά παραδείγματα μιας *άλγεβρας κόμβων*. Μια άλγεβρα κόμβων είναι μια άλγεβρα η οποία στην παράστασή της εμπεριέχει τις σχέσεις των πλεξίδων (braid relations), οι οποίες χρησιμοποιούνται στην κατανόηση της ταξινόμησης των κόμβων. Πιο συγκεκριμένα, μια άλγεβρα κόμβων αποτελείται από μια τριάδα (A, τ, π) όπου π είναι μια αναπαράσταση της ομάδας των πλεξίδων στην A και τ είναι μια συνάρτηση ίχνους Markov ορισμένη πάνω στην A.

ii. Η άλγεβρα Yokonuma–Hecke

Η άλγεβρα Yokonuma–Hecke, $\mathrm{Y}_{d,n}(u)$, ορίστηκε αρχικά στο [37], ενώ στη συνέχεια στο [18], ορίστηκε ως το πηλίκο της modular ομάδας των πλαισιωμένων πλεξίδων $\mathcal{F}_{d,n}$ (στις οποίες οι κλωστές έχουν framing modulo d), πάνω από μια τετραγωνική σχέση που εμπεριέχει τους γεννήτορες framing t_i μέσω συγκεκριμένων ιδιοδύναμων στοιχείων e_i. Πιο αναλυτικά, είναι η ακόλουθη $\mathbb{C}(u)$-άλγεβρα πηλίκο:

$$\mathrm{Y}_{d,n}(u) = \frac{\mathbb{C}\mathcal{F}_{d,n}}{\langle g_i^2 - 1 - (u-1)e_i - (u-1)e_i\, g_i \rangle}$$

όπου $e_i = \frac{1}{d}\sum_{m=0}^{d-1} t_i^m t_{i+1}^{-m}$. Από τα παραπάνω, η άλγεβρα $\mathrm{Y}_{d,n}(u)$ μπορεί να παρασταθεί από τους γεννήτορες $t_1, \ldots, t_n, g_1, \ldots, g_{n-1}$ και τις ακόλουθες σχέσεις μεταξύ αυτών:

$$\begin{aligned}
g_i g_j &= g_j g_i, \quad |i-j| > 1 \\
g_{i+1} g_i g_{i+1} &= g_i g_{i+1} g_i, \\
g_i^2 &= 1 + (u-1)e_i + (u-1)e_i g_i \\
t_i t_j &= t_j t_i, \quad \text{για κάθε } i, j \\
t_i^d &= 1, \quad \text{για κάθε } i \\
g_i t_i &= t_{i+1} g_i \\
g_i t_{i+1} &= t_i g_i \\
g_i t_j &= t_j g_i, \quad \text{όπου } j \neq i, \text{ και } j \neq i+1
\end{aligned}$$

Τα στοιχεία e_i είναι ιδιοδύναμα και μπορούν να γενικευτούν ως εξής. Για οποιοσδήποτε δείκτες i, j και για οποιοδήποτε $m \in \mathbb{Z}/d\mathbb{Z}$, ορίζουμε τα ακόλουθα στοιχεία μέσα στην $\mathbb{C}\mathcal{F}_{d,n}$:

$$e_{i,j} := \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_j^{-s}, \tag{5}$$

και:

$$e_i^{(m)} := \frac{1}{d}\sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{-s}. \tag{6}$$

Επίσης, παρατηρούμε ότι: $e_i = e_{i,i+1} = e_i^{(0)}$. Θέτοντας $d = 1$, η άλγεβρα $\mathrm{Y}_{1,n}(u)$ ταυτίζεται με την άλγεβρα Iwahori–Hecke $\mathrm{H}_n(u)$. Οι άλγεβρες Yokonuma–Hecke έχουν μελετηθεί στα [37, 18, 23, 20, 24, 33, 4]. Επιπλέον, στο [18] ο Juyumaya κατασκεύασε μια επαγωγική γραμμική βάση για τις άλγεβρες $\mathrm{Y}_{d,n}(u)$ και μέσω αυτής όρισε μια μοναδική γραμμική απεικόνιση ίχνους Markov, tr, πάνω σ' αυτές τις άλγεβρες, η οποία εξαρτάται από παραμέτρους $z, x_1, \ldots, x_{d-1}$. Πιο συγκεκριμένα:

Θεώρημα 1 ([18] Θεώρημα 12). *Έστω d ένας θετικός ακέραιος. Για τις απροσδιόριστες z, x_1, $\ldots, x_{d-1}$ υπάρχει μια μοναδική γραμμική απεικόνιση ίχνους Markov* tr*:*

$$\mathrm{tr} : \cup_{n=1}^{\infty} \mathrm{Y}_{d,n}(u) \longrightarrow \mathbb{C}(u)[z, x_1, \ldots, x_{d-1}]$$

η οποία ορίζεται με επαγωγή στο n σύμφωνα με τους ακόλουθους κανόνες:

$$\begin{array}{rcll}
\mathrm{tr}(ab) &=& \mathrm{tr}(ba) & \\
\mathrm{tr}(1) &=& 1 & \\
\mathrm{tr}(ag_n) &=& z\,\mathrm{tr}(a) & (\textit{ιδιότητα Markov}) \\
\mathrm{tr}(at_{n+1}^s) &=& x_s \mathrm{tr}(a) & (s = 1, \ldots, d-1)
\end{array}$$

όπου $a, b \in \mathrm{Y}_{d,n}(u)$.

Στην προσπάθεια τους, οι Juyumaya και Λαμπροπούλου, να ορίσουν τοπολογικές αναλλοίωτες για πλαισιωμένους κόμβους μέσω του tr, όπως προέκυψε στο [23], το tr δεν μπορεί να κανονικοποιηθεί άμεσα σύμφωνα με τις σχέσεις ισοδυναμίας για πλαισιωμένες πλεξίδες. Το γεγονός αυτό οδήγησε σε συνθήκες που θα έπρεπε να επιβληθούν στις παραμέτρους $x_1, \ldots, x_{d-1}$ της απεικόνισης ίχνους tr. Συγκεκριμένα, οι παράμετροι $x_1, \ldots, x_{d-1}$ της απεικονίσης ίχνους tr θα πρέπει να ικανοποιούν ένα σύστημα μη γραμμικών εξισώσεων, το λεγόμενο E–*σύστημα*:

$$\sum_{s=0}^{d-1} x_{m+s} x_{-s} = x_m \sum_{s=0}^{d-1} x_s x_{-s}$$

Μια προφανής λύση είναι όταν τα x_i είναι d–ρίζες της μονάδας. Ο P. Gérardin στο[23, Appendix] βρήκε το πλήρες σύνολο λύσεων για το E–σύστημα και έδειξε ότι παραμετροποιούνται από τα μη–κενά υποσύνολα της $\mathbb{Z}/d\mathbb{Z}$. Δεδομένης μιας λύσης του E–συστήματος, πολυωνυμικές αναλλοίωτες ισοτοπίας 2–μεταβλητών για πλαισιωμένους κόμβους, κλασικούς κόμβους και κόμβους singular κατασκευάστηκαν στα [20, 22, 21] αντίστοιχα και μελετηθήκαν περαιτέρω στο [2, 5].

iii. Η άλγεβρα Yokonuma–Temperley–Lieb

iii.α. Η άλγεβρα $\mathrm{YTL}_{d,n}(u)$

Η πρώτη άλγεβρα που ορίσαμε ως πιθανό framization της άλγεβρας Temperley–Lieb είναι η λεγόμενη άλγεβρα Yokonuma–Temperely–Lieb. Συγκεκριμένα θεωρούμε το ιδεώδες που ορίζουν τα στοιχεία Steinberg στην περίπτωση της άλγεβρας $\mathrm{Y}_{d,n}(u)$. Πιο αναλυτικά, θέτουμε:

$$g_{i,j} := g_i g_j g_i + g_j g_i + g_i g_j + g_i + g_j + 1$$

με $i, j \in \{1, \ldots, d-1\}$ και $|i-j| = 1$. Επιπρόσθετα, θεωρούμε το αμφίπλευρο ιδεώδες $I_1 = \langle g_{i,j} \rangle$. Έχουμε τον ακόλουθο ορισμό:

Ορισμός 1. Για $n \geq 3$, η άλγεβρα *Yokonuma–Temperley–Lieb*, $\mathrm{YTL}_{d,n}(u)$, ορίζεται ως το πηλίκο:

$$\mathrm{YTL}_{d,n}(u) := \frac{\mathrm{Y}_{d,n}(u)}{I_1}.$$

Με άλλα λόγια η άλγεβρα $\mathrm{YTL}_{d,n}(u)$ μπορεί να παρασταθεί από τους γεννήτορες $g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$, οι οποίοι ικανοποιούν τις σχέσεις:

$$g_i g_j = g_j g_i, \quad |i-j| > 1 \tag{7}$$

$$g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i \tag{8}$$

$$g_i^2 = 1 + (u-1)e_i + (u-1)e_i g_i \tag{9}$$

$$t_i t_j = t_j t_i, \quad \text{για κάθε } i, j \tag{10}$$

$$t_i^d = 1, \quad \text{για κάθε } i \tag{11}$$

$$g_i t_i = t_{i+1} g_i \tag{12}$$

$$g_i t_{i+1} = t_i g_i \tag{13}$$

$$g_i t_j = t_j g_i, \quad \text{όπου } j \neq i, \text{ και } j \neq i+1 \tag{14}$$

$$g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1 = 0 \tag{15}$$

Αποδεικνύεται πως το ιδεώδες I_1 είναι κύριο και πως παράγεται από το στοιχείο $g_{1,2}$. Θέτοντας $d = 1$ στην παραπάνω παράσταση, παρατηρούμε πως η άλγεβρα $\mathrm{YTL}_{1,n}(u)$ ταυτίζεται με την άλγεβρα Temperley–Lieb, $\mathrm{TL}_n(u)$.

Θεωρώντας τώρα το μετασχηματισμό $\ell_i := \frac{1}{u+1}(g_i + 1)$ μπορούμε να πάρουμε μια παράσταση για την άλγεβρα $\mathrm{YTL}_{d,n}(u)$ με μη αντιστρέψιμους γεννήτορες. Πράγματι, έχουμε την ακόλουθη πρόταση:

Πρόταση 1. *Η άλγεβρα* $\mathrm{YTL}_{d,n}(u)$ *μπορεί να θεωρηθεί ως η άλγεβρα που έχει ως γεννήτορες τα στοιχεία*

$$\ell_1, \ldots \ell_{n-1}, t_1, \ldots, t_n$$

και τις παρακάτω σχέσεις:

$$\begin{aligned}
t_i^d &= 1, \quad \text{για κάθε } i & (16)\\
t_i t_j &= t_j t_i, \quad \text{για κάθε } i, j & (17)\\
\ell_i t_j &= t_j \ell_i, \quad \text{όπου } j \neq i \text{ και } j \neq i+1 & (18)\\
\ell_i t_i &= t_{i+1}\ell_i + \frac{1}{u+1}(t_i - t_{i+1}) & (19)\\
\ell_i t_{i+1} &= t_i \ell_i + \frac{1}{u+1}(t_{i+1} - t_i) & (20)\\
\ell_i^2 &= \frac{(u-1)e_i + 2}{u+1}\,\ell_i & (21)\\
\ell_i \ell_j &= \ell_j \ell_i, \quad |i-j| > 1 & (22)\\
\ell_i \ell_{i\pm1} \ell_i &= \frac{(u-1)e_i + 1}{(u+1)^2}\,\ell_i & (23)
\end{aligned}$$

Στη συνέχεια, δείχνουμε πως κάθε στοιχείο στην $\mathrm{YTL}_{d,n}(u)$ μπορεί να γραφεί ως γραμμικό συνδυασμός μονονύμων, καθένα από τα οποία περιέχει το μέγιστο και τον ελάχιστο γεννήτορα το πολύ μια φορά και βρήκαμε ένα παράγωγο σύνολο για την άλγεβρα $\mathrm{YTL}_{d,n}(u)$. Πιο συγκεκριμένα, μια λέξη στην $\mathrm{YTL}_{d,n}(u)$ λέγεται *ανηγμένη* όταν έχει το ελάχιστο μήκος ως προς τις σχέσεις (7)–(15). Έχουμε λοιπόν την ακόλουθη πρόταση:

Πρόταση 2. *Το σύνολο των ανηγμένων λέξεων*

$$\Sigma_{d,n} \;=\; \left\{ t^a (g_{i_1} g_{i_1-1} \ldots g_{i_1-k_1})(g_{i_2} g_{i_2-1} \ldots g_{i_2-k_2}) \ldots (g_{i_p} g_{i_p-1} \ldots g_{i_p-k_p}) \right\}, \quad (24)$$

όπου

$$t^a = t_1^{a_1} \ldots t_n^{a_n} \in C_d^n, \quad 1 \leq i_1 < i_2 < \ldots < i_p \leq n-1,$$

και

$$1 \leq i_1 - k_1 < i_2 - k_2 < \ldots < i_p - k_p,$$

παράγει γραμμικά την άλγεβρα Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$. *Ο μέγιστος γεννήτορας είναι ο* g_{i_p} *του δεξιότερου κύκλου και ο ελάχιστος γεννήτορας είναι ο* $g_{i_1-k_1}$ *και βρίσκεται στον πιο αριστερό κύκλο του* $\Sigma_{d,n}$.

Οι Χλουβεράκη και Pouchin χρησιμοποιώντας εργαλεία της Θεωρίας Αναπαραστάσεων κατάφεραν στο [3] να υπολογίσουν τη διάσταση της άλγεβρας $\mathrm{YTL}_{d,n}(u)$:

Πρόταση 3. *Η διάσταση της άλγεβρας Yokonuma–Temperley–Lieb είναι:*

$$\dim(\mathrm{YTL}_{d,n}(u)) = dc_n + \frac{d(d-1)}{2}\sum_{k=1}^{n-1}\binom{n}{k}^2,$$

όπου c_n είναι ο n–οστός αριθμός Catalan.

Επίσης από το σύνολο $\Sigma_{d,n}$ κατάφεραν να εξάγουν μια γραμμική βάση για κάθε d και κάθε n. Χρησιμοποιώντας την Πρόταση 3 καταφέραμε να βρούμε μια γραμμική βάση για την περίπτωση $d = 2$ και $n = 3$, διαφορετική από αυτή των Χλουβεράκη και Pouchin.

Πρόταση 4. *Το σύνολο*

$$\begin{aligned}\mathcal{S}_{2,3} = \{&1, t_1, t_2, t_1t_2, g_1, t_2g_1, t_3g_1, t_2t_3g_1, g_2, t_1g_2, t_3g_2, t_1t_3g_2,\\ &g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,\\ &g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\}.\end{aligned}$$

αποτελεί μια γραμμική βάση για την άλγεβρα $\mathrm{YTL}_{2,3}(u)$.

iii.β. Ένα γραμμικό ίχνος Markov πάνω στην άλγεβρα $\mathrm{YTL}_{d,n}(u)$

Το επόμενο ερώτημα που προκύπτει είναι εάν και υπό ποιές συνθήκες η απεικόνιση ίχνους tr του Juyumaya που ορίστηκε πάνω στην άλγεβρα $\mathrm{Y}_{\mathrm{d,n}}(\mathrm{u})$ περνά στην άλγεβρα πηλίκο $\mathrm{YTL}_{d,n}(u)$. Κατ' αναλογία με την περίπτωση της κλασικής άλγεβρας Temperley–Lieb, το tr θα πρέπει να μηδενίζει το γεννήτορα του ιδεώδους I_1 και κατ' επέκταση κάθε στοιχείο που βρίσκεται μέσα σ' αυτό. Έτσι λοιπόν προκύπτει πως ο γεννήτορας του ιδεώδους I_1, το στοιχείο $g_{1,2}$ μηδενίζεται για τις ακόλουθες τιμές της παραμέτρου z του tr:

$$z_{\pm} = \frac{-\left((u-1)E+3\right) \pm \sqrt{\left((u-1)E+3\right)^2 - 4(u+1)}}{2(u+1)}, \tag{25}$$

όπου $E := \mathrm{tr}(e_i)$ για κάθε i. Επιπλέον για $m \in \mathbb{Z}/d\mathbb{Z}$ συμβολίζουμε με $E^{(m)} := \mathrm{tr}(e_i^{(m)})$, για κάθε i. Θέλοντας να εξετάσουμε αν αυτές οι τιμές της παραμέτρου z της απεικόνισης tr μηδενίζουν κάθε στοιχείο του I_1, δουλέψαμε χρησιμοποιώντας επαγωγή στο n. Πιο αναλυτικά, για $n = 3$, υπολογίσαμε το ίχνος κάθε στοιχείου του I_1. Αυτά είναι της μορφής $\mathfrak{m}g_{1,2}$, όπου το $\mathfrak{m}$ βρίσκεται στην κανονική βάση της $\mathrm{Y}_{d,3}(u)$. Έτσι προκύπτει ένα σύστημα πολυωνυμικών εξισώσεων Σ:

$$(\Sigma)\begin{cases} Z_{a,b,c} = 0 & (26\alpha')\\ V_{a,b+c} = 0 & (26\beta')\\ W_{a,b,c} = 0, & (26\gamma')\end{cases}$$

όπου:

$$\begin{aligned}
Z_{a,b,c} &:= (u+1)z^2x_{a+b+c} + \left((u-1)E^{(a+b+c)} + x_ax_{b+c} + x_bx_{a+c} + x_cx_{a+b}\right)z + x_ax_bx_c\\
V_{a,b+c} &:= (u+1)z^2x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\,x_ax_{b+c} + x_aE^{(b+c)}\\
W_{a,b,c} &:= (u+1)z^2x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}\left(e_1^{(a+b+c)}e_2\right).
\end{aligned}$$

Στη συνέχεια, απαιτήσαμε το (Σ) να έχει λύσεις τις τιμές $z_\pm$ της (25). Διακρίνουμε δύο περιπτώσεις: Είτε το σύστημα των εξισώσεων έχει και τις δύο τιμές $z_\pm$ ως λύσεις, είτε τουλάχιστον μια απο αυτές. Στην πρώτη περίπτωση έχουμε μόνο αναγκαίες συνθήκες για να περνάει το tr στην άλγεβρα πηλίκο. Πράγματι:

Θεώρημα 2. *Για $n \geq 3$, το ίχνος* tr *που ορίζεται στην* $\mathrm{Y}_{d,n}(u)$ *περνάει στην άλγεβρα πηλίκο* $\mathrm{YTL}_{d,n}(u)$ *αν οι παράμετροι x_i είναι d–ρίζες της μονάδας ($x_i = x_1^i$, $1 \leq i \leq d-1$) και $z = -\frac{1}{u+1}$ ή $z = -1$.*

Στην περίπτωση που τα x_i είναι d–ρίζες της μονάδας προκύπτει ότι $E = 1$. Αντικαθιστώντας στην (25) προκύπτουν οι τιμές για το z του Θεωρήματος 2.

Στη δεύτερη περίπτωση ικανές και αναγκαίες συνθήκες προκύπτουν έτσι ώστε το tr να περνά στην άλγεβρα πηλίκο. Πριν προχωρήσουμε στη διατύπωση του Θεωρήματος, εισάγουμε τον ακόλουθο συμβολισμο: $\exp_s(k) := \exp(2i\pi sk/d)$, όπου $0 \leq k \leq d-1$.

Θεώρημα 3. *Το ίχνος* tr *περνά στην άλγεβρα πηλίκο* $\mathrm{YTL}_{d,n}(u)$ *αν και μόνο αν x_i είναι λύσεις του* Ε*–συστήματος και μια από τις δύο περιπτώσεις ισχύουν:*

(*i*) *Για κάποιο $0 \leq m_1 \leq d-1$ τα x_ℓ εκφράζονται ως:*

$$x_\ell = \exp_{m_1}(\ell) \quad (0 \leq \ell \leq d-1).$$

Σ' αυτή την περίπτωση τα x_ℓ είναι d–ρίζες της μονάδας και $z = -\frac{1}{u+1}$ ή $z = -1$.

(*ii*) *Για κάποια $0 \leq m_1, m_2 \leq d-1$ τα x_ℓ εκφράζονται ως:*

$$x_\ell = \frac{1}{2}\left(\exp_{m_1}(\ell) + \exp_{m_2}(\ell)\right) \quad (0 \leq \ell \leq d-1).$$

Σ' αυτή την περίπτωση έχουμε $z = -\frac{1}{2}$.

Απόδειξη. Παρατηρούμε αρχικά ότι τα x_ℓ που εμφανίζονται στο (*i*) είναι πράγματι λύσεις του συστήματος (Σ). Υποθέτουμε τώρα πως οι λύσεις μας δεν είναι αυτής της μορφής. Από αυτό έπεται ότι $x_a \neq E^{(a)}$ για κάποιο $0 \leq a \leq d-1$.

Θα χρησιμοποιήσουμε επαγωγή στο n. Θα δείξουμε αρχικά την περίπτωση $n = 3$. Υποθέτουμε ότι το ίχνος tr περνά στην άλγεβρα πηλίκο $\mathrm{YTL}_{d,3}(u)$. Αυτό σημαίνει ότι το σύστημα (Σ) έχει λύσεις για το z οποιαδήποτε από αυτές που δίδονται από τις εξισώσεις 25, για οποιοδήποτε $a, b, c \in \mathbb{Z}/d\mathbb{Z}$. Αφαιρώντας την εξίσωση (26α΄) από την (26β΄) έχουμε ότι:

$$\left(x_a x_{b+c} + x_b x_{a+c} - 2E^{(a+b+c)}\right) z = -\left(x_a x_b x_c - x_c E^{(a+b)}\right). \tag{27}$$

Επιλέγοντας $b = c = 0$ στην εξίσωση (27) και εφόσον υποθέσαμε ότι υπάρχει ένα a τέτοιο ώστε $x_a \neq E^{(a)}$, έχουμε ότι: $z = -\frac{1}{2}$. Από την άλλη, αφαιρώντας τις εξισώσεις (26α΄) και από την εξίσωση (26γ΄) έχουμε ότι:

$$\left(3E^{(a+b+c)} - x_a x_{b+c} - x_b x_{a+c} - x_c x_{a+b}\right) z = x_a x_b x_c + x_c - \mathrm{tr}(e_1^{(a+b+c)} e_2). \tag{28}$$

Για την τιμή a για την οποία ισχύει $x_a - E^{(a)} \neq 0$

και θέτοντας $b = c = 0$ στην εξίσωση 28 έχουμε ότι:

$$z = -\frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}. \tag{29}$$

Συνδυάζοντας τώρα τις εξισώσεις (27) και (29) έχουμε ότι:

$$\frac{1}{2} = \frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}$$

ή ισοδύναμα:

$$3(x_a - E^{(a)}) = 2(x_a - \mathrm{tr}(e_1^{(a)} e_2)).$$

Χρησιμοποιώντας το Λήμμα 1.4, τα παραπάνω είναι ισοδύναμα με:

$$3x - \frac{3}{d} x * x = 2x - \frac{2}{d^2} x * x * x.$$

Παίρνοντας το μετασχηματισμό Fourier (βλ. Λήμμα 1.5) καταλήγουμε στην ακόλουθη:

$$\frac{2}{d^2}\widehat{x}^3 - \frac{3}{d}\widehat{x}^2 + \widehat{x} = 0.$$

Υποθέτοντας τώρα ότι $\widehat{x} = \sum_{0 \le \ell \le d-1} y_\ell t^\ell$ έχουμε την ακόλουθη έκφραση για τους συντελεστές y_ℓ στο ανάπτυγμα της $\widehat{x}$:

$$y_\ell \left(\frac{2}{d^2} y_\ell^2 - \frac{3}{d} y_\ell + 1 \right) = 0.$$

Οπότε είτε $y_\ell = 0$ είτε $y_\ell = d$ είτε $y_\ell = \frac{1}{2}d$. Επομένως, αν πάρουμε τη διαμέριση του συνόλου $\{\ell : 0 \le \ell \le d-1\}$ στα σύνολα S_0, S_1, $S_{\frac{1}{2}}$ έτσι ώστε τα y_ℓ να παίρνουν την τιμή $i \cdot d$ στο σύνολο S_i ($i = 0, 1, \frac{1}{2}$), από το Λήμμα 1.5 έχουμε ότι:

$$x = \sum_{m \in S_1} \mathbf{i}_{-m} + \frac{1}{2} \sum_{m \in S_{\frac{1}{2}}} \mathbf{i}_{-m}.$$

Από τη συνθήκη $x_0 = 1$ προκύπτουν και οι ακόλουθες συνθήκες :

$$1 = x(0) = |S_1| + \frac{1}{2}|S_{\frac{1}{2}}|.$$

Αυτό σημάνει πως είτε το σύνολο S_1 είναι μονοσύνολο και $S_{\frac{1}{2}} = \emptyset$ είτε ότι το $S_1 = \emptyset$ και το σύνολο $S_{\frac{1}{2}}$ έχει δύο στοιχεία. Η πρώτη περίπτωση είναι αντιστοιχή της περίπτωσης (i) όπου τα x_ℓ είναι d ρίζες τις μονάδας. Στη δεύτερη περίπτωση, αν $S_{\frac{1}{2}} = \{m_1, m_2\}$ τότε παίρνουμε την ακόλουθη λύση του E–συστήματος:

$$x_\ell = \frac{1}{2}\left(\exp_{m_1}(\ell) + \exp_{m_2}(\ell)\right), \quad (0 \le \ell \le d-1) \tag{30}$$

που αντιστοιχεί στην τιμή $z = -\frac{1}{2}$.

Μπορούμε πλέον να επαληθεύσουμε το γεγονός ότι αυτές οι λύσεις ικανοποιούν το σύστημα (Σ). Εφόσον $z = -\frac{1}{2}$ και $E = \frac{1}{2}$, έχουμε ότι $E^{(\ell)} = x_\ell/2$, $V_{c,a+b} = W_{a,b,c} = 0$, και επίσης πως η εξίσωση (26α′) ισούται με :

$$x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b} = x_{a+b+c} + 2x_a x_b x_c,$$

η οποία ικανοποείται από τις τιμές των x_ℓ που δίνονται στην (30). Συνεχίζουμε με την επαγωγή στο n. Υποθέτουμε ότι η υπόθεση ισχύει για όλες τις άλγεβρες πηλίκα $\mathrm{YTL}_{d,k}(u)$, όπου $k \leq n$, δηλαδή:

$$\mathrm{tr}(a_k \, g_{1,2}) = 0$$

για κάθε $a_k \in \mathrm{Y}_{d,k}(u)$, $k \leq n$. Θα αποδείξουμε την υπόθεση για $k = n+1$. Αρκεί να απόδείξουμε ότι η απεικόνιση ίχνους είναι ίση με μηδέν για οποιοδήποτε στοιχείο βρίσκεται στη μορφή $a_{n+1}g_{1,2}$, όπου το a_{n+1} ανήκει στην επαγωγική βάση της $\mathrm{Y}_{d,n+1}(u)$, δεδομένων των συνθηκών του Θεωρήματος. Αυτό σημαίνει ότι:

$$\mathrm{tr}(a_{n+1} \, g_{1,2}) = 0.$$

Εφόσον το στοιχείο a_{n+1} βρίσκεται στην επαγωγική βάση της $\mathrm{Y}_{d,n+1}(u)$, τότε θα έχει μια από τις ακόλουθες μορφές:

$$a_{n+1} = a_n g_n \ldots g_i t_i^k \quad ή \quad a_{n+1} = a_n t_{n+1}^k,$$

όπου το a_n βρίσκεται στην επαγωγική βάση της $\mathrm{Y}_{d,n}(u)$. Για την πρώτη περίπτωση έχουμε ότι:

$$\mathrm{tr}(a_{n+1} \, g_{1,2}) = \mathrm{tr}(a_n g_n \ldots g_i t_i^k \, g_{1,2}) = z \, \mathrm{tr}(a_n g_{n-1} \ldots g_i t_i^k \, r_{1,2}) = z \, \mathrm{tr}(\tilde{a} g_{1,2}),$$

και το αποτέλεσμα προκύπτει μέσω επαγωγής. Η δεύτερη περίπτωση αποδεικνύεται με ανάλογο τρόπο.

$\square$

iii.γ. Αναλλοίωτες κόμβων από τις άλγεβρες $\mathrm{YTL}_{d,n}(u)$

Δεδομένης μια λύσης $X_S = \{x_1, \ldots, x_{d-1}\}$ του E–συστήματος, η οποία παραμετροποιείται από το μη–κενό υποσύνολο S της $\mathbb{Z}/d\mathbb{Z}$, οι Juyumaya και Λαμπροπούλου στο [23] όρισαν την ακόλουθη απεικόνιση, για κάθε $\alpha \in \cup_\infty \mathcal{F}_n$:

$$\Gamma_{d,S}(w,u)(\hat{\alpha}) = \left(-\frac{(1-wu)}{\sqrt{w}(1-u)E} \right)^{n-1} \left(\sqrt{w}\right)^{\varepsilon(\alpha)} \mathrm{tr}\left(\gamma(\alpha)\right), \tag{31}$$

όπου $w = \frac{z+(1-u)E}{uz}$, γ ο φυσικός επιμορφισμός της άλγεβρας που αντιστοιχεί στην ομάδα των πλαισιωμένων πλεξίδων $\mathbb{C}\mathcal{F}_n$ επί της άλγεβρας $\mathrm{Y}_{d,n}(u)$. Όπως απέδειξαν, η απεικόνιση $\Gamma_{d,S}(w,u)$ είναι τοπολογική αναλλοίωτη πλαισιωμένων κρίκων.

Επιπλέον, οι Juyumaya και Λαμπροπούλου περιορίζοντας την $\Gamma_{d,S}(w,u)$ στην περίπτωση των κλασσικών πλεξίδων, οι οποίες μπορούν να θεωρηθούν ως πλαισιωμένες πλεξίδες με μηδενικό framing, όρισαν μια αναλλοίωτη για κλασικούς κρίκους, τη $\Delta_{d,S}(w,u)$. Στο [21] η αναλλοίωτη $\Delta_{d,S}(w,u)$ επεκτάθηκε σε μια αναλλοίωτη για singular κρίκους.

Στο [2] αποδεικνύεται πως για γενικές τιμές των παραμέτρων u, z οι αναλλοίωτες $\Delta_{d,S}(w,u)$ δεν συμπίπτουν με το πολυώνυμο Homflypt, εκτός για τις τετριμμένες περιπτώσεις $u = 1$ ή $E = 1$. Παρ' όλα αυτά όμως, τα υπολογιστικά δεδομένα [5] δείχνουν πως αυτές οι αναλλοίωτες δεν ξεχωρίζουν ζευγάρια κόμβων τα οποία το πολυώνυμο Homflypt επίσης δεν ξεχωρίζει, επομένως μπορεί να είναι τοπολογικά ισοδύναμες με το πολυώνυμο Homflypt.

Από τις προϋποθέσεις του Θεωρήματος 3 απορρίπτουμε τις περιπτώσεις όπου: $z = -1$ (και τα x_i είναι d–ρίζες της μονάδας) και $z = -\frac{1}{2}$ (και $x_\ell = \frac{1}{2}\left(\exp(\ell m_1) + \exp(\ell m_2)\right)$) καθώς σημαντική τοπολογική πληροφορία χάνεται. Πράγματι, για παράδειγμα, το ίχνος tr δίνει την ίδια τιμή για όλες τις άρτιες (αντιστ. περιττές) δυνάμεις των g_i, για $m \in \mathbb{Z}^{>0}$ [23]. Οπότε οι (διαφορετικοί) κρίκοι που αντιστοιχούν στις παραπάνω δυνάμεις των g_i θα παίρνουν την ίδια τιμή της αναλλοίωτης $\Gamma_{d,S}(w,u)$. Επομένως, από τις τιμές του Θεωρήματος 3 που απομένουν προκύπτει ότι $E = 1$ και $w = u$. Έχουμε λοιπόν το ακόλουθο πόρισμα:

Πόρισμα 1. *Οι αναλλοίωτες $V_{d,S}(u) := \Delta_{d,S}(u,u)$ ταυτίζονται με το πολυώνυμο Jones.*

iv. Η άλγεβρα $\mathrm{FTL}_{d,n}(u)$

Όπως αναφέραμε και προηγουμένως, υπάρχουν τρεις πιθανοί υποψήφιοι για τον ορισμό του framization της άλγεβρας Temperley–Lieb. Η δεύτερη άλγεβρα που ορίζουμε είναι η άλγεβρα πηλίκο $\mathrm{FTL}_{d,n}(u)$. Πιο συγκεκριμένα, ορίζουμε αυτήν την άλγεβρα ως το πηλίκο της άλγεβρας Yokonuma–Hecke πάνω από ένα αμφίπλευρο ιδεώδες που κατασκευάζεται από την ακόλουθη υποομάδα της ομάδας $C_{d,n} := C_d^n \rtimes S_n$, όπου $C_d = \langle t \mid t^d = 1 \rangle$ η κυκλική ομάδα τάξης d και $t_i = (1, \ldots 1, t, 1, \ldots, 1)$:

$$H_{i,j} := \langle t_i t_{i+1}^{-1}, t_j t_{j+1}^{-1} \rangle \rtimes \langle s_i, s_j \rangle \quad \text{με} \quad |i-j| = 1.$$

Σημειώνουμε ότι, για $j = i+1$, έχουμε ότι κάθε $x \in H_{i,i+1}$ μπορεί να γραφεί στη μορφή:

$$x = t_i^\alpha t_{i+1}^\beta t_{i+2}^\gamma w, \tag{32}$$

όπου $\alpha + \beta + \gamma = 0$ και $w \in \langle s_i, s_{i+1} \rangle$. Έχουμε τον ακόλουθο ορισμό:

Ορισμός 2. Για $n \geq 3$, το *Framization της άλγεβρας Temperley–Lieb*, $\mathrm{FTL}_{d,n}(u)$, ορίζεται ως το πηλίκο της άλγεβρας $\mathrm{Y}_{d,n}(u)$ πάνω από το αμφίπλευρο ιδεώδες J που παράγεται από τα στοιχεία:

$$r_{i,i+1} := \sum_{x \in H_{i,i+1}} g_x = \sum_{\substack{\alpha+\beta+\gamma=0 \\ w \in \langle s_i, s_{i+1} \rangle}} t_i^\alpha t_{i+1}^\beta t_{i+2}^\gamma \, g_w \qquad (i = 1, \ldots, n-2). \tag{33}$$

Ομοίως, όπως και στις περιπτώσεις των αλγεβρών Temperley–Lieb και $\mathrm{YTL}_{d,n}(u)$, αποδεικνύεται ότι το ιδεώδες J είναι κύριο και παράγεται από το στοιχείο $r_{1,2}$:

Θεώρημα 4. *Η άλγεβρα $\mathrm{FTL}_{d,n}(u)$ είναι το πηλίκο της άλγεβρας $\mathrm{Y}_{d,n}(u)$ πάνω από το αμφίπλευρο ιδεώδες J που παράγεται από το στοιχείο:*

$$r_{1,2} = \sum_{x \in H_{1,2}} g_x = \sum_{\alpha+\beta+\gamma=0} t_1^\alpha t_2^\beta t_3^\gamma g_{1,2}.$$

Η άλγεβρα $\mathrm{FTL}_{d,n}(u)$ μπορεί να παρασταθεί από τους γεννήτορες $t_1, \ldots, t_n, g_1, \ldots, g_{n-1}$

και τις ακόλουθες σχέσεις μεταξύ αυτών:

$$
\begin{aligned}
g_i g_j &= g_j g_i, \quad |i-j|>1 \\
g_{i+1} g_i g_{i+1} &= g_i g_{i+1} g_i \\
g_i^2 &= 1 + (u-1)e_i + (u-1)e_i g_i \\
t_i t_j &= t_j t_i, \quad \text{για κάθε } i, j \\
t_i^d &= 1, \quad \text{για κάθε } i \\
g_i t_i &= t_{i+1} g_i \\
g_i t_{i+1} &= t_i g_i \\
g_i t_j &= t_j g_i, \quad \text{όπου } j \neq i, \text{ και } j \neq i+1 \\
\sum_{\substack{\alpha+\beta+\gamma=0 \\ w \in \langle s_i, s_{i+1}\rangle}} t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma}\, g_w &= 0 \qquad (i = 1, \ldots, n-2).
\end{aligned}
$$

Όπως απέδειξαν οι Χλουβεράκη και Pouchin [3], η διάσταση της άλγεβρας $\mathrm{FTL}_{d,n}(u)$ είναι :

$$
\dim \mathrm{FTL}_{d,n}(u) = \sum_{|k_1|+|k_2|+\ldots+|k_d|=n} \left(\frac{n!}{k_1! \ldots k_d!} \right)^2 c_{k_1} \ldots c_{k_d},
$$

όπου c_n είναι ο n–οστός αριθμός Catalan.

Η άλγεβρα $\mathrm{FTL}_{d,n}(u)$ μπορεί να παρασταθεί και με μη-αντιστρέψιμους γεννήτορες. Πράγματι, έχουμε την ακόλουθη πρόταση:

Πρόταση 5. *Η άλγεβρα $\mathrm{FTL}_{d,n}(u)$ μπορεί να παρασταθεί με τους γεννήτορες:*

$$
\ell_1, \ldots, \ell_{n-1}, t_1, \ldots, t_n
$$

οι οποίοι ικανοποιούν τις σχέσεις (16) – (22) καθώς και τις ακόλουθες δύο σχέσεις:

$$
\ell_i \ell_{i+1} \ell_i - \frac{(u-1)e_i + 1}{(u+1)^2}\, \ell_i = \ell_{i+1} \ell_i \ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\, \ell_{i+1}
$$

$$
e_i e_{i+1} \ell_i \ell_{i+1} \ell_i = \frac{u}{(u+1)^2}\, e_i e_{i+1} \ell_i
$$

iv.α. Ένα γραμμικό ίχνος Markov πάνω στην άλγεβρα $\mathrm{FTL}_{d,n}(u)$

Ακολουθώντας το ίδιο σκεπτικό όπως και στην περίπτωση της άλγεβρας $\mathrm{YTL}_{d,n}(u)$, υπολογίσαμε τις συνθήκες υπό τις οποίες το ίχνος tr της άλγεβρας $\mathrm{Y}_{d,n}(u)$ περνά στην άλγεβρα πηλίκο. Σε αυτήν την περίπτωση ο γεννήτορας του ιδεώδους J μηδενίζεται για τις ακόλουθες τιμές της παραμέτρου z:

$$
z_{\pm} := \frac{-(u+2)E \pm \sqrt{(u+2)^2 E^2 - 4(u+1)A}}{2(u+1)} \tag{34}
$$

όπου $A := \mathrm{tr}(e_1 e_2)$. Αυτή τη φορά έχουμε το ακόλουθο γραμμικό σύστημα d εξισώσεων:

$$
(\Sigma) \begin{cases} (u+1)z^2 x_0 + (u+2)E^{(0)} z + \mathrm{tr}(e_1^{(0)} e_2) = 0 \\ (u+1)z^2 x_l + (u+2)E^{(l)} z + \mathrm{tr}(e_1^{(l)} e_2) = 0 \quad (1 \leq l \leq d-1), \end{cases}
$$

το οποίο απαιτούμε να έχει λύσεις ως προς z τις τιμες της (34). Αν το σύστημα έχει και τις δύο τιμές του z ως κοινές λύσεις τότε έχουμε αναγκαίες συνθήκες ώστε το tr να περνά στην άλγεβρα πηλίκο:

Θεώρημα 5. *Για $n \geq 3$, το γραμμικό ίχνος tr της $\mathrm{Y}_{d,n}(u)$ περνά στην άλγεβρα πηλίκο $\mathrm{FTL}_{d,n}(u)$ αν οι παράμετροι $x_1, \ldots, x_{d-1}$ αποτελούν λύσεις του E–συστήματος και το παίρνει z μια από τις ακόλουθες τιμές:*

$$z_{S,+} = -\frac{1}{u+1}E \quad ή \quad z_{S,-} = -E.$$

Αν το σύστημα έχει τουλάχιστον μια από τις τιμές της (34) ως κοινή λύση τότε έχουμε ικανές και αναγκαίες συνθήκες ώστε το tr να περνά στην άλγεβρα πηλίκο:

Θεώρημα 6. *Το γραμμικό ίχνος tr της $\mathrm{Y}_{d,n}(u)$ περνά στην άλγεβρα πηλίκο $\mathrm{FTL}_{d,n}(u)$ αν και μόνο αν οι παράμετροι x_i είναι της μορφής:*

$$x = -z\left(\sum_{s\in S_1}\mathbf{i}_{-s} + (u+1)\sum_{s\in S_2}\mathbf{i}_{-s}\right),$$

όπου x είναι η μιγαδική συνάρτηση πάνω από το $\mathbb{Z}/d\mathbb{Z}$, που στέλνει το 0 στο 1 και το k στην παράμετρο x_k της συνάρτησης ίχνους, ενώ $\mathrm{Sup}_1 \cup \mathrm{Sup}_2$ (ξένη ένωση) είναι ο φορέας (support) του μετασχηματισμού Fourier της συνάρτησης x.

