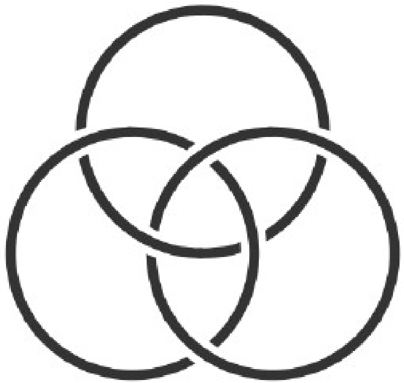
Combinatorial Construction of the *HOMLYPT* polynomial

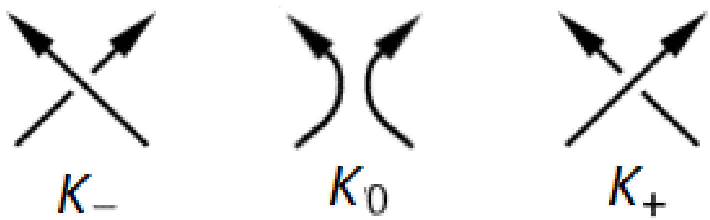


Diamantis Yannis

Main Theorem

- There is a unique function $\mathscr{P}$ which associates to each *K* in $\mathcal{L}$ an element $\mathscr{P}$*(K)=K(l,m)* in $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ which depends only on the isotopy class of the oriented link such that:

- If K_-, K_0, K_+ are identical except for a single crossing:



- Then:

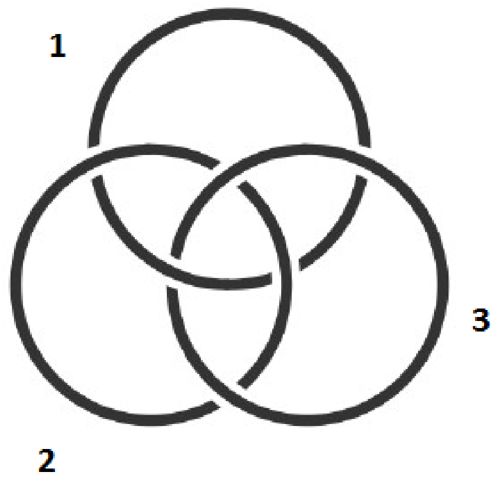
(I) $$lK_+(l,m) + l^{-1}K_-(l,m) + mK_0(l,m) = 0$$

- And if $\mathcal{U}$ denotes the unknot (of 1 component), then

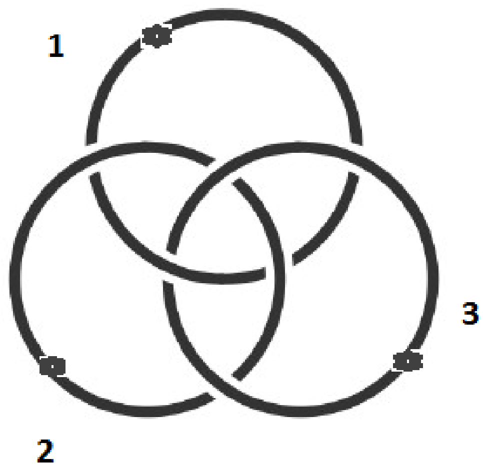
(II) $\mathcal{U}(l,m)=1$.

Definitions

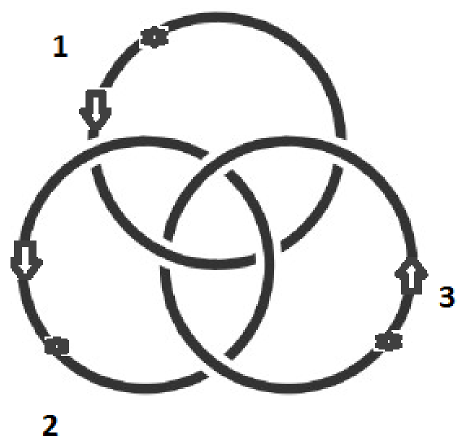
- A link is *ordered* if an order is given to its components.



- A link is *based* if a base point is specified on each component.



- A link is *oriented* if an orientation of each component is specified.



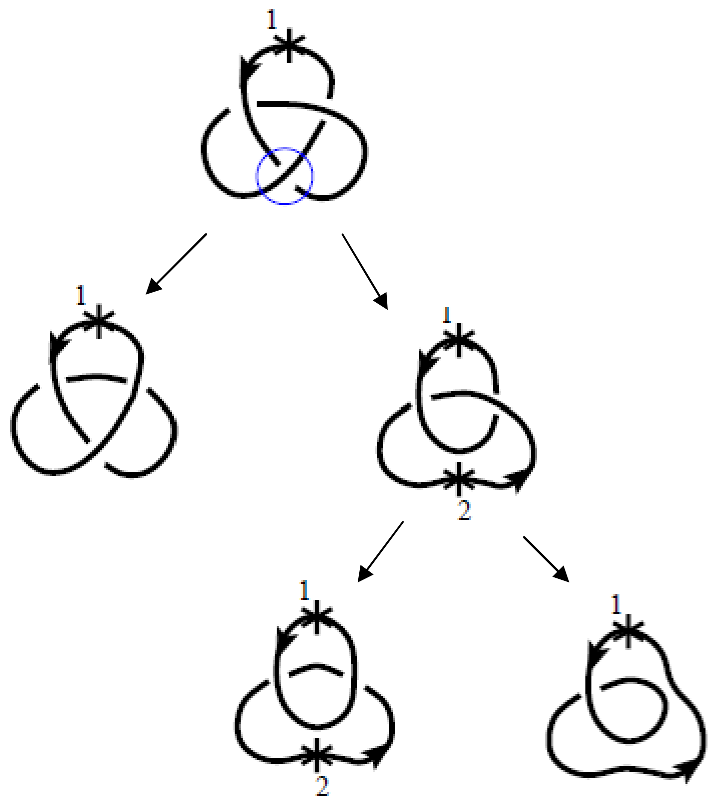
- A projection of a based link is *generic* if the projection defines an immersion of the link into the plane having no triple points and only transverse double points such that the base points are distinct from the double points.

- $\mathcal{L}_n$: The set of generic, based, ordered, oriented link projections with at most *n* crossings.

- Let $\mathcal{L} = \bigcup \mathcal{L}_n$