Απόδειξη. Με επαγωγή στο n. Αρχικά θα αποδείξουμε την περίπτωση $n = 3$. Θέλουμε να επιλύσουμε το ακόλουθο σύστημα εξισώσεων:

$$(u+1)z^2x_\ell + (u+2)zE^{(\ell)} + \mathrm{tr}(e_1^{(\ell)}e_2) = 0, \quad \text{για κάθε } 0 \leq \ell \leq d-1.$$

Αφαιρώντας την πρώτη εξίσωση από τις υπόλοιπες προκύπτουν οι ακόλουθες:

$$z(u+2)\big(E^{(\ell)} - x_lE\big) = -\Big(\mathrm{tr}(e_1^{(\ell)}e_2) - x_\ell\,\mathrm{tr}(e_1e_2)\Big) \text{ για κάθε} 0 \leq l \leq d-1. \quad (36)$$

Υποθέτουμε πως τα x_i δεν αποτελούν λύση του E–συστήματος. Σε αυτή την περίπτωση θα λύσουμε το σύστημα που δίνεται από τις εξισώσεις (36). Αν τα παραπάνω τα δούμε μέσα στην άλγεβρα $\mathbb{C}[\mathbb{Z}/d\mathbb{Z}]$ τότε τα παραπάνω παίρνουν την παρακάτω μορφή:

$$(u+2)z\sum_{0\leq l\leq d-1}\left(E^{(\ell)} - x_\ell E\right)t^\ell = -\sum_{0\leq \ell\leq d-1}\Big(\mathrm{tr}(e_1^{(\ell)}e_2 + x_\ell\,\mathrm{tr}(e_1e_2))\Big)t^\ell$$

Χρησιμοποιώντας τον συναρτησιακό συμβολισμό της Παραγράφου 1.11.1 και έχοντας υπόψη το Λήμμα 1.4, έπεται ότι οι εξισώσεις (36) μπορούν να γραφούν ως:

$$(u+2)z\left(\frac{1}{d}x * x - Ex\right) = -\left(\frac{1}{d^2}x * x * x - \mathrm{tr}(e_1e_2)x\right)$$

εφαρμόζοντας τώρα τον μετασχηματισμό Fourier στην παραπάνω ισότητα συναρτήσεων προκύπτει ότι:

$$(u+2)z\left(\frac{\widehat{x}^2}{d} - E\widehat{x}\right) = -\left(\frac{\widehat{x}^3}{d^2} - \mathrm{tr}(e_1e_2)\widehat{x}\right) \quad (37)$$

Έστω τώρα $\widehat{x} = \sum_{m=0}^{d-1} y_mt^m$. Τότε η εξίσωση (37) μετατρέπεται στην ακόλουθη:

$$(u+2)z\left(\frac{y_m^2}{d} - Ey_m\right) = -\left(\frac{y_m^3}{d^2} - \mathrm{tr}(e_1e_2)y_m\right)$$

Επομένως:

$$y_m\left(\frac{y_m^2}{d^2}+(u+2)z\frac{y_m}{d}-(u+2)zE-\mathrm{tr}(e_1e_2)\right)=0 \tag{38}$$

Από την εξίσωση $\mathbb{E}_0 = 0$, τώρα, έχουμε ότι $-(u+2)zE = (u+1)z^2 + \mathrm{tr}(e_1e_2)$. Αντικαθιστώντας την παραπάνω έκφραση για το $-(u+2)zE$ στην εξίσωση (38) έχουμε ότι:

$$y_m\left(\frac{y_m^2}{d^2}+(u+2)z\frac{y_m}{d}+(u+1)z^2\right)=0$$

ή ισοδύναμα:

$$y_m\left(y_m+dz\right)\left(y_m+dz(u+1)\right)=0 \tag{39}$$

Συμβολίζουμε με $\mathrm{Sup}_1\cup\mathrm{Sup}_2$ το φορέα της $\widehat{x}$, όπου:

$$\mathrm{Sup}_1 := \{m\in\mathbb{Z}/d\mathbb{Z}\,;\, y_m=-dz\} \quad \text{και} \quad \mathrm{Sup}_2 := \{m\in\mathbb{Z}/d\mathbb{Z}\,;\, y_m=-dz(u+1)\}$$

κι έτσι έχουμε ότι:

$$\widehat{x}=\sum_{m\in\mathrm{Sup}_1}-dzt^m+\sum_{m\in\mathrm{Sup}_2}-dz(u+1)t^m$$

Επομένως:

$$\widehat{\widehat{x}}=-dz\sum_{m\in\mathrm{Sup}_1}\widehat{\delta}_m-dz(u+1)\sum_{m\in\mathrm{Sup}_2}\widehat{\delta}_m$$

και άρα από την Πρόταση 1.5 προκύπτει ότι:

$$\widehat{\widehat{x}}=-z\left(\sum_{m\in\mathrm{Sup}_1}\mathbf{i}_{-m}+(u+1)\sum_{m\in\mathrm{Sup}_2}\mathbf{i}_{-m}\right)$$

Χρησιμοποιώντας την Πρόταση 1.5, συμπεραίνουμε ότι:

$$x_k=-z\left(\sum_{m\in\mathrm{Sup}_1}\chi(km)+(u+1)\sum_{m\in\mathrm{Sup}_2}\chi(km)\right) \tag{40}$$

Έχοντας υπόψη τη συνθήκη $x_0 = 1$, μπορούμε τώρα να υπολογίσουμε τις τιμές για την παράμετρο z. Πράγματι, από την εξίσωση (40), έχουμε:

$$1=x_0=-z(|\mathrm{Sup}_1|+(u+1)|\mathrm{Sup}_2|)$$

ή ισοδύναμα:

$$z=-\frac{1}{|\mathrm{Sup}_1|+(u+1)|\mathrm{Sup}_2|}. \tag{41}$$

Η υπόλοιπη απόδειξη (η επαγωγή στο n) ακολουθεί την απόδειξη του Θεωρήματος 3.
$\square$

Προτού συζητήσουμε την εφαρμογή του Θεωρήματος 6 πάνω σε τοπολογικές αναλλοίωτες κόμβων θα εισάγουμε και την τρίτη υποψήφια άλγεβρα ως το framization της άλγεβρας Temperley–Lieb.

iv.β. Αναλλοίωτες κόμβων από τις άλγεβρες $\mathrm{FTL}_{d,n}(u)$

Σχετικά με τις άλγεβρες $\mathrm{FTL}_{d,n}(u)$, όπως προκύπτει από το Θεώρημα 6, αν $|S_1| = 0$ ή $|S_2| = 0$ τότε τα x_i αποτελούν λύση του E–συστήματος και $z = -\frac{1}{(u+1)|S|}$ ή $z = -\frac{1}{|S|}$. Από την άλλη, η περίπτωση κατά την οποία $|S_1| \neq 0$ και $|S_2| \neq 0$, απορρίπτεται καθώς τα x_i δεν αποτελούν λύση του E–συστήματος και επομένως δεν πληρείται η ικανή και αναγκαία συνθήκη για τον ορισμό τοπολογικών αναλλοίωτων. Επιπλέον, απορρίπτεται και η περίπτωση $z = -\frac{1}{|S|}$, γιατί όπως και στην περίπτωση της άλγεβρας $\mathrm{YTL}_{d,n}(u)$, σημαντική τοπολογική πληροφορία χάνεται. Για την περίπτωση που απομένει, κατά την οποία τα x_i είναι λύσεις του E–συστήματος και το $z = -\frac{1}{(u+1)|S|}$, προκύπτει ότι $w = u$ από την (31). Οπότε, έχουμε τον ακόλουθο ορισμό:

Ορισμός 3. *Έστω $X_S\{x_1, \ldots, x_{d-1}\}$ μια λύση του E–συστήματος, που παραμετροποιείται από το μη–κενό υποσύνολο S της $\mathbb{Z}/d\mathbb{Z}$, και έστω $z = -\frac{1}{(u+1)|S|}$.*
Από την αναλλοίωτη $\Gamma_{d,S}(w,u)$ προκύπτει η ακόλουθη αναλλοίωτη για πλαισιωμένους κρίκους, όπου $\alpha \in \cup_\infty \mathcal{F}_n$:

$$\vartheta_{d,S}(u)(\hat{\alpha}) := \left(-\frac{(1+u)|S|}{\sqrt{u}}\right)^{n-1} \left(\sqrt{u}\right)^{\varepsilon(\alpha)} \operatorname{tr}\left(\gamma(\alpha)\right) = \Gamma_{d,S}(u,u)(\hat{\alpha}),$$

όπου, γ ο φυσικός επιμορφισμός της άλγεβρας που αντιστοιχεί στην ομάδα των πλαισιωμένων πλεξίδων $\mathbb{C}\mathcal{F}_n$ επί της άλγεβρας $\mathrm{Y}_{d,n}(u)$.

Κατ΄ αναλογία με την περίπτωση της αναλλοίωτης $\Gamma_{d,S}(w,u)$, αν περιοριστούμε στις πλαισιωμένες πλεξίδες με μηδενικό framing, προκύπτει μια αναλλοίωτη για κλασσικούς κόμβους, η οποία συμβολίζεται με: $\theta_S(u) := \Delta_S(u,u)$.

v. Η άλγεβρα $\mathrm{CTL}_{d,n}(u)$

Η τελευταία άλγεβρα πηλίκο που ορίζουμε είναι η άλγεβρα $\mathrm{CTL}_{d,n}(u)$. Πιο συγκεκριμένα, θεωρούμε το ακόλουθο στοιχείο της $\mathrm{Y}_{d,n}(u)$:

$$c_{1,2} := \sum_{w \in C_{d,3}} g_w = \sum_{\alpha,\beta,\gamma \in C_d} t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_{1,2}$$

και το ιδεώδες $I_2 = \langle c_{1,2} \rangle$. Έχουμε λοιπόν τον ακόλουθο ορισμό:

Ορισμός 4. *Για $n \geq 3$, ορίζουμε την άλγεβρα πηλίκο $\mathrm{CTL}_{d,n}(u)$:*

$$\mathrm{CTL}_{d,n}(u) := \frac{\mathrm{Y}_{d,n}(u)}{\langle c_{1,2} \rangle}.$$

Η άλγεβρα $\mathrm{FTL}_{d,n}(u)$ μπορεί να παρασταθεί από τους γεννήτορες $t_1, \ldots, t_n, g_1, \ldots, g_{n-1}$

και τις ακόλουθες σχέσεις μεταξύ αυτών:

$$\begin{aligned}
g_i g_j &= g_j g_i, \quad |i-j|>1 \\
g_{i+1} g_i g_{i+1} &= g_i g_{i+1} g_i \\
g_i^2 &= 1 + (u-1)e_i + (u-1)e_i g_i \\
t_i t_j &= t_j t_i, \quad \text{για κάθε } i, j \\
t_i^d &= 1, \quad \text{για κάθε } i \\
g_i t_i &= t_{i+1} g_i \\
g_i t_{i+1} &= t_i g_i \\
g_i t_j &= t_j g_i, \quad \text{όπου } j \neq i, \text{ και } j \neq i+1 \\
\sum_{w \in \langle s_i, s_{i+1} \rangle} t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma}\, g_w &= 0 \qquad (i = 1, \ldots, n-2).
\end{aligned}$$

Επιπλέον, υπάρχει παράσταση με μη-αντιστρέψιμους γεννήτορες και για την άλγεβρα $\mathrm{CTL}_{d,n}(u)$. Πράγματι, έχουμε την ακόλουθη πρόταση:

Πρόταση 6. *Η άλγεβρα $\mathrm{CTL}_{d,n}(u)$ μπορεί να παρασταθεί με τους γεννήτορες:*

$$\ell_1, \ldots, \ell_{n-1}, t_1, \ldots, t_n$$

οι οποίοι ικανοποιούν τις σχέσεις (16) – (22) καθώς και τις ακόλουθες δύο σχέσεις:

$$\begin{aligned}
\ell_i \ell_{i+1} \ell_i - \frac{(u-1)e_i + 1}{(u+1)^2}\, \ell_i &= \ell_{i+1} \ell_i \ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\, \ell_{i+1} \\
\sum_{k=0}^{d-1} e_i^{(k)} e_{i+1} \ell_i \ell_{i+1} \ell_i &= \sum_{k=0}^{d-1} e_i^{(k)} e_{i+1} \frac{u}{(u+1)^2}\, \ell_i
\end{aligned}$$

Ακολουθώντας ανάλογες μεθόδους με τις προηγούμενες άλγεβρες έχουμε το ακόλουθο θεώρημα:

Θεώρημα 7. *Το ίχνος* tr *περνά στην άλγεβρα πηλίκο $\mathrm{CTL}_{d,n}(u)$ αν και μόνο αν οι παράμετροι z και x_i σχετίζονται μέσω της εξίσωσης:*

$$(u+1)z^2 \sum_{k \in \mathbb{Z}/d\mathbb{Z}} x_k + (u+2)z \sum_{k \in \mathbb{Z}/d\mathbb{Z}} E^{(k)} + \sum_{k \in \mathbb{Z}} \mathrm{tr}(e_1^{(k)} e_2) = 0.$$

v.α. Τοπολογικές αναλλοίωτες από τις άλγεβρες $\mathrm{CTL}_{d,n}(u)$

Οι συνθήκες του Θεωρήματος 7 δεν εμπριέχουν καμία λύση του E–συστήματος, επομένως για να μπορεί να ορισθεί μια αναλλοίωτη κρίκων στο επίπεδο της άλγεβρας $\mathrm{CTL}_{d,n}(u)$ θα πρέπει να επιβάλλουμε αυτή τη συνθήκη στις παραμέτρους x_i. Οι λύσεις του E–συστήματος μπορούν να εκφραστούν στη μορφή:

$$x_S = \frac{1}{|S|} \sum_{k \in S} \mathbf{i}_k \in \mathbb{C}C_d,$$

όπου $\mathbf{i}_k = \sum_{j=0}^{d-1} \chi_k(j) t^j$, χ_k είναι ο χαρακτήρας που στέλνει $m \mapsto \cos \frac{2\pi k m}{d} + i \sin \frac{2\pi k m}{d}$ και S είναι το μη–κενό υποσύνολο της ομάδας$\mathbb{Z}/d\mathbb{Z}$ που παραμετροποιεί μια λύση του

E–συστήματος. Έστω τώρα ε να είναι η απεικόνιση της άλγεβρας που αντιστοιχεί στην κυκλική ομάδα τάξης d, $\mathbb{C}C_d$, που στέλνει το άθροισμα $\sum_{j=0}^{d-1} x_j t^j$ στο $\sum_{j=0}^{d-1} x_j$. Έχουμε ότι:

$$\varepsilon(x_{_S}) = \frac{1}{|S|}\sum_{k\in S}\varepsilon(\mathbf{i}_k) = \frac{1}{|S|}\sum_{j=0}^{d-1}\sum_{k\in S}\chi_j(k) = \begin{cases} \frac{d}{|S|}, & \text{αν} \quad 0\in S \\ 0 & \text{αν} \quad 0\notin S \end{cases}. \tag{42}$$

Από αυτό προκύπτει ότι:

$$\sum_{j=0}^{d-1} E^{(j)} = \varepsilon\left(\frac{x*x}{d}\right) = \frac{1}{d|S|^2}\sum_{k\in S}\varepsilon(\mathbf{i}_k*\mathbf{i}_k) = \frac{1}{|S|^2}\sum_{k\in S}\varepsilon(\mathbf{i}_k) = \begin{cases} \frac{d}{|D|^2}, & \text{αν} \quad 0\in S \\ 0 & \text{αν} \quad 0\notin S \end{cases} \tag{43}$$

και επίσης έχουμε ότι:

$$\sum_{j=0}^{d-1} \mathrm{tr}(e_1^{(j)}e_2) = \varepsilon\left(\frac{x*x*x}{d^2}\right) = \frac{1}{d^2|S|^3}\sum_{k\in S}\varepsilon(\mathbf{i}_k*\mathbf{i}_k*\mathbf{i}_k) = \frac{1}{|S|^3}\sum_{k\in S}\varepsilon(\mathbf{i}_k) = \begin{cases} \frac{d}{|S|^3}, & \text{αν} \quad 0\in S \\ 0 & \text{αν} \quad 0\notin S \end{cases}. \tag{44}$$

Χρησιμοποιώντας τώρα τις εξισώσεις. (42), (43) και(44), η εξίσωση (7) γίνεται για την περίπτωση που $0\in S$:

$$\frac{d}{|S|}\left((u+1)z^2 + \frac{(u+2)}{|S|}z + \frac{1}{|S|^2}\right) = 0.$$

Επομένως, το ίχνος tr περνά στην άλγεβρα πηλίκο για τις ακόλουθες τιμές του z:

$$z = -\frac{1}{(u+1)|S|} \quad \text{ή} \quad z = -\frac{1}{|S|}.$$

Η τιμή $z = -\frac{1}{|S|}$ απορρίπτεται, εφόσον το ίχνος tr δίνει την ίδια τιμή για όλες τις άρτιες (αντιστ. περιττές) δυνάμεις των g_i. Επομένως, οι αναλλοίωτες που προκύπτουν από το tr στο επίπεδο της άλγεβρας πηλίκο $\mathrm{CTL}_{d,n}(u)$ ταυτίζονται με τις αναλλοίωτες ϑ_D και θ_D στο επίπεδο της άλγεβρας πηλίκο $\mathrm{FTL}_{d,n}(u)$. Πιο συγκεκριμένα, οι συνθήκες που πρέπει να επιβληθούν στις παραμέτρους της απεικόνισης ίχνους είναι ίδιες και στις δύο περιπτώσεις και επομένως οι αναλλοίωτες που προκύπτον ταυτίζοντα.

Επιπλέον, οι λύσεις το E–συστήματος (οι οποίες είναι ικανές και αναγκαίες συνθήκες έτσι ώστε ο ορισμός τοπολογικών αναλλοίωτων για πλαισιωμένες πλεξίδες να είναι εφικτός) εμπεριέχονται στις συνθήκες του Θεωρήματος 5, ενώ στην περίπτωση της άλγεβρας πηλίκο $\mathrm{CTL}_{d,n}(u)$ θα πρέπει να επιβληθούν εφόσον το ίχνος περάσει στην άλγεβρα πηλίκο. Αυτοί είναι και οι κύριοι λόγοι που μας οδηγούν στο να θεωρήσουμε την άλγεβρα πηλίκο $\mathrm{FTL}_{d,n}(u)$ ως το πιο φυσιολογικό ανάλογο της άλγεβρας Temperley–Lieb για την περίπτωση των πλαισιωμένων κρίκων.

vi. Η σχέση μεταξύ των τριών αλγεβρών

Συνεχίζοντας, συγκρίναμε τις άλγεβρες $\mathrm{FTL}_{d,n}(u)$, $\mathrm{YTL}_{d,n}(u)$ και $\mathrm{CTL}_{d,n}(u)$ και αποδείξαμε την ακόλουθη πρόταση:

Πρόταση 7. *Το ακόλουθο διάγραμμα είναι αντιμεταθετικό:*

$$\begin{tikzcd} \mathrm{Y}_{d,n}(u) \arrow[r] \arrow[d] & \mathrm{CTL}_{d,n}(u) \arrow[r] \arrow[d] & \mathrm{FTL}_{d,n}(u) \arrow[r] \arrow[dl] & \mathrm{YTL}_{d,n}(u) \arrow[dll] \\ \mathrm{H}_n(u) \arrow[r] & \mathrm{TL}_n(u) & & \end{tikzcd}$$

vii. Συμπεράσματα

Οι αναλλοίωτες κόμβων από τις άλγεβρες $\mathrm{FTL}_{d,n}(u)$ και $\mathrm{CTL}_{d,n}(u)$ παραμένουν υπό διερεύνηση. Αν οι αναλλοίωτες απο τις άλγεβρες Yokonuma–Hecke αποδειχθούν οτι είναι τοπολογικά ισοδύναμες με το πολυώνυμο Homflypt, τότε οι αναλλοίωτες από τις άλγεβρες πηλίκα $\mathrm{FTL}_{d,n}(u)$ και $\mathrm{CTL}_{d,n}(u)$ θα είναι τοπολογικά ισοδύναμες με το πολυώνυμο Jones. Αν όχι, τότε θα είναι χρήσιμο να μελετηθούν οι αντίστοιχες αναλλοίωτες για 3–πολλαπλότητες (όπως προκύπτουν από την εργασία του Wenzl [36]). Σε αυτή την περίπτωση, από τις άλγεβρες $\mathrm{YTL}_{d,n}(u)$ μπορούμε να ανακτήσουμε τις αναλλοίωτες Witten για 3–πολλαπλότητες, εφόσον οι αντίστοιχες αναλλοίωτες κόμβων ανακτούν το πολυώνυμο Jones [8].

Τέλος μερικά από τα ποιό σημαντικά ερωτήματα που προκύπτουν από αυτή τη διατριβή είναι η διαγραμματική ερμηνεία των παραστάσεων με μη–αντιστρέψιμους γεννήτορες για κάθε μια από τις άλγεβρες πηλίκα και η μελέτη της Θεωρίας Αναπαραστάσεων των αλγεβρών $\mathrm{FTL}_{d,n}(u)$ και $\mathrm{CTL}_{d,n}(u)$.

vii. Δημοσιεύσεις και παρουσιάσεις σε συνέδρια

Τα παραπάνω αποτελέσματα έχουν συγγραφεί με τη μορφή άρθρων με τους ακόλουθους τίτλους:
I. "The Yokonuma–Temperley–Lieb algebra", που θα δημοσιευτεί στο έγκριτο επιστημονικό περιοδικό *Banach Center Publications* **103**.
II. "Framization of the Temperley–Lieb algebra", το οποίο κατατέθηκε προς δημοσίευση σε έγκριτο επιστημονικό περιοδικό.

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This thesis is dedicated to my family and to the
memory of my uncle Yorgos Goudaroulis
(1945 – 1996)

D. Gkountaroulis
Athens, 27 January 2014

Introduction

The Temperley–Lieb algebra appeared originally in Statistical Mechanics and is important in several areas of Mathematics. Since the original construction of the Jones polynomial by V.F.R. Jones in his seminal work [15], where he constructed a unique Markov trace on the Temperley–Lieb algebra, the Temperley–Lieb algebra has become a cornerstone in the fertile interaction of Knot theory and Representation theory. The Temperley–Lieb algebra was introduced by Temperley and Lieb [34] and was rediscovered by Jones [15]. In algebraic terms, the Temperley–Lieb algebra can be defined as a quotient of the Iwahori–Hecke algebra.

The Temperley–Lieb algebra, the Hecke algebra and the BMW algebra are the most important examples of knot algebras. A knot algebra is an algebra that is typically defined by generators and relations including the braiding relations, which can be used in the understanding of the classification of knots.

The framization is a mechanism designed by Juyumaya and Lambropoulou and consists of a generalization of a knot algebra via the addition of framing generators. In this way we obtain a new algebra which is related to framed knots.

More precisely, the framization procedure can roughly be regarded as the procedure of adding framing generators to the generating set of a knot algebra, of defining interacting relations between the framing generators and the original generators of the algebra and of applying framing on the original defining relations of the algebra. The resulting framed relations should be topologically consistent. The most difficult problem in this procedure is to apply the 'framization' on the relations of polynomial type.

The Yokonuma–Hecke algebra can be regarded as the basic example of a framization of the Hecke algebra [20, 24]. This framization of the Hecke algebra gives the recipe of how to apply the framization technique on the quadratic Hecke relation. Further, in [18] the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ (defined originally in [37]) has been defined as a quotient of the modular framed braid group $\mathcal{F}_{d,n}$, which comprises framed braids with framings modulo d, over a quadratic relation (Eq. 1.27) involving the framing generators t_i by means of certain weighted idempotents e_i (Eq. 1.22). Setting $d = 1$, the algebra $\mathrm{Y}_{1,n}(u)$ coincides with the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. The Yokonuma–Hecke algebras have been studied in [37, 18, 23, 33, 4]. Further, in [18] the Juyumaya found an inductive linear basis for the algebras $\mathrm{Y}_{d,n}(u)$ and constructed a unique Markov trace tr on these algebras depending on parameters $z, x_1, \ldots, x_{d-1}$. Aiming to extracting framed link invariants from tr, as it turned out in [20], tr does not re-scale directly according to the framed braid equivalence, leading to conditions that have to be imposed on the trace parameters $x_1, \ldots, x_{d-1}$; namely, they had to satisfy a non-linear system of equations, the E–*system* (Eq. 1.39). The x_i's being d^{th} roots of unity is one obvious solution.

Gérardin found in [20, Appendix] the full set of solutions of the E–system. Given now any solution of the E–system, 2–variable isotopy invariants for framed, classical and singular links were constructed in [20, 22, 21] respectively, which are studied further in [2, 5].

In this thesis we propose three framizations of the Temperley–Lieb algebra as a quotient of the Yokonuma–Hecke algebra over appropriate two–sided ideals. The possible quotient algebras that arise are three: the *Framization of the Temperley–Lieb algebra* ($\mathrm{FTL}_{d,n}(u)$), the *Yokonuma–Temperley–Lieb* algebra ($\mathrm{YTL}_{d,n}(u)$) and the *Complex Reflection Temperely–Lieb* algebra ($\mathrm{CTL}_{d,n}(u)$). From these we choose the algebra $\mathrm{FTL}_{d,n}(u)$ as the analogue of the Temperley–Lieb algebra in the context of framing, since it reflects the construction of a "framed Jones Polynomial" in the most natural way.

The outline of the thesis is as follows. Chapter 1 is dedicated to providing necessary background material from the literature including the Iwahori–Hecke algebra, $\mathrm{H}_n(u)$, the Temperley–Lieb algebra, $\mathrm{TL}_n(u)$, the construction of the Ocneanu trace, τ, on $\mathrm{H}_n(u)$, the passing of τ to $\mathrm{TL}_n(u)$, and the representation theories of $\mathrm{H}_n(u)$ and $\mathrm{TL}_n(u)$. Further, we discuss the work of Juyumaya and Lambropoulou on the Yokonuma–Hecke algebras, $\mathrm{Y}_{d,n}(u)$, [18],[23]. More precisely, we give the definition of $\mathrm{Y}_{d,n}(u)$ as a quotient of the d modular framed braid group $\mathcal{F}_{d,n}(u)$ (Definition 1.10),[18]. Next, we present an inductive basis for $\mathrm{Y}_{d,n}(u)$ (Proposition 1.9) [18] and we also give the construction of a unique linear Markov trace, with parameters $z, x_1, \ldots, x_{d-1}$, by Juyumaya on the algebras $\mathrm{Y}_{d,n}(u)$ (Theorem 1.5), [18]. Furthermore, we discuss the set of solutions of the E–system found by P. Gérardin (Theorem 1.7) and how this applies to the normalization and rescaling of the trace tr. One (trivial) solution is that the x_i's are d^{th} roots of unity, but this case is not of a great topological importance. Moreover we discuss the invariants for framed links that are derived from the trace tr and we show that these invariants coincide with the Homflypt polynomial only for the trivial cases where the x_i's are d^{th} roots of unity or $u = 1$ [2]. However, these invariants are conjectured to be topologically equivalent to the Homflypt polynomial [5]. We conclude this chapter with the results of Chlouveraki and Poulan [4] on the representation theory of $\mathrm{Y}_{d,n}(u)$.

The motivation behind this thesis is given in Chapter 2 where we present three possible quotients of $\mathrm{Y}_{d,n}(u)$ that could possibly lead to the framization of the Temperley–Lieb algebra (Section 2.1) and we analyze the connection between these quotients of $\mathrm{Y}_{d,n}(u)$ (Proposition 2.1). We then present our results on the first quotient algebra that emerged through this process, *the Yokonuma–Temperley–Lieb algebra* $\mathrm{YTL}_{d,n}(u)$ (Definition (2.1)). For $d = 1$ the algebra $\mathrm{YTL}_{1,n}(u)$ coincides with the Temperley–Lieb algebra. We first show that the defining ideal of this quotient algebra is principal (Corollary 2.1) and we give a presentation for $\mathrm{YTL}_{d,n}(u)$ with non-invertible generators analogous to the classical case (Proposition 2.2). We then give a spanning set $\Sigma_{d,n}$ for $\mathrm{YTL}_{d,n}(u)$, where each word in $\Sigma_{d,n}$ contains the highest and lowest index braiding generator exactly once (Proposition 2.4). Moreover, any word in $\Sigma_{d,n}$ inherits the splitting property from $\mathrm{Y}_{d,n}(u)$, that is, it splits into the framing part and the braiding part. We also present the results of Chlouveraki and Pouchin [3] on the dimension (Proposition 2.5) and a linear basis for $\mathrm{YTL}_{d,n}(u)$ (Theorem 2.1). From the spanning set $\Sigma_{d,n}$, they extracted an explicit basis for $\mathrm{YTL}_{d,n}(u)$ by describing a set of linear dependence relations among the framing parts for each fixed element in the braiding part. Furthermore, using the dimension results of [3] we find a basis for $\mathrm{YTL}_{2,3}(u)$ different than the basis in [3]. We conclude this chapter by discussing the results of Chlouveraki and Pouchin [3] on the representation theory of $\mathrm{YTL}_{d,n}(u)$. Next, we focus on the quotient algebras *Framization of Temperley–Lieb algebra*, $\mathrm{FTL}_{d,n}(u)$ and *Complex Reflection Temperley–Lieb algebra*, $\mathrm{CTL}_{d,n}(u)$. We show that the defining ideal for each of these two quotient algebras is principal (Theorem 2.3) and we give presentations in terms of generators and relations with invertible and non–invertible generators for both of the algebras. Further, we provide a linear basis for the case $d = 2$, $n = 3$ and present the formula for the dimension of $\mathrm{FTL}_{d,n}(u)$ by M. Chlouveraki and G. Pouchin [3] (Section 2.4.3).

Chapter 3 is dedicated to the determination of the conditions that the trace parameters $z, x_1, \ldots, x_{d-1}$ should satisfy so that the trace tr, defined on the algebras $\mathrm{Y}_{d,n}(u)$, passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$, $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ respectively. More precisely, the trace tr passes to each one the quotient algebras above, if it annihilates the generator of the defining ideal that corresponds to each quotient algebra. From this we extract conditions for the trace parameter z (Eqs. 3.3 and 3.20 for the cases of $\mathrm{YTL}_{d,n}(u)$ and $\mathrm{FTL}_{d,n}(u)$ respectively, and Theorem 3.7 for the case of $\mathrm{CTL}_{d,n}(u)$). Unfortunately, this condition is not enough for the cases of $\mathrm{YTL}_{d,n}(u)$ and $\mathrm{FTL}_{d,n}(u)$. Therefore, we have to seek extra conditions for the trace parameters $x_1, \ldots, x_{d-1}$. Our method consists of annihilating every element in the defining ideal of each the two quotient algebras using the trace tr, for the case $n = 3$. From this we extract a system of polynomial equations which we solve on the level of the group algebra by expressing the polynomials by means of the convolution product of the group algebra, and by applying an appropriate Fourier transformation (Sections 3.1 and 3.2).

More precisely, for the case of the quotient algebra $\mathrm{YTL}_{d,n}(u)$, we compute first the values of the trace parameter z that annihilate the generator of the defining ideal, which are the roots of a quadratic equation (Eq. 3.2). Then we annihilate the traces of all elements of $\mathrm{Y}_{d,n}(u)$ that lie in the defining ideal of $\mathrm{YTL}_{d,n}(u)$ and so we end up with a system (Σ) of quadratic equations in z (Eqs. 3.10a–3.10c). If we demand that (Σ) has both roots of Eq. 3.2 as common solutions, which is essential for discussing link invariants, we end up with necessary conditions for the trace tr to pass to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ (Theorem 3.4). In particular, Theorem 3.4 states that the trace tr passes to the quotient algebra $\mathrm{YTL}_{d,n}(u)$ if the trace parameters are d^{th} roots of unity $x_1, \ldots, x_{d-1}$ and $z = -\frac{1}{u+1}$ and $z = -1$. Note that these two values for z are precisely the ones that Jones computed such that the Ocneanu trace on $\mathrm{H}_n(u)$ passes to the quotient, the Temperley–Lieb algebra $\mathrm{TL}_n(u)$. If we also let (Σ) to have one common solution for z we obtain the necessary and sufficient conditions for the trace tr to pass through to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ (Theorem 3.3). To be more precise, Theorem 3.3 states that the trace tr passes to the quotient algebras $\mathrm{YTL}_{d,n}(u)$ if and only if either the conditions of Theorem 3.4 are satisfied or the trace parameters $x_1, \ldots, x_{d-1}$ comprise a solution of the E–system (other than d^{th} roots of unity) and $z = -\frac{1}{2}$. Thus, we obtain the conditions for the x_i's, so that the trace tr passes to the quotient algebra.

For the case of the quotient algebra $\mathrm{FTL}_{d,n}(u)$ we use the same reasoning. Solving the corresponding system (Eqs. 3.27 and 3.28) we deduce that the sufficient conditions such that the trace passes to quotient algebra $\mathrm{FTL}_{d,n}(u)$ are that the trace parameters $x_1, \ldots, x_{d-1}$ comprise solutions of the E–system and z takes one of the following two values: $z = -\frac{1}{(u+1)}E$ or $z = -E$ (Theorem 3.4). Furthermore, the necessary and sufficient conditions for the case $n = 3$ (Theorem 3.5) state that the trace parameters $x_1, \ldots, x_{d-1}$ should be the value at k, $1 \leq k \leq d-1$, of the following complex function over $\mathbb{Z}/d\mathbb{Z}$:

$$x_k = -z \left(\sum_{m \in \mathrm{Sup}_1} \chi(km) + (u+1) \sum_{m \in \mathrm{Sup}_2} \chi(km) \right),$$

where $\mathrm{Sup}_1 \cup \mathrm{Sup}_2$ (disjoint union) is the support of the Fourier transform of x and z takes the value:

$$z = -\frac{1}{|\mathrm{Sup}_1| + (u+1)|\mathrm{Sup}_2|}.$$

We then generalize by using induction on n (3.6). These are our main results.

Finally, in Chapter 4 we discuss the invariants that can be constructed through the trace tr on each one of the three quotient algerbas. As already mentioned, the x_i's comprising a solution of the E–system is a necessary and sufficient condition so that invariants for (classical, framed, etc.) links can be defined on the algebras $\mathrm{Y}_{d,n}(u)$. By further specializing the trace parameter z to each of the values that tr passes to the corresponding algebra quotient and thus invariants for (framed, classical) links emerge on the quotient algebras.

More precisely, for the case of $\mathrm{YTL}_{d,n}(u)$, in [2] it is shown that if the trace parameters $x_1, \ldots, x_{d-1}$ are d^{th} roots of unity, then the classical link invariants derived from the algebra $\mathrm{Y}_{d,n}(u)$ coincide with the 2–variable Jones or Homflypt polynomial. Using Theorem 3.3 and the results in [2], we obtain from the invariants for framed and classical links in [20, 22] related to $\mathrm{Y}_{d,n}(u)$, the 1–variable invariants $\mathcal{V}_{d,s}(u)$ for framed links and $V_{d,s}(u)$ for classical links, through the algebras $\mathrm{YTL}_{d,n}(u)$. As we show, the invariants coincide with the Jones polynomial for the case of classical links and they are framed analogues of the Jones polynomial for the case of framed links (Corollary 4.1).

In the case of $\mathrm{FTL}_{d,n}(u)$, we obtain from the invariants for framed and classical links in [20, 22] related to $\mathrm{Y}_{d,n}(u)$, the 1–variable invariants $\vartheta_{d,s}(u)$ for framed links and $\theta_{d,s}(u)$ for classical links. In the case of $\mathrm{CTL}_{d,n}(u)$, the parameters x_i are free, so in order to obtain a link invariant of any kind related to this algerbra, we must impose the condition of the x_i's comprising a solution of the E–system. The advantage of the quotient algebra $\mathrm{FTL}_{d,n}(u)$ over the quotient algebra $\mathrm{CTL}_{d,n}(u)$ is that the conditions of Theorem 3.5 so that the trace tr passes to the quotient algebra, include the solutions of the E–system. For this reason, we propose the algebra $\mathrm{FTL}_{d,n}(u)$ as the most natural analogue of the Temperley–Lieb algebra in the context of framed links.

Chapter 1

Preliminaries

1.1 Notations

Throughout this thesis we shall fix the following notation. By the term algebra we mean an associative unital (with unity 1) algebra over the field $\mathbb{C}(u)$, where u is an indeterminate. The following two positive integers are also fixed: d and n.

We denote S_n the symmetric group on n symbols. Let s_i be the elementary transposition $(i, i+1)$. We denote by l the length function on S_n with respect to the s_i's.

Denote by $C_d = \langle t \,|\, t^d = 1 \rangle$ the cyclic group of order d. Let $t_i = (1, \ldots, t, 1, \ldots, 1) \in C_d^n$, where t is in the i^{th} position.

Finally, we denote $C_{d,n} := C_d^n \rtimes S_n$, where the action is defined by permutation on the indices of the t_i's, namely: $s_i t_j = t_{s_i(j)} s_i$.

1.2 The Iwahori–Hecke algebra

In this section we will present the Iwahori–Hecke algebra $\mathrm{H}_n(u)$ and we will discuss some of its basic properties such as linear basis, dimension and the existence of a unique Markov trace function on it. We start by giving the connection between the braid group and the algebra $\mathrm{H}_n(u)$. Consider n points on a horizontal plane and n points on another horizontal plane directly below the first one. A braid is formed when these $2n$ points are connected by n strands that are not allowed to go back up. We have the following definition [16]:

Definition 1.1. *The braid group on n–strands, B_n, is the group with following presentation:*

$$B_n = \left\langle 1, \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i \sigma_j = \sigma_j \sigma_i & |i-j| > 1 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & |i-j| = 1 \end{array} \right\rangle$$

The generators σ_i are called the elementary crossings and are the positive crossing between the i^{th} and the $i+1^{st}$ strand.

A geometric interpretation of the elements σ_i and their inverses can be seen in Figure 1.1.

The algebra $\mathrm{H}_n(u)$ can be seen as a u–deformation of $\mathbb{C}S_n$. That is, the $\mathbb{C}(u)$–algebra that is generated by the elements h_w, where $w \in S_n$ and the following rules

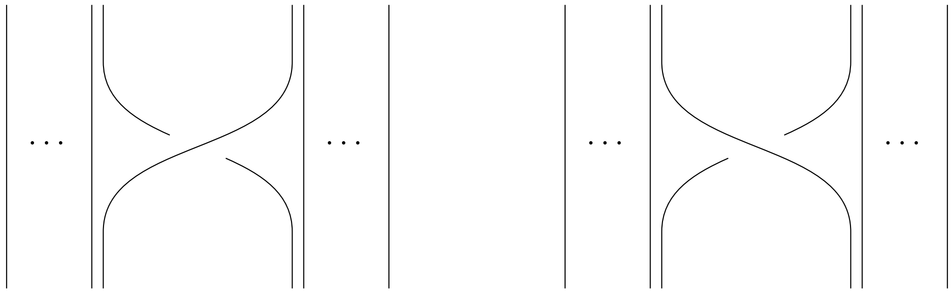


Figure 1.1: *The elements σ_i and σ_i^{-1}*

of multiplication.

$$h_{s_i}h_w = \begin{cases} h_{s_iw} & \text{for } l(s_iw) > l(w) \\ uh_{s_iw} + (u-1)h_w & \text{for } l(s_iw) < l(w) \end{cases}$$

where l is the length function defined on S_n. Setting $h_i := h_{s_i}$, the algebra $\mathrm{H}_n(u)$ can be presented in terms of generators and relations by $h_1, \ldots, h_{n-1}$ subject to the following relations:

$$\begin{aligned} h_ih_j &= h_jh_i \quad \text{for all} \quad |i-j|>1 && (1.1)\\ h_ih_jh_i &= h_jh_ih_j \quad \text{for all} \quad |i-j|=1 && (1.2)\\ h_i^2 &= u\cdot 1 + (u-1)h_i && (1.3)\end{aligned}$$

Remark 1.1. The first two relations in the presentation of $\mathrm{H}_n(u)$ are exactly the braid relations. Thus, there exists a natural epimorphism: $\mathbb{C}B_n \rightarrow \mathrm{H}_n(u)$, that sends $\sigma_i \mapsto h_i$.

Remark 1.2. The generators h_i are invertible. Indeed from the quadratic relation in the presentation above we obtain:

$$h_i^{-1} = u^{-1}(h_i+1) - 1 \tag{1.4}$$

Moreover, for $m \in \mathbb{N}$ we have the following computation formulas in $\mathrm{H}_n(u)$ (see [2]):

$$h_i^m = \left(\frac{u^m-1}{u+1}\right)h_i + \left(\frac{u^m-1}{u+1} + (-1)^m\right) \tag{1.5}$$

Next, we present a linear basis for $\mathrm{H}_n(u)$ (see [12], [16]). We start with the following property of $\mathrm{H}_n(u)$:

Proposition 1.1. *Every word in $\mathrm{H}_{n+1}(u)$ can be written as a sum of monomials that contain the generator h_n at most once.*

Having that property in mind, we introduce the following sets that will be used in constructing an appropriate spanning set of $\mathrm{H}_n(u)$:

$$\mathcal{H}_i = \{1, h_i, h_ih_{i-1}, \ldots, h_ih_{i-1}\ldots h_1\} \qquad (1 \le i \le n) \tag{1.6}$$

Notice that if $v_i \in \mathcal{H}_i$ then $h_{i+1}v_i \in \mathcal{H}_{i+1}$, for any $1 \le i \le n-1$.

Proposition 1.2 ([12])**.** *The following set of monomials spans the algebra $\mathrm{H}_n(u)$:*

$$\mathcal{U} = \{u_1u_2\ldots u_{n-1} \,|\, u_i \in \mathcal{H}_i,\ 1 \le i \le n-1\}$$

We say that the elements of $\mathcal{U}$ are in normal form.

Remark 1.3. An analogous argument holds for the symmetric group S_n. Let $w \in S_n$ then it can be written as a product $w_1 w_2 \dots w_{n-1}$, where:

$$w_i \in \{1, s_i, s_i s_{i-1}, \dots, s_i s_{i-1} \dots s_1\}, \quad 1 \leq i \leq n-1.$$

In order to show that the elements of $\mathcal{U}$ are linearly independent and thus they furnish a linear basis for $\mathrm{H}_n(u)$, one should use the method by J. Tits (see [12], [14, §7]). We start by defining the following endomorphisms of $\mathrm{End}(\mathbb{C}S_n)$ for $w \in S_n$:

$$\mathcal{L}_i(w) = \begin{cases} s_i w, & l(s_i w) > l(w) \\ u s_i w + (u-1)w, & l(s_i w) < l(w) \end{cases} \tag{1.7}$$

and

$$\mathcal{R}_j(w) = \begin{cases} w s_j, & l(w s_j) > l(w) \\ u w s_j + (u-1)w, & l(w s_j) < l(w) \end{cases} \tag{1.8}$$

Using (1.7) and (1.8) one can show that there exists an algebra homomorphism ϕ from $\mathrm{H}_n(u)$ to $\mathbb{C}S_n$, that sends $h_i \mapsto \mathcal{L}_i$, by proving that $\mathcal{L}_i$ and $\mathcal{R}_i$ commute, and that the $\mathcal{L}_i$'s satisfy the relations of $\mathrm{H}_n(u)$. Finally, by showing that ϕ is a bijection and using Remark 1.3, one proves that any element of $\mathcal{U}$ maps to an element of the linear basis of $\mathbb{C}S_n$ and thus we have:

Proposition 1.3. *The elements in normal form $m = u_1 u_2 \dots u_{n-1}$, where $u_i \in \mathcal{H}_i$, $1 \leq i \leq n-1$, are linearly independent. Moreover,* $\dim(\mathrm{H}_n(u)) = n!$.

Remark 1.4. The linear basis $\mathcal{U}$ of Proposition 1.3 indicate that the following natural inclusions exist:

$$\mathrm{H}_{n-1}(u) \subseteq \mathrm{H}_n(u).$$

Thus, $\mathrm{H}_n(u)$ can be considered a $\mathrm{H}_{n-1}(u) - \mathrm{H}_{n-1}(u)$ bimodule.

One of the most important properties of the Iwahori–Hecke algebra, is that it supports a unique Markov trace function which commutes with the inclusions of Remark 1.4. It was first proved in [13, 29] by Ocneanu. We give first the definition of a trace on an algerba:

Definition 1.2. *A linear function τ from an algebra to some module is called a trace if it satisfies $\tau(xy) = \tau(yx)$ for any x, y in the algebra.*

We now have the following:

Theorem 1.1 (Ocneanu)**.** *For any $\zeta \in \mathbb{C}^\times$ there exists a linear trace τ on $\cup_{n=1}^{\infty} \mathrm{H}_n(u)$ uniquely defined by the inductive rules:*

1. $\tau(1) = 1$
2. $\tau(ab) = \tau(ba), \quad a, b \in \mathrm{H}_n(u) \qquad$ (*Conjugation*).
3. $\tau(a h_n) = \zeta\, \tau(a), \quad a \in \mathrm{H}_n(u) \qquad$ (*Markov property*).

1.3 The Homflypt polynomial

From a topological point of view, closing a braid α, that is, connecting corresponding ends in pairs, gives rise to an oriented link. The closed braid is denoted by $\hat{\alpha}$ and is called *the closure of the braid* α. For the converse, we have the following theorem by Alexander:

Theorem 1.2 (Alexander)**.** *Any oriented link is isotopic to the closure of a braid.*

Further, isotopy classes of oriented links are in one–to–one correspondence with the equivalence classes of braids. More precisely, we have the following theorem:

Theorem 1.3 (Markov)**.** *Isotopy classes of links are in bijection with equivalence classes of braids in* $\cup_\infty B_n$*. The equivalence relation is generated by the two following moves:*

I. *Conjugation:* $\alpha\beta \sim \beta\alpha$, α, $\beta \in B_n$.

II. *Markov move:* $\alpha \sim \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n$.

The two–variable Jones or Homflypt polynomial is one of the most important invariants in Knot Theory and it can be defined through the Ocneanu trace τ. It was discovered independently by Lickorish and Millett, Freyd and Yetter, Ocneanu and Hoste in [13]. The addition of PT recognizes independent work carried out by Józef H. Przytycki and Paweł Traczyk [29]. Postal delays prevented Przytycki and Traczyk from receiving full recognition alongside the other six discoverers. Before moving to the construction of the invariant, we note the analogy between the Markov moves of type II of Theorem 1.3 and the third relation of Theorem 1.1. Therefore, the most natural way to define the invariant is to normalize τ so that the generators h_i and h_i^{-1} yield the same trace value [16]. Let $\vartheta \in \mathbb{C}$ such that:

$$\tau(\vartheta h_i) = \tau\left((\vartheta h_i)^{-1}\right)$$

or equivalently using Eq 1.4:

$$\vartheta^2 = \frac{\tau\left(h_i^{-1}\right)}{\tau(h_i)} = \frac{u^{-1}\tau(h_i) + u^{-1} - 1}{\zeta} = \frac{1 - u + \zeta}{u\zeta}. \tag{1.9}$$

Denote $\lambda := \frac{1-u+\zeta}{u\zeta}$. We have the following definitions [16]:

Definition 1.3. *The two–variable invariant* $X_L(u, \lambda)$ *of the oriented link* L *is the function:*

$$X_L(u, \lambda)(\hat{\alpha}) = \left(-\frac{1 - \lambda u}{\sqrt{\lambda}(1 - u)}\right)^{n-1} \left(\sqrt{\lambda}\right)^{\epsilon(\alpha)} \tau(\pi(\alpha)), \tag{1.10}$$

where $\alpha \in B_n$, $\epsilon(\alpha)$ *is the algebraic sum of the exponents of the* σ_i*'s in* α *and* π *is the natural epimorphism of* $\mathbb{C}B_n$ *into* $\mathrm{H}_n(u)$ *that sends* σ_i *to* h_i.

Definition 1.4. *Three oriented links* L_+, L_- *and* L_0 *are skein related if they have diagrams that are identical except in the neighbourhood of one crossing point where they look exactly as in Figure 1.2.*

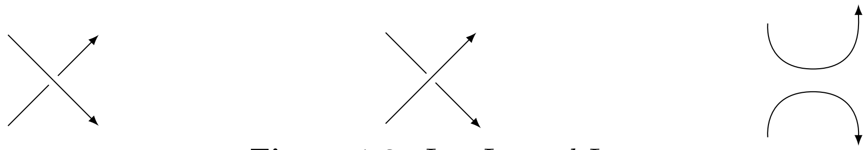


Figure 1.2: *L_+, L_- and L_0*

For certain values of the parameters u and λ, the polynomial invariant $X_L(u,\lambda)$ coincides with the two–variable Jones polynomial $P_L(t,x)$. We have the following proposition [16]:

Proposition 1.4. *To each oriented link L (up to isotopy) there is a Laurent polynomial $P_L(t,x)$ in two variables t and x that, if λ and u satisfy $t = \sqrt{\lambda}\sqrt{u}$, $x = \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)$ then:*

$$P_L(t,x) = X_L(u,\lambda).$$

Moreover, $P_L(t,x)$ is uniquely defined by the following Skein relation:

$$t^{-1}P_{L_+} - tP_{L_-} = xP_{L_0}$$

1.4 The representation theory of the algebra $\mathrm{H}_n(u)$

In this section will briefly discuss the representation theory of the symmetric group S_n which will help us define inductively the representations of the algebra $\mathrm{H}_n(u)$. These representations will play an important role in the computation of the dimension of the Temperley–Lieb algebra (see next section). We start with the following definitions.

Definition 1.5. *Let G be a finite group and let V be a finite dimensional $\mathbb{C}$–vector space. We say that V is a representation of G if there exists a group homorphism*

$$\rho : G \to GL(V).$$

That is, the following relation holds: $\rho(g_1 * g_2) = \rho(g_1)\rho(g_2)$, for any $g_1,\ g_2, \in\ G$.

Definition 1.6. *1. Let $W \subseteq V$ be a subspace of the representation V. The subspace W is a subrepresentation of G if:*

$$\rho(g)(w) \in W, \quad \forall g \in G\, w \in W$$

2. The representation V is called irreducible if $V \neq \{0\}$ and the only subrepresentations of V are the following two: $\{0\}$ and V.

It is known that any representation V is a direct sum of irreducible representations and also that the irreducible representations of a finite group are in 1–1 correspondence with its conjugacy classes [7]. Since a braid induces a permutation in an obvious way, it is expected that some family of representations of B_n will be related to the representations of the symmetric group S_n [16]. We have the following definition:

Definition 1.7. *A partition is a family of positive integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. We set $|\lambda| := \lambda_1 + \ldots + \lambda_r$,and we call $|\lambda|$ the size of λ.*

The conjugacy classes of S_n are indexed by the partitions of size n that correspond to the length of the disjoint cycles of the permutation. These can be represented by a Young diagram. From now on we will refer to the partitions of size n as *the partitions of* n. We have the following [16]:

Definition 1.8. *A Young diagram is a finite collection of boxes arranged in left-justified rows, with row sizes weakly decreasing. The Yound diagram associated to the partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ *is the one that has* r *rows and* λ_i *boxes in the* i^{th} *row (see Figure 1.3).*

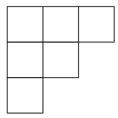
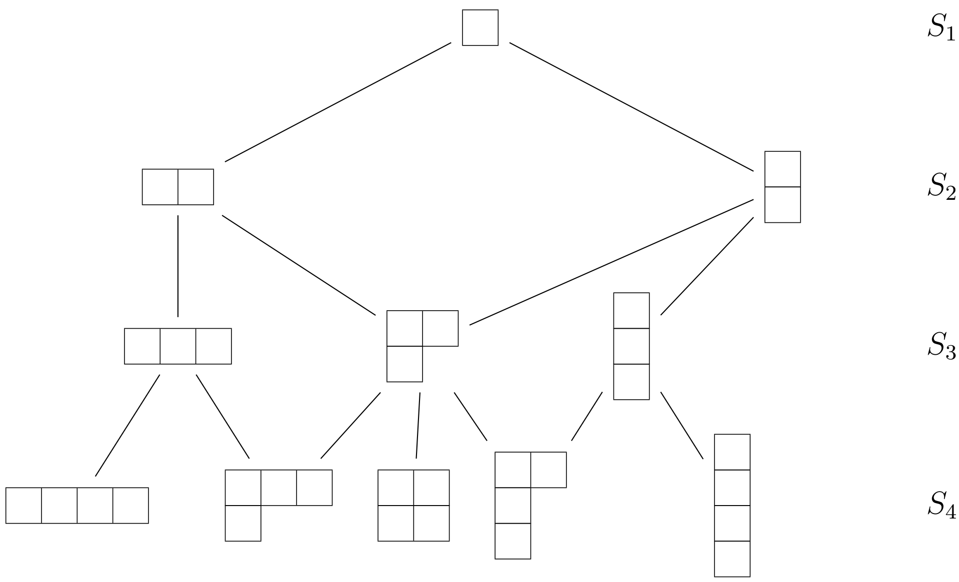


Figure 1.3: *The Young diagram that corresponds to the partition* $\lambda = (3, 2, 1)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition that is associated to a given Young diagram and let V_λ be the corresponding representation. The dimension of the representation V_λ is given by the following formula, known as the "hook length" formula. For each box v we compute its hook length $h_\lambda(v)$, that is, the number of boxes horizontally to the right and vertically below the box in question, including the box itself. The dimension of the representation is then $n!$ divided by the product of the hook lengths, namely:

$$V_\lambda = \frac{n!}{\prod_{v \in h_\lambda(v)}}$$

We shall describe now, following [16], how a given irreducible representation of S_n is decomposed when it is restricted to S_{n-1}, thus giving us a rule to construct the representation inductively. Given an irreducible representation π of S_n with Young diagram Y, its restriction to S_{n-1} is the direct sum of each representation of S_{n-1} obtained from S_n by removing a box so as to obtain a Young diagram. We have the following situation that describes inductively all irreducible representations of all symmetric groups:



One can imagine the diagram to continue indefinitely downwards. The lines connecting different row represent the restrictions of the representations. For example, the restriction of $\ydiagram{3,1}$ of S_4 restricts to the direct sum of $\ydiagram{2,1}$ and $\ydiagram{3}$ of S_3.

On the other hand, since the Iwahori–Hecke algebra $\mathrm{H}_n(u)$ can be regarded as a u–deformation of $\mathbb{C}S_n$, it is semisimple. Choosing a u close to 1, the whole structure of $\mathrm{H}_n(u)$ and the inclusions $\mathrm{H}_n(u) \subseteq \mathrm{H}_{n+1}(u)$ remain the same under the deformations. We have the following theorem [16, 35]:

Theorem 1.4. *If $u \neq 0$ and u is not a root of unity, the irreducible representations of $\mathrm{H}_n(u)$ are in one–to–one correspondence with Young diagrams. The decomposition rule and hence their dimensions are the same as for S_n.*

1.5 The Temperley–Lieb algebra

The Temperley–Lieb algebra over $\mathbb{C}$ was originally defined by generators $f_1, \dots, f_{n-1}$ subject to the following relations:

$$\begin{aligned} f_i^2 &= f_i \\ f_i f_j f_i &= \delta f_i, \quad |i-j| = 1 \\ f_i f_j &= f_j f_i, \quad |i-j| > 1 \end{aligned}$$

where δ is an indeterminate (see [10],[16],[15]). The generators f_i are non-invertible; one can define the Temperley–Lieb algebra with the following invertible generators (see [16]):

$$h_i := (u+1) f_i - 1 \tag{1.11}$$

where u is defined via the relation $\delta^{-1} = 2 + u + u^{-1}$.

The Temperley algebra $\mathrm{TL}_n(u)$, over $\mathbb{C}(u)$, is defined by generators $h_1, \dots, h_{n-1}$ under the relations:

$$h_i h_j h_i = h_j h_i h_j, \quad |i-j| = 1 \tag{1.12}$$

$$h_i h_j = h_j h_i, \quad |i-j| > 1 \tag{1.13}$$

$$h_i^2 = (u-1) h_i + u \tag{1.14}$$

$$h_i h_j h_i + h_j h_i + h_i h_j + h_i + h_j + 1 = 0, \quad |i-j| = 1. \tag{1.15}$$

Note that relations (1.15) are symmetric with respect to the indices i, j, so relations (1.12) follow from relations (1.15). Relations (1.12)–(1.14) are the well–known defining relations of the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. Therefore, $\mathrm{TL}_n(u)$ can be considered as a quotient of $\mathrm{H}_n(u)$ over the two–sided ideal generated by relations (1.15).

Definition 1.9. The *Temperley–Lieb algebra* $\mathrm{TL}_n(u)$ can be defined as the quotient of the algebra $\mathrm{H}_n(u)$ over the two–sided ideal generated by the *Steinberg elements* $h_{i,j}$:

$$h_{i,j} := \sum_{w \in \langle s_i, s_j \rangle} h_w, \quad \text{for all} \quad |i-j| = 1. \tag{1.16}$$

It is not difficult to show that the ideal of Definition 1.9 is a principal ideal and is generated by the element $h_{1,2}$. Indeed, we have the following:

Proposition 1.5. *The algebra* $\mathrm{TL}_n(u)$ *is the quotient of the algebra* $\mathrm{H}_n(u)$ *over the two–sided ideal that is generated by the element* $h_{1,2}$*, namely:*

$$\mathrm{TL}_n(u) = \frac{\mathrm{H}_n(u)}{\langle h_{1,2} \rangle}$$

In the remaining part of this section we will give the results of [10, 16] on a linear basis for $\mathrm{TL}_n(u)$ and an upper bound for the dimension of the algebra $\mathrm{TL}_n(u)$. We have the following proposition [10]:

Proposition 1.6. *The following is a spanning set for* $\mathrm{TL}_n(u)$*:*

$$\mathcal{B}_n = \left\{ (h_{j_1} h_{j_1-1} \ldots h_{j_1-k_1})\,(h_{j_2} h_{j_2-1} \ldots h_{j_2-k_2}) \ldots \left(h_{j_p} h_{j_p-1} \ldots h_{j_p-k_p}\right) \right\}$$

where $1 \leq j_1 < j_2 < \ldots < j_p \leq n-1$ *and* $1 \leq j_1 - k_1 < j_2 - k_2 < \ldots < j_p - k_p$*,*

One can count the monomials in $\mathcal{B}_n$ by corresponding them to the paths from $(0,0)$ to (n,n), that do not cross the diagonal, in the lattice $\mathbb{Z}^2$ [15, 10]. For example, the element $(h_4 h_3 h_2)(h_5 h_4 h_3) \in \mathcal{B}_6$ corresponds to the following path in $\mathbb{Z}^2$.

$$(0,0) \to (4,0) \to (4,2) \to (5,2) \to (5,3) \to (6,3) \to (6,6).$$

Such paths are counted by the *Catalan numbers*, $c_n := \frac{1}{n+1}\binom{2n}{n}$. Thus we have that: $\dim(\mathrm{TL}_n(u)) \leq c_n$.

1.6 The Jones Polynomial

Jones' methods for redefining his Markov trace on the Temperley–Lieb algebra as factoring of the Ocneanu trace on the Iwahori–Hecke algebra [16] tells us that the least requirement is that the Ocneanu trace respects the defining relations (1.15), that is, τ annihilates the generator of the defining ideal of the quotient algebra $\mathrm{TL}_n(u)$, namely:

$$\tau(h_{1,2}) = 0$$

or equivalently:

$$(u+1)\zeta^2 + (u+2)\zeta + 1 = 0, \tag{1.17}$$

which has the two following solutions for ζ:

$$\zeta = -\frac{1}{u+1} \quad \text{and} \quad \zeta = -1. \tag{1.18}$$

The value $\zeta = -1$ is of no topological interest since from Eq. 1.5 we have that:

$$\tau(h_i^m) = \left(\frac{u^m - 1}{u+1} \right) \zeta + \left(\frac{u^m - 1}{u+1} + (-1)^m \right).$$

Hence, the trace τ gives the same value for any even (resp. odd) power of the generators h_i and therefore important topological information is lost. Thus, when we discuss topological invariants of links, we consider that τ passes through to the quotient algebra $\mathrm{TL}_n(u)$ only for the case where $\zeta = -\frac{1}{u+1}$.

In Section 1.3 we discussed the construction of the Homflypt polynomial through the Ocneanu trace [16]. By specializing the trace parameter ζ to $-\frac{1}{u+1}$ the Jones polynomial is recovered. Indeed, for $\zeta = -\frac{1}{u+1}$ in Eq. 1.9 we have that:

$$\lambda = \frac{1-u+\zeta}{u\zeta} = -\frac{1-u-\frac{1}{u+1}}{\frac{u}{u+1}} = \frac{(u-1)(u+1)+1}{u} = \frac{u^2}{u} = u$$

and therefore Eq. 1.10 becomes:

$$X(u,u)(\hat{\alpha}) = \left(-\frac{1+u}{\sqrt{u}}\right)^{n-1} \left(\sqrt{u}\right)^{\epsilon(\alpha)} \tau(\pi(\alpha)) \tag{1.19}$$

Denoting $V(u)(\hat{\alpha}) := X(u,u)(\hat{\alpha})$ one obtains the one-variable Jones Polynomial as a specialization of the Homflypt polynomial (see also [16], [10], [12]).

1.7 The representation theory of the algebra $\mathrm{TL}_n(u)$

In this Section we will discuss the representation theory of the Temperley–Lieb algebra by refering to the discussion of Section 1.4, and then use it to compute the dimension of $\mathrm{TL}_n(u)$. From Definition 1.9 we have that $\mathrm{TL}_n(u)$ can be considered as a quotient of the algebra $\mathrm{H}_n(u)$ over the two–sided ideal that is generated by the relations:

$$h_i h_{i+1} h_i + h_{i+1} h_i + h_i h_{i+1} + h_{i+1} + h_i + 1 = 0.$$

It is known [16] that the representations of the symmetric group for which

$$s_i s_{i+1} s_i + s_{i+1} s_i + s_i s_{i+1} + s_{i+1} + s_i + 1 = 0$$

are those whose Young diagrams have at most two columns. Indeed, let

$$\begin{aligned} \pi_\lambda : S_n &\to \mathrm{End}(V_\lambda) \\ s_i &\mapsto \pi_\lambda(s_i) \end{aligned}$$

be the representation of S_n assosiated to the partition λ of n. For $n = 3$ we associate $\ydiagram{3}$ to the identity representation, $\ydiagram{2,1}$ to the representation that sends s_1 to $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and s_2 to $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ and finally, we associate $\ydiagram{1,1,1}$ to the sign representation. For the first case we have that:

$$\pi_{\ydiagram{3}}(s_i s_{i+1} s_i + s_{i+1} s_i + s_i s_{i+1} + s_{i+1} + s_i + 1) = 1+1+1+1+1+1 = 6,$$

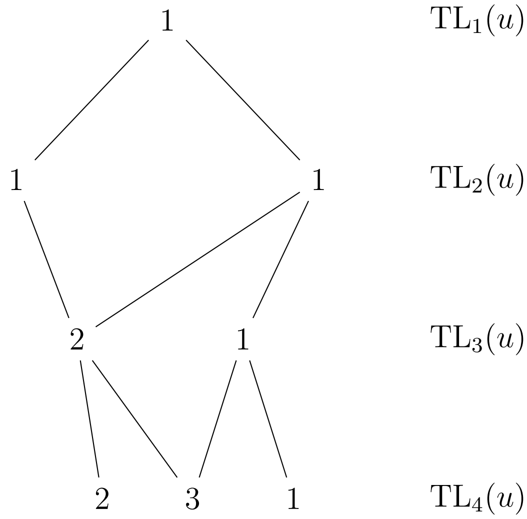
which is not equal to zero and thus it is discarded.
For the second case we have that:

$$\begin{aligned} \pi_{\ydiagram{2,1}}(s_i s_{i+1} s_i + s_{i+1} s_i + s_i s_{i+1} + s_{i+1} + s_i + 1) &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \\ + \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

For the last case we have that:

$$\pi_{\substack{\square\\\square\\\square}}(s_is_{i+1}s_i + s_{i+1}s_i + s_is_{i+1} + s_{i+1} + s_i + 1) = -1 + 1 + 1 - 1 - 1 + 1 = 0.$$

Notice that the last two cases have at most two columns. Therefore, the irreducible representations of $\mathrm{TL}_n(u)$ are the irreducible representations of $\mathrm{H}_n(u)$ that their restriction to S_3 does not contain the trivial representation. In other words, the irreducible representations of $\mathrm{TL}_n(u)$ are the irreducible representations of $\mathrm{H}_n(u)$ that have at most two columns in their Young diagram. One obtains the following diagram for $\mathrm{TL}_n(u)$, whose meaning is the same as for the case of Hecke algebras of Section 1.4, where the Young diagrams are replaced by the dimensions of the corresponding representations:



The sum of the squares of the dimensions of the irreducible representations in the k^{th} row gives the dimension of the algebra $\mathrm{TL}_k(u)$. Therefore we have that:

$$\dim \mathrm{TL}_n(u) = \frac{1}{n+1}\binom{2n}{n}$$

For an explicit proof the reader should refer to [10, §2.8, §2.11]. Thus, we have also the following:

Corollary 1.1. *The set $\mathcal{B}$ is a linear basis for* $\mathrm{TL}_n(u)$*.*

For the rest of this section we shall present the results of Juyumaya and Lambropoulou regarding invariants of framed links (see [18], [17],[23], [20], [21], [22]).

1.8 Framed braids

The group $\mathbb{Z}^n$ is generated by the "framing generators" $t_1, \dots, t_n$, the standard multiplicative generators of $\mathbb{Z}^n$. In this notation an element $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ in the additive notation can be expressed as $t_1^{a_1} \dots t_n^{a_n}$. The *framed braid group* on n strands is then defined as:

$$\mathcal{F}_n = \mathbb{Z}^n \rtimes B_n$$

where the action of B_n on $\mathbb{Z}^n$ is given by the permutation induced by a braid on the indices:

$$\sigma_i t_j = t_{s_i(j)} \sigma_i. \tag{1.20}$$

In particular, $\sigma_i t_i = t_{i+1}\sigma_i$ and $t_{i+1}\sigma_i = \sigma_i t_i$. A word w in $\mathcal{F}_n$ has thus the "splitting property", that is, it splits into the "framing" part and the "braiding" part:

$$w = t_1^{a_1} \dots t_n^{a_n}\, \sigma$$

where $\sigma \in B_n$ and $a_i \in \mathbb{Z}$. So w is a classical braid with an integer attached to each strand. Topologically, an element of $\mathbb{Z}^n$ is identified with a framed identity braid on n strands, while a classical braid in B_n is viewed as a framed braid with all framings zero. The multiplication in $\mathcal{F}_n$ is defined by placing one braid on top of the other and collecting the total framing of each strand to the top.

For a fixed positive integer d, the *d-modular framed braid group* on n strands, $\mathcal{F}_{d,n}$, is defined as the quotient of $\mathcal{F}_n$ over the *modular relations*:

$$t_i^d = 1 \quad (i = 1, \dots, n). \tag{1.21}$$

Thus, $\mathcal{F}_{d,n} = C_d^n \rtimes B_n$, where C_d^n is isomorphic to $(\mathbb{Z}/d\mathbb{Z})^n$ but with multiplicative notation. Note that there is an obvious embedding of $\mathbb{C}C_d^n$ in $\mathcal{F}_{d,n}$. Framed braids in $\mathcal{F}_{d,n}$ have framings modulo d.

Passing now to the group algebra $\mathbb{C}\mathcal{F}_{d,n}$, we have the following elements e_i (see Figure 1.4 diagrammatic interpretations), which are idempotents (cf. [20, Lemma 4]):

$$e_i := \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s}, \qquad i = 1, \dots, n-1. \tag{1.22}$$

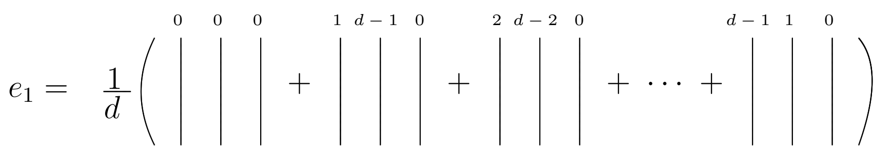


Figure 1.4: *The element $e_1 \in \mathbb{C}\mathcal{F}_{d,3}$.*

The definition of the idempotent e_i can be generalized in the following way. For any indices i, j and any $m \in \mathbb{Z}/d\mathbb{Z}$, we define the following elements in $\mathbb{C}\mathcal{F}_{d,n}$:

$$e_{i,j} := \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_j^{-s}, \tag{1.23}$$

and:

$$e_i^{(m)} := \frac{1}{d}\sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{-s}. \tag{1.24}$$

(notice that $e_i = e_{i,i+1} = e_i^{(0)}$). The following lemma collects some of the relations among the e_i's, the t_i's and the σ_i's. These relations will be used in this thesis.

Lemma 1.1. *For the idempotents e_i and for $1 \leq i,\ j \leq n-1$ the following relations hold:*

$$\begin{array}{rcl} t_j e_i & = & e_i t_j \\ e_{i+1}\sigma_i & = & \sigma_i e_{i,i+2} \\ e_i \sigma_j & = & \sigma_j e_i, \quad \textit{for } j \neq i-1, i+1 \\ e_j \sigma_i \sigma_j & = & \sigma_i \sigma_j e_i \quad \textit{for } |i-j| = 1 \\ e_i e_{i+1} & = & e_i e_{i,i+2} \\ e_i e_{i+1} & = & e_{i,i+2} e_{i+1}. \end{array}$$

Proof. All relations are immediate consequences of the definitions. The proofs for the first four relations can be found, for example, in [21, Lemma 2.1]. For the fifth relation we have:

$$\begin{aligned} e_i e_{i+1} &= \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+1}^{-s} \frac{1}{d}\sum_{m=0}^{d-1} t_{i+1}^m t_{i+2}^{-m} \\ &= \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{m=0}^{d-1} t_i^s t_{i+1}^{m-s} t_{i+2}^{-m}. \end{aligned} \tag{1.25}$$

Setting now $k = m - s$ we obtain:

$$\begin{aligned} (1.25) &= \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} t_i^s t_{i+1}^{k} t_{i+2}^{-k-s} \\ &= \frac{1}{d}\sum_{s=0}^{d-1} t_i^s t_{i+2}^{-s} \frac{1}{d}\sum_{k=0}^{d-1} t_{i+1}^k t_{i+2}^{-k} \\ &= e_{i,i+2} e_{i+1}. \end{aligned}$$

The sixth relation is proved in an analogous way. □

We conclude this section with two more technical lemmas that will be used extensively in the rest of this thesis.

Lemma 1.2. *For $1 \leq i \leq n-1$ and $m \in \mathbb{Z}/d\mathbb{Z}$ we have the following:*

$$e_i^{(m)} e_{i+1} = e_i e_{i+1}^{(m)}$$

Proof. We have that:

$$e_i^{(m)} e_{i+1} = \frac{1}{d}\sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{-s} \frac{1}{d}\sum_{k=0}^{d-1} t_{i+1}^k t_{i+2}^{-k} = \frac{1}{d^2}\sum_{s,k=0}^{d-1} t_i^{m+s} t_{i+1}^{-s+k} t_{i+2}^{-k}. \tag{1.26}$$

Setting now $r = m + s$ we have that:

$$(1.26) = \frac{1}{d^2}\sum_{r,k=0}^{d-1} t_i^r t_{i+1}^{m-r+k} t_{i+2}^{-k} = \frac{1}{d}\sum_{r=0}^{d-1} t_i^r t_{i+1}^{-r} \frac{1}{d}\sum_{k=0}^{d-1} t_{i+1}^{m+k} t_{i+2}^{-k} = e_i e_{i+1}^{(m)}.$$

□

In a complete analogous way we can also prove that:

Lemma 1.3. *For $1 \leq i \leq n$ the following holds $m \in \mathbb{Z}/d\mathbb{Z}$:*

$$\frac{1}{d}\sum_{s=0}^{d-1} t_i^{m+s} t_{i+1}^{-s} = e_i^{(m)} = \frac{1}{d}\sum_{s=0}^{d-1} t_i^{s} t_{i+1}^{m-s}.$$

1.9 The Yokonuma–Hecke algebra

The *Yokonuma–Hecke algebra* $\mathrm{Y}_{d,n}(u)$ is defined [18, 20] as the quotient of the group algebra $\mathbb{C}(u)\mathcal{F}_{d,n}$ over the two-sided ideal generated by the elements:

$$\sigma_i^2 - 1 - (u-1)\, e_i - (u-1)\, e_i\, \sigma_i, \quad \text{for all } i,$$

which give rise to the following quadratic relations in $\mathrm{Y}_{d,n}(u)$:

$$g_i^2 = 1 + (u-1)\, e_i + (u-1)\, e_i\, g_i \tag{1.27}$$

where g_i corresponds to σ_i. Since the quadratic relations do not change the framing we have that $\mathbb{C}C_d^n \subset \mathbb{C}(u)C_d^n \subset \mathrm{Y}_{d,n}(u)$ and we keep the same notation for the elements of $\mathbb{C}C_d^n$ and for the elements e_i in $\mathrm{Y}_{d,n}(u)$. We have the following definition [18], [23]:

Definition 1.10. *The algebra* $\mathrm{Y}_{d,n}(u)$ *has a presentation with generators* $t_1, \ldots, t_n, g_1, g_2, \ldots, g_{n-1}$*, subject to the following relations:*

$$\begin{aligned}
g_i g_j &= g_j g_i \quad |i-j| > 1 &(1.28)\\
g_i g_j g_i &= g_j g_i g_j \quad |i-j| = 1 &(1.29)\\
t_i^d &= 1 &(1.30)\\
t_i t_j &= t_j t_i &(1.31)\\
g_i t_j &= t_{s_i(j)} g_i &(1.32)\\
g_i^2 &= 1 + (u-1)e_i + (u-1)e_i g_i &(1.33)
\end{aligned}$$

The elements g_i are invertible (see Figure 1.5 for diagrammatic interpretations):

$$g_i^{-1} = g_i + (u^{-1}-1)\, e_i + (u^{-1}-1)\, e_i\, g_i.$$

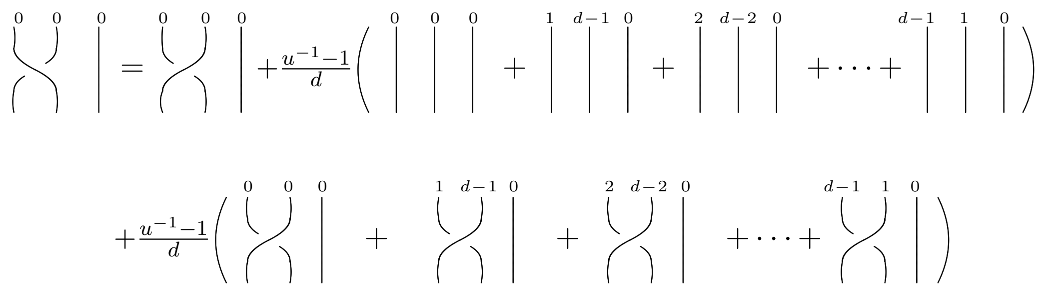


Figure 1.5: *The element* $g_1^{-1} \in \mathrm{Y}_{d,3}(u)$.

Remark 1.5. For $d = 1$ we have $t_j = 1$ and $e_i = 1$, and in this case the quadratic relations (1.27) become $g_i^2 = (u-1)g_i + u$, which are the quadratic relations of the Iwahori–Hecke algebra $\mathrm{H}_n(u)$. So, $\mathrm{Y}_{1,n}(u)$ coincides with the algebra $\mathrm{H}_n(u)$. Further, there is an obvious epimorphism of the Yokonuma-Hecke algebra $\mathrm{Y}_{d,n}(u)$ onto the algebra $\mathrm{H}_n(u)$ via the map:

$$\begin{array}{rcl} g_i & \mapsto & h_i \\ t_j & \mapsto & 1. \end{array} \tag{1.34}$$

We can alternatively define the algebra $\mathrm{Y}_{d,n}(u)$ as a u–deformation of the algebra $\mathbb{C}C_{d,n}$ (recall that $C_{d,n}$ is the semidirect product $C_d^n \rtimes S_n$). More precisely, let $w \in S_n$ and let $w = s_{i_1} \ldots s_{i_k}$ be a reduced expression for w. Since the generators $g_i := g_{s_i}$ of $\mathrm{Y}_{d,n}(u)$ satisfy the same braiding relations as the generators of S_n, then together with the well-known theorem of Matsumoto [28], it follows that the element $g_w := g_{i_1} \ldots g_{i_k}$ is well defined. We have the following multiplication rule in $\mathrm{Y}_{d,n}(u)$ (see Proposition 2.4[17]):

$$g_{s_i} g_w = \begin{cases} g_{s_i w} & \text{for } l(s_i w) > l(w) \\ g_{s_i w} + (u-1)e_i g_{s_i w} + (u-1)e_i g_w & \text{for } l(s_i w) < l(w) \end{cases} \tag{1.35}$$

We also correspond g_{t_i} to t_i and we define: $g_{t_i w} = g_{t_i} g_w = t_i g_w$. Next we present an inductive basis for the Yokonuma–Hecke algebra. We will need the following proposition [18]:

Proposition 1.7. *Every word in $\mathrm{Y}_{d,n}(u)$ can be written as a linear combination of words in $t_1, \ldots, t_n, g_1 \ldots, g_{n-1}$ having at most one $\alpha_{n-1} \in \{g_{n-1} t_n^m \,|\, 1 \leq m \leq d-1\}$. Words containing α_{n-1} at most once are called reduced.*

We have the following definition:

Definition 1.11 ([18])**.** *The normal words of $\mathrm{Y}_{d,n}(u)$ are the words that are of the form:*

$$v_0 v_1 \ldots v_{n-1}$$

where $v_i \in R_i$, such that $R_i := \{1, t_{i+1}^m, g_i v \,|\, v \in R_{i-1}, 1 \leq m \leq n-1\}$ and $R_0 := \{1, t_1^m \,|\, 1 \leq m \leq n-1\}$.

Remark 1.6. An analogous argument holds for the algebra $\mathbb{C}C_{d,n}$. Indeed, if we change the quadratic relation of Definition 1.10 to $s_i^2 = 1$, one can easily prove that any word in $\mathbb{C}C_{d,n}$ can be written the form:

$$w_0 w_1 \ldots w_{n-1}$$

where $w_i \in W_i := \{1, t_{i+1}^s, s_i w \,|\, w \in W_{i-1}, 0 \leq s \leq d-1\}$ and $W_0 := \{1, t_1^s \,|\, 0 \leq s \leq d-1\}$.

Proposition 1.8 ([18])**.** *The algebra $\mathrm{Y}_{d,n}(u)$ is linearly spanned by the normal words.*

What remains now to show is that the set of the normal words in linearly independent and thus it furnishes a linear basis for the algebra $\mathrm{Y}_{d,n}(u)$. For this one has to use the method of J. Tits [14].

Proposition 1.9 ([18])**.** *The set of normal words is a $\mathbb{C}(u)$–basis for $\mathrm{Y}_{d,n}(u)$. Moreover, the dimension of $\mathrm{Y}_{d,n}(u)$ is $d^n n!$.*

Sketch of proof. For further details the reader should refer to [18]. Let $V = \mathbb{C}(u)C_{d,n}$ and let $g \in V$.We define the following algebra homomorphism ρ and λ from the algebra $\mathrm{Y}_{d,n}(u)$ to the algebra $\mathrm{End}(V)$:

$$\rho_{g_i} g := \begin{cases} g s_i & \text{if} \quad l(g s_i) > l(g) \\ g s_i + (u-1)e_i + (u-1) g s_i e_i & \text{if} \quad l(g s_i) < l(g) \end{cases}$$

$$\lambda_{g_i} g := \begin{cases} s_i g & \text{if} \quad l(s_i g) > l(g) \\ s_i g + (u-1)e_i + (u-1)e_i s_i g & \text{if} \quad l(s_i g) < l(g) \end{cases}$$

For any $t_i \in C_d^n$ we can define λ_{t_i} and ρ_{t_i} as $\lambda_{t_i} g = t_i g$ and $\rho_{t_i} g = g t_i$ respectively. Moreover, it is obvious that $\lambda_{t_i}\rho_{t_j} = \rho_{t_j}\lambda_{t_i}$ for any $t_i, t_j \in C_d^n$. Next, one has to prove that $\lambda_{g_i}\rho_{g_j} = \rho_{g_j}\lambda_{g_i}$, by distinguishing cases according to the lengths of g, $s_i g$, $g s_j$ and $s_i g s_j$. Further, the map $t \mapsto \lambda_t$, $g_i \mapsto \lambda_{g_i}$ defines a homomorphism of $\mathbb{C}(u)$–algebras $\lambda : \mathrm{Y}_{d,n}(u) \to \mathrm{End}(V)$. Finally, we deduce that:

$$\lambda(v_0 \dots v_{n-1})(1) = w_1 \dots w_{n-1}.$$

Thus, any linear combination $\sum_g \alpha_g g$ (where g runs the normals words in $\mathrm{Y}_{d,n}(u)$) becomes a linear combination $\sum_w \alpha_g w$, where now w runs the normal words in $C_{d,n}$. Therefore $\alpha_g = 0$, for all g, and thus the set of normal words in $\mathrm{Y}_{d,n}(u)$ is linearly independent. Two final remarks are now due.

Remark 1.7. The normal words of Proposition 1.9 can be rewritten in the form:

$$w_{n-1} g_{n-1} \dots g_i t_i^k \quad \text{or} \quad w_{n-1} t_n^k \tag{1.36}$$

where w_{n-1} is in the inductive basis of $\mathrm{Y}_{d,n-1}(u)$ and $0 \le k \le d-1$. We shall use this notation for the rest of this thesis.

Remark 1.8. In analogy to Remark 1.4, the set of normal words indicate that the following natural inclusions exist:

$$\mathrm{Y}_{d,n}(u) \subseteq \mathrm{Y}_{d,n+1}(u).$$

Thus, $\mathrm{Y}_{d,n+1}(u)$ can be considered a $\mathrm{Y}_{d,n}(u) - \mathrm{Y}_{d,n}(u)$ bimodule.

□

1.10 A Markov trace on $\mathrm{Y}_{d,n}(u)$

Using the inductive basis of Proposition 1.9, J. Juyumaya constructed in [18] a unique linear Markov trace on the algebra $\mathrm{Y}_{d,n}(u)$. Namely:

Theorem 1.5 ([18] Theorem 12)**.** *Let d a positive integer. For $x_0 := 1$ and indeterminates z, $x_1, \dots, x_{d-1}$ there exists a unique linear Markov trace* tr*:*

$$\mathrm{tr} : \cup_{n=1}^{\infty} \mathrm{Y}_{d,n}(u) \longrightarrow \mathbb{C}(u)[z, x_1, \dots, x_{d-1}]$$

defined inductively on n by the following rules:

$$\begin{array}{llcll} (i) & \mathrm{tr}(ab) & = & \mathrm{tr}(ba) & \\ (ii) & \mathrm{tr}(1) & = & 1 & \\ (iii) & \mathrm{tr}(a g_n) & = & z\,\mathrm{tr}(a) & (\textit{Markov property}) \\ (iv) & \mathrm{tr}(a t_{n+1}^s) & = & x_s \mathrm{tr}(a) & (s = 1, \dots, d-1) \end{array}$$

where $a, b \in \mathrm{Y}_{d,n}(u)$.

Remark 1.9. By direct computation, $\mathrm{tr}(e_i)$ takes the same value for all i. We denote this value by E, that is:

$$E := \mathrm{tr}(e_i) = \frac{1}{d}\sum_{s=0}^{d-1} \mathrm{tr}(t_i^s t_{i+1}^{-s}) = \frac{1}{d}\sum_{s=0}^{d-1} x_s x_{d-s},$$

For all $0 \le m \le d-1$, we also define:

$$E^{(m)} := \mathrm{tr}(e_i^{(m)}) = \frac{1}{d}\sum_{s=0}^{d-1} \mathrm{tr}(t_i^{m+s} t_{i+1}^{-s}) = \frac{1}{d}\sum_{s=0}^{d-1} x_{m+s} x_{d-s},$$

Notice that $E = E^{(0)}$.

1.11 Framed link invariants through $\mathrm{Y}_{d,n}(u)$

In analogy to the classical case, closing a framed braid gives rise to an oriented *framed link*. Conversely, by adapting Theorem 1.2 to the framed link context, an oriented framed link can be isotoped to the closure of a framed braid.

Further, adapting Theorem 1.3 to the modular framed braid context, isotopy classes of oriented modular framed links are in one–to–one correspondence with equivalence classes of modular framed braids. More precisely, we have the following result from [23]:

Theorem 1.6 (Markov equivalence for framed braids and modular framed braids)**.** *Isotopy classes of oriented framed links (resp. modular framed links) are in bijection with equivalence classes of framed braids in $\cup_{n\in\mathbb{N}}\mathcal{F}_n$ (resp. $\cup_{n\in\mathbb{N}}\mathcal{F}_{d,n}$). The equivalence relation is generated by the two moves:*

(i) Conjugation: $\alpha\beta \sim \beta\alpha$, $\quad \alpha, \beta \in \mathcal{F}_n$ (resp. $\mathcal{F}_{d,n}$)

(ii) Markov move: $\alpha \sim \alpha{\sigma_n}^{\pm 1}$, $\quad \alpha \in \mathcal{F}_n$ (resp. $\mathcal{F}_{d,n}$)

The case of classical framed links is well known (see for example [26]).

1.11.1 The E–system

In analogy to the discussion of Section 1.3 and according to Theorem 1.6, any invariant of oriented framed links has to agree on the closures of the braids α, $\alpha\sigma_n$ and $\alpha{\sigma_n}^{-1}$. Note the resemblance of the conjugation rule and the Markov property in Theorem 1.5 with moves (i) and (ii) of Theorem 1.6. Following [16] Juyumaya and Lambropoulou defined an invariant by re–scaling and normalizing the trace tr. In order to do that one needs that the expression $\mathrm{tr}(\alpha g_n^{-1})$, for $\alpha \in \mathrm{Y}_{d,n}(u)$, factors through $\mathrm{tr}(\alpha)$, just like $\mathrm{tr}(\alpha g_n)$ does from the Markov property of the trace [23]. Yet, we have:

$$\mathrm{tr}(\alpha g_n^{-1}) = \mathrm{tr}(\alpha g_n) + (u^{-1}-1)\mathrm{tr}(\alpha e_n) + (u^{-1}-1)\mathrm{tr}(\alpha e_n g_n) \tag{1.37}$$

Note that the following holds for any $y \in \mathrm{Y}_{d,n}(u)$:

$$\mathrm{tr}(y\, e_n g_n) = \mathrm{tr}(y g_n) = z\,\mathrm{tr}(y)$$

since:

$$y\, e_n g_n = \frac{1}{d}\sum_{m=0}^{d-1} y\, t_n^m t_{n+1}^{-m} g_n = \frac{1}{d}\sum_{m=0}^{d-1} y\, t_n^m g_n t_n^{-m}.$$

Thus, we only need further that the trace tr satisfies the multiplicative property:

$$\mathrm{tr}_d(\alpha e_n) = \mathrm{tr}_d(\alpha)\,\mathrm{tr}_d(e_n) \quad \alpha \in \mathrm{Y}_{d,n}(u) \tag{1.38}$$

With these properties we could then define framed link invariants using the same method as for defining the Jones polynomial[16]. Unfortunately, we do not have a nice formula for $\mathrm{tr}(\alpha\, e_{d,n})$. The reason is that the element e_n involves the nth strand of the braid α.

Remark 1.10. One obvious solution of the E–system is that the x_i's are d^{th} roots of unity but this of no topological importance. Indeed, if we let the x_i's be d^{th} roots of unity, we have the following example:

$$\mathrm{tr}(t_1^k t_2^l) = x_k x_l = x_{k+l} = \mathrm{tr}(t_1^{k+l} t_2^0)$$

but the closures of these two 2-stranded braids are not isotopic as framed (un)links of two components. In [23] it is proved that such a rescaling is possible if the trace parameters x_i are solutions of a non-linear system of equations, the so–called E–system.

Definition 1.12. [23] We say that the complex numbers $(x_0, x_1, \ldots, x_{d-1})$ (where x_0 is always equal to 1) satisfy the E*–condition* if $x_1, \ldots, x_{d-1}$ satisfy the following E*–system* of non–linear equations in $\mathbb{C}$:

$$E^{(m)} = x_m E \qquad (1 \leq m \leq d-1)$$

or equivalently:

$$\sum_{s=0}^{d-1} x_{m+s} x_{d-s} = x_m \sum_{s=0}^{d-1} x_s x_{d-s} \qquad (1 \leq m \leq d-1). \tag{1.39}$$

In [23, Appendix] it is proved that the solutions of the E–system are the complex functions $x : s \mapsto x_s$ over $\mathbb{Z}/d\mathbb{Z}$, parametrized by the non–empty subsets S of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ as follows:

$$x_S = \frac{1}{|S|}\sum_{s\in S} \exp_s \tag{1.40}$$

where $\exp_s(k) = \exp(2i\pi sk/d)$, for $0\ leqk \leq d-1$ (see Theorem 1.7 below).

Remark 1.11. It is worth noting that the solution of the E–system can be interpreted as a generalization of the Ramanujan's sum. Indeed, by taking the subset P of $\mathbb{Z}/d\mathbb{Z}$ consisting of the numbers coprimes to d, then the solution parametrized by P is, up to the factor $|P|$, the Ramanujan's sum $c_d(k)$ (see [30]).

Equivalently, x_S can be seen as an element in $\mathbb{C}C_d$, namely:

$$x_S = \sum_{k=0}^{d-1} x_k t^k \tag{1.41}$$

where $x_k = \frac{1}{|S|}\sum_{s\in S}\chi_s(k)$, $k = 0,\ldots,d-1$ and χ_s is the character of C_d defined as $\chi_s : t^m \mapsto \exp_s(m)$. So, the coefficient x_k of t^k in (1.41) corresponds to $x_S(k)$ in (1.40).

Recall now that on the group algebra $\mathbb{C}G$ of the finite group G, we have two products, one of them is the multiplication by coordinates, also called the multiplications of the values, which is defined as:

$$\left(\sum_{g\in G} a_g g\right)\cdot\left(\sum_{g\in G} b_g g\right) = \sum_{g\in G} a_g b_g g.$$

and the other product is the convolution product:

$$\left(\sum_{g\in G} a_g g\right) * \left(\sum_{h\in G} b_h h\right) = \sum_{g\in G}\sum_{h\in G} a_g b_h gh = \sum_{g\in G}\left(\sum_{h\in G} a_h b_{gh^{-1}}\right) g. \tag{1.42}$$

By taking $G = C_d$ and considering the element $x = \sum_{0\le k\le d-1} x_k t^k$, we have the following lemma:

Lemma 1.4. *In $\mathbb{C}C_d$ we have:*

$$x * x = d\sum_{0\le\ell\le d-1} E^{(\ell)} t^\ell$$

and

$$x * x * x = d^2\sum_{0\le\ell\le d-1} \mathrm{tr}(e_1^\ell e_2) t^\ell.$$

Proof. The expression for $x*x$ follows immediately by direct computation. Indeed, from the definition of the convolution product we have that:

$$x * x = \sum_{k=0}^{d-1} a_k t^k \sum_{\ell=0}^{d-1} a_\ell t^\ell = \sum_{k=0}^{d-1}\sum_{\ell=0}^{d-1} a_k a_\ell t^{k+\ell} = \sum_{\ell=0}^{d-1}\left(\sum_{k=0}^{d-1} a_k a_{\ell-k}\right) t^\ell = d\sum_{\ell=0}^{d-1} E^{(\ell)} t^\ell$$

For the second expression we have that:

$$\begin{aligned}
x * x * x &= d\sum_{0\le\ell\le d-1} E^{(\ell)} t^\ell * x \\
&= d\sum_{0\le\ell\le d-1} E^{(\ell)} t^\ell * \sum_{0\le k\le d-1} a_k t^k \\
&= d\sum_{0\le\ell,k\le d-1} E^{(\ell)} a_k t^{\ell+k} \\
&= d\sum_{0\le\ell,k,s\le d-1} a_s a_{\ell-s} a_k t^{\ell+k} \\
&= d\sum_{0\le\ell,k,s\le d-1} a_s a_{\ell-s-k} a_k t^{\ell} \\
&= d^2 \mathrm{tr}(e_1^{(\ell)} e_2).
\end{aligned}$$

$\square$

For each $a \in \mathbb{Z}/d\mathbb{Z}$ the character χ_a defines, with respect to the convolution product, an element $\mathbf{i}_a$ of $\mathbb{C}C_d$,

$$\mathbf{i}_a := \sum_{0 \leq s \leq d-1} \chi_a(s) t^s.$$

One can verify that

$$\mathbf{i}_a * \mathbf{i}_b = \begin{cases} d\,\mathbf{i}_a & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

that is, $\mathbf{i}_a/d$ is an idempotent element. On the other hand, regarding $\delta_a := t^a$ as element in $\mathbb{C}C_d$, it is clear that,

$$\delta_a \cdot \delta_b = \begin{cases} \delta_a & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$

The connection between the two products on $\mathbb{C}C_d$ is given by the *Fourier transform*. More precisely, the Fourier transform is the linear automorphism on $\mathbb{C}C_d$, defined as:

$$x := \sum_{0 \leq s \leq d-1} a_s t^s \mapsto \widehat{x} := \sum_{0 \leq s \leq d-1} (x * \mathbf{i}_s)(0) t^s \tag{1.43}$$

With the above notation we have:

Lemma 1.5. *The following hold in $\mathbb{C}C_d$:*

$$\widehat{x * y} = \widehat{x} \cdot \widehat{y}, \qquad \widehat{x \cdot y} = d^{-1} \widehat{x} * \widehat{y},$$

$$\widehat{\delta}_a = \mathbf{i}_{-a}, \qquad \widehat{\mathbf{i}}_a = d\delta_a, \qquad \widehat{\widehat{x}}(u) = dx(-u).$$

Proof. The proof is just a straightforward computation (see [32]). □

We can now prove the following theorem using the notation that we introduced above (see [23, Appendix] for the original proof):

Theorem 1.7 (P. Gérardin)**.** *The solutions of the* E*–system of Eq. 1.39 are of the form:*

$$x_S = \frac{1}{|S|} \sum_{s \in S} \exp_s$$

where $x(0) := 1$.

Proof. The E–condition (1.39) can be written using Lemma 1.4 as:

$$x * x = (x * x)(0)\, x.$$

To solve the E–system we use Fourier transform to obtain:

$$\widehat{x}^{\,2} = (x * x)(0)\, \widehat{x}.$$

If $(x * x)(0) = 0$, then $\widehat{x}^{\,2} = 0$ so $\widehat{x}$ is 0 and also is x, which is excluded by the condition $x(0) = 1$. Now, the equation says that the function $\widehat{x}$ is constant on its support S where it is $(x * x)(0)$. As the characteristic function of S is, up to the factor d, the Fourier transform of the sum of $\mathbf{i}_a, a \in S$, we have shown that

$$x = (x * x)(0)\frac{1}{d}\sum_{s\in S}\mathbf{i}_s = E\sum_{s\in S}\mathbf{i}_s \tag{1.44}$$

For $x(0) = 1$, we have $(x * x)(0)\frac{1}{d}|S| = 1$, with $|S|$ the cardinality of S. Applying Lemma 1.4 we deduce that:

$$E = \frac{1}{|S|} \tag{1.45}$$

We have proved that the solutions of the E–system are the functions x_S parametrized by the non–empty subsets S of the cyclic group C_d of order d as follows:

$$x_S = \frac{1}{|S|}\sum_{s\in S}\mathbf{i}_s$$

□

We have the following remark due to [23, Appendix]

Remark 1.12. For $S = C_d$, it is the trivial solution $\mathbf{i}_0$. The complement of the support of any non trivial solution is another solution. In particular, each element $a \in C_d$ defines two solutions of the E–system : one is the character χ_a, the other is given by $\dfrac{\chi_a}{1-d}$ outside 0. When the order d is even, we can take $a = d/2$, this gives the solution $u \mapsto \dfrac{(-1)^u}{1-d}, u \neq 0$.

We have the following theorem [23]:

Theorem 1.8. *If the trace parameters $x_1, \ldots, x_{d-1}$ satisfy the* E*–condition then for all $\alpha \in \mathrm{Y}_{d,n}(u)$ we have:*

$$\mathrm{tr}(\alpha e_{d,n}) = \mathrm{tr}(\alpha)\,\mathrm{tr}(e_{d,n})$$

1.11.2 Link invariants from tr

Let $d \in \mathbb{N}$ and let $X_{d,S} = \{x_1, \ldots, x_{d-1}\}$ be a solution of the E–system parametrized by a non–empty subset S of $\mathbb{Z}/d\mathbb{Z}$. Then, using Theorem 1.8 we can proceed with the factorization of $\mathrm{tr}(\alpha e_n)$ in (1.37). Indeed, we have from (1.37):

$$\begin{aligned}\mathrm{tr}(\alpha g_n^{-1}) &= \left[z + (u^{-1}-1)E + (u^{-1}-1)z\right]\mathrm{tr}(\alpha)\\ &= \frac{z-(u-1)}{u|S|}\,\mathrm{tr}(\alpha) = \mathrm{tr}(g_n^{-1})\,\mathrm{tr}(\alpha)\end{aligned}$$

where E was defined in Remark 1.9. For the value of E under the E–condition, recall from (1.45) that $E = \frac{1}{|S|}$. In analogy to the construction of the Homflypt polynomial (see Section 1.3), αg_n and αg_n^{-1} must be assigned to the same trace value for any $\alpha \in \mathrm{Y}_{d,n}(u)$. For this, we define

$$w := \frac{z-(u-1)}{uz|S|} \tag{1.46}$$

so

$$\mathrm{tr}_d(g_n^{-1}) = \frac{z-(u-1)}{u|S|} = wz. \tag{1.47}$$

Denote now by *exponential sum* $\epsilon(\alpha)$ of α as the algebraic sum of the exponents of the σ_i's in α, and let γ is the natural epimorphism of $\mathcal{F}_{d,n}$ onto $\mathrm{Y}_{d,n}(u)$. We then have the following definition [23, 22]:

Definition 1.13. Given a solution of the E–system parametrized by a non–empty subset S of $\mathbb{Z}/d\mathbb{Z}$, for any framed braid $\alpha \in \mathcal{F}_n$ we define for its closure $\widehat{\alpha}$:

$$\Gamma_{d,S}(w,u)(\widehat{\alpha}) = \left(\frac{(1-wu)|S|}{\sqrt{w}(u-1)}\right)^{n-1} (\sqrt{w})^{\epsilon(\alpha)}\mathrm{tr}(\gamma(\alpha))$$

Let now $\mathcal{L}$ denote the set of oriented framed links and let $\mathbb{C}(u)(z, x_1, \ldots, x_{d-1})$ be, as usual, the ring of rational functions on $z, X_{d,S}$ over $\mathbb{C}(u)$. Then we have the following [23]:

Theorem 1.9. *If the set $X_{d,S}$ satisfies the E–condition then the map $\Gamma_{d,S}(w,u)$ is an isotopy invariant of (modular) oriented framed links:*

$$\begin{array}{rcccl} \Gamma_{d,S}(w,u) & : & \mathcal{L} & \longrightarrow & \mathbb{C}(u)(z,x_1,\ldots,x_{d-1}) \\ & & \widehat{\alpha} & \mapsto & \Gamma_{d,S}(\widehat{\alpha}) \end{array}$$

Further, by restricting $\Gamma_S(w,u)$ to classical links, seen as framed links with all framings zero, to an invariant of classical oriented links is obtained, denoted $\Delta_{d,S}(w,u)$.

In [2] it is proved that the invariants $\Delta_{d,S}(w,u)$ coincide with the Homflypt polynomial, $P(u,\lambda)$ for the trivial cases where $u = 1$ and $E = 1$. In order to compare these two invariants, Chlouveraki and Lambropoulou specialized the indeterminates $x_1, \ldots, x_{d-1}$, thus constructing a *specialized Juyumaya trace.*

Definition 1.14. Let $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_{d-1} \in \mathbb{C}\backslash\{0\}$ and consider the ring homomorphism

$$\begin{array}{rcll} \theta : \mathbb{C}(u)[z, x_1, \ldots, x_{d-1}] & \longrightarrow & \mathbb{C}(u)[z] & \\ z & \mapsto & z & \\ x_m & \mapsto & \mathrm{x}_m & (1 \leq m \leq d-1) \end{array}$$

The map θ shall be called the *specialization map.* We will call the composition

$$\theta \circ \mathrm{tr} : \bigcup_{n \geq 0} \mathrm{Y}_{d,n}(u) \longrightarrow \mathbb{C}(u)[z]$$

the *specialized Juyumaya trace* with parameter z.

In the case where $X_{d,S} = \{x_1, \ldots, x_{d-1}\}$ is a solution of the E–system we denote: $\theta \circ \mathrm{tr} = \mathrm{tr}_{_S}$. In particular for the case where $E = 1$ we have the following:

Proposition 1.10. *Let $X_{d,S}$ be a solution of the* E*-system such that $E = 1$. Let* tr *be the Markov trace on $\mathrm{Y}_{d,n}(u)$ with parameters z, $X_{d,S}$, and let τ be the Ocneanu trace on $\mathrm{H}_n(u)$ with parameter ζ. If we take $z = \zeta$, then*

$$(\tau \circ \pi)(\alpha) = (\mathrm{tr} \circ \delta)(\alpha) \quad (\alpha \in B_n)$$

for all $n \in \mathbb{N}$.

By the assumptions of Proposition 1.10 we have that $w = \lambda$ and thus:

Corollary 1.2. *Let $X_{d,S}$ be a solution of the* E*-system such that $E = 1$. Let* tr *be the Markov trace on* $\mathrm{Y}_{d,n}(u)$ *with parameters z, $X_{d,S}$, and let τ be the Ocneanu trace on* $\mathrm{H}_n(q)$ *with parameter ζ. If we take $\zeta = z$, then*

$$P(u,\lambda)(\widehat{\alpha}) = \Delta_{d,S}(w,u)(\widehat{\alpha}) \quad (\alpha \in B_n)$$

for all $n \in \mathbb{N}$.

We conclude this chapter with a few words on the representation theory of the Yokonuma–Hecke algebra.

1.12 The representation theory of $\mathrm{Y}_{d,n}(u)$

The representation theory of the Yokonuma–Hecke algebra has initially been studied by Thiem in the general context of unipotent Hecke algebras [33]. Chlouveraki and Poulain d'Adency [4] developed an inductive and highly combinatorial approach to the representation of the $\mathrm{Y}_{d,n}(u)$. We know that $\mathrm{Y}_{d,n}(u)$ is a u–deformation of $\mathbb{C}C_{d,n}$. Since the group $C_{d,n}$ is isomorphic to the complex reflection group $G(d,1,n)$, the representations of the complex reflection group are studied instead. We start with the following definition [4]:

Definition 1.15. *Let $d \in \mathbb{N}$. A d–partition λ, or a Young d–diagram, of size n is a d–tuple of partitions such that the total number of nodes in the associated Young diagrams is equal to n. That is, we have $\lambda(1), \ldots, \lambda(d-1)$ usual partitions such that*

$$|\lambda(0)| + |\lambda(1)| + \ldots + |\lambda(d-1)| = n.$$

We also say that λ is a d–partition of n. We denote by $\mathcal{P}(d,n)$ the set of d–partitions of n. We have $P(1,n) = P(n)$.

A standard d–tableau of shape $\lambda \in \mathcal{P}(d,n)$ is a way of filling the boxes of the Young d–diagram of λ with the numbers $1, 2, \ldots, n$ such that the entries strictly increase down the columns and along the rows.

For example for the standard d–tableaux $\mathcal{T}$ of size 3 we have :

$$\mathcal{T} = \left(\boxed{1\,|\,3},\ \varnothing,\ \boxed{2} \right)$$

We write $\theta = (x,y)$ for the node in row x and column y. A pair (θ, k) consisting of a node θ and an integer $k \in \{1, \ldots, d\}$ is called *a d–node*. The integer k is called *the position of θ*. A d–partition is then the set of d–nodes such that the subset consisting of the d–nodes having position k forms a usual partition, for any $k \in \{1, \ldots, d\}$.

For a d–node lying in the line x and column y of the k^{th} diagram of λ (that is $\theta = (x,y,k)$), define $p(\theta) := k$, the *position* of θ and $c(\theta) := q^{2(y-x)}$, the *quantum content* of θ.

For a d–tableaux $\mathcal{T}$, we denote by $p(\mathcal{T}\,|\,i)$ and $q(\mathcal{T}\,|\,i)$ the position and the quantum content of the d–node with the number i in it.

Let $\{\xi_1, \ldots, \xi_{d-1}\}$ be the set of all roots of unity ordered arbitrarily. Let also V_λ be a $\mathbb{C}(u)$–vector space with basis $\{\mathbf{v}_\mathcal{T}\}$ indexed by the standard d–tableaux of shape λ, and set $\mathbf{v}_\mathcal{T} := 0$ for any non-standard d–tableaux $\mathcal{T}$ of shape λ. We have the following proposition [3]:

Proposition 1.11. *Let $\mathcal{T}$ be a standard d–tableaux of shape $\lambda \in \mathcal{P}(d,n)$. The vector space V_λ is a representation of $\mathrm{Y}_{d,n}(u)$ with the action of the generators on the basis elements $\mathbf{v}_\mathcal{T}$ defined as follows for $i = 1, \ldots, n$:*

$$t_j(\mathbf{v}_\mathcal{T}) = \xi_{p_j}\mathbf{v}_\mathcal{T}$$

for $i = 1, \ldots, n-1$ if $p_i \neq p_{i+1}$ then:

$$g_i(\mathbf{v}_\mathcal{T}) = \mathbf{v}_{\mathcal{T}^{s_i}}$$

and if $p_i = p_{i+1}$ then:

$$g_i(\mathbf{v}_\mathcal{T}) = \frac{c_{i+1}(u-u^{-1})}{c_{i+1}-c_i} + \frac{uc_{i+1}-u^{-1}c_i}{c_{i+1}-c_i}\mathbf{v}_{\mathcal{T}^{s_i}}$$

where s_i is the transposition $(i, i+1)$.

The following Theorem describes the irreducible representations of $\mathrm{Y}_{d,n}(u)$ [4].

Theorem 1.10. *For any $\lambda \in \mathcal{P}(d,n)$, let V_λ denote the representation of $\mathrm{Y}_{d,n}(u)$ constructed in Proposition 1.11. Then*

(a) If V_λ is isomorphic to $V_{\lambda'}$ for some $\lambda' \in \mathcal{P}(d,n)$ then $\lambda = \lambda'$.

(b) The representation V_λ is irreducible.

(c) The set $\{V_\lambda \,|\, \lambda \in \mathcal{P}(d,n)\}$ is a complete set of pairwise non–isomorphic irreducible representations of $\mathrm{Y}_{d,n}(u)$.

Remark 1.13. Since $\mathrm{Y}_{d,n}(u)$ is a u–deformation of $\mathbb{C}C_{d,n}$, the specialization $u \mapsto 1$ induces a bijection between the irreducible representations of $\mathrm{Y}_{d,n}(u)$ and the following set of irreducible representations of $G(d,1,n)$ [4]:

$$\mathrm{Irr}(G(d,1,n)) = \{E^\lambda \,|\, \lambda \in \mathcal{P}(d,\lambda)\}. \tag{1.48}$$

Chapter 2

Framization of the Temperley–Lieb Algebra

2.1 Three possible candidates

Recall that the algebra $\mathrm{H}_n(u)$ is a u–deformation of the group algebra $\mathbb{C}S_n$, while $\mathrm{TL}_n(u)$ is the quotient of $\mathrm{H}_n(u)$ over the two–sided ideal that is generated by the Steinberg elements $h_{i,i+1}$, given in Definition 1.9, namely:

$$h_{i,i+1} = \sum_{w \in \langle s_i, s_{i+1} \rangle} h_w.$$

As mentioned in the Introduction, the algebra $\mathrm{Y}_{d,n}(u)$ is the basic example of a framization of the Iwahori–Hecke alebra, $\mathrm{H}_n(u)$. Moreover, from the discussion in Section 1.9, $\mathrm{Y}_{d,n}(u)$ can be regarded as a u–deformation of the group algebra $\mathbb{C}C_{d,n}$. Following the construction of the classical Temperley–Lieb algebra we would like to introduce an analogue of $\mathrm{TL}_n(u)$ in the 'framing context', that is, in the context of framed knot algebras. Namely, to define a quotient of $\mathrm{Y}_{d,n}(u)$ over a two–sided ideal that is constructed from an appropriately chosen subgroup of the underlying group $C_{d,n}$ of $\mathrm{Y}_{d,n}(u)$. At this point two such subgroups emerge naturally. More precisely, the subgroups $\langle s_i, s_{i+1} \rangle$ of S_n that are related to the defining ideal of $\mathrm{TL}_n(u)$ can be also considered as subgroups of $C_{d,n}$. Thus, we consider the following elements in $\mathrm{Y}_{d,n}(u)$:

$$g_{i,i+1} := \sum_{w \in \langle s_i, s_{i+1} \rangle} g_w = 1 + g_i + g_{i+1} + gig_{i+1} + g_{i+1}g_i + g_i g_{i+1} g_i \tag{2.1}$$

The second possibility is to let the framing generators t_i be involved in the generating set of such a subgroup. Thus, we consider the following subgroup of $C_{d,n}$:

$$C_{i,i+1} := \langle t_i, t_{i+1}, t_{i+2} \rangle \rtimes \langle s_i, s_{i+1} \rangle.$$

We also consider the following elements in $\mathrm{Y}_{d,n}(u)$:

$$c_{i,i+1} := \sum_{w \in C_{i,i+1}} g_w = \sum_{\alpha,\beta,\gamma \in C_d} t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma}\, g_{i,i+1} \tag{2.2}$$

The second equalities in (2.1) and (2.2) follow from the multiplication defined on $\mathrm{Y}_{d,n}(u)$ (recall Eq. 1.35). Therefore we can define at least two types of algebras which could be considered as analogues of the Temperley–Lieb algebras in the context of knot algebras with framing.

Definition 2.1. For $n \geq 3$, the *Yokonuma–Temperley–Lieb algebra*, denoted by $\mathrm{YTL}_{d,n}(u)$, is defined as the quotient of $\mathrm{Y}_{d,n}(u)$ over the two–sided ideal I_1 generated by the elements $g_{i,i+1}$, namely:

$$\mathrm{YTL}_{d,n}(u)\frac{\mathrm{Y}_{d,n}(u)}{\langle g_{i,i+1}\text{ , for all } i\rangle} = \frac{\mathrm{Y}_{d,n}(u)}{I_1}.$$

Definition 2.2. For $n \geq 3$, we define the *Complex Reflection Temperley–Lieb* algebra, denoted by $\mathrm{CTL}_{d,n}(u)$, as the quotient of the algebra $\mathrm{Y}_{d,n}(u)$ over the ideal that is generated by the elements $c_{i,i+1}$, that is:

$$\mathrm{CTL}_{d,n}(u) := \frac{\mathrm{Y}_{d,n}(u)}{I_2}.$$

As mentioned in Section 1.11.2, invariants for framed knots and links are defined through the trace tr on the Yokonuma–Hecke algebra, by imposing the E–system on the parameters $x_1, \ldots, x_{d-1}$ [23]. Hence, we expect that the 'framization' of the Temperley–Lieb algebra will allow us to define a 'framization' of the Jones polynomial, analogous to that of the Jones polynomial. Unfortunately, both quotients above are not satisfactory for this purpose. More precisely, in the case of I_1, very strong conditions on the trace parameters must be applied in order for tr to pass through to the quotient algebra. This leads to a great loss of topological information, thus $\mathrm{YTL}_{d,n}(u)$ is also not a good candidate. However, the original Jones polynomial can be recovered from this quotient algebra. In the case of I_2, the algebra $\mathrm{CTL}_{d,n}(u)$ is large enough so that only conditions on the trace parameters z are needed in order that tr passes to the quotient algebra $\mathrm{CTL}_{d,n}(u)$ (see Theorem 3.7). However, in order to obtain knot invariants we would still need to impose the E–system on the trace parameters $x_1, \ldots, x_{d-1}$ as in the case of $\mathrm{Y}_{d,n}(u)$.

The trace considerations indicate a third alternative of definition for the analogue of the Temperley–Lieb algebra as knot algebra with framing: the *Framization of the Temperley–Lieb algebra*. More precisely, we define this framization as a quotient of the Yokonuma–Hecke algebra over an ideal that is constructed from the following subgroup of $C_{d,n}$:

$$H_{i,i+1} := \langle t_i t_{i+1}^{-1}, t_{i+1} t_{i+2}^{-1}\rangle \rtimes \langle s_i, s_i + 1\rangle \quad \text{for all } i.$$

Each x in $H_{i,i+1}$ can be written in the form:

$$x = t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} w, \tag{2.3}$$

where $\alpha + \beta + \gamma = 0$ and $w \in \langle s_i, s_{i+1}\rangle$.

We have the following definition:

Definition 2.3. For $n \geq 3$, the *Framization of the Temperley–Lieb algebra*, denoted by $\mathrm{FTL}_{d,n}(u)$, is defined as the quotient $\mathrm{Y}_{d,n}(u)$ over the two–sided ideal J generated by the elements

$$r_{i,i+1} := \sum_{x \in H_{i,i+1}} g_x = \sum_{\substack{\alpha+\beta+\gamma=0 \\ w \in \langle s_i, s_{i+1}\rangle}} t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma}\, g_w \qquad (i = 1, \ldots, n-2), \tag{2.4}$$

where the second equality follows from the multiplication defined on $\mathrm{Y}_{d,n}(u)$.

The Yokonuma–Temperley–Lieb algebra was introduced and studied in [8], while the algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ in [9].

2.2 Comparison of the three quotient algebras

We will now give a relation between these algebras. Following the discussion above, we have the following interesting quotients of $\mathrm{Y}_{d,n}(u)$ to consider:

$$\mathrm{CTL}_{d,n}(u), \qquad \mathrm{FTL}_{d,n}(u), \qquad \mathrm{YTL}_{d,n}(u).$$

Notice now that the following hold:

$$\langle s_i, s_{i+1}\rangle \leq H_{i,i+1} \leq C_{i,i+1} \implies I_2 \lhd J \lhd I_1$$

The second inclusion of the ideals, $J \lhd I_1$ is clear. Indeed, every x in $H_{i,i+1}$ can be written in the form:

$$x = t_i^a t_{i+1}^{-a} t_{i+1}^b t_{i+2}^{-b} w = t_i^a t_{i+1}^{b-a} t_{i+2}^{-b} w, \quad \text{where } w \in \langle s_i, s_{i+1}\rangle.$$

Therefore, from the multiplication rule of Eq. 1.35, we have that $g_x = t_i^a t_{i+1}^{b-a} t_{i+2}^{-b} g_w$. Thus we can rewrite the elements $r_{i,i+1}$ in the following form:,

$$r_{i,i+1} = \sum_{\substack{a,b\mathbb{Z}/d\mathbb{Z} \\ w\in\langle s_i,s_{i+1}\rangle}}^{d-1} t_i^a t_{i+1}^{b-a} t_{i+2}^{-b} g_w = \left(\sum_{a,b\in\mathbb{Z}/d\mathbb{Z}}^{d-1} t_i^a t_{i+1}^{b-a} t_{i+2}^{-b}\right)\left(\sum_{w\in\langle s_i,s_{i+1}\rangle} g_w\right) = e_i e_{i+1} g_{i,i+1} \tag{2.5}$$

We shall proceed with the proof for the first inclusion of ideals. We have that:

$$C_{i,i+1} = H_{i,i+1} \rtimes C_d.$$

Indeed, let $x = t_i^\alpha t_{i+1}^\beta t_{i+2}^\gamma\, w \in C_{i,i+1}$, where $w \in \langle s_i, s_{i+1}\rangle$, and let ϕ be the following homorphism:

$$\begin{aligned} \phi : C_{i,i+1} &\to C_d \\ x &\mapsto t_i^{\alpha+\beta+\gamma} \end{aligned}$$

where $C_d \simeq \langle t_i \,|\, t_i^d = 1\rangle$. Observe that $\ker\phi = H_{i,i+1}$ and also that $\phi\big|_{H_{i,i+1}} = \mathrm{id}_{C_d}$. This implies that $C_{i,i+1} = H_{i,i+1} \rtimes C_d$. Thus, given $x \in C_{i,i+1}$ we have an unique decomposition $x = t_i^k y$, where $k \in \mathbb{Z}/d\mathbb{Z}$ and $y \in H_{i,i+1}$. Now, from the multiplication rule of $\mathrm{Y}_{d,n}(u)$, we have:

$$c_{i,i+1} = \sum_{x\in C_{i,i+1}} g_x = \sum_{k\in\mathbb{Z}/d\mathbb{Z}} t_i^k \sum_{y\in H_{i,i+1}} g_y. \tag{2.6}$$

Hence $I_2 \lhd J$.

We have proved the following proposition:

Proposition 2.1. *The inclusions of ideals above yield the following natural commutative diagram of epimorphisms:*

$$\begin{tikzcd} \mathrm{Y}_{d,n}(u) \arrow[r] \arrow[d] & \mathrm{CTL}_{d,n}(u) \arrow[r] \arrow[d] & \mathrm{FTL}_{d,n}(u) \arrow[r] \arrow[dl] & \mathrm{YTL}_{d,n}(u) \arrow[dll] \\ \mathrm{H}_n(u) \arrow[r] & \mathrm{TL}_n(u) & & \end{tikzcd}$$

2.3 The Yokonuma–Temperley–Lieb algebra

The rest of this section will be dedicated to our results on the Yokonuma–Temperley–Lieb algerba.

2.3.1 The definition of $\mathrm{YTL}_{d,n}(u)$

We will first give a definition of $\mathrm{YTL}_{d,n}(u)$ in terms of generators and relations. We have the following:

Definition 2.4. For $n \geq 3$, the *Yokonuma–Temperley–Lieb* algebra can be presented by the generators $g_1, \dots, g_{n-1}$, $t_1, \dots, t_n$ (by abuse of notation), subject to the following relations:

$$g_i g_j = g_j g_i, \quad |i-j| > 1 \tag{2.7}$$
$$g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i \tag{2.8}$$
$$g_i^2 = 1 + (u-1)e_i + (u-1)e_i g_i \tag{2.9}$$
$$t_i t_j = t_j t_i, \quad \text{for all } i, j \tag{2.10}$$
$$t_i^d = 1, \quad \text{for all } i \tag{2.11}$$
$$g_i t_i = t_{i+1} g_i \tag{2.12}$$
$$g_i t_{i+1} = t_i g_i \tag{2.13}$$
$$g_i t_j = t_j g_i, \quad \text{for } j \neq i, \text{ and } j \neq i+1 \tag{2.14}$$
$$g_i g_{i+1} g_i + g_i g_{i+1} + g_{i+1} g_i + g_i + g_{i+1} + 1 = 0 \tag{2.15}$$

We shall refer to relations (2.15) as *the Steinberg relations of* $\mathrm{YTL}_{d,n}(u)$.

Note that relations (2.15) are symmetric with respect to the indices i, $i+1$, i.e.:

$$g_i g_{i+1} g_i = -g_i g_{i+1} - g_{i+1} g_i - g_{i+1} - g_i - 1 = g_{i+1} g_i g_{i+1}.$$

so relations (2.8) follow from relations (2.15).

Remark 2.1. In analogy to the Yokonuma–Hecke algebra, $\mathrm{YTL}_{1,n}(u)$ coincides with the algebra $\mathrm{TL}_n(u)$. Further, the epimorphism (1.34) induces an epimorphism of the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$ onto the algebra $\mathrm{TL}_n(u)$. Clearly, by relations (2.12) and (2.13), any monomial in $\mathrm{YTL}_{d,n}(u)$ inherits the *splitting property* of $\mathrm{Y}_{d,n}(u)$, that is, it can be written in the form:

$$w = t_1^{a_1} \dots t_n^{a_n} g_{i_1} \dots g_{i_k}, \tag{2.16}$$

where: $a_1, \dots, a_n \in \mathbb{Z}/d\mathbb{Z}$.

We shall now prove that the ideal I_1 is in fact principal.

Lemma 2.1. *The following hold in* $\mathrm{Y}_{d,n}(u)$ *for all* $i = 1, \dots, n-2$*:*

$$\begin{array}{llcl}
(1) & g_i & = & (g_1 \dots g_{n-1})^{i-1}\, g_1\, (g_1 \dots g_{n-1})^{-(i-1)} \\
(2) & g_{i+1} & = & (g_1 \dots g_{n-1})^{i-1}\, g_2\, (g_1 \dots g_{n-1})^{-(i-1)} \\
(3) & g_i g_{i+1} & = & (g_1 \dots g_{n-1})^{i-1}\, g_1 g_2\, (g_1 \dots g_{n-1})^{-(i-1)} \\
(4) & g_{i+1} g_i & = & (g_1 \dots g_{n-1})^{i-1}\, g_2 g_1\, (g_1 \dots g_{n-1})^{-(i-1)} \\
(5) & g_i g_{i+1} g_i & = & (g_1 \dots g_{n-1})^{i-1}\, g_1 g_2 g_1\, (g_1 \dots g_{n-1})^{-(i-1)}
\end{array}$$

Proof. We will demonstrate the proof for the cases (1) and (5). The rest of the cases are proved in an analogous manner. For case (1) we have that the statement is true for $i = 2$. Indeed:

$$\begin{aligned}(g_1 \dots g_{n-1})\, g_1\, (g_1 \dots g_{n-1})^{-1} &= g_1 g_2 g_1 g_3 \dots g_{n-1} (g_1 \dots g_{n-1})^{-1} \\ &= g_2 (g_1 g_2 \dots g_{n-1})(g_1 \dots g_{n-1})^{-1} \\ &= g_2.\end{aligned}$$

Suppose that the statement is true for $i = k$. We will show that the statement holds for $i = k + 1$. We have:

$$\begin{aligned}(g_1 \dots g_{n-1})^k g_1 (g_1 \dots g_{n-1})^{-k} &= (g_1 \dots g_{n-1})(g_1 \dots g_{n-1})^{k-1} g_1 (g_1 \dots g_{n-1})^{-(k-1)} (g_1 \dots g_{n-1})^{-1} \\ &= (g_1 \dots g_{n-1}) g_k (g_1 \dots g_{n-1})^{-1} \\ &= g_1 \dots g_{k-1} g_k g_{k+1} g_k g_{k+2} \dots g_{n-1} (g_1 \dots g_{n-1})^{-1} \\ &= g_1 \dots g_{k-1} g_{k+1} g_k g_{k+1} \dots g_{n-1} (g_1 \dots g_{n-1})^{-1} \\ &= g_{k+1} (g_1 \dots g_{n-1})(g_1 \dots g_{n-1})^{-1} \\ &= g_{k+1}.\end{aligned}$$

For case (5) we have from (1):

$$\begin{aligned}g_i g_{i+1} g_i &= (g_1 \dots g_{n-1})^{i-1} g_1 (g_1 \dots g_{n-1})^{-(i-1)} (g_1 \dots g_{n-1})^{i} g_1 (g_1 \dots g_{n-1})^{-i} \\ &\quad \cdot (g_1 \dots g_{n-1})^{i-1} g_1 (g_1 \dots g_{n-1})^{-(i-1)} \\ &= (g_1 \dots g_{n-1})^{i-1} g_1 (g_1 \dots g_{n-1})^{-(i-1)} (g_1 \dots g_{n-1})^{i-1} (g_1 \dots g_{n-1}) \\ &\quad \cdot g_1 (g_1 \dots g_{n-1})^{-1} (g_1 \dots g_{n-1})^{-(i-1)} (g_1 \dots g_{n-1})^{i-1} g_1 (g_1 \dots g_{n-1})^{-(i-1)} \\ &= (g_1 \dots g_{n-1})^{i-1} g_1 g_2 g_1 (g_1 \dots g_{n-1})^{-(i-1)}.\end{aligned}$$

□

Corollary 2.1. $\mathrm{YTL}_{d,n}(u)$ *is the* $\mathbb{C}(u)$*–algebra generated by the set* $\{t_1, \dots, t_n, g_1, \dots, g_{n-1}\}$ *whose elements are subject to the defining relations of* $\mathrm{Y}_{d,n}(u)$ *and the relation:*

$$g_{1,2} = 0.$$

Proof. The result follows using the multiplication rule (1.35) defined on $\mathrm{Y}_{d,n}(u)$ and Lemma 2.1. □

2.3.2 A presentation with non-invertible generators for $\mathrm{YTL}_{d,n}(u)$

In analogy with Eq. 1.11 one can obtain a presentation for the Yokonuma–Temperley–Lieb algebra with the non-invertible generators:

$$\ell_i := \frac{1}{u+1}(g_i + 1). \tag{2.17}$$

In particular we have:

Proposition 2.2. $\mathrm{YTL}_{d,n}(u)$ *can be viewed as the algebra generated by the elements:*

$$\ell_1, \dots \ell_{n-1}, t_1, \dots, t_n,$$

which satisfy the following defining relations:

$$t_i^d = 1, \quad \textit{for all } i \tag{2.18}$$
$$t_it_j = t_jt_i, \quad \textit{for all } i,j \tag{2.19}$$
$$\ell_it_j = t_j\ell_i, \quad \textit{for } j \neq i \textit{ and } j \neq i+1 \tag{2.20}$$
$$\ell_it_i = t_{i+1}\ell_i + \frac{1}{u+1}(t_i - t_{i+1}) \tag{2.21}$$
$$\ell_it_{i+1} = t_i\ell_i + \frac{1}{u+1}(t_{i+1} - t_i) \tag{2.22}$$
$$\ell_i^2 = \frac{(u-1)e_i + 2}{u+1}\ell_i \tag{2.23}$$
$$\ell_i\ell_j = \ell_j\ell_i, \quad |i-j| > 1 \tag{2.24}$$
$$\ell_i\ell_{i+1}l_i = \frac{(u-1)e_i + 1}{(u+1)^2}\ell_i \tag{2.25}$$

Proof. Obviously, $\mathrm{YTL}_{d,n}(u)$ is generated by the ℓ_i's and the t_i's. It is a straightforward computation to see that relations (2.7)–(2.15) are transformed into the relations (2.18) – (2.25). Indeed, for Eq. 2.20 we have, for $|i-j| > 1$, that:

$$\ell_it_j = \frac{1}{u+1}(g_i+1)t_j = \frac{1}{u+1}(g_it_j + t_j) + \frac{1}{u+1}(t_jg_i + t_j) = t_j\frac{1}{u+1}(g_i+1) = t_j\ell_i.$$

For Eq. 2.21 we have that:

$$\ell_it_i = \frac{1}{u+1}(g_i+1)t_i = \frac{1}{u+1}(g_it_i + 1) = \frac{1}{u+1}(t_{i+1}g_i + 1). \tag{2.26}$$

On the other hand, multiplying Eq. 2.17 with t_{i+1} from the left gives:

$$t_{i+1}\ell_i = \frac{1}{u+1}t_{i+1}g_i + \frac{1}{u+1}t_{i+1}. \tag{2.27}$$

From Eqs. 2.26 and 2.27 we deduce that:

$$\ell_it_i = t_{i+1}\ell_i + \frac{1}{u+1}(t_i - t_{i+1}).$$

Equation 2.22 is proved in an analogous way. For the quadratic relation, we have from Eq. 2.17 that:

$$g_i = (u+1)\ell_i - 1. \tag{2.28}$$

We then have that:

$$g_i^2 = \left((u+1)^2\ell_i - 1\right)^2$$

which is equivalent to:

$$1 + (u-1)e_i + (u-1)e_ig_i = (u+1)^2\ell_i^2 - 2(u-1)\ell_i + 1$$

or equivalently:

$$(u-1)(u+1)e_i\ell_i = (u+1)^2\ell_i^2 - 2(u+1)\ell_i$$

which leads to:

$$\ell_i^2 = \frac{(u-1)e_i + 2}{u+1}\ell_i.$$

which is Eq. 2.23. For Eq. 2.24 we have, for $|i-j|>1$, that:

$$\begin{aligned}\ell_i\ell_j &= \frac{1}{(u+1)^2}(g_i+1)(g_j+1) = \frac{1}{(u+1)^2}(g_ig_j+g_i+g_j+1)\\ &= \frac{1}{(u+1)^2}(g_jg_i+g_j+g_i+1) = \frac{1}{(u+1)^2}(g_j+1)(g_i+1) = \ell_j\ell_i.\end{aligned}$$

Finally, for the Steinberg elements $g_{i,i\pm1}$ using Eq. 2.28 we have that:

$$g_{i,i+1} = g_ig_{i+1}g_i+g_{i+1}g_i+g_ig_{i+1}+g_{i+1}+g_i+1 = (u+1)^3\ell_i\ell_{i+1}\ell_i-(u+1)^2\ell_i^2+(u+1)\ell_i$$

From the Steinberg relation (2.15) and Eq. 2.23 we have that:

$$(u+1)^2\ell_i\ell_{i+1}\ell_i = ((u-1)e_i+1)\ell_i$$

or equivalently:

$$\ell_i\ell_{i+1}\ell_i = \frac{(u-1)e_i+1}{(u+1)^2}\,\ell_i,$$

which is Eq. 2.25. □

Remark 2.2. Setting $d=1$ in the presentation of $\mathrm{YTL}_{d,n}(u)$ in Proposition 2.2, one obtains the classical presentation of $\mathrm{TL}_n(u)$, as discussed in Subsection 1.5. Note also that, substituting in the braid relation (2.8) the g_i's using Eq. 2.28, we obtain the equation:

$$\ell_i\ell_{i+1}\ell_i - \frac{(u-1)e_i+1}{(u+1)^2}\,\ell_i = \ell_{i+1}\ell_i\ell_{i+1} - \frac{(u-1)e_{i+1}+1}{(u+1)^2}\,\ell_{i+1}$$

which becomes superfluous, since it can be deduced from Eq. 2.25. This was to be expected, since the braid relations (2.8) were also superfluous.

2.3.3 A spanning set for $\mathrm{YTL}_{d,n}(u)$

In this section we discuss various properties of a word in $\mathrm{YTL}_{d,n}(u)$ and we present a spanning set for $\mathrm{YTL}_{d,n}(u)$ (Proposition 2.4). Furthermore, we give the dimension of $\mathrm{YTL}_{d,n}(u)$ (Proposition 2.5) as computed by Chlouveraki and Pouchin in [3] and we also present their results on the linear basis of $\mathrm{YTL}_{d,n}(u)$ (Theorem 2.1). We finally compute a basis for $\mathrm{YTL}_{2,3}(u)$ different than the one of Theorem 2.1. We have the following definition:

Definition 2.5. In $\mathrm{YTL}_{d,n}(u)$ we define a length function l as follows:

$$l(t^a g_{i_1}\dots g_{i_k}) := l'(s_{i_1}\dots s_{i_k}),$$

where l' is the usual *length function* of S_n and $t^a := t_1^{a_1}\dots t_n^{a_n} \in C_d^n$. A word in $\mathrm{YTL}_{d,n}(u)$ of the form (2.16) shall be called *reduced* if it is of minimal length with respect to relations (2.7)–(2.9), (2.15).

Proposition 2.3. *Each word in* $\mathrm{YTL}_{d,n}(u)$ *can be written as a sum of monomials, where the highest and lowest index of the generators* g_i *appear at most once.*

Proof. Since $\mathrm{YTL}_{d,n}(u)$ is a quotient of the algebra $\mathrm{Y}_{d,n}(u)$ the highest index property (Proposition 1.7) passes through to the algebra $\mathrm{YTL}_{d,n}(u)$. The idea is analogous to [15, Lemma 4.1.2] and it is based on induction on the length of reduced words, use of the braid relations and reduction of length using the quadratic relations (2.9). For the case of the lowest index generator g_i we shall use induction on the length of words and the Steinberg relations (2.15). Indeed, clearly, the statement is true for all words of length ≤ 2, namely for words of the form t^a, $t^a g_1$ $t^a g_1 g_2$ and $t^a g_2 g_1$.

For words of length 3: Let $w = t^a g_1 g_2 g_1$. Applying relation (2.8) will violate the highest index property of the word, so we must use the Steinberg relation (2.15) and we have:

$$t^a g_1 g_2 g_1 \quad = \quad -t^a g_2 g_1 - t^a g_1 g_2 - t^a g_2 - t^a g_1 - t^a.$$

We assume that the lowest index generator appears at most once in all words of length $\leq r$, and we will show the lowest index property for words of length $r+1$. Let $w = t^a g_{i_1} g_{i_2} \ldots g_{i_k}$ be a reduced word in $\mathrm{YTL}_{d,n}(u)$ of length $r+1$, and $l = \min\{i_1, \ldots, i_k\}$.

Let first $w = t^a w_1 g_l w_2 g_l w_3$, and suppose that w_2 does not contain g_l. We then have two possibilities:

If w_2 does not contain g_{l+1}, then g_l commutes with all the g_i's in w_2 so the length of w can be reduced using the quadratic relations (2.9) for g_l^2 and we use the induction hypothesis:

$$\begin{aligned} w &= t^a w_1 g_l w_2 g_l w_3 \\ &= t^a w_1 w_2 g_l^2 w_3 \\ &= t^a w_1 w_2 (1 + (u-1)e_l + (u-1)e_l g_l) w_3 \\ &= t^a w_1 w_2 w_3 + (u-1) t^a w_1 w_2 e_l w_3 + (u-1) t^a w_1 w_2 e_l g_l w_3. \end{aligned}$$

If w_2 does contain g_{l+1}, then, by the induction hypothesis w_2 has the form $w_2 = v_1 g_{l+1} v_2$, where in v_1, v_2 the lowest index generator is at least g_{l+2}, hence:

$$\begin{aligned} w &= t^a w_1 g_l v_1 g_{l+1} v_2 g_l w_3 \\ &= t^a w_1 v_1 g_l g_{l+1} g_l v_2 w_3 \\ &= t^a w_1 v_1 g_{l+1} g_l g_{l+1} v_2 w_3, \end{aligned}$$

and there is one less occurrence of g_l in w. In the case where $l+1 = m$, where $m = max\,\{i_1, \ldots i_k\}$, we apply instead the Steinberg relation (2.15), so no contradiction is caused with respect to the highest index generator. Continuing in the same manner for all possible pairs of g_l in the word we reduce to having g_l at most once. $\square$

The following proposition gives us a precise spanning set for $\mathrm{YTL}_{d,n}(u)$.

Proposition 2.4. *The following set of reduced words*

$$\Sigma_{d,n} \quad = \quad \left\{ t^a (g_{i_1} g_{i_1 - 1} \ldots g_{i_1 - k_1})(g_{i_2} g_{i_2 - 1} \ldots g_{i_2 - k_2}) \ldots (g_{i_p} g_{i_p - 1} \ldots g_{i_p - k_p}) \right\} \quad (2.29)$$

where

$$t^a = t_1^{a_1} \ldots t_n^{a_n} \in C_d^n, \quad 1 \leq i_1 < i_2 < \ldots < i_p \leq n-1,$$

and

$$1 \leq i_1 - k_1 < i_2 - k_2 < \ldots < i_p - k_p,$$

spans the Yokonuma–Temperley–Lieb algebra $\mathrm{YTL}_{d,n}(u)$. *The highest index generator is* g_{i_p} *of the rightmost cycle and the lowest index generator is* $g_{i_1-k_1}$ *of the leftmost cycle of a word in* $\Sigma_{d,n}$.

Proof. We will prove the statement by induction on the length of a word starting from the linear basis of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}(u)$ [18, Proposition 8]. Namely,

$$\mathcal{B}_{\mathrm{Y}_{d,n}} = \left\{ t^a (g_{i_1} g_{i_1-1} \ldots g_{i_1-k_1})(g_{i_2} g_{i_2-1} \ldots g_{i_2-k_2}) \ldots (g_{i_p} g_{i_p-1} \ldots g_{i_p-k_p}) \right\}, \quad (2.30)$$

where:

$$a \in (\mathbb{Z}/d\mathbb{Z})^n, \quad 1 \leq i_1 < i_2 < \ldots < i_p \leq n-1.$$

Note that $\mathcal{B}_{\mathrm{Y}_{d,n}}$ spans linearly the quotient $\mathrm{YTL}_{d,n}(u)$ since it is a quotient of $\mathrm{Y}_{d,n}(u)$ and also that in $\mathcal{B}_{\mathrm{Y}_{d,n}}$ there is no restriction on the indices $i_1-k_1, \ldots, i_p-k_p$. Starting now with a word in the set $\mathcal{B}_{\mathrm{Y}_{d,n}}$, we will show that it is a linear combination of words in the subset $\Sigma_{d,n}$. The statement holds trivially for words of length 0,1 and 2, since such words are in $\Sigma_{d,n}$. For length 3 consider the representative case of the word $t^a g_1 g_2 g_1$ which is not in $\Sigma_{d,n}$. Applying the Steinberg relation (2.15) a linear combination of words in $\Sigma_{d,n}$ is obtained (see Eq. 2.29). Suppose now that the statement holds for all words of length $\leq q$, namely, that any word in $\mathcal{B}_{\mathrm{Y}_{d,n}}$ of length q can be written as a linear combination of words in $\Sigma_{d,n}$. Let w be a word in $\mathcal{B}_{\mathrm{Y}_{d,n}}$ of length $q+1$ which is not contained in $\Sigma_{d,n}$. Then w must contain a pair of consecutive cycles:

$$(g_{i_1} g_{i_1-1} \ldots g_k)(g_{i_2} g_{i_2-1} \ldots g_l),$$

where $k \geq l$. It suffices to consider the situation where $i_2 = i_1 + 1$, otherwise the generators of higher index may pass temporarily to the left of the word. Next, we move the term g_k as far to the right as possible obtaining:

$$(g_{i_1} \ldots g_{k+1})(g_{i_2} \ldots g_{k+2} \underline{g_k g_{k+1} g_k} g_{k-1} \ldots g_l).$$

We now apply the Steinberg relation (2.15) and we obtain five terms, all of length $< q+1$, and we apply the induction hypothesis. More precisely, we have the following five terms:

$(g_{i_1} \ldots g_{k+1})(g_{i_2} \ldots g_{k+2} \underline{g_{k+1} g_k} g_{k-1} \ldots g_l)$,
$(g_{i_1} \ldots g_{k+1})(g_{i_2} \ldots g_{k+2} \underline{g_{k+1}} g_{k-1} \ldots g_l)$,
$(g_{i_1} \ldots g_{k+1})(g_{i_2} \ldots g_{k+2} \underline{g_k} g_{k-1} \ldots g_l)$,
$(g_{i_1} \ldots g_{k+1})(g_{i_2} \ldots g_{k+2} \underline{g_k g_{k+1}} g_{k-1} \ldots g_l)$,
$(g_{i_1} \ldots g_{k+1})(g_{i_2} \ldots g_{k+2} g_{k-1} \ldots g_l)$.

Proposition 2.3 guarantees that the highest and lowest index generator will appear at most once in each word of $\Sigma_{d,n}$. To see the exact position of the highest and lowest index generators in the words of $\Sigma_{d,n}$ one can observe that the position of the highest index generator g_i is already clear in the set $\mathcal{B}_{\mathrm{Y}_{d,n}}$ (cf. [18] [16]). To establish the position of the lowest index generator in the words of $\Sigma_{d,n}$ we shall analyze each of the five terms above. In the first term the lowest indices of the two cycles are not in the desired form. To resolve this, we move g_{k+1} to the right in order to create the term $g_{k+1} g_{k+2} g_{k+1}$, we apply the Steinberg relation once more and we use the induction hypothesis. In the second term the subword $(g_{k-1} \dots g_l)$ may pass to the left (since the generator g_k has disappeared), so we obtain the following word:

$$(g_{k-1} \dots g_l)(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+1}). \tag{2.31}$$

This word contains two cycles with the same lowest index generators, hence we need to apply the Steinberg relation (2.15) and use the induction hypothesis as above. In the third term, g_k returns to its original position and the subword $(g_{k-1} \dots g_l)$ may pass to the left, obtaining a word in the set $\Sigma_{d,n}$, namely:

$$(g_{i_1} \dots g_{k+1} g_k g_{k-1} \dots g_l)(g_{i_2} \dots g_{k+2}). \tag{2.32}$$

The same holds for the forth term, which can be rewritten as:

$$(g_{i_1} \dots g_{k+1} g_k g_{k-1} \dots g_l)(g_{i_2} \dots g_{k+1}). \tag{2.33}$$

Finally, in the fifth term, the subword $(g_{k-1} \dots g_l)$ may pass to the far left, namely:

$$(g_{k-1} \dots g_l)(g_{i_1} \dots g_{k+1})(g_{i_2} \dots g_{k+2}), \tag{2.34}$$

which is a word in the set $\Sigma_{d,n}$. The fact that the lowest index generator g_i appears in the leftmost cycle of the monomial in $\Sigma_{d,n}$ is now clear from (2.31), (2.32), (2.33) and (2.34). Concluding, in each application of the Steinberg relation (2.15) the length of w is reduced by at least one, so, from the above and by the induction hypothesis the proof that $\Sigma_{d,n}$ is a spanning set is concluded. □

Remark 2.3. An alternative proof of the above proposition would be the following. An element w in a group is called fully commutative if any reduced expression for w can be obtained from any other by means of braid relations that only involve commuting generators. Through relations (2.7)–(2.15) any word is a linear combination of words of the form $t^a g_{i_1} \dots g_{i_k}$, where $g_{i_1} \dots g_{i_k}$ is the image of a fully commutative word of the braid monoid and it is well–known that a fully commutative word can be written under the form given in the statement of Proposition 2.4. For facts about fully commutative elements the reader is referred to [31], [11], [6], [1].

2.3.4 A linear basis and the dimension of $\mathrm{YTL}_{d,n}(u)$

M. Chlouveraki and G. Pouchin in [3] have computed the dimension for $\mathrm{YTL}_{d,n}(u)$ by using the representation theory of the Yokonuma–Hecke algebra [4]. More precisely, they proved the following result.

Proposition 2.5 (cf. Proposition 4 [3]). *The dimension of the Yokonuma–Temperley–Lieb algebra is:*

$$\dim(\mathrm{YTL}_{d,n}(u)) = dc_n + \frac{d(d-1)}{2} \sum_{k=1}^{n-1} \binom{n}{k}^2,$$

where c_n is the n^{th} Catalan number.

To find an explicit basis for $\mathrm{YTL}_{d,n}(u)$ Chlouveraki and Pouchin in [3] worked as follows: As mentioned in Remark 2.1 each word in $\mathrm{YTL}_{d,n}(u)$ inherits the splitting property. For each fixed element in the braiding part, they described a set of linear dependence relations among the framing parts (see [3, Proposition 5]). Using these relations they extracted from $\Sigma_{d,n}$ (recall Eq. 2.29) a smaller spanning set for $\mathrm{YTL}_{d,n}(u)$ and showed that the cardinality of this smaller spanning set is equal to the dimension of the algebra. Thus, it is a basis for $\mathrm{YTL}_{d,n}(u)$. Before describing this basis, we will need the following notations:

Let $\underline{i}$ and $\underline{k}$ be the following p–tuples:

$$\underline{i} = (i_1, \ldots, i_p) \quad \text{and} \quad \underline{k} = (k_1, \ldots, k_p)$$

and let $\mathcal{I}$ be the set of pairs $(\underline{i}, \underline{k})$ such that:

$$1 \leq i_1 < \ldots < i_p \leq n-1 \quad \text{and} \quad 1 \leq i_1 - k_1 < \ldots < i_p - k_p \leq n-1$$

We also denote by $g_{\underline{i},\underline{k}}$ the element:

$$g_{\underline{i},\underline{k}} := (g_{i_1} g_{i_1 -1} \ldots g_{i_1 - k_1})(g_{i_2} g_{i_2 -1} \ldots g_{i_2 - k_2}) \ldots (g_{i_p} g_{i_p -1} \ldots g_{i_p - k_p})$$

Under these notations the set $\Sigma_{d,n}$ can be written as:

$$\Sigma_{d,n} = \{ t_1^{r_1} \ldots t_n^{r_n} \, g_{\underline{i},\underline{k}} \,|\, r_1, \ldots, r_n \in \mathbb{Z}/d\mathbb{Z}, (\underline{i}, \underline{k}) \in \mathcal{I} \}.$$

The *degree of a word* $w = t_1^{r_1} \ldots t_n^{r_n} g_{i_1} \ldots g_{i_m}$ in $\mathrm{Y}_{d,n}(u)$, denoted $deg(w)$, is defined to be the integer m. Set:

$$\Sigma_{d,n}^{<w} := \{ s \in \Sigma_{d,n} \,|\, deg(s) < deg(w) \}.$$

The group algebra $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ is isomorphic to the subalgebra of $\mathrm{Y}_{d,n}(u)$ that is generated by the t_i's but not to the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. Further, the group algebra $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ has a natural basis, $B_{d,n}$, given by monomials in $t_1, \ldots, t_n$, the following:

$$B_{d,n} = \{ t_1^{r_1} \ldots t_n^{r_n} \,|\, r_1, \ldots, r_n \in \mathbb{Z}/d\mathbb{Z} \}.$$

Thus, any element of $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ can be written as a linear combination of words in $B_{d,n}$. There is a surjective algebra morphism from $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$ to the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's. We will denote the image of an element $b \in B_{d,n}$ into the subalgebra of $\mathrm{YTL}_{d,n}(u)$ that is generated by the t_i's with $\bar{b}$. Before stating the final theorem of this section, we shall introduce the following notation. Let w be any word in $\mathrm{YTL}_{d,n}(u)$. We denote by $R(w)$ the following ideal of $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n$:

$$R(w) = \{ \mathfrak{m} \in \mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n \,|\, \bar{\mathfrak{m}} \, w \in Span_{\mathbb{C}(u)}(\Sigma_{d,n}^{<w}) \}.$$

We consider the set $\mathcal{B}_{d,n}(g_{\underline{i},\underline{k}})$, which is a proper subset of $B_{d,n}$ such that

$$\{ b_{\underline{i},\underline{k}} + R(g_{\underline{i},\underline{k}}) \,|\, b_{\underline{i},\underline{k}} \in \mathcal{B}(g_{\underline{i},\underline{k}}) \}$$

is a basis of the quotient space $\mathbb{C}(u)(\mathbb{Z}/d\mathbb{Z})^n / R(g_{\underline{i},\underline{k}})$. We then have the following theorem:

Theorem 2.1 (Chlouveraki and Pouchin, cf. [3], Theorem 2)**.** *The following set is a linear basis for* $\mathrm{YTL}_{d,n}(u)$*:*

$$S_{d,n} = \{ \bar{b}_{\underline{i},\underline{k}} \, g_{\underline{i},\underline{k}} \,|\, (\underline{i}, \underline{k}) \in \mathcal{I}, \; b_{\underline{i},\underline{k}} \in \mathcal{B}_{d,n}(g_{\underline{i},\underline{k}}) \},$$

2.3.5 A basis for $\mathrm{YTL}_{2,3}(u)$

For $d = 2$, $n = 3$ it is not difficult to find a basis for $\mathrm{YTL}_{2,3}(u)$. We will give here a basis different than the one in Theorem 2.1. To find a basis for $\mathrm{YTL}_{2,3}(u)$ we work as follows: From Proposition 2.5 we have that $\dim(\mathrm{YTL}_{2,3}(u)) = 28$. On the other hand the spanning set $\Sigma_{2,3}$ of $\mathrm{YTL}_{2,3}(u)$ of Proposition 2.4, contains 40 elements. Thus, any relation $w_1 g_{1,2} w_2 = 0$ with $w_1, w_2 \in \mathrm{Y}_{2,3}(u)$ reduces to having $w_1, w_2 \in \Sigma_{2,3}$. Further, if any of w_1, w_2 contain braiding generators, then by Lemma 2.4 (after pushing framing generators in w_2 to the right) these get absorbed by $g_{1,2}$. Thus, and since $e_{i,j} = \frac{1}{2}(1 + t_i t_j)$ for $d = 2$, it suffices to consider the following system of equations:

$$w_1\, g_{1,2} w_2 = 0 \qquad w_1, w_2 \in \mathcal{T}, \tag{2.35}$$

where $\mathcal{T} := \{1, t_1, t_2, t_3, t_1t_2, t_1t_3, t_2t_3, t_1t_2t_3\}$. For finding all possible linear dependencies in $\Sigma_{2,3}$, after substituting $g_1g_2g_1$ with $-1 - g_1 - g_2 - g_1g_2 - g_2g_1$ in Eq. 2.35, note that some of these 64 equations reduce trivially to $g_{1,2} = 0$; for example if $w_2 = 1$ or $w_2 = t_1t_2t_3$ (since it commutes with $g_{1,2}$). From the rest one can extract 12 linearly independent equations which, applied on the spanning set $\Sigma_{2,3}$ lead to the following basis for $\mathrm{YTL}_{2,3}(u)$:

$$\begin{aligned}\mathcal{S}_{2,3} = \{&1, t_1, t_2, t_1t_2, g_1, t_2g_1, t_3g_1, t_2t_3g_1, g_2, t_1g_2, t_3g_2, t_1t_3g_2,\\ &g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,\\ &g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\}\,.\end{aligned}$$

For the case $d = 2$ and $n = 3$ we impose the following ordering in the elements of the spanning set (2.29):

$$\begin{aligned}\Sigma_{2,3} \;=\; \{&1, t_1, t_2, t_3, t_1t_2, t_1t_3, t_2t_3, t_1t_2t_3,\\ &g_1, t_1g_1, t_2g_1, t_3g_1, t_1t_2g_1, t_1t_3g_1, t_2t_3g_1, t_1t_2t_3g_1,\\ &g_2, t_1g_2, t_2g_2, t_3g_2, t_1t_2g_2, t_1t_3g_2, t_2t_3g_2, t_1t_2t_3g_2,\\ &g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,\\ &g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\}\end{aligned} \tag{2.36}$$

We now have the following:

Proposition 2.6. *The following set is a linear basis for* $\mathrm{YTL}_{2,3}(u)$*:*

$$\begin{aligned}\{&1, t_1, t_2, t_1t_2, g_1, t_2g_1, t_3g_1, t_2t_3g_1, g_2, t_1g_2, t_3g_2, t_1t_3g_2,\\ &g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,\\ &g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\}\end{aligned}$$

Proof. We begin from the relation $g_{1,2} = 0$ which generates the ideal I_1 and we compute all possible expressions of the form:

$$w_1 g_{1,2} w_2 = 0, \tag{2.37}$$

where $w_1, w_2 \in \mathcal{T} = \{1, t_1, t_2, t_3, t_1t_2, t_1t_3, t_2t_3, t_1t_2t_3\}$. Note that from Lemma 2.4 it suffices to check (2.37) only for the elements of $\mathcal{T}$. Note also that we always substitute the term $g_1g_2g_1$ of $g_{1,2}$ with $-1-g_1-g_2-g_1g_2-g_2g_1$. For example for the case where $w_1=1$ and $w_2=t_1$, we have:

$$\begin{aligned}(1+g_1+g_2+g_1g_2+g_2g_1+g_1g_2g_1)t_1=0 &\Leftrightarrow\\ t_1+t_2g_1+t_1g_2+t_2g_1g_2+t_3g_1g_2g_1=0 &\Leftrightarrow\\ t_1+t_2g_1+t_1g_2+t_2g_1g_2-t_3-t_3g_1-t_3g_2-t_3g_1g_2=0&\end{aligned}$$

Equation 2.37 yields 64 expressions in total from which 16 are trivial, i.e. the cases where $w_1\in\mathcal{T}$ and $w_2=1$, and the cases where $w_1\in\mathcal{T}$ and $w_2=t_1t_2t_3$, because $t_1t_2t_3$ passes intact to the left of $g_{1,2}$. For the remaining 48 cases we observe that any expression of the form (2.37) is equal to one of the form:

$$w_1'g_{1,2}w_2'=0,$$

such that $w_1w_1'=t_1t_2t_3$ (resp. $w_2w_2'=t_1t_2t_3$).
Indeed, recall first that $t_1t_2t_3$ is the only monomial that can freely move from the left to the right of $g_{1,2}$ without changing the expression, i.e.:

$$g_{1,2}t_1t_2t_3=t_1t_2t_3g_{1,2}.$$

So, starting from $g_{1,2}w_2$, where $w_2\in\mathcal{T}$ we have that:

$$g_{1,2}w_2=g_{1,2}(t_1t_2t_3)w_2'=t_1t_2t_3g_{1,2}w_2', \tag{2.38}$$

where $w_2'\in\mathcal{T}$ and $w_2w_2'=t_1t_2t_3$. For example we have that:

$$g_{1,2}t_1t_2=g_{1,2}(t_1t_2t_3)t_3=t_1t_2t_3g_{1,2}t_3.$$

Now let $w_1g_{1,2}w_2=0$, where $w_1,w_2\in\mathcal{T}$. From Eq. 2.38 we have that:

$$w_1g_{1,2}w_2=w_1(g_{1,2}(t_1t_2t_3)w_2')=w_1(t_1t_2t_3)g_{1,2}w_2'=w_1'g_{1,2}w_2',$$

where $w_1'\in\mathcal{T}$ and $w_1w_1'=t_1t_2t_3$. For example we have that:

$$t_1g_{1,2}t_1t_3=t_1(g_{1,2}(t_1t_2t_3)t_2)=t_1(t_1t_2t_3)g_{1,2}t_2=t_2t_3g_{1,2}t_2.$$

Therefore there are only 24 apparently different equations between the monomials of the spanning set (2.36), the following:

1. $t_1t_2-t_2t_3+t_1t_2g_1-t_2t_3g_1+t_1t_3g_2-t_2t_3g_2+t_1t_3g_2g_1-t_2t_3g_2g_1=0$
2. $t_1-t_3+t_2g_1-t_3g_1+t_1g_2-t_3g_2+t_2g_1g_2-t_3g_1g_2=0$
3. $t_1g_1-t_2g_1-t_2g_2+t_3g_2-t_2g_1g_2+t_3g_1g_2+t_1g_2g_1-t_2g_2g_1=0$
4. $-t_1+t_3-t_1g_1+t_3g_1-t_1g_2+t_2g_2-t_1g_2g_1+t_2g_2g_1=0$
5. $-t_1t_3g_1+t_2t_3g_1+t_1t_2g_2-t_1t_3g_2+t_1t_2g_1g_2-t_1t_3g_1g_2-t_1t_3g_2g_1+t_2t_3g_2g_1=0$
6. $-t_1t_2+t_2t_3-t_1t_2g_1+t_1t_3g_1-t_1t_2g_2+t_2t_3g_2-t_1t_2g_1g_2+t_1t_3g_1g_2=0$
7. $1-t_1t_3+t_1t_2g_1-t_1t_3g_1+g_2-t_1t_3g_2+t_1t_2g_1g_2-t_1t_3g_1g_2=0$

8. $g_1 - t_1t_2g_1 - t_1t_2g_2 + t_1t_3g_2 - t_1g_2g_1g_2 + t_1t_3g_1g_2 + g_2g_1 - t_1t_2g_2g_1 = 0$

9. $-1 + t_1t_3 - g_1 + t_1t_3g_1 - g_2 + t_1t_2g_2 - g_2g_1 + t_1t_2g_2g_1 = 0$

10. $t_1t_2 - t_2t_3 + g_1 - t_2t_3g_1 + t_1t_2g_2 - t_2t_3g_2 + g_1g_2 - t_2t_3g_1g_2 = 0$

11. $-g_1 + t_1t_2g_1 - g_2 + t_2t_3g_2 - g_1g_2 + t_2t_3g_1g_2 - g_2g_1 + t_1t_2g_2g_1 = 0$

12. $-t_1t_2 + t_2t_3 - t_1t_2g_1 + t_2t_3g_1 + g_2 - t_1t_2g_2 + g_2g_1 - t_1t_2g_2g_1 = 0$

13. $-1 + t_1t_3 - g_1 + t_2t_3g_1 - g_2 + t_1t_3g_2 - g_1g_2 + t_2t_3g_1g_2 = 0$

14. $t_1t_3g_1 - t_2t_3g_1 + g_2 - t_2t_3g_2 + g_1g_2 - t_2t_3g_1g_2 + t_1t_3g_2g_1 - t_2t_3g_2g_1 = 0$

15. $1 - t_1t_3 + g_1 - t_1t_3g_1 - t_1t_3g_2 + t_2t_3g_2 - t_1t_3g_2g_1 + t_2t_3g_2g_1 = 0$

16. $t_2 - t_1t_2t_3 + t_1g_1 - t_1t_2t_3g_1 + t_2g_2 - t_1t_2t_3g_2 + t_1g_1g_2 - t_1t_2t_3g_1g_2 = 0$

17. $-t_1g_1 + t_2g_1 - t_1g_2 + t_1t_2t_3g_2 - t_1g_1g_2 + t_1t_2t_3g_1g_2 - t_1g_2g_1 + t_2g_2g_1 = 0$

18. $-t_2 + t_1t_2t_3 - t_2g_1 + t_1t_2t_3g_1 + t_1g_2 - t_2g_2 + t_1g_2g_1 - t_2g_2g_1 = 0$

19. $-t_2 + t_1t_2t_3 - t_2g_1 + t_3g_1 - t_2g_2 + t_1t_2t_3g_2 - t_2g_1g_2 + t_3g_1g_2 = 0$

20. $-t_3g_1 + t_1t_2t_3g_1 + t_2g_2 - t_3g_2 + t_2g_1g_2 - t_3g_1g_2 - t_3g_2g_1 + t_1t_2t_3g_2g_1 = 0$

21. $t_2 - t_1t_2t_3 + t_2g_1 - t_1t_2t_3g_1 + t_3g_2 - t_1t_2t_3g_2 + t_3g_2g_1 - t_1t_2t_3g_2g_1 = 0$

22. $-t_1 + t_3 - t_1g_1 + t_1t_2t_3g_1 - t_1g_2 + t_3g_2 - t_1g_1g_2 + t_1t_2t_3g_1g_2 = 0$

23. $t_3g_1 - t_1t_2t_3g_1 + t_1g_2 - t_1t_2t_3g_2 + t_1g_1g_2 - t_1t_2t_3g_1g_2 + t_3g_2g_1 - t_1t_2t_3g_2g_1 = 0$

24. $t_1 - t_3 + t_1g_1 - t_3g_1 - t_3g_2 + t_1t_2t_3g_2 - t_3g_2g_1 + t_1t_2t_3g_2g_1 = 0$

We then group them according to the leading term of each equation in the sense of the ordering (2.36) (i.e. we take all equations that start with 1, then all equations that start with t_1 and so on) and then we perform Gauss elimination on this system. From these we deduce the following system of 12 intrinsically different linear relations:

(i) $1 - t_1t_3 + t_1t_2g_1 - t_1t_3g_1 - t_1t_3g_2 + t_2t_3g_2 - \frac{1}{2}g_1g_2 + \frac{1}{2}t_1t_2g_1g_2 - \frac{1}{2}t_1t_3g_1g_2 + \frac{1}{2}t_2t_3g_1g_2 - \frac{1}{2}g_2g_1 + \frac{1}{2}t_1t_2g_2g_1 - \frac{1}{2}t_1t_3g_2g_1 + \frac{1}{2}t_2t_3g_2g_1 = 0$

(ii) $t_1 - t_3 + t_2g_1 - t_1t_2t_3g_1 - t_3g_2 + t_1t_2t_3g_2 - t_1g_2g_1 + t_2g_2g_1 = 0$

(iii) $t_2 - t_1t_2t_3 + t_2g_1 - t_1t_2t_3g_1 + t_3g_2 - t_1t_2t_3g_2 + t_3g_2g_1 - t_1t_2t_3g_2g_1 = 0$

(iv) $t_1t_2 - t_2t_3 + t_1t_2g_1 - t_2t_3g_1 + t_1t_3g_2 - t_2t_3g_2 - \frac{1}{2}g_1g_2 + \frac{1}{2}t_1t_2g_1g_2 - \frac{1}{2}t_1t_3g_1g_2 + \frac{1}{2}t_2t_3g_1g_2 + \frac{1}{2}g_2g_1 - \frac{1}{2}t_1t_2g_2g_1 + \frac{1}{2}t_1t_3g_2g_1 - \frac{1}{2}t_2t_3g_2g_1 = 0$

(v) $g_1 - t_1t_2g_1 + \frac{1}{2}g_1g_2 - \frac{1}{2}t_1t_2g_1g_2 + \frac{1}{2}t_1t_3g_1g_2 - \frac{1}{2}t_2t_3g_1g_2 + \frac{1}{2}g_2g_1 - \frac{1}{2}t_1t_2g_2g_1 - \frac{1}{2}t_1t_3g_2g_1 + \frac{1}{2}t_2t_3g_2g_1 = 0$

(vi) $t_1g_1 - t_2g_1 + \frac{1}{2}t_1g_1g_2 - \frac{1}{2}t_2g_1g_2 + \frac{1}{2}t_3g_1g_2 - \frac{1}{2}t_1t_2t_3g_1g_2 + \frac{1}{2}t_1g_2g_1 - \frac{1}{2}t_2g_2g_1 - \frac{1}{2}t_3g_2g_1 + \frac{1}{2}t_1t_2t_3g_2g_1 = 0$

(vii) $t_3g_1 - t_1t_2t_3g_1 + \frac{1}{2}t_1g_1g_2 - \frac{1}{2}t_2g_1g_2 + \frac{1}{2}t_3g_1g_2 - \frac{1}{2}t_1t_2t_3g_1g_2 - \frac{1}{2}t_1g_2g_1 + \frac{1}{2}t_2g_2g_1 + \frac{1}{2}t_3g_2g_1 - \frac{1}{2}t_1t_2t_3g_2g_1 = 0$

(viii) $t_1t_3g_1 - t_2t_3g_1 + \frac{1}{2}g_1g_2 - \frac{1}{2}t_1t_2g_1g_2 + \frac{1}{2}t_1t_3g_1g_2 - \frac{1}{2}t_2t_3g_1g_2 - \frac{1}{2}g_2g_1 + \frac{1}{2}t_1t_2g_2g_1 + \frac{1}{2}t_1t_3g_2g_1 - \frac{1}{2}t_2t_3g_2g_1 = 0$

(ix) $g_2 - t_2t_3g_2 + \frac{1}{2}g_1g_2 + \frac{1}{2}t_1t_2g_1g_2 - \frac{1}{2}t_1t_3g_1g_2 - \frac{1}{2}t_2t_3g_1g_2 + \frac{1}{2}g_2g_1 - \frac{1}{2}t_1t_2g_2g_1 + \frac{1}{2}t_1t_3g_2g_1 - \frac{1}{2}t_2t_3g_2g_1 = 0$

(x) $t_1g_2 - t_1t_2t_3g_2 + \frac{1}{2}t_1g_1g_2 + \frac{1}{2}t_2g_1g_2 - \frac{1}{2}t_3g_1g_2 - \frac{1}{2}t_1t_2t_3g_1g_2 + \frac{1}{2}t_1g_2g_1 - \frac{1}{2}t_2g_2g_1 + \frac{1}{2}t_3g_2g_1 - \frac{1}{2}t_1t_2t_3g_2g_1 = 0$

(xi) $t_2g_2 - t_3g_2 + \frac{1}{2}t_1g_1g_2 + \frac{1}{2}t_2g_1g_2 - \frac{1}{2}t_3g_1g_2 - \frac{1}{2}t_1t_2t_3g_1g_2 - \frac{1}{2}t_1g_2g_1 + \frac{1}{2}t_2g_2g_1 - \frac{1}{2}t_3g_2g_1 + \frac{1}{2}t_1t_2t_3g_2g_1 = 0$

(xii) $t_1t_2g_2 - t_1t_3g_2 + \frac{1}{2}g_1g_2 + \frac{1}{2}t_1t_2g_1g_2 - \frac{1}{2}t_1t_3g_1g_2 - \frac{1}{2}t_2t_3g_1g_2 - \frac{1}{2}g_2g_1 + \frac{1}{2}t_1t_2g_2g_1 - \frac{1}{2}t_1t_3g_2g_1 + \frac{1}{2}t_2t_3g_2g_1 = 0$

These give rise the following set of linearly independent monomials:

$$\begin{aligned}&\{1, t_1, t_2, t_1t_2, g_1, t_2g_1, t_3g_1, t_2t_3g_1, g_2, t_1g_2, t_3g_2, t_1t_3g_2,\\ &g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,\\ &g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\}\end{aligned}$$

and so the proof of the proposition is concluded. □

We conclude this section with the representation theory of $\mathrm{Y}_{d,n}(u)$.

2.3.6 The representation theory of the algebra $\mathrm{YTL}_{d,n}(u)$

In this section we will present briefly the results of Chlouveraki and Pouchin [3] on the representation theory of the algebra $\mathrm{YTL}_{d,n}(u)$. By definition, $\mathrm{YTL}_{d,n}(u)$ is a quotient of $\mathrm{Y}_{d,n}(u)$ and so, by standard results in representation theory we have that the irreducible representations of $\mathrm{YTL}_{d,n}(u)$ are in bijection with the irreducible representations ρ^λ of $\mathrm{Y}_{d,n}(u)$, such that:

$$\rho^\lambda(g_{1,2}) = 0 \tag{2.39}$$

Recall now that $\mathcal{P}(d,n)$ is the set of d–partitions of n and denote by $\mathcal{R}(d,n)$ the set of d–partitions λ of n such that (2.39) is satisfied. We denote also:

$$\mathrm{Irr}(\mathrm{YTL}_{d,n}(u)) = \{\rho^\lambda \mid \lambda \in \mathcal{R}(d,n)\}$$

the set of irreducible representations of $\mathrm{YTL}_{d,n}(u)$. For every $\lambda \in \mathcal{R}(d,n)$ we have:

$$\rho^\lambda \circ \omega = \rho^\lambda$$

where ω is the natural surjective homomorphism from $\mathrm{Y}_{d,n}(u)$ onto $\mathrm{YTL}_{d,n}(u)$. We have the following [3]:

Proposition 2.7. *We have that $\lambda \in \mathcal{R}(d,n)$ if and only if the trivial representation is not a direct summand of $\mathrm{Res}^{G(d,1,n)}_{\langle s_1,s_2\rangle}(E^\lambda)$, the restriction of the irreducible representation E^λ of $G(d,1,n)$ to $\langle s_1, s_2\rangle$.*

Since $\langle s_1, s_2\rangle$ is isomorphic to S_3, the problem of determining the irreducible representations of $\mathrm{YTL}_{d,n}(u)$, transforms to the problem of finding the irreducible representations that appear in the restriction of a representation from $G(d,1,n)$ to S_3. Following [3], we see that for $d = 1$, the restriction of an irreducible representation labelled by a partition λ corresponds to the removal of nodes from the Young diagram of λ in a way consistent to the definition of Young diagrams. More precisely, if λ is a partition of n, then $\mathrm{Res}^{S_n}_{S_{n-1}}(E^\lambda)$ is the direct sum of all representations labelled by the partitions of $n-1$, whose diagrams are obtained from the Young diagram of λ by removing one node. Consequently, $\mathrm{Res}^{S_n}_{S_k}(E^\lambda)$, where $k < n$, is a direct sum of all representations labelled by the partitions of k whose Young diagrams are obtained from the Young diagram of λ by removing $n-k$ nodes. In particular, $\mathrm{Res}^{S_n}_{S_3}(E^\lambda)$ is a direct sum of all representations labelled by the partitions of 3 whose Young diagrams are obtained from the Young diagram of λ by removing $n-3$ nodes. Hence the trivial representation is a direct summand of $\mathrm{Res}^{S_n}_{S_3}(E^\lambda)$ if and only if the Young diagram of λ has more than two columns. We, thus, have the following corollary [3]:

Corollary 2.2. *We have $\lambda \in \mathcal{R}(d,n)$ if and only if all direct summands of $\mathrm{Res}^{G(d,1,n)}_{S_3}(E^\lambda)$ are labelled by the partitions whose Young diagrams have at most two columns.*

For $d = 1$, this, in turn, yields the characterization of the classical Temperley–Lieb algebra $\mathrm{TL}_n(u)$ (see Section 1.5).

Corollary 2.3. *We have:*

$$\mathcal{R}(1,n) = \{\lambda \in \mathcal{P}(n) \mid \lambda_1 \leq 2\}$$

that is, $E^\lambda \in \mathrm{Irr}(\mathrm{TL}_n(u))$ if and only if the Young diagram of λ has at most two columns.

In order to obtain a description of $\mathcal{R}(d,n)$ we will use the following proposition [3]:

Proposition 2.8. *Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)}) \in \mathcal{P}(d,n)$. The Young diagrams of all direct summands of $\mathrm{Res}^{G(d,1,n)}_{S_n}(E^\lambda)$ have at most two columns if and only if $\sum_{i=0}^{d-1} \lambda_1^{(i)} \leq 2$.*

Combining this with together with Corollary 2.2, we have the following [3]:

Theorem 2.2. *For $n \geq 3$, We have that:*

$$\mathcal{R}(d,n) = \left\{ \lambda \in \mathcal{P}(d,n) | \sum_{i=0}^{d-1} \lambda_1^{(i)} \leq 2 \right\}.$$

That is, $E^\lambda \in \mathrm{Irr}(\mathrm{YTL}_{d,n}(u))$ if and only if the Yound d–diagram of λ has at most two columns in total.

2.4 The algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$

For the rest of this chapter we will concentrate on our results for the algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ (recall definitions 2.2 and 2.3).

2.4.1 Defining relations for $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$

We will start by showing that the defining ideal for each one of the quotient algebras is principal.

Theorem 2.3. *The algebra* $\mathrm{FTL}_{d,n}(u)$ *is the quotient of* $\mathrm{Y}_{d,n}(u)$ *over the two–sided ideal generated by the single element:*

$$r_{1,2} = \sum_{x \in H_{1,2}} g_x = \sum_{\alpha+\beta+\gamma=0} t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_{1,2}.$$

For the proof of the theorem we will need the following two lemmas:

Lemma 2.2. *The following hold in* $\mathrm{Y}_{d,n}(u)$ *for all* $i = 1, \ldots, n-1$ *and* $j = 1, \ldots, n$*:*

1. $t_j = (g_1 \ldots g_{n-1})^{j-1}\, t_1\, (g_1 \ldots g_{n-1})^{-(j-1)}$
2. $g_i = (g_1 \ldots g_{n-1})^{i-1}\, g_1\, (g_1 \ldots g_{n-1})^{-(i-1)}$

Proof. For case (1) we have that the statement is true for $j = 2$. Indeed:

$$\begin{aligned}(g_1 \ldots g_{n-1})\, t_1\, (g_1 \ldots g_{n-1})^{-1} &= g_1 t_1 g_2 \ldots g_{n-1} (g_1 \ldots g_{n-1})^{-1} \\ &= t_2 (g_1 \ldots g_{n-1})(g_1 \ldots g_{n-1})^{-1} \\ &= t_2.\end{aligned}$$

Suppose that the statement is true for $j = k$. We will show that the statement holds for $j = k+1$. We have:

$$\begin{aligned}(g_1 \ldots g_{n-1})^k t_1 (g_1 \ldots g_{n-1})^{-k} &= (g_1 \ldots g_{n-1})(g_1 \ldots g_{n-1})^{k-1} t_1 (g_1 \ldots g_{n-1})^{-(k-1)} (g_1 \ldots g_{n-1})^{-1} \\ &= (g_1 \ldots g_{n-1}) t_k (g_1 \ldots g_{n-1})^{-1} \\ &= g_1 \ldots g_{k-1} g_k\, t_k\, g_{k+1} \ldots g_{n-1} (g_1 \ldots g_{n-1})^{-1} \\ &= t_{k+1} (g_1 \ldots g_{n-1})(g_1 \ldots g_{n-1})^{-1} \\ &= t_{k+1}.\end{aligned}$$

The proof of case (2) follows also by induction. We have that the statement is true for $i = 2$. Indeed:

$$\begin{aligned}(g_1 \ldots g_{n-1})\, g_1\, (g_1 \ldots g_{n-1})^{-1} &= g_1 g_2 g_1 g_3 \ldots g_{n-1} (g_1 \ldots g_{n-1})^{-1} \\ &= g_2 (g_1 g_2 \ldots g_{n-1})(g_1 \ldots g_{n-1})^{-1} \\ &= g_2.\end{aligned}$$

Suppose that the statement is true for $i = k$. We will show that the statement holds for $i = k+1$. We have:

$$\begin{aligned}(g_1 \ldots g_{n-1})^k g_1 (g_1 \ldots g_{n-1})^{-k} &= (g_1 \ldots g_{n-1})(g_1 \ldots g_{n-1})^{k-1} g_1 (g_1 \ldots g_{n-1})^{-(k-1)} (g_1 \ldots g_{n-1})^{-1} \\ &= (g_1 \ldots g_{n-1}) g_k (g_1 \ldots g_{n-1})^{-1} \\ &= g_1 \ldots g_{k-1} g_k g_{k+1} g_k g_{k+2} \ldots g_{n-1} (g_1 \ldots g_{n-1})^{-1} \\ &= g_1 \ldots g_{k-1} g_{k+1} g_k g_{k+1} \ldots g_{n-1} (g_1 \ldots g_{n-1})^{-1} \\ &= g_{k+1} (g_1 \ldots g_{n-1})(g_1 \ldots g_{n-1})^{-1} \\ &= g_{k+1}.\end{aligned}$$

□

Lemma 2.3. *The following hold in* $\mathrm{Y}_{d,n}(u)$ *for all* $i = 1, \ldots, n-2$ *and* $\alpha, \beta, \gamma \in \mathbb{Z}/d\mathbb{Z}$*:*

$$\begin{array}{lrcl}
(1) & t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} & = & (g_1 \ldots g_{n-1})^{i-1}\, t_1^{\alpha} t_2^{\beta} t_3^{\gamma}\, (g_1 \ldots g_{n-1})^{-(i-1)} \\
(2) & t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} g_i & = & (g_1 \ldots g_{n-1})^{i-1}\, t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_1\, (g_1 \ldots g_{n-1})^{-(i-1)} \\
(3) & t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} g_{i+1} & = & (g_1 \ldots g_{n-1})^{i-1}\, t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_2\, (g_1 \ldots g_{n-1})^{-(i-1)} \\
(4) & t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} g_i g_{i+1} & = & (g_1 \ldots g_{n-1})^{i-1}\, t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_1 g_2\, (g_1 \ldots g_{n-1})^{-(i-1)} \\
(5) & t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} g_{i+1} g_i & = & (g_1 \ldots g_{n-1})^{i-1}\, t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_2 g_1\, (g_1 \ldots g_{n-1})^{-(i-1)} \\
(6) & t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} g_i g_{i+1} g_i & = & (g_1 \ldots g_{n-1})^{i-1}\, t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_1 g_2 g_1\, (g_1 \ldots g_{n-1})^{-(i-1)}
\end{array}$$

Proof. We will make extensive use of Lemma 2.2. For case (1) we have:

$$\begin{aligned}
t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} &= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{i} t_1^{\beta} (g_1 \ldots g_{n-1})^{-i} \\
&\quad \cdot (g_1 \ldots g_{n-1})^{i+1} t_1^{\gamma} (g_1 \ldots g_{n-1})^{-(i+1)} \\
&= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} (g_1 \ldots g_{n-1}) t_1^{\beta} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-(i-1)} \\
&\quad \cdot (g_1 \ldots g_{n-1})^{i-1} (g_1 \ldots g_{n-1})^{2} t_1^{\gamma} (g_1 \ldots g_{n-1})^{-2} (g_1 \ldots g_{n-1})^{-(i-1)} \\
&= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} t_2^{\beta} t_3^{\gamma} (g_1 \ldots g_{n-1})^{-(i-1)}.
\end{aligned}$$

For case (2) we have:

$$\begin{aligned}
t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} g_i &= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{i} t_1^{\beta} (g_1 \ldots g_{n-1})^{-i} \\
&\quad \cdot (g_1 \ldots g_{n-1})^{i+1} t_1^{\gamma} (g_1 \ldots g_{n-1})^{-(i+1)} (g_1 \ldots g_{n-1})^{i-1} g_1 (g_1 \ldots g_{n-1})^{-(i-1)} \\
&= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} (g_1 \ldots g_{n-1}) t_1^{\beta} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-(i-1)} \\
&\quad \cdot (g_1 \ldots g_{n-1})^{i-1} (g_1 \ldots g_{n-1})^{2} t_1^{\gamma} (g_1 \ldots g_{n-1})^{-2} (g_1 \ldots g_{n-1})^{-(i-1)} \\
&\quad \cdot (g_1 \ldots g_{n-1})^{i-1} g_1 (g_1 \ldots g_{n-1})^{-(i-1)} \\
&= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_1 (g_1 \ldots g_{n-1})^{-(i-1)}
\end{aligned}$$

For case (3) we have:

$$\begin{aligned}
t_i^{\alpha} t_{i+1}^{\beta} t_{i+2}^{\gamma} g_{i+1} &= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} (g_1 \ldots g_{n-1})^{-(i-1)} (g_1 \ldots g_{n-1})^{i} t_1^{\beta} (g_1 \ldots g_{n-1})^{-i} \\
&\quad \cdot (g_1 \ldots g_{n-1})^{i+1} t_1^{\gamma} (g_1 \ldots g_{n-1})^{-(i+1)} (g_1 \ldots g_{n-1})^{i} g_1 (g_1 \ldots g_{n-1})^{-i} \\
&= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} (g_1 \ldots g_{n-1}) t_1^{\beta} (g_1 \ldots g_{n-1})^{-1} (g_1 \ldots g_{n-1})^{-(i-1)} \\
&\quad \cdot (g_1 \ldots g_{n-1})^{i-1} (g_1 \ldots g_{n-1})^{2} t_1^{\gamma} (g_1 \ldots g_{n-1})^{-2} (g_1 \ldots g_{n-1})^{-(i-1)} \\
&\quad \cdot (g_1 \ldots g_{n-1})^{i-1} g_2 (g_1 \ldots g_{n-1})^{-(i-1)} \\
&= (g_1 \ldots g_{n-1})^{i-1} t_1^{\alpha} t_2^{\beta} t_3^{\gamma} g_2 (g_1 \ldots g_{n-1})^{-(i-1)}
\end{aligned}$$

For case (4) we have:

$$\begin{aligned}
t_i^{\alpha}t_{i+1}^{\beta}t_{i+2}^{\gamma}g_ig_{i+1} &= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}(g_1\dots g_{n-1})^{-(i-1)}(g_1\dots g_{n-1})^{i}t_1^{\beta}(g_1\dots g_{n-1})^{-i}\\
&\quad\cdot(g_1\dots g_{n-1})^{i+1}t_1^{\gamma}(g_1\dots g_{n-1})^{-(i+1)}(g_1\dots g_{n-1})^{i-1}g_1(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i}g_1(g_1\dots g_{n-1})^{-i}\\
&= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}(g_1\dots g_{n-1})t_1^{\beta}(g_1\dots g_{n-1})^{-1}(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i-1}(g_1\dots g_{n-1})^{2}t_1^{\gamma}(g_1\dots g_{n-1})^{-2}(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i-1}g_1(g_1\dots g_{n-1})^{-(i-1)}(g_1\dots g_{n-1})^{i-1}(g_1\dots g_{n-1})\\
&\quad\cdot g_1(g_1\dots g_{n-1})^{-1}(g_1\dots g_{n-1})^{-(i-1)}\\
&= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}t_2^{\beta}t_3^{\gamma}g_1g_2(g_1\dots g_{n-1})^{-(i-1)}.
\end{aligned}$$

For case (5) we have:

$$\begin{aligned}
t_i^{\alpha}t_{i+1}^{\beta}t_{i+2}^{\gamma}g_{i+1}g_i &= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}(g_1\dots g_{n-1})^{-(i-1)}(g_1\dots g_{n-1})^{i}t_1^{\beta}(g_1\dots g_{n-1})^{-i}\\
&\quad\cdot(g_1\dots g_{n-1})^{i+1}t_1^{\gamma}(g_1\dots g_{n-1})^{-(i+1)}(g_1\dots g_{n-1})^{i}g_1(g_1\dots g_{n-1})^{-i}\\
&\quad\cdot(g_1\dots g_{n-1})^{i-1}g_1(g_1\dots g_{n-1})^{-(i-1)}\\
&= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}(g_1\dots g_{n-1})t_1^{\beta}(g_1\dots g_{n-1})^{-1}(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i-1}(g_1\dots g_{n-1})^{2}t_1^{\gamma}(g_1\dots g_{n-1})^{-2}(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i-1}(g_1\dots g_{n-1})g_1(g_1\dots g_{n-1})^{-1}(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i-1}g_1(g_1\dots g_{n-1})^{-(i-1)}\\
&= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}t_2^{\beta}t_3^{\gamma}g_2g_1(g_1\dots g_{n-1})^{-(i-1)}.
\end{aligned}$$

Finally, for case (6) we have:

$$\begin{aligned}
t_i^{\alpha}t_{i+1}^{\beta}t_{i+2}^{\gamma}g_ig_{i+1}g_i &= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}(g_1\dots g_{n-1})^{-(i-1)}(g_1\dots g_{n-1})^{i}t_1^{\beta}(g_1\dots g_{n-1})^{-i}\\
&\quad\cdot(g_1\dots g_{n-1})^{i+1}t_1^{\gamma}(g_1\dots g_{n-1})^{-(i+1)}(g_1\dots g_{n-1})^{i-1}g_1(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i}g_1(g_1\dots g_{n-1})^{-i}(g_1\dots g_{n-1})^{i-1}g_1(g_1\dots g_{n-1})^{-(i-1)}\\
&= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}(g_1\dots g_{n-1})t_1^{\beta}(g_1\dots g_{n-1})^{-1}(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i-1}(g_1\dots g_{n-1})^{2}t_1^{\gamma}(g_1\dots g_{n-1})^{-2}(g_1\dots g_{n-1})^{-(i-1)}\\
&\quad\cdot(g_1\dots g_{n-1})^{i-1}g_1(g_1\dots g_{n-1})^{-(i-1)}(g_1\dots g_{n-1})^{i-1}(g_1\dots g_{n-1})\\
&\quad\cdot g_1(g_1\dots g_{n-1})^{-1}(g_1\dots g_{n-1})^{-(i-1)}(g_1\dots g_{n-1})^{i-1}g_1(g_1\dots g_{n-1})^{-(i-1)}\\
&= (g_1\dots g_{n-1})^{i-1}t_1^{\alpha}t_2^{\beta}t_3^{\gamma}g_1g_2g_1(g_1\dots g_{n-1})^{-(i-1)}.
\end{aligned}$$

□

Proof of Theorem 2.3. Expanding Eq. 2.4 and applying Lemma 2.3 we obtain:

$$\begin{aligned}
\sum_{\substack{\alpha+\beta+\gamma=0\\ w\in\langle s_i,s_{i+1}\rangle}} t_i^{\alpha}t_{i+1}^{\beta}t_{i+2}^{\gamma}\,g_w &= \sum_{\substack{\alpha+\beta+\gamma=0\\ w\in S_3}} (g_1\dots g_{n-1})^{i-1}\,t_1^{\alpha}t_2^{\beta}t_3^{\gamma}\,g_w\,(g_1\dots g_{n-1})^{-(i-1)}\\
&= (g_1\dots g_{n-1})^{i-1}\left(\sum_{\alpha+\beta+\gamma=0} t_1^{\alpha}t_2^{\beta}t_3^{\gamma}g_{1,2}\right)(g_1\dots g_{n-1})^{-(i-1)},
\end{aligned}$$

or equivalently:

$$r_{i,i+1} = (g_1 \ldots g_{n-1})^{i-1} r_{1,2} (g_1 \ldots g_{n-1})^{-(i-1)} \tag{2.40}$$

Therefore, the two–sided ideal J is generated by the single element:

$$r_{1,2} = \sum_{\alpha+\beta+\gamma=0} t_1^{\alpha} t_2^{\beta} t_3^{\gamma}\, g_{1,2}.$$

and so the proof of the Theorem is concluded. □

Remark 2.4. From Eq. 2.5 we have that the element $r_{1,2}$ can be rewritten as:

$$r_{1,2} = \sum_{\alpha,\gamma\in\mathbb{Z}/d\mathbb{Z}} t_1^{\alpha} t_2^{-\alpha-\gamma} t_3^{\gamma} g_{1,2} = d\, e_1 e_2 g_{1,2}.$$

Therefore we deduce that:

$$J = \langle r_{1,2} \rangle = \langle e_1 e_2 g_{1,2} \rangle.$$

From Theorem 2.3 we have the following corollary:

Corollary 2.4. $\mathrm{FTL}_{d,n}(u)$ *is the* $\mathbb{C}(u)$*–algebra generated by the set* $\{t_1, \ldots, t_n, g_1, \ldots, g_{n-1}\}$ *whose elements are subject to the defining relations of* $\mathrm{Y}_{d,n}(u)$ *and the relation:*

$$r_{1,2} = 0.$$

Subsequently, by Definition 2.3, we have that the relations $r_{i,i+1} = 0$ *also hold in* $\mathrm{FTL}_{d,n}(u)$*.*

From the above we obtain an alternative definition of the algebra $\mathrm{FTL}_{d,n}(u)$ in terms of generators and relations:

Definition 2.6. *For* $n \geq 3$*, the algebra* $\mathrm{FTL}_{d,n}(u)$ *can be presented by the generators* $g_1, \ldots, g_{n-1}$*,* $t_1, \ldots, t_n$*, subject to the following relations:*

$$\begin{aligned}
&g_i g_j = g_j g_i, \quad |i-j| > 1 \\
&g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i \\
&g_i^2 = 1 + (u-1)e_i + (u-1)e_i g_i \\
&t_i t_j = t_j t_i, \quad \textit{for all } i, j \\
&t_i^d = 1, \quad \textit{for all } i \\
&g_i t_i = t_{i+1} g_i \\
&g_i t_{i+1} = t_i g_i \\
&g_i t_j = t_j g_i, \quad \textit{for } j \neq i, \textit{ and } j \neq i+1 \\
&e_i e_{i+1} g_{i,i+1} = 0
\end{aligned} \tag{2.41}$$

In analogy to Theorem 2.3 we have the following for the algebra $\mathrm{CTL}_{d,n}(u)$:

Theorem 2.4. *The algebra* $\mathrm{CTL}_{d,n}(u)$ *is the quotient of* $\mathrm{Y}_{d,n}(u)$ *over the two–sided ideal generated by the single element:*

$$c_{1,2} := \sum_{w\in C_{1,2}} g_w = \sum_{\alpha,\beta,\gamma\in C_d} t_1^{\alpha} t_2^{\beta} t_3^{\gamma}\, g_{1,2}.$$

Proof. In analogy to the proof of Theorem 2.3, it is enough to prove that $c_{i,i+1} = \gamma c_{1,2}\gamma^{-1}$ where $\gamma := (g_1 \dots g_{n-1})^{i-1}$. From Eq. 2.6, we have

$$\gamma c_{1,2}\gamma^{-1} = \left(\sum_{0\leq k\leq d-1} \gamma t_1^k \gamma^{-1} \right) \gamma r_{1,2}\gamma^{-1}$$

By using now Lemma 2.2 and Eq. 2.40, it follows that $\gamma c_{1,2}\gamma^{-1} = c_{i,i+1}$. The rest of the statement is now clear. □

Remark 2.5. Note that from Eq. 2.3 and 2.6 we have that:

$$c_{i,i+1} = \sum_{k\in\mathbb{Z}/d\mathbb{Z}} t_i^k \sum_{x\in H_{i,i+1}} g_x = \sum_{k\in\mathbb{Z}/d\mathbb{Z}} t_i^k r_{i,i+1} = \sum_{k\in\mathbb{Z}/d\mathbb{Z}} t_i^k e_i e_{i+1} g_{i,i+1} = \sum_{k\in\mathbb{Z}/d\mathbb{Z}} e_i^{(k)} e_{i,i+1} g_{i,i+1}$$

The following corollary provides a presentation for $\mathrm{CTL}_{d,n}(u)$ in terms of generators and relations.

Corollary 2.5. *The $\mathbb{C}(u)$–algebra $\mathrm{CTL}_{d,n}(u)$ can be presented by the elements $t_1, \dots, t_n$, $g_1, \dots, g_{n-1}$ who are subject to the defining relations of $\mathrm{Y}_{d,n}(u)$ and the relation:*

$$c_{1,2} = 0. \tag{2.42}$$

Thus, the algebra $\mathrm{CTL}_{d,n}(u)$ can be alternatively be defined as follows:

Definition 2.7. *For $n \geq 3$, the algebra $\mathrm{CTL}_{d,n}(u)$ can be presented by the generators $g_1, \dots, g_{n-1}$, $t_1, \dots, t_n$, subject to the following relations:*

$$\begin{aligned}
&g_i g_j = g_j g_i, \quad |i-j| > 1 \\
&g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i \\
&g_i^2 = 1 + (u-1)e_i + (u-1)e_i g_i \\
&t_i t_j = t_j t_i, \quad \textit{for all } i, j \\
&t_i^d = 1, \quad \textit{for all } i \\
&g_i t_i = t_{i+1} g_i \\
&g_i t_{i+1} = t_i g_i \\
&g_i t_j = t_j g_i, \quad \textit{for } j \neq i, \textit{ and } j \neq i+1 \\
&\sum_{k\in\mathbb{Z}/d\mathbb{Z}} e_i^{(k)} e_{i+1} g_{i,i+1} = 0
\end{aligned} \tag{2.43}$$

We conclude this section with the following remark.

Remark 2.6. For $d = 1$, the algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ coincide with the algebra $\mathrm{TL}_n(q)$.

2.4.2 Presentations with non–invertible generators

In complete analogy to the case of $\mathrm{YTL}_{d,n}(u)$, one can obtain presentations with non–invertible generators for both of the algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ using the transformation (2.17). We have the following propositions:

Proposition 2.9. *The algebra* $\mathrm{FTL}_{d,n}(u)$ *can be presented with generators:*

$$\ell_1, \ldots, \ell_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$\begin{aligned}
t_i^d = 1, \quad t_i t_j &= t_j t_i, \quad \text{for all } i, j \\
\ell_i t_j = t_j \ell_i, \quad \ell_i t_j &= t_j \ell_i, \quad \text{for} \quad |i-j| > 1 \\
\ell_i \ell_j &= \ell_j \ell_i, \quad \text{for} \quad |i-j| > 1 \\
\ell_i t_i &= t_{i+1} \ell_i + \frac{1}{u+1}(t_i - t_{i+1}) \\
\ell_i t_{i+1} &= t_i \ell_i + \frac{1}{u+1}(t_{i+1} - t_i) \\
\ell_i^2 &= \frac{(u-1)e_i + 2}{u+1}\, \ell_i \\
\ell_i \ell_{i+1} \ell_i - \frac{(u-1)e_i + 1}{(u+1)^2}\, \ell_i &= \ell_{i+1} \ell_i \ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\, \ell_{i+1} \\
e_i e_{i+1} \ell_i \ell_{i+1} \ell_i &= \frac{u}{(u+1)^2}\, e_i e_{i+1} \ell_i.
\end{aligned}$$

Proof. Obviously, $\mathrm{FTL}_{d,n}(u)$ is generated by the ℓ_i's and the t_i's. All the equations, except the last one, can be proved in total analogy to those of Proposition 2.2. Note also that, contrary to the case of $\mathrm{YTL}_{d,n}(u)$, the equation:

$$\ell_i \ell_{i+1} \ell_i - \frac{(u-1)e_i + 1}{(u+1)^2}\, \ell_i = \ell_{i+1} \ell_i \ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\, \ell_{i+1}, \quad 1 \leq i \leq n-2.$$

which corresponds to the braid relation $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, is not superfluous.

For the elements $e_i e_{i+1} g_{i,i+1}$ using Eq. 2.28 we have for $1 \leq i \leq n-2$ that:

$$\begin{aligned}
e_i e_{i+1} g_{i,i+1} &= e_i e_{i+1} \left(g_i g_{i+1} g_i + g_{i+1} g_i + g_i g_{i+1} + g_{i+1} + g_i + 1 \right) \\
&= e_i e_{i+1} \left((u+1)^3 \ell_i \ell_{i+1} \ell_i - (u+1)^2 \ell_i^2 + (u+1) \ell_i \right)
\end{aligned}$$

From Eqs. 2.41 and the quadratic relation for the ℓ_i's we have that:

$$e_i e_{i+1} \left((u+1)^2 \ell_i \ell_{i+1} \ell_i \right) = e_i e_{i+1} \big((u-1) e_i + 1 \big) \ell_i$$

or equivalently:

$$e_i e_{i+1} \ell_i \ell_{i+1} \ell_i = \frac{u}{(u+1)^2}\, e_i e_{i+1} \ell_i,$$

which is Eq. 2.25. ∎

Proposition 2.10. *The algebra* $\mathrm{CTL}_{d,n}(u)$ *can be presented with generators:*

$$\ell_1, \ldots, \ell_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$\begin{aligned}
t_i^d &= 1, \quad t_i t_j = t_j t_i, \quad \textit{for all } i, j \\
\ell_i t_j &= t_j \ell_i, \quad \ell_i t_j = t_j \ell_i, \quad \textit{for} \quad |i-j| > 1 \\
\ell_i \ell_j &= \ell_j \ell_i, \quad \textit{for} \quad |i-j| > 1 \\
\ell_i t_i &= t_{i+1} \ell_i + \frac{1}{u+1}(t_i - t_{i+1}) \\
\ell_i t_{i+1} &= t_i \ell_i + \frac{1}{u+1}(t_{i+1} - t_i) \\
\ell_i^2 &= \frac{(u-1)e_i + 2}{u+1}\, \ell_i, \quad 1 \le i \le n-1 \\
\ell_i \ell_{i+1} \ell_i - \frac{(u-1)e_i + 1}{(u+1)^2}\, \ell_i &= \ell_{i+1} \ell_i \ell_{i+1} - \frac{(u-1)e_{i+1} + 1}{(u+1)^2}\, \ell_{i+1} \\
\sum_{k=0}^{d-1} e_i^{(k)} e_{i+1} \ell_i \ell_{i+1} \ell_i &= \sum_{k=0}^{d-1} e_i^{(k)} e_{i+1} \frac{u}{(u+1)^2}\, \ell_i.
\end{aligned}$$

Proof. The proof is a straight forward computation and totally analogous to the proof of Proposition 2.9. □

2.4.3 A basis for $\mathrm{FTL}_{2,3}(u)$

For a complete description of $\mathrm{FTL}_{d,n}(u)$ a basis needs to be presented. Unfortunately, it is very difficult to compute a concrete basis for $\mathrm{FTL}_{d,n}(u)$, for any d and any n. The problem lies in the fact that the dimension grows rapidly for $n \geq 4$. Using tools from representation theory Chlouveraki and Pouchin [3] were able to provide a formula for the dimension of $\mathrm{FTL}_{d,n}(u)$. More precisely, they proved that:

$$\dim \mathrm{FTL}_{d,n}(u) = \sum_{|k_1|+|k_2|+\ldots+|k_d|=n} \left(\frac{n!}{k_1! \ldots k_d!} \right)^2 c_{k_1} \ldots c_{k_d}, \tag{2.44}$$

where k_i is a partition with at most two columns and c_k is the k–th Catalan number. Having said that, the only case that can be computed relatively easy is $\mathrm{FTL}_{2,3}$. For that, we will need the following lemmas that will also be used in the proof of Theorem 3.4.

Lemma 2.4 (cf. Lemma 7.5 [19]). *For the element $g_{1,2}$ we have in $\mathrm{Y}_{d,n}(u)$ (recall (1.23) for $e_{1,3}$):*

$$\begin{array}{llcl}
(1) & g_1 g_{1,2} & = & [1 + (u-1)e_1] g_{1,2} \\
(2) & g_2 g_{1,2} & = & [1 + (u-1)e_2] g_{1,2} \\
(3) & g_1 g_2 g_{1,2} & = & [1 + (u-1)e_1 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2] g_{1,2} \\
(4) & g_2 g_1 g_{1,2} & = & [1 + (u-1)\, e_2 + (u-1)e_{1,3} + (u-1)^2 e_1 e_2] g_{1,2} \\
(5) & g_1 g_2 g_1 g_{1,2} & = & [1 + (u-1)(e_1 + e_2 + e_{1,3}) + (u-1)^2 (u+2)\, e_1 e_2] g_{1,2}
\end{array}$$

Analogous relations hold for multiplications with $g_{1,2}$ from the right.

Proof. The idea is to expand the left–hand side of each equation and then use Eq. 1.33 and Lemma 1.1. For case (1) we have:

$$\begin{aligned} g_1g_{1,2} &= g_1 + g_1^2 + g_1g_2 + g_1^2g_2 + g_1g_2g_1 + g_1^2g_2g_1 \\ &= g_1 + [1 + (u-1)e_1 + (u-1)e_1g_1] \\ &\quad + g_1g_2 + [g_2 + (u-1)e_1g_2 + (u-1)e_1g_1g_2] \\ &\quad + g_1g_2g_1 + [g_2g_1 + (u-1)e_1g_2g_1 + (u-1)e_1g_1g_2g_1] \\ &= g_{1,2} + (u-1)e_1g_{1,2}. \end{aligned}$$

Case (2) is completely analogous. We have that:

$$\begin{aligned} g_2g_{1,2} &= g_2 + g_2g_1 + g_2^2 + g_2g_1g_2 + g_2^2g_1 + g_2g_1g_2g_1 \\ &= g_2 + +g_2g_1\,[1 + (u-1)e_2 + (u-1)e_2g_2] \\ &\quad + g_1g_2g_1 + [g_1 + (u-1)e_2g_1 + (u-1)e_2g_2g_1] \\ &\quad + g_1g_2g_1 + [g_1g_2 + (u-1)e_2g_1g_2 + (u-1)e_2g_1g_2g_1] \\ &= g_{1,2} + (u-1)e_2g_{1,2}. \end{aligned}$$

In order to prove Case (3) we will use Case (1):

$$\begin{aligned} g_1g_2g_{1,2} &= g_1\left(g_{1,2} + (u-1)e_2g_{1,2}\right) \\ &= g_1g_{1,2} + (u-1)e_{1,3}g_1g_{1,2} \quad \text{(Lemma 1.1)} \\ &= [1 + (u-1)e_1]\,g_{1,2} + (u-1)e_{1,3}(1 + (u-1)e_1)g_{1,2} \\ &= [1 + (u-1)e_1]\,g_{1,2} + (u-1)e_{1,3}g_{1,2} + (u-1)^2e_{1,3}e_1g_{1,2} \quad \text{(Lemma 1.1)} \\ &= \left[1 + (u-1)\,e_1 + (u-1)e_{1,3} + (u-1)^2e_1e_2\right]\,g_{1,2}. \end{aligned}$$

Case (4) is completely analogous.

$$\begin{aligned} g_2g_1g_{1,2} &= g_2\left(g_{1,2} + (u-1)e_1g_{1,2}\right) \\ &= g_2g_{1,2} + (u-1)e_{1,3}g_2g_{1,2} \quad \text{(Lemma 1.1)} \\ &= [1 + (u-1)e_2]\,g_{1,2} + (u-1)e_{1,3}(1 + (u-1)e_2)g_{1,2} \\ &= [1 + (u-1)e_2]\,g_{1,2} + (u-1)e_{1,3}g_{1,2} + (u-1)^2e_{1,3}e_2g_{1,2} \quad \text{(Lemma 1.1)} \\ &= \left[1 + (u-1)\,e_2 + (u-1)e_{1,3} + (u-1)^2e_1e_2\right]\,g_{1,2}. \end{aligned}$$

Finally, for Case (5) we shall use Cases (1) and (4):

$$\begin{aligned} g_1g_2g_1g_{1,2} &= g_1[1 + (u-1)e_2 + (u-1)e_{1,3} + (u-1)^2e_1e_2]g_{1,2} \\ &= [g_1 + (u-1)e_{1,3}g_1 + (u-1)e_2g_1 + (u-1)^2e_1e_2g_1]g_{1,2} \\ &= [1 + (u-1)e_2 + (u-1)e_{1,3} + (u-1)^2e_1e_2]g_1g_{1,2} \\ &= [1 + (u-1)e_2 + (u-1)e_{1,3} + (u-1)^2e_1e_2][g_{1,2} + (u-1)e_1g_{1,2}] \\ &= [1 + (u-1)e_2 + (u-1)e_{1,3} + (u-1)^2e_1e_2]g_{1,2} \\ &\quad + [(u-1)e_1 + (u-1)^2e_1e_2 + (u-1)^2e_1e_2 + (u-1)^2e_1e_2]g_{1,2} \\ &= [1 + (u-1)(e_1 + e_2 + e_{1,3}) + (u-1)^2(u-2)e_1e_2]g_{1,2} \end{aligned}$$

□

Lemma 2.5. *For the element $r_{1,2}$ we have in $\mathrm{Y}_{d,n}(u)$:*

$$\begin{array}{lrcl}
(1) & g_1 r_{1,2} & = & [1+(u-1)e_1]r_{1,2} \\
(2) & g_2 r_{1,2} & = & [1+(u-1)e_2]r_{1,2} \\
(3) & g_1 g_2 r_{1,2} & = & [1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2 e_1e_2]r_{1,2} \\
(4) & g_2 g_1 r_{1,2} & = & [1+(u-1)\,e_2+(u-1)e_{1,3}+(u-1)^2 e_1e_2]r_{1,2} \\
(5) & g_1g_2g_1 r_{1,2} & = & [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1e_2]r_{1,2}
\end{array}$$

Proof. In order to prove this lemma we will make extensive use of Lemmas 2.4 and 1.1. For case (1) we have:

$$\begin{aligned}
g_1 r_{1,2} &= g_1 e_1 e_2 g_{1,2} = e_1 e_{1,3} g_1 g_{1,2} \\
&= e_1 e_2 [1+(u-1)e_1] g_{1,2} \\
&= [1+(u-1)e_1] e_1 e_2 g_{1,2} \\
&= [1+(u-1)e_1] r_{1,2}
\end{aligned}$$

In an analogous way we prove case (2). For case (3) we have that:

$$\begin{aligned}
g_1 g_2 r_{1,2} &= g_1 g_2 e_1 e_2 g_{1,2} = e_2 e_{1,3} g_1 g_2 g_{1,2} \\
&= e_1 e_2 [1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2 e_1 e_2] g_{1,2} \\
&= [1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2 e_1 e_2] e_1 e_2 g_{1,2} \\
&= [1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2 e_1 e_2] r_{1,2}
\end{aligned}$$

In an analogous way we prove case(4). Finally, we have for case (5):

$$\begin{aligned}
g_1 g_2 g_1 r_{1,2} &= g_1 g_2 g_1 e_1 e_2 g_{1,2} \\
&= e_1 e_2 g_1 g_2 g_1 g_{1,2} \\
&= e_1 e_2 [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1 e_2] g_{1,2} \\
&= [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1 e_2] e_1 e_2 g_{1,2} \\
&= [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1 e_2] r_{1,2}
\end{aligned}$$

□

To compute a basis for $\mathrm{FTL}_{2,3}(u)$, we start from the linear basis of $\mathrm{Y}_{2,3}(u)$ which also spans $\mathrm{FTL}_{2,3}(u)$. We have that $\dim(\mathrm{Y}_{2,3}(u)) = 48$, while from Eq. 2.44 we deduce that: $\dim(\mathrm{FTL}_{2,3}(u)) = 46$. Therefore, we have to find two relations of linear dependency among the elements of $\mathrm{Y}_{2,3}(u)$. We work as follows. From the defining relation of the ideal J we have:

$$r_{1,2} = e_1 e_2 g_{1,2} = 0 \tag{2.45}$$

In order to find linear dependencies we consider the following system of equations:

$$w_1 r_{1,2} w_2 = 0 \tag{2.46}$$

where $w_1, w_2 \in \mathrm{Y}_{d,3}(u)$. From Lemma 2.5 we have that whenever w_1 or w_2 contain a braiding generator, it is absorbed by $r_{1,2}$ (after pushing the framing generators in w_2 to the right). Therefore it suffices to consider that w_1 and w_2 contain only framing monomials in Eq. 2.46. Notice also that any framing monomial commutes with the element $e_1e_2g_{1,2}$. We have the following system:

$$e_1e_2g_{1,2} = 0$$
$$e_1^{(1)}e_2g_{1,2} = 0$$

which is equivalent to:

$$(1 + t_1t_2 + t_1t_3 + t_2t_3)g_{1,2} = 0 \tag{2.47}$$
$$(t_1 + t_2 + t_3 + t_1t_2t_3)g_{1,2} = 0 \tag{2.48}$$

Notice that each framing monomial appears once in the system of equations (2.47) and (2.48). Choosing one element of each equation and expressing it as a linear combination of the rest of the elements of that equation will result to a basis for $\mathrm{FTL}_{2,3}(u)$. Choosing now the elements $g_1g_2g_1$ and $t_1t_2t_3g_1g_2g_1$ to be linearly dependent. Thus, we have proved the following:

Proposition 2.11. *The following set is a linear basis for* $\mathrm{FTL}_{2,3}(u)$*:*

$$\begin{aligned}\{&1, t_1, t_2, t_3, t_1t_2, , t_1t_3, t_2t_3, t_1t_2t_3,\\ &g_1, t_1g_1, t_2g_1, t_3g_1, t_1t_2g_1, t_1t_3g_1, t_2t_3g_1, t_1t_2t_3g_1,\\ &g_2, t_1g_2, t_2g_2t_3g_2, t_1t_2g_2, t_1t_3g_2, t_2t_3g_2t_1t_2t_3g_2,\\ &g_1g_2, t_1g_1g_2, t_2g_1g_2, t_3g_1g_2, t_1t_2g_1g_2, t_1t_3g_1g_2, t_2t_3g_1g_2, t_1t_2t_3g_1g_2,\\ &g_2g_1, t_1g_2g_1, t_2g_2g_1, t_3g_2g_1, t_1t_2g_2g_1, t_1t_3g_2g_1, t_2t_3g_2g_1, t_1t_2t_3g_2g_1\\ &t_1g_1g_2g_2, t_2g_1g_2g_1, t_3g_1g_2g_1, t_1t_2g_1g_2g_1, t_1t_3g_1g_2g_1, t_2t_3g_1g_2g_1\ \}\,.\end{aligned}$$

Chapter 3

Markov traces on the three algebras

The following chapter is dedicated to the determination of the necessary and sufficient conditions for the trace tr on $\mathrm{Y}_{d,n}(u)$ to pass to the each one of the three quotient algebras $\mathrm{YTL}_{d,n}(u)$, $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$, in analogy to the classical case, where the Ocneanu trace on $\mathrm{H}_n(u)$ passes to the quotient algebra $\mathrm{TL}_n(u)$ under the condition that the trace parameter ζ takes specific values.

It should be clear by now that tr will pass to the quotient algebra if it kills the generator of the defining ideal of each quotient algebra. We will treat each case separately and then we will do a comparison of the derived conditions for each quotient algebra. We start with the Yokonuma–Temperley–Lieb algebra.

3.1 A Markov trace on $\mathrm{YTL}_{d,n}(u)$

We shall find the values of the trace parameter z that annihilate the generator of the defining ideal of $\mathrm{YTL}_{d,n}(u)$. We have the following lemma:

Lemma 3.1. *For the element $g_{1,2}$ we have:*

$$\mathrm{tr}(g_{1,2}) = (u+1)z^2 + \left((u-1)E+3\right)z + 1. \tag{3.1}$$

Proof. The proof is a straightforward computation:

$$\begin{aligned}\mathrm{tr}(g_{1,2}) &= \mathrm{tr}(1) + \mathrm{tr}(g_1) + \mathrm{tr}(g_2) + \mathrm{tr}(g_1g_2) + \mathrm{tr}(g_2g_1) + \mathrm{tr}(g_1g_2g_1)\\ &= 1 + 2z + 2z^2 + z + (u-1)Ez + (u-1)z^2\\ &= (u+1)z^2 + \left((u-1)E+3\right)z + 1.\end{aligned}$$

□

Lemma 3.1, together with the equation:

$$\mathrm{tr}(g_{1,2}) = 0 \tag{3.2}$$

give the following values for z:

$$z_{\pm} = \frac{-\left((u-1)E+3\right) \pm \sqrt{\left((u-1)E+3\right)^2 - 4(u+1)}}{2(u+1)}. \tag{3.3}$$

We shall do now the analysis for all conditions that must be imposed on the trace parameters $x_1, \ldots, x_{d-1}$ so that tr passes to $\mathrm{YTL}_{d,n}(u)$. Having in mind Corollary 2.1 and the linearity of tr, it follows that tr passes to $\mathrm{YTL}_{d,n}(u)$ if and only if the following equations are satisfied for all monomials $\mathfrak{m}$ in the inductive basis of $\mathrm{Y}_{d,n}(u)$. Namely:

$$\mathrm{tr}(\mathfrak{m}\, g_{1,2}) = 0. \tag{3.4}$$

Let us first consider the case $n = 3$. By (1.36) the elements in the inductive basis of $\mathrm{Y}_{d,3}(u)$ are of the following forms:

$$t_1^a t_2^b t_3^c, \quad t_1^a g_1 t_1^b t_3^c, \quad t_1^a t_2^b g_2 t_2^c, \quad t_1^a g_1 t_1^b g_2 t_2^c, \quad t_1^a t_2^b g_2 g_1 t_1^c, \quad t_1^a g_1 t_1^b g_2 g_1 t_1^c \tag{3.5}$$

Using Lemma 2.4 and the following notations:

$$\begin{aligned}
Z_{a,b,c} &:= (u+1)z^2 x_{a+b+c} + \left((u-1)E^{(a+b+c)} + x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b}\right) z + x_a x_b x_c\\
V_{a,b+c} &:= (u+1)z^2 x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\, x_a x_{b+c} + x_a E^{(b+c)}\\
V_{b,a+c} &:= (u+1)z^2 x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\, x_b x_{a+c} + x_b E^{(a+c)}\\
V_{c,a+b} &:= (u+1)z^2 x_{a+b+c} + (u+1)zE^{(a+b+c)} + z\, x_c x_{a+b} + x_c E^{(a+b)}\\
W_{a,b,c} &:= (u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}\left(e_1^{(a+b+c)} e_2\right)
\end{aligned}$$

From (3.4) and (3.5) we obtain the following equations, for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$:

$$Z_{a,b,c} = 0 \tag{3.6}$$
$$Z_{a,b,c} + (u-1)V_{a,b+c} = 0 \tag{3.7}$$
$$Z_{a,b,c} + (u-1)\left[V_{a,b+c} + V_{b,a+c} + W_{a,b,c}\right] = 0 \tag{3.8}$$
$$Z_{a,b,c} + (u-1)\left[V_{a,b+c} + V_{b,a+c} + V_{c,a+b} + W_{a,b,c}\right] = 0 \tag{3.9}$$

Equations 3.6–3.9 reduce to the following system of equations of $z, x_1, \ldots, x_{d-1}$ for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$:

$$(\Sigma)\begin{cases} Z_{a,b,c} = 0 & (3.10a)\\ V_{a,b+c} = 0 & (3.10b)\\ W_{a,b,c} = 0 & (3.10c)\end{cases}$$

Notice that for $a = b = c = 0$ Eq. 3.6 becomes Eq. 3.2. If, now, we require both solutions in (3.3) to participate in the solutions of (Σ), then we are led to sufficient conditions for tr to pass to $\mathrm{YTL}_{2,3}(u)$ (Section 4.2). If not then we are led to necessary and sufficient conditions for tr to pass to $\mathrm{YTL}_{2,3}(u)$ (Section 4.3).

Suppose that both solutions for z from Eq. 3.3 participate in the solution set of (Σ). We have the following proposition:

Proposition 3.1. *The trace* tr *defined on* $\mathrm{Y}_{d,3}(u)$ *passes to the quotient* $\mathrm{YTL}_{d,3}(u)$ *if the trace parameters x_i are d^{th} roots of unity ($x_i = x_1^i$, $1 \leq i \leq d-1$) and $z = -\frac{1}{u+1}$ or $z = -1$.*

Proof. Suppose that tr passes to $\mathrm{YTL}_{d,3}(u)$ and that (Σ) has both solutions for z from Eq. 3.3. This implies that there exists a λ in $\mathbb{C}(u)(x_1 \ldots, x_{d-1})$ such that:

$$Z_{a,b,c} = \lambda Z_{0,0,0}$$

From this we deduce that:

$$\begin{aligned} \lambda &= x_{a+b+c} \\ x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b} &= 3x_{a+b+c} \\ E^{(a+b+c)} &= x_{a+b+c}E \qquad (3.11) \\ x_{a+b+c} &= x_a x_b x_c. \qquad (3.12) \end{aligned}$$

Since this holds for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$, by taking $b = c = 0$ in Eq. 3.11 we have that:

$$E^{(a)} = x_a E \tag{3.13}$$

which is exactly the E–system. Moreover, by taking $c = 0$ in Eq. 3.12 we obtain:

$$x_a x_b = x_{a+b} \tag{3.14}$$

This implies that the x_i's are d^{th} roots of unity which is equivalent to $E = 1$ [23, Appendix]. In order to conclude the proof it is enough to verify that these conditions for the x_i's satisfy also (3.10b)–(3.10c) of (Σ). Since the x_i's are solutions of the E–system, Eq. 3.10b is immediately satisfied. We will finally check Eq. 3.10c. One has that $\mathrm{tr}(e_1^{(m)}e_2) = x_m E^2$ as soon as the x_m satisfy the E–system. Once this has been noticed, Eq. 3.10c becomes the same as Eq. 3.6 using Eq. 3.12 and $E = 1$. □

Using induction on n one can prove the general case of the sufficient conditions for tr to pass to $\mathrm{YTL}_{d,n}(u)$. Indeed we have:

Theorem 3.1. *If the trace passes to the quotient for $n = 3$ then it passes for all $n > 3$.*

Proof. We shall use induction on n. In Proposition 3.1 we proved the case where $n = 3$. Assume that the statement holds for all $\mathrm{YTL}_{d,k}(u)$, where $k \leq n$, that is:

$$\mathrm{tr}(a_k\, g_{1,2}) = 0$$

for all $a_k \in \mathrm{Y}_{d,k}(u)$, $k \leq n$. We will show the statement for $k = n + 1$. It suffices to prove that the trace vanishes on any element in the form $a_{n+1}g_{1,2}$, where a_{n+1} belongs to the inductive basis of $\mathrm{Y}_{d,n+1}(u)$ (recall (1.36)), given the conditions of the Theorem. Namely:

$$\mathrm{tr}(a_{n+1}\, g_{1,2}) = 0.$$

Since a_{n+1} is in the inductive basis of $\mathrm{Y}_{d,n+1}(u)$, it is of one of the following forms:

$$a_{n+1} = a_n g_n \dots g_i t_i^k \quad \text{or} \quad a_{n+1} = a_n t_{n+1}^k,$$

where a_n is in the inductive basis of $\mathrm{Y}_{d,n}(u)$. For the first case we have:

$$\mathrm{tr}(a_{n+1}\, g_{1,2}) = \mathrm{tr}(a_n g_n \dots g_i t_i^k\, g_{1,2}) = z\, \mathrm{tr}(a_n g_{n-1} \dots g_i t_i^k\, g_{1,2}) = z\, \mathrm{tr}(\tilde{a} g_{1,2}),$$

where $\tilde{a} := a_n g_{n-1} \dots g_i t_i^k$. The result follows by induction and thus the statement is proved. The second case is proved similarly. Hence, the proof is concluded. □

The above theorem allows us to state the following:

Theorem 3.2. *For $n \geq 3$, if the trace parameters x_i are d^{th} roots of unity, $x_i = x_1^i$, $1 \leq i \leq d-1$, and $z = -\frac{1}{u+1}$ or $z = -1$, then the trace* tr *defined on* $\mathrm{Y}_{d,n}(u)$ *passes to the quotient* $\mathrm{YTL}_{d,n}(u)$.

In the proofs of Proposition 3.1 and Theorem 3.1 became apparent that the x_i's are d^{th} roots of unity if and only if the values of z_+ and z_- satisfy all equations of (Σ). Clearly, if we loosen this last condition, then other solutions for the x_i's may appear such that the trace tr passes to the quotient $\mathrm{YTL}_{d,n}(u)$. Indeed, we have the following:

Theorem 3.3. *The trace* tr *passes to the quotient* $\mathrm{YTL}_{d,n}(u)$ *if and only if the x_i's are solutions of the* E*–system and one of the two cases holds:*

(i) For some $0 \leq m_1 \leq d-1$ the x_ℓ's are expressed as:

$$x_\ell = \exp_{m_1}(\ell) \quad (0 \leq \ell \leq d-1).$$

In this case the x_ℓ's are d^{th} roots of unity and $z = -\frac{1}{u+1}$ or $z = -1$.

(ii) For some $0 \leq m_1, m_2 \leq d-1$, $m_1 \neq m_2$, the x_ℓ's are expressed as:

$$x_\ell = \frac{1}{2}\left(\exp_{m_1}(\ell) + \exp_{m_2}(\ell)\right) \quad (0 \leq \ell \leq d-1).$$

In this case we have $z = -\frac{1}{2}$.

Note that case (i) captures Theorem 3.2.

Proof. Observe that the x_ℓ's expressed by (i) are indeed solutions of the system (Σ). We will now assume that our solutions are not of this form. This implies that $x_a \neq E^{(a)}$ for some $0 \leq a \leq d-1$. This will allow us to have this quantity in denominators later.

We will use induction on n. We will first prove the case $n = 3$. Suppose that trace tr passes to the quotient algebra $\mathrm{YTL}_{d,3}(u)$. This means that (Σ) has solutions for z any one of those in Eq. 3.3, for any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$. Subtracting Eq. 3.10a from Eq. 3.10b we obtain:

$$\left(x_a x_{b+c} + x_b x_{a+c} - 2E^{(a+b+c)}\right) z = -\left(x_a x_b x_c - x_c E^{(a+b)}\right). \tag{3.15}$$

For $b = c = 0$ in Eq. 3.15 and since we assumed that there is an a such that $x_a \neq E^{(a)}$ we obtain: $z = -\frac{1}{2}$. On the other hand, subtracting Eqs. 3.10a from Eq. 3.10c we have:

$$\left(3E^{(a+b+c)} - x_a x_{b+c} - x_b x_{a+c} - x_c x_{a+b}\right) z = x_a x_b x_c - \mathrm{tr}(e_1^{(a+b+c)} e_2). \tag{3.16}$$

For the value a such that $x_a - E^{(a)} \neq 0$ and for $b = c = 0$ in Eq. 3.16 we obtain:

$$z = -\frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}. \tag{3.17}$$

By combining Eqs. 3.15 and 3.17 we have that:

$$\frac{1}{2} = \frac{x_a - \mathrm{tr}(e_1^{(a)} e_2)}{3(x_a - E^{(a)})}$$

or equivalently:

$$3(x_a - E^{(a)}) = 2(x_a - \mathrm{tr}(e_1^{(a)} e_2)).$$

Using Lemma 1.4, this is equivalent to:

$$3x - \frac{3}{d} x * x = 2x - \frac{2}{d^2} x * x * x.$$

By taking the Fourier transform (see Lemma 1.5) we arrive at:

$$\frac{2}{d^2}\widehat{x}^3 - \frac{3}{d}\widehat{x}^2 + \widehat{x} = 0.$$

Assuming that $\widehat{x} = \sum_{0 \leq \ell \leq d-1} y_\ell t^\ell$ we have the following expression for the cofficients y_ℓ in the expansion of $\widehat{x}$:

$$y_\ell \left(\frac{2}{d^2} y_\ell^2 - \frac{3}{d} y_\ell + 1 \right) = 0.$$

So either $y_\ell = 0$ or $y_\ell = d$ or $y_\ell = \frac{1}{2}d$. So if we take a partition of the set $\{\ell : 0 \leq \ell \leq d-1\}$ into sets S_0, S_1, $S_{\frac{1}{2}}$ such that y_ℓ takes the value $i \cdot d$ on S_i $(i = 0, 1, \frac{1}{2})$. We have from Lemma 1.5 that:

$$x = \sum_{m \in S_1} \mathbf{i}_{-m} + \frac{1}{2} \sum_{m \in S_{\frac{1}{2}}} \mathbf{i}_{-m}.$$

From $x_0 = 1$ we obtain the conditions:

$$1 = x(0) = |S_1| + \frac{1}{2}|S_{\frac{1}{2}}|.$$

This means that either S_1 has only one element and $S_{\frac{1}{2}} = \emptyset$ or $S_1 = \emptyset$ and $S_{\frac{1}{2}}$ has two elements. The first case corresponds to the case (i) where the x_ℓ's are d^{th} roots of unity. In the second case, if $S_{\frac{1}{2}} = \{m_1, m_2\}$ we obtain the following solution of the E–system:

$$x_\ell = \frac{1}{2}\left(\exp_{m_1}(\ell) + \exp_{m_2}(\ell)\right), \quad (0 \leq \ell \leq d-1) \tag{3.18}$$

which corresponds to $z = -\frac{1}{2}$.

We can now check that these solutions satisfy the system (Σ). Since $z = -\frac{1}{2}$ and $E = \frac{1}{2}$, we have that $E^{(\ell)} = x_\ell/2$, $V_{c,a+b} = W_{a,b,c} = 0$, and that $Z_{a,b,c} = 0$ (Eq. 3.6) reduces to:

$$x_a x_{b+c} + x_b x_{a+c} + x_c x_{a+b} = x_{a+b+c} + 2 x_a x_b x_c,$$

which can be checked to be satisfied by the values x_ℓ given in Eq. (3.18). The rest of the proof (the induction on n) follows by Theorem 3.1. □

Remark 3.1. The values for the trace parameter z in Theorems 3.2 and 3.3, $z = -\frac{1}{u+1}$ and $z = -1$, in order that tr on $\mathrm{Y}_{d,n}(u)$ passes to the quotient $\mathrm{YTL}_{d,n}(u)$ are the same as the values in Eq. 1.18 for ζ of the Ocneanu trace τ on $\mathrm{H}_n(u)$, so that τ passes to the quotient $\mathrm{TL}_n(u)$ (recall Section 1.2).

3.2 A Markov trace on $\mathrm{FTL}_{d,n}(u)$

By Definition 2.3 we have that, if the trace tr passes to the quotient algebra $\mathrm{FTL}_{d,n}(u)$, then $\mathrm{tr}(r_{i,i+1}) = 0$ for all i, and in virtue of Corollary 2.4 it suffices that $\mathrm{tr}(r_{1,2}) = 0$. In the following lemma we compute the expression for $\mathrm{tr}(r_{1,2})$.

Lemma 3.2. *For the elements $r_{1,2} \in \mathrm{Y}_{d,n}(u)$ we have:*

$$\mathrm{tr}(r_{1,2}) = (u+1)z^2 + (u+2)E\,z + \mathrm{tr}(e_1e_2).$$

Proof. By direct computation we have that:

$$\begin{aligned}
\mathrm{tr}(r_{1,2}) &= \mathrm{tr}\left(e_1e_2\,g_{1,2}\right) = \mathrm{tr}(e_1e_2) + \mathrm{tr}(e_1e_2g_1) + \mathrm{tr}(e_1e_2g_2) \\
&\quad + \mathrm{tr}(e_1e_2g_1g_2) + \mathrm{tr}(e_1e_2g_2g_1) + \mathrm{tr}(e_1e_2g_1g_2g_1) \\
&= \mathrm{tr}(e_1e_2) + \mathrm{tr}(g_1e_{1,3}) + \mathrm{tr}(e_{1,3}g_2) + 2z^2 + \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} t_1^s t_2^{-s+k} t_3^{-k} g_1g_2g_1 \\
&= \mathrm{tr}(e_1e_2) + 2zE + 2z^2 + zE + (u-1)zE + (u-1)z^2 \\
&= (u+1)z^2 + (u+2)E\,z + \mathrm{tr}(e_1e_2).
\end{aligned}$$

□

Thus, the lemma above together with $\mathrm{tr}(r_{1,2}) = 0$ imply that the parameters z, u and $x_1, \ldots, x_{d-1}$ must satisfy the following equation:

$$(u+1)z^2 + (u+2)E\,z + \mathrm{tr}(e_1e_2) = 0. \tag{3.19}$$

The solutions of Eq. 3.19 are:

$$z_{\pm} := \frac{-(u+2)E \pm \sqrt{(u+2)^2E^2 - 4(u+1)A}}{2(u+1)} \tag{3.20}$$

where $A := \mathrm{tr}(e_1e_2)$.

Remark 3.2. For $d = 1$ we obtain from Eq. 3.19 that $z = -\frac{1}{u+1}$ or $z = -1$. Indeed, $d = 1$ implies $E = 1$ and $\mathrm{tr}(e_1e_2) = 1$, thus Eq. 3.19 becomes $(u+1)z^2 + (u+2)z + 1 = 0$ which yields the values of Eq. 1.18.

We will see that the above equation involving z, u and $x_1, \ldots, x_{d-1}$ is not a sufficient condition for tr to pass to $\mathrm{FTL}_{d,n}(u)$. Thus, the main purpose of the rest of this section is to find the necessary and sufficient conditions for trace tr to pass through to the quotient $\mathrm{FTL}_{d,n}(u)$. For this we recall the discussion in Section 2.5. We have following result:

Theorem 3.4. *For $n \geq 3$, the trace* tr *on $\mathrm{Y}_{d,n}(u)$ passes through to the quotient $\mathrm{FTL}_{d,n}(u)$ if the trace parameters $x_1, \ldots, x_{d-1}$ are solutions of the E–system and z takes one of the following values:*

$$z_{S,+} = -\frac{1}{u+1}E \quad \text{or} \quad z_{S,-} = -E.$$

To prove Theorem 3.4 we will need the following lemma:

Lemma 3.3. *The following holds in* $\mathrm{Y}_{d,n}(u)$ *for* $l \in \mathbb{Z}/d\mathbb{Z}$:

$$\mathrm{tr}\left(e_1^{(l)} e_2 g_{1,2}\right) = (u+1)z^2 x_l + (u+2)z\, E^{(l)} + \mathrm{tr}(e_1^{(l)} e_2)$$

Proof. By direct computation we have:

$$\begin{aligned}
\mathrm{tr}\left(e_1^{(l)} e_2 g_{1,2}\right) &= \mathrm{tr}\left(e_1^{(l)} e_2\right) + \mathrm{tr}\left(e_1^{(l)} e_2 g_1\right) + \mathrm{tr}\left(e_1^{(l)} e_2 g_2\right) \\
&\quad + \mathrm{tr}\left(e_1^{(l)} e_2 g_1 g_2\right) \quad + \mathrm{tr}\left(e_1^{(l)} e_2 g_2 g_1\right) + \mathrm{tr}\left(e_1^{(l)} e_2 g_1 g_2 g_1\right) \\
&= \quad \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \mathrm{tr}(t_1^{l+s} t_2^{-s+k} t_3^{-k} g_1) + \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \mathrm{tr}(t_1^{l+s} t_2^{-s+k} t_3^{-k} g_2) \\
&\quad + \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \mathrm{tr}(t_1^{l+s} t_2^{-s+k} t_3^{-k} g_1 g_2) + \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \mathrm{tr}(t_1^{l+s} t_2^{-s+k} t_3^{-k} g_2 g_1) \\
&\quad + \frac{1}{d^2}\sum_{s=0}^{d-1}\sum_{k=0}^{d-1} \mathrm{tr}(t_1^{l+s} t_2^{-s+k} t_3^{-k} g_1 g_2 g_1) + \mathrm{tr}\left(e_1^{(l)} e_2\right) \\
&= 2zE^{(l)} 2z^2 x_l + +zE^{(l)} + (u-1)zE^{(l)} + (u-1)z^2 x_l \\
&= (u+1)z^2 x_l + (u+2)zE^{(l)} + \mathrm{tr}\left(e_1^{(l)} e_2\right)
\end{aligned}$$

□

The strategy of proving Theorem 3.4 is by proving it first for $n = 3$ and then using induction on n.

For the general case, having in mind Corollary 2.4 and the linearity of tr, it follows that tr passes to $\mathrm{FTL}_{d,n}(u)$ if and only if the following equations are satisfied for all monomials $\mathfrak{m}$ in the inductive basis of $\mathrm{Y}_{d,n}(u)$. Namely:

$$\mathrm{tr}(\mathfrak{m}\, r_{1,2}) = 0. \tag{3.21}$$

As usual, we consider first the case $n = 3$. From Eqs. 3.5, Eq. 3.21 and from Lemma 3.3, for $l = a+b+c$, we obtain from four sets of equations, which all reduce to one single type of equation. Indeed, we have the following lemma:

Lemma 3.4. *(a) The case* $\mathfrak{m} = t_1^a t_2^b t_3^c$. *This case yields the equation:*

$$\mathrm{tr}(\mathfrak{m} r_{1,2}) = (u+1)z^2 x_{a+b+c} + (u+2)E^{(a+b+c)} z + \mathrm{tr}(e_1^{(a+b+c)} e_2). \tag{3.22}$$

(b) The cases $\mathfrak{m} = t_1^a g_1 t_1^b t_3^c$ *and* $\mathfrak{m} = t_1^a t_2^b g_2 t_2^c$. *These cases yield the same equation:*

$$\mathrm{tr}(\mathfrak{m} r_{1,2}) = u\left[(u+1)z^2 x_{a+b+c} + (u+2)E^{(a+b+c)} z + \mathrm{tr}(e_1^{(a+b+c)} e_2)\right]. \tag{3.23}$$

(c) The cases $\mathfrak{m} = t_1^a t_2^b g_2 g_1 t_1^c$ *and* $\mathfrak{m} = t_1^a g_1 t_1^b g_2 t_2^c$. *From these cases we obtain the equation:*

$$\mathrm{tr}(\mathfrak{m} r_{1,2}) = u^2\left[(u+1)z^2 x_{a+b+c} + (u+2)E^{(a+b+c)} z + \mathrm{tr}(e_1^{(a+b+c)} e_2)\right]. \tag{3.24}$$

(d) The case $\mathfrak{m} = t_1^a g_1 t_1^b g_2 g_1 t_1^c$. *This case yields the equation:*

$$\mathrm{tr}(\mathfrak{m} r_{1,2}) = u^3\left[(u+1)z^2 x_{a+b+c} + (u+2)E^{(a+b+c)} z + \mathrm{tr}(e_1^{(a+b+c)} e_2)\right]. \tag{3.25}$$

Proof. Case (a) follows immediately from Lemma 3.3 for $l = a+b+c$. For Case (b) we have that:

$$\mathrm{tr}(t_1^a g_1 t_1^b t_3^c r_{1,2}) = \mathrm{tr}(t_1^a t_2^b t_3^c g_1 r_{1,2})$$

Using now Lemmas 2.5 and 3.3 we have:

$$\begin{aligned}\mathrm{tr}(t_1^a t_2^b t_3^c[1+(u-1)e_1]r_{1,2}) &= \mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right) + (u-1)\mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right)\\ &= u\left[(u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}(e_1^{(a+b+c)}e_2)\right]\end{aligned}$$

In an analogous way, we have for $\mathfrak{m} = t_1^a t_2^b g_2 t_2^c$

$$\mathrm{tr}(t_1^a t_2^b g_2 t_2^c r_{1,2}) = \mathrm{tr}(t_1^a t_2^b t_3^c g_2 r_{1,2})$$

which is equivalent to:

$$\begin{aligned}\mathrm{tr}(t_1^a t_2^b t_3^c[1+(u-1)e_2]r_{1,2}) &= \mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right) + (u-1)\mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right)\\ &= u\left[(u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}(e_1^{(a+b+c)}e_2)\right]\end{aligned}$$

For case (c) we have that:

$$\mathrm{tr}(t_1^a g_1 t_1^b g_2 t_2^c r_{1,2}) = \mathrm{tr}(t_1^a t_2^b t_3^c g_1 g_2 r_{1,2}).$$

This is equal to:

$$\begin{aligned}&\mathrm{tr}(t_1^a t_2^b t_3^c[1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2 e_1 e_2]r_{1,2}) = \mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right)\\ &\quad + 2(u-1)\mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right) + (u-1)^2\mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right)\\ &= u^2\left[(u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}(e_1^{(a+b+c)}e_2)\right].\end{aligned}$$

If $\mathfrak{m} = t_1^a t_2^b g_2 g_1 t_1^c$ that:

$$\mathrm{tr}(t_1^a t_2^b g_2 g_1 t_1^c r_{1,2}) = \mathrm{tr}(t_1^a t_2^b t_3^c g_2 g_1 r_{1,2}).$$

This is equal to:

$$\begin{aligned}&\mathrm{tr}(t_1^a t_2^b t_3^c[1+(u-1)e_2+(u-1)e_{1,3}+(u-1)^2 e_1 e_2]r_{1,2}) = \mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right)\\ &\quad + 2(u-1)\mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right) + (u-1)^2\mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right)\\ &= u^2\left[(u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}(e_1^{(a+b+c)}e_2)\right].\end{aligned}$$

Finally, for case (d) we have:

$$\mathrm{tr}(t_1^a g_1 t_1^b g_2 g_1 t_1^c r_{1,2}) = \mathrm{tr}(t_1^a t_2^b t_3^c g_1 g_2 g_1 r_{1,2}).$$

Using now Lemmas 2.5 and 3.3 we have:

$$\begin{aligned}&\mathrm{tr}(t_1^a t_2^b t_3^c[1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1 e_2]r_{1,2}) = \mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right)\\ &\quad + 3(u-1)\mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right) + (u-1)^2(u+2)\mathrm{tr}\left(e_1^{(a+b+c)}e_2 g_{1,2}\right)\\ &= u^3\left[(u+1)z^2 x_{a+b+c} + (u+2)zE^{(a+b+c)} + \mathrm{tr}(e_1^{(a+b+c)}e_2)\right].\end{aligned}$$

□

We are now able to prove Theorem 3.4.

Proof of Theorem 3.4. We start by proving the statement for $n = 3$. As we mentioned at the beggining of this section, if tr passes to $\mathrm{FTL}_{d,n}(u)$ then tr vanishes on the generator of the principal ideal $J = \langle r_{1,2}\rangle$. The equation $\mathrm{tr}(r_1, 2) = 0$ implies that z takes one of the values z_+ or z_- of Eq. 3.20. Suppose now that tr passes to the quotient $\mathrm{FTL}_{d,n}(u)$. This means that tr vanishes on all elements of J. Using Lemma 3.4 we see that all elements yield Eq. 3.22. Let $l = a + b + c$, with $0 \leq l \leq d-1$, and let (Σ) be the following equivalent to (3.22) system of equations, with unknowns $z, x_1, \ldots, x_{d-1}$:

$$(\Sigma)\begin{cases} (u+1)z^2x_0 + (u+2)E^{(0)}z + \mathrm{tr}(e_1^{(0)}e_2) = 0 \\ (u+1)z^2x_l + (u+2)E^{(l)}z + \mathrm{tr}(e_1^{(l)}e_2) = 0 \quad (1 \leq l \leq d-1) \end{cases}$$

Recall that $x_0 := 1$, $E^{(0)} = E$ and $e_i^{(0)} = e_i$, for all i, and denote $A^{(l)} := \mathrm{tr}(e_1^{(l)}e_2)$, for $0 \leq l \leq d-1$. Note that $A^{(0)} = \mathrm{tr}(e_1e_2) =: A$. We then equivalently have the following system:

$$(\Sigma)\begin{cases} (u+1)z^2 + (u+2)E\,z + A = 0 & (3.27) \\ (u+1)z^2x_l + (u+2)E^{(l)}z + A^{(l)} = 0 \quad (1 \leq l \leq d-1) & (3.28) \end{cases}$$

Note that Eq. 3.27 is exactly Eq. 3.19. This implies that (Σ) has z_+ or z_- (or both) of Eq. 3.20 as solutions for z. If we require that (Σ) has both z_+ and z_- as solutions for z, then each one of Eqs. 3.28 should be a multiple of Eq. 3.27. In other words the following should hold:

$$(u+1)z^2x_l + (u+2)zE^{(l)} + A^{(l)} = \lambda_l\left[(u+1)z^2 + (u+2)zE + A\right],$$

where λ_l is in $\mathbb{C}(u)(x_1, \ldots, x_{d-1})$ and $0 \leq l \leq d-1$. This is true if and only if the following conditions hold:

$$\lambda_l = x_l \tag{3.29}$$
$$E^{(l)} = x_lE \tag{3.30}$$
$$A^{(l)} = A\,x_l \tag{3.31}$$

Equation 3.30 is precisely the E–system which, in turn, implies that Eq. 3.31 also holds. Indeed, if the x_i's are solutions of the E–system we have from Theorem 1.8, for $0 \leq l \leq d-1$ and $\alpha = e_1^{(l)}$, that:

$$\mathrm{tr}(e_1^{(l)}e_2) = \mathrm{tr}(e_1^{(l)})\mathrm{tr}(e_2) = E^{(l)}E = x_lE^2. \tag{3.32}$$

On the other hand we also have:

$$A = \mathrm{tr}(e_1e_2) = \mathrm{tr}(e_1)\mathrm{tr}(e_2) = E^2. \tag{3.33}$$

From Eqs. 3.32 and 3.33 we deduce immediately Eq. 3.31. From the above (Σ) has the two solutions z_+ and z_- for z if and only if the x_i's are solutions of the E–system. Then, given a solution $X_{d,S}$ of the E–system, Eq. 3.22 becomes:

$$x_{a+b+c}\left[(u+1)z^2 + ((u+2)E)\,z + E^2\right] = 0,$$

or equivalently:

$$x_{a+b+c}\left[((u+1)z+E)(z+E)\right]=0.$$

That is, given the E–condition, z must take one of the values $z_{S,+}$ or $z_{S,-}$ of the statement.

This concludes the proof for the case $n=3$. The rest of the proof (induction on n) follows the proof of Theorem 3.1. $\square$

Remark 3.3. In the proof of Theorem 3.4 it became apparent that the x_i's satisfy the E–system and also the system (Σ) if and only if the values z_+ and z_- satisfy all equations of (Σ). Clearly, if we loosen this last condition, then other values of the x_i's may appear in a solution of (Σ), other than a solution of the E–system. This means that the trace tr also passes to $\mathrm{FTL}_{d,n}(u)$ for these other values for the x_i's. However, for the purpose of constructing framed knot invariants from tr we have to exclude these values for the x_i's, since they do not permit the rescaling and normalization of tr (recall Section 1.11.1). In the following theorem we compute all possible solutions for the system of equations (Σ). In fact, the set of the solutions of the E–system is a subset of the set of solutions of (Σ) for the x_i's.

Theorem 3.5. *The trace* tr *passes to* $\mathrm{FTL}_{d,3}$ *if and only if the parameters of the trace* tr *satisfy:*

$$x_k=-z\left(\sum_{m\in\mathrm{Sup}_1}\chi(km)+(u+1)\sum_{m\in\mathrm{Sup}_2}\chi(km)\right)\quad \textit{and}\quad z=-\frac{1}{|\mathrm{Sup}_1|+(u+1)|\mathrm{Sup}_2|},$$

where $\mathrm{Sup}_1\cup\mathrm{Sup}_2$ *(disjoint union) is the support of the Fourier transform of* x *and* x *is the function complex function on* $\mathbb{Z}/d\mathbb{Z}$*, that maps* 0 *to* 1 *and* k *to the trace parameter* x_k*.*

Proof. We would like solve the system of equations:

$$(u+1)z^2x_\ell+(u+2)zE^{(\ell)}+\mathrm{tr}(e_1^{(\ell)}e_2)=0,\quad \text{for all } 0\le\ell\le d-1. \tag{3.34}$$

By subtracting the first quadratic equation from the others we have the equations:

$$z(u+2)\big(E^{(\ell)}-x_lE\big)=-\Big(\mathrm{tr}(e_1^{(\ell)}e_2)-x_\ell\,\mathrm{tr}(e_1e_2)\Big)\ \text{for all } 0\le l\le d-1. \tag{3.35}$$

Keep in mind that in the special case we have a solution of the E-system then all of the above conditions vanish and all equations have two common roots in z.

Assume that this is not the case. In this case we are going to solve the system given in equation (3.35). In terms of the group algebra it is transformed to the equation:

$$(u+2)z\sum_{0\le l\le d-1}\big(E^{(\ell)}-x_\ell E\big)\,t^\ell=-\sum_{0\le\ell\le d-1}\Big(\mathrm{tr}(e_1^{(\ell)}e_2+x_\ell\,\mathrm{tr}(e_1e_2))\Big)\,t^\ell$$

Interpreting now the above equation in the functional notation of Section 1.11.1 and having in mind Lemma 1.4, it follows that Eq. 3.35 can be rewritten as:

$$(u+2)z\left(\frac{1}{d}x*x-Ex\right)=-\left(\frac{1}{d^2}x*x*x-\mathrm{tr}(e_1e_2)x\right)$$

applying now the Fourier transform on the above functional equality and using Proposition 1.5, we obtain:

$$(u+2)z\left(\frac{\widehat{x}^2}{d}-E\widehat{x}\right)=-\left(\frac{\widehat{x}^3}{d^2}-\mathrm{tr}(e_1e_2)\widehat{x}\right) \tag{3.36}$$

Let now $\widehat{x}=\sum_{m=0}^{d-1}y_m t^m$. Then Eq. 3.36 becomes:

$$(u+2)z\left(\frac{y_m^2}{d}-Ey_m\right)=-\left(\frac{y_m^3}{d^2}-\mathrm{tr}(e_1e_2)y_m\right)$$

Hence

$$y_m\left(\frac{y_m^2}{d^2}+(u+2)z\frac{y_m}{d}-(u+2)zE-\mathrm{tr}(e_1e_2)\right)=0 \tag{3.37}$$

Now, from (3.34), for $l=0$, we have $-(u+2)zE=(u+1)z^2+\mathrm{tr}(e_1e_2)$. Replacing this expression of $-(u+2)zE$ in Eq. 3.37 we have that:

$$y_m\left(\frac{y_m^2}{d^2}+(u+2)z\frac{y_m}{d}+(u+1)z^2\right)=0$$

or equivalently:

$$y_m\left(y_m+dz\right)\left(y_m+dz(u+1)\right)=0 \tag{3.38}$$

Denote $\mathrm{Sup}_1\cup\mathrm{Sup}_2$ the support of $\widehat{x}$, where

$$\mathrm{Sup}_1:=\{m\in\mathbb{Z}/d\mathbb{Z}\,|\,y_m=-dz\}\quad\text{and}\quad\mathrm{Sup}_2:=\{m\in\mathbb{Z}/d\mathbb{Z}\,|\,y_m=-dz(u+1)\}$$

hence

$$\widehat{x}=\sum_{m\in\mathrm{Sup}_1}-dzt^m+\sum_{m\in\mathrm{Sup}_2}-dz(u+1)t^m$$

Then

$$\widehat{\widehat{x}}=-dz\sum_{m\in\mathrm{Sup}_1}\widehat{\delta}_m-dz(u+1)\sum_{m\in\mathrm{Sup}_2}\widehat{\delta}_m$$

thus from Proposition 1.5 we have:

$$\widehat{\widehat{x}}=-z\left(\sum_{m\in\mathrm{Sup}_1}\mathbf{i}_{-m}+(u+1)\sum_{m\in\mathrm{Sup}_2}\mathbf{i}_{-m}\right)$$

Therefore, having in mind again Proposition 1.5, we deduce that:

$$x_k=-z\left(\sum_{m\in\mathrm{Sup}_1}\chi(km)+(u+1)\sum_{m\in\mathrm{Sup}_2}\chi(km)\right) \tag{3.39}$$

Having in mind that $x_0=1$, one can determine the values of z. Indeed, from Eq. 3.39, we have that:

$$1=x_0=-z(|\mathrm{Sup}_1|+(u+1)|\mathrm{Sup}_2|)$$

or equivalently:

$$z=-\frac{1}{|\mathrm{Sup}_1|+(u+1)|\mathrm{Sup}_2|}. \tag{3.40}$$

□

Remark 3.4. We can now determine the values of z, $E = d^{-1}x*x(0)$ and $\mathrm{tr}(e_1e_2) = d^{-2}x * x * x(0)$ using the information $x_0 = x(0) = 1$. From Eq. 3.40 we have that:

$$E = z^2(|S_1| + (u+1)^2|S_2|) = \frac{|S_1| + |S_2|(1+u)^2}{(|S_1| + (u+1)|S_2|)^2}$$

$$A = -z^3(|S_1| + (u+1)^3|S_2|) = \frac{|S_1| + (u+1)^3|S_2|}{(|S_1| + (u+1)|S_2|)^3}.$$

A simple calculation shows that z, E, A as given above satisfy eq. (3.19).

Remark 3.5. For $|S_1| = 0$ and $E = \frac{1}{|S_2|}$ one obtains the value $z = -\frac{1}{u+1}E$, while for $E = \frac{1}{|S_1|}$ and $|S_2| = 0$ the value $z = -E$ of Theorem 3.4.

Using induction on n one can prove the general case of the necessary conditions for tr to pass to $\mathrm{FTL}_{d,n}(u)$. Indeed we have:

Theorem 3.6. *For $n \geq 3$, the trace* tr *defined on $\mathrm{Y}_{d,n}(u)$ passes to the quotient algebra $\mathrm{FTL}_{d,n}(u)$ if and only if the trace parameters z and x_i satisfy the conditions of Theorem 3.5.*

Proof. The proof is completely analogous to the proof of Theorem 3.1. □

3.3 A Markov trace on $\mathrm{CTL}_{d,n}(u)$

We will now present the conditions for tr to pass through to $\mathrm{CTL}_{d,n}(u)$. In virtue of the definition of $\mathrm{CTL}_{d,n}(u)$, tr passes to the quotient if $\mathrm{tr}(c_{1,2}) = 0$. We have the following lemma:

Lemma 3.5. *For the elements $c_{1,2}$ we have:*

$$\mathrm{tr}(c_{1,2}) = \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \Big((u+1)z^2 x_k + (u+2)zE^{(k)} + \mathrm{tr}(e_1^{(k)}e_2) \Big)$$

Proof. Using Remark 2.5 we have that:

$$\begin{aligned}
\mathrm{tr}(c_{1,2}) &= \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \Big(\mathrm{tr}(e_1^{(k)}e_2g_{1,2}) \Big) \\
&= \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \Big(\mathrm{tr}(e_1^{(k)}e_2) + \mathrm{tr}(e_1^{(k)}e_2g_1) + \mathrm{tr}(e_1^{(k)}e_2g_2) + \mathrm{tr}(e_1^{(k)}e_2g_1g_2) \\
&\quad + \mathrm{tr}(e_1^{(k)}e_2g_2g_1) + \mathrm{tr}(e_1^{(k)}e_2g_1g_2g_1) \Big) \\
&= \sum_{k \in \mathbb{Z}/d\mathbb{Z}} \Big((u+1)z^2x_k + (u+2)zE^{(k)} + \mathrm{tr}(e_1^{(k)}e_2) \Big)
\end{aligned}$$

□

In order to prove that tr passes to $\mathrm{CTL}_{d,n}(u)$, we will need the following lemmas.

Lemma 3.6. *For the element $r_{1,2}$ we have in $\mathrm{Y}_{d,n}(u)$:*

$$
\begin{array}{lrcl}
(1) & g_1c_{1,2} & = & [1+(u-1)e_1]c_{1,2} \\
(2) & g_2c_{1,2} & = & [1+(u-1)e_2]c_{1,2} \\
(3) & g_1g_2c_{1,2} & = & [1+(u-1)e_1+(u-1)e_{1,3}+(u-1)^2e_1e_2]c_{1,2} \\
(4) & g_2g_1c_{1,2} & = & [1+(u-1)\,e_2+(u-1)e_{1,3}+(u-1)^2e_1e_2]c_{1,2} \\
(5) & g_1g_2g_1c_{1,2} & = & [1+(u-1)(e_1+e_2+e_{1,3})+(u-1)^2(u+2)\,e_1e_2]c_{1,2}
\end{array}
$$

Proof. The proof is completely analogous to the proof of Lemma 2.5. □

As we mentioned before, the trace tr passes to the quotient if and only if $\mathrm{tr}(\alpha\, c_{1,2}) = 0$, for all α in the inductive basis of $\mathrm{Y}_{d,n}(u)$. Considering for a moment the case $n = 3$, we have, using Eq. 3.5 and Lemmas 3.5 and 3.6, the following:

Lemma 3.7. *For any $a, b, c \in \mathbb{Z}/d\mathbb{Z}$, and $\mathfrak{m}$ in the canonical basis of $\mathrm{Y}_{d,3}(u)$, the equation $\mathrm{tr}(\mathfrak{m}c_{1,2}) = 0$ yields the following: (a) For $\mathfrak{m} = t_1^a t_2^b t_3^c$:*

$$\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \Big((u+1)z^2x_k + (u+2)zE^{(k)} + \mathrm{tr}(e_1^{(k)}e_2)\Big) = 0 \tag{3.41}$$

(b) For $\mathfrak{m} = t_1^a g_1 t_1^b t_3^c$ and $\mathfrak{m} = t_1^a t_2^b g_2 t_2^c$:

$$u\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \Big((u+1)z^2x_k + (u+2)zE^{(k)} + \mathrm{tr}(e_1^{(k)}e_2)\Big) = 0$$

(c) For $\mathfrak{m} = t_1^a t_2^b g_2 g_1 t_1^c$ and $\mathfrak{m} = t_1^a g_1 t_1^b g_2 t_2^c$:

$$u^2\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \Big((u+1)z^2x_k + (u+2)zE^{(k)} + \mathrm{tr}(e_1^{(k)}e_2)\Big) = 0$$

(d) For $\mathfrak{m} = t_1^a g_1 t_1^b g_2 g_1 t_1^c$:

$$u^3\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \Big((u+1)z^2x_k + (u+2)zE^{(k)} + \mathrm{tr}(e_1^{(k)}e_2)\Big) = 0$$

Proof. The proof is analogous to the proof of Lemma 3.4. □

Concluding, we have the following theorem:

Theorem 3.7. *The trace* tr *passes to the quotient if and only if the parameter z and the x_i's are related through the equation:*

$$(u+1)z^2\sum_{k\in\mathbb{Z}/d\mathbb{Z}} x_k + (u+2)z\sum_{k\in\mathbb{Z}/d\mathbb{Z}} E^{(k)} + \sum_{k\in\mathbb{Z}} \mathrm{tr}(e_1^{(k)}e_2) = 0. \tag{3.42}$$

Proof. The proof follows using the same reasoning that was used to prove Theorem 3.1 and having in mind Eq. 3.21, Corollary 2.5, Lemmas 3.6 and 3.7. □

3.4 Comparison of the three trace conditions

In this section we will compare the conditions that need to be applied to the trace paramaters z and x_i, $i = 1, \ldots, d-1$ so that tr passes to each of the quotient algebras.

In Theorem 3.3 (see also [8]) we found the necessary and sufficient conditions so that tr passes to $\mathrm{YTL}_{d,n}(u)$. The conditions for the x_i's in this case are particular solutions of the E–system. Thus, the conditions such that tr passes to $\mathrm{YTL}_{d,n}(u)$ are contained in those of Theorem 3.5.

Moreover, Theorem 3.5 can be rephrased in the following way:

Theorem 3.8. *The trace* tr *passes to the quotient algebra* $\mathrm{FTL}_{d,n}(u)$ *if and only if the parameter* z *and the* x_i*'s are related through the equation:*

$$(u+1)z^2 x_k + (u+2)zE^{(k)} + \mathrm{tr}(e_1^{(k)} e_2) = 0, \qquad \forall k \in \mathbb{Z}/d\mathbb{Z}$$

This implies that the conditions such that the trace passes to the quotient algebra $\mathrm{FTL}_{d,n}(u)$ are contained in those of Theorem 3.7. All of the above can be summarised in the following table:

	$\mathrm{Y}_{d,n}(u)$	$\twoheadrightarrow$	$\mathrm{CTL}_{d,n}(u)$	$\twoheadrightarrow$	$\mathrm{FTL}_{d,n}(u)$	$\twoheadrightarrow$	$\mathrm{YTL}_{d,n}(u)$
z	free	$\hookleftarrow$	Theorem 3.7	$\hookleftarrow$	Theorem 3.8	$\hookleftarrow$	Theorem 3.3
x_i	free						

The first row includes the projections between the algebras while the second shows the inclusions of the trace conditions for each case.

Chapter 4

Link invariants

We recall now the discussion in Chapter 1 regarding the construction of the Homflypt and the Jones polynomials. One can define the 2–variable Jones or Homflypt polynomial, $P(\lambda, u)$ [16], by re–scaling and normalizing the Ocneanu trace τ on $\mathrm{H}_n(u)$. We have:

$$P(\lambda, u)(\widehat{\alpha}) = \left(-\frac{1-\lambda u}{\sqrt{\lambda}(1-u)}\right)^{n-1} \left(\sqrt{\lambda}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)),$$

where: $\alpha \in \cup_\infty B_n$, $\lambda = \frac{1-u+\zeta}{u\zeta}$, π is the natural epimorphism of $\mathbb{C}(u)B_n$ onto $\mathrm{H}_n(u)$ that sends the braid generator σ_i to h_i and $\varepsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in α. Further, by specializing ζ to $-\frac{1}{u+1}$, the non–trivial value for which the Ocneanu trace τ passes to the quotient algebra $\mathrm{TL}_n(u)$, the Jones polynomial, $V(u)$, can be defined through the Homflypt polynomial [16]. Namely:

$$V(u)(\widehat{\alpha}) = \left(-\frac{1+u}{\sqrt{u}}\right)^{n-1} \left(\sqrt{u}\right)^{\varepsilon(\alpha)} \tau(\pi(\alpha)) = P(u, u)(\widehat{\alpha}).$$

In Chapter 2 we discussed about the way that the trace tr defined on $\mathrm{Y}_{d,n}(u)$ could be re–scaled according to the braid equivalence corresponding to isotopic framed links [23]. This can be achieved, if and only if the x_i's furnish a solution of the E–system (recall discussion in Section 1.11.1). Let $X_{d,S} = (\mathrm{x_1}, \ldots, \mathrm{x_{d-1}})$ be a solution of the E–system parametrized by the non–empty set S of $\mathbb{Z}/d\mathbb{Z}$.

Further, by restricting $\Gamma_{d,S}(w, u)$ to classical links, seen as framed links with all framings zero, an invariant of classical oriented links is obtained, denoted by $\Delta_{d,S}(w, u)$ [22]. Moreover, in [21] the invariant $\Delta_{d,S}(w, u)$ was extended to an invariant for singular links. Recall also that for $d = 1$ the Juyumaya trace tr and the specialized Juyumaya tracetr_S coincide with the Ocneanu trace. For generic values of the parameters u, z the invariants $\Delta_{d,S}(w, u)$ do not coincide with the Homflypt polynomial except in the trivial cases $u = 1$ and $E = 1$ [2]. Yet, computational data [5] indicate that these invariants may be topologically equivalent to the Homflypt polynomial.

4.1 Link Invariants from $\mathrm{YTL}_{d,n}(u)$

We shall now define framed and classical link invariants related to the algebra $\mathrm{YTL}_{d,n}(u)$. Recalling now the conditions of Theorem 3.3 for the trace tr to pass to the quotient $\mathrm{YTL}_{d,n}(u)$, we note that in both cases the x_i's are solutions of the E–system, as required by [20], in order to proceed with defining link invariants. Further, we do not take into consideration case (i) where $z = -1$ and the x_i's are roots of unity (which implies $E = 1$), and case (ii) where $z = -\frac{1}{2}$ and $E = \frac{1}{2}$, since crucial braiding information is lost and therefore they are of no topological interest. Indeed, the trace tr gives the same value for all even (resp. odd) powers of the g_i's, for $m \in \mathbb{Z}^{>0}$ [23]:

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m - 1}{u+1}\right) z + \left(\frac{u^m - 1}{u+1}\right) E + 1 \qquad \text{if } m \text{ is even} \tag{4.1}$$

and

$$\mathrm{tr}(g_i^m) = \left(\frac{u^m - 1}{u+1}\right) z + \left(\frac{u^m - 1}{u+1}\right) E - E \qquad \text{if } m \text{ is odd.} \tag{4.2}$$

The only remaining case of interest is case (i) of Theorem 3.3, where the x_ℓ's are the d^{th} roots of unity and $z = -\frac{1}{u+1}$. This implies that $E = 1$ and $w = u$ in Eq. 1.9. We know from [23, Remark 5] that the invariant $\Gamma_{d,S}(w, u)$ is not very interesting for framed links when the x_i's are d^{th} roots of unity because basic pairs of framed links are not distinguished. For classical links, as mentioned earlier, we know from [2, Corollary 1] that the invariants $\Delta_{d,S}(w, u)$ coincide with the Homflypt polynomial (case $E = 1$). More precisely, for $E = 1$ an algebra homomorphism can be defined, $h : \mathrm{Y}_{d,n}(u) \longrightarrow \mathrm{H}_n(u)$, and the composition $\tau \circ h$ is a Markov trace on $\mathrm{Y}_{d,n}(u)$ which takes the same values as the specialized trace tr, whereby the x_i's are specialized to d^{th} roots of unity ($x_i = x_1^i$, $1 \leq m \leq d - 1$). For details see [2, §3]. The above discussion leads to the following corollary:

Corollary 4.1. *The invariants* $V_{d,S}(u) := \Delta_{d,S}(u, u)$ *coincide with the Jones polynomial.*

4.2 Link Invariants from $\mathrm{FTL}_{d,n}(u)$

In Section 3.2 we proved that tr passes to $\mathrm{FTL}_{d,n}(u)$ if and only if the trace parameters satisfy certain conditions. It is only natural now to discuss the connection between tr on $\mathrm{FTL}_{d,n}(u)$ and knot invariants. As it has already been stated, the trace parameters x_i should be solutions of the E–system so that a link invariant through tr is well–defined. Moreover, the conditions of Theorem 3.6 include these solutions for the x_i's. In order to define a link invariant on the level of the quotient algebra $\mathrm{FTL}_{d,n}(u)$, we discard any value of the x_i's that does not comprise a solution of the E–system. Using Remark 3.5 we choose a solution of the E–system and denote with S the subset of $\mathbb{Z}/d\mathbb{Z}$ that parametrizes the said solution. This leads to the following values for z:

$$z = -\frac{1}{(u+1)|S|} \quad \text{or} \quad z = -\frac{1}{|S|}.$$

In analogy to the invariants from the algebra $\mathrm{YTL}_{d,n}(u)$, we do not take into consideration the case where $z = -\frac{1}{|S|}$ and $E = \frac{1}{|S|}$, since important topological information is lost. From the remaining case where the x_i's are solutions of the E–system and $z = -\frac{1}{(u+1)|S|}$ we deduce that $w = u$ in Eq. 1.9. We then have the following definition:

Definition 4.1. Let $X_{d,S}$ be a solution of the E–system, parametrized byt the non–empty subset S of $\mathbb{Z}/d\mathbb{Z}$ and let $z = -\frac{1}{(u+1)|S|}$. We obtain from $\Gamma_{d,S}(w,u)$ the following invariant for $\alpha \in \cup_\infty \mathcal{F}_n$:

$$\vartheta_{d,S}(u)(\widehat{\alpha}) \ := \ \left(-\tfrac{(1+u)|S|}{\sqrt{u}}\right)^{n-1} (\sqrt{u})^{\varepsilon(\alpha)} \operatorname{tr}_S(\gamma(\alpha)) = \Gamma_S(u,u)(\widehat{\alpha})$$

In analogy to the case of $\Gamma_S(w,u)$, if we restrict to framed links with all framings zero, we obtain an invariant of classical oriented links, denoted by $\theta_{d,S}(u) := \Delta_{d,S}(u,u)$.

Remark 4.1. If the invariants $\Delta_{d,S}(w,u)$ on the level of the Yokonuma–Hecke algebras turn out to be topologically equivalent to the Homflypt polynomial [5] then the invariants $\theta_{d,S}(u)$ will be topologically equivalent to the Jones polynomial, and the invariants $\vartheta_{d,S}(u)$ framed analogues of the Jones polynomial.

4.3 Link Invariants from $\mathrm{CTL}_{d,n}(u)$

The conditions of Theorem 3.7 do not involve the solutions of the E–system at all, so in order to obtain a well–defined link invariant on the level of $\mathrm{CTL}_{d,n}(u)$ we must impose this condition on the x_i's. Recall that the solutions of the E–system can be expressed in the form:

$$x_S = \frac{1}{|S|}\sum_{k\in S} \mathbf{i}_k \in \mathbb{C}C_d,$$

where $\mathbf{i}_k = \sum_{j=0}^{d-1} \chi_k(j)t^j$, and χ_k is the character that sends $m \mapsto \cos\frac{2\pi km}{d} + i\sin\frac{2\pi km}{d}$ and S is the subset of $\mathbb{Z}/d\mathbb{Z}$ that parametrizes a solution of the E–system. Let now ε be the augmentation function of the group algebra $\mathbb{C}C_d$, sending $\sum_{j=0}^{d-1} x_j t^j$ to $\sum_{j=0}^{d-1} x_j$. We have that:

$$\varepsilon(x_S) = \frac{1}{|S|}\sum_{k\in S}\varepsilon(\mathbf{i}_k) = \frac{1}{|S|}\sum_{j=0}^{d-1}\sum_{k\in S}\chi_j(k) = \begin{cases} \frac{d}{|S|}, & \text{if } 0\in S \\ 0 & \text{if } 0\notin S \end{cases}. \tag{4.3}$$

From this we deduce that:

$$\sum_{j=0}^{d-1} E^{(j)} = \varepsilon\left(\frac{x*x}{d}\right) = \frac{1}{d|S|^2}\sum_{k\in S}\varepsilon(\mathbf{i}_k*\mathbf{i}_k) = \frac{1}{|S|^2}\sum_{k\in S}\varepsilon(\mathbf{i}_k) = \begin{cases} \frac{d}{|D|^2}, & \text{if } 0\in S \\ 0 & \text{if } 0\notin S \end{cases} \tag{4.4}$$

and also that:

$$\sum_{j=0}^{d-1}\operatorname{tr}(e_1^{(j)}e_2) = \varepsilon\left(\frac{x*x*x}{d^2}\right) = \frac{1}{d^2|S|^3}\sum_{k\in S}\varepsilon(\mathbf{i}_k*\mathbf{i}_k*\mathbf{i}_k) = \frac{1}{|S|^3}\sum_{k\in S}\varepsilon(\mathbf{i}_k) = \begin{cases} \frac{d}{|S|^3}, & \text{if } 0\in S \\ 0 & \text{if } 0\notin S \end{cases}. \tag{4.5}$$

Using now Eqs. 4.3, 4.4 and 4.5, Eq. 3.42 becomes for the case where $0\in S$:

$$\frac{d}{|S|}\left((u+1)z^2 + \frac{(u+2)}{|S|}z + \frac{1}{|S|^2}\right) = 0.$$

Therefore, the trace tr passes to the quotient for the following values of z:

$$z = -\frac{1}{(u+1)|S|} \quad \text{or} \quad z = -\frac{1}{|S|}.$$

The value $z = -\frac{1}{|S|}$ is not taken into consideration, since from Eq. 3.42 we deduce that $E = \frac{1}{|S|}$ and therefore from Eqs. 4.1 and 4.2 the trace tr gives the same value for all even (resp. odd) powers of the g_i's. Thus, the invariants that are obtained from tr on the level of the quotient algebra $\mathrm{CTL}_{d,n}(u)$ coincide with the invariants ϑ_D and θ_D on the level of $\mathrm{FTL}_{d,n}(u)$. More precisely, the conditions that are applied to the trace parameters are the same for both of the quotient algebras and, consequently, so are the related invariants.

Furthermore, the solutions of the E–system (which are the necessary and sufficient conditions so that topological invariants for framed links can be defined) are included in the conditions of Theorem 3.4, while for the case of $\mathrm{CTL}_{d,n}(u)$ we still have to impose them. These are the main reasons that lead us to consider the quotient algebra $\mathrm{FTL}_{d,n}(u)$ as the most natural non–trivial analogue of the Temperley–Lieb algebra in the context of framed links. The following table gives a full overview of the invariants for each quotient algebra:

d, D	$\mathcal{F}_{d,n}$	u	w
$\mathrm{Y}_{d,n}(u)$	Γ_D	u	w
$\mathrm{YTL}_{d,n}(u)$	$\mathcal{V}_D$	u	u
$\mathrm{FTL}_{d,n}(u)$	ϑ_D	u	u
$\mathrm{CTL}_{d,n}(u)$	ϑ_D	u	u

$d = 1$	B_n	u	w
$\mathrm{H}_n(u)$	P	u	λ
$\mathrm{TL}_n(u)$	V	u	u

$d, \lvert D\rvert = 1$	$\mathcal{F}_{d,n}$	u	w
$\mathrm{Y}_{d,n}(u)$	Γ_D	u	λ
$\mathrm{YTL}_{d,n}(u)$	$\mathcal{V}_D$	u	u
$\mathrm{FTL}_{d,n}(u)$	$\mathcal{V}_D$	u	u
$\mathrm{CTL}_{d,n}(u)$	$\mathcal{V}_D$	u	u

$d, \lvert D\rvert > 1$	$\mathcal{F}_{d,n}$	u	w
$\mathrm{Y}_{d,n}(u)$	Γ_D	u	λ
$\mathrm{YTL}_{d,n}(u)$	no	–	–
$\mathrm{FTL}_{d,n}(u)$	ϑ_D	u	u
$\mathrm{CTL}_{d,n}(u)$	ϑ_D	u	u

Table 4.1: *Overview of the invariants for each algebra.*

4.4 Conclusions

In this thesis we proposed three possible quotients of the Yokonuma–Hecke algebra, $\mathrm{Y}_{d,n}(u)$, as framizations of the Temperley–Lieb algebra, the algebras $\mathrm{YTL}_{d,n}(u)$, $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$. We proved the necessary and sufficient condition such that the Markov trace tr defined on $\mathrm{Y}_{d,n}(u)$ passes to each of the quotient algebras, and we defined the link invariants that correspond to each case.

The knot invariants from the algebras $\mathrm{YTL}_{d,n}(u)$, when restricted to classical knots, recover the Jones polynomial while the knot invariants for the algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ still remain under investigation. If the invariants from the Yokonuma–Hecke algebras prove to be topologically equivalent to the Homflypt polynomial, then the invariants from $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$ will be topologically equivalent to the Jones polynomial. If not, it would be then meaningfull to consider the corresponding 3–manifold invariants (as obtained from work of Wenzl [36]). In the case of the algebras $\mathrm{YTL}_{d,n}(u)$ the Witten invariants of 3–manifolds can be recovered, since the related knot invariants recover the Jones polynomial [8].

Finally, some of the most interesting problems that arise from this thesis are the diagrammatic interpretation of the presentation with non-invertible generators for each of the three quotient algebras and the study of the Representation Theory of the algebras $\mathrm{FTL}_{d,n}(u)$ and $\mathrm{CTL}_{d,n}(u)$.

4.5 Connection with 3–manifolds

It is remarkable that while the trace tr is not local (i.e. multiplicative) in principle, it becomes so when the x_i's are chosen to be solutions of the E–system. This implies that there exists a 3–manifold invariant that corresponds to the trace tr. Indeed, from the following Proposition by H. Wenzl [36] we have that:

Proposition 4.1. *(i) There exists a 1–1 correspondence between multiplicative invariants of framed links $\mathcal{L}$ such that $\mathcal{L}(0-\textit{unknot}) \neq 0$ and local Markov trace* tr *on $\mathcal{F}_\infty$, for all $a \in \mathcal{F}_n$. It is given by:*

$$\mathrm{tr}(a) = \frac{\mathcal{L}(\hat{a})}{\mathcal{L}(0-\textit{unknot})^n}$$

(ii) There exists a 1–1 correspondence between multiplicative invariants $\mathfrak{F}$ of closed connected oriented 3–manifolds with $\mathfrak{F}(S^1 \times S^2) = C_0 \neq 0$, where multiplicative here means that $\mathfrak{F}(M_1 \# M_2) = \mathfrak{F}(M_1)\mathfrak{F}(M_2)$, and local Markov trace on $\mathcal{F}_\infty$ satisfying for any $a \in \mathcal{F}_n$ and any $m \leq n+1$:

$$\mathrm{tr}(\alpha(\sigma_{n-m+1}\dots\sigma_{n+1})\dots(\sigma_{n+1}\dots\sigma_{n+m})) = \mathrm{tr}(\alpha)C_0^{-m-1} = \mathrm{tr}(\alpha)\mathrm{tr}(\sigma_1^{(m)})$$

and

$$\mathrm{tr}(\sigma_1) = \mathrm{tr}(\sigma_1^{-1}) = \frac{1}{C_0^{-2}}$$

and such that the corresponding invariant of framed links does not depend on the choice of orientation of the links. The correspondence is given by:

$$\mathrm{tr}(\alpha) = \frac{\mathfrak{F}(M(\alpha,n))}{\mathfrak{F}(S^1 \times S^2)^n}$$

where $\mathfrak{F}(S^1 \times S^2) = C_0 = \frac{\mathrm{tr}(t_1)}{\mathrm{tr}(\sigma_1)}$.

We have the following corollary:

Corollary 4.2. *Applying Wenzl's result on the case of the invariants $V_{d,S}(u)$ from the* $\mathrm{YTL}_{d,n}(u)$ *algebras, we recover the Witten invariants for 3–manifolds, since the invariants $V_{d,S}(u)$ recover the Jones polynomial.*

A final remark is now due:

Remark 4.2. Applying Wenzl's result on the case of the invariants $\vartheta_{d,S}(u)$ from the $\mathrm{FTL}_{d,n}(u)$ algebras, we recover topological invariants for 3–manifolds, which remain to be studied.

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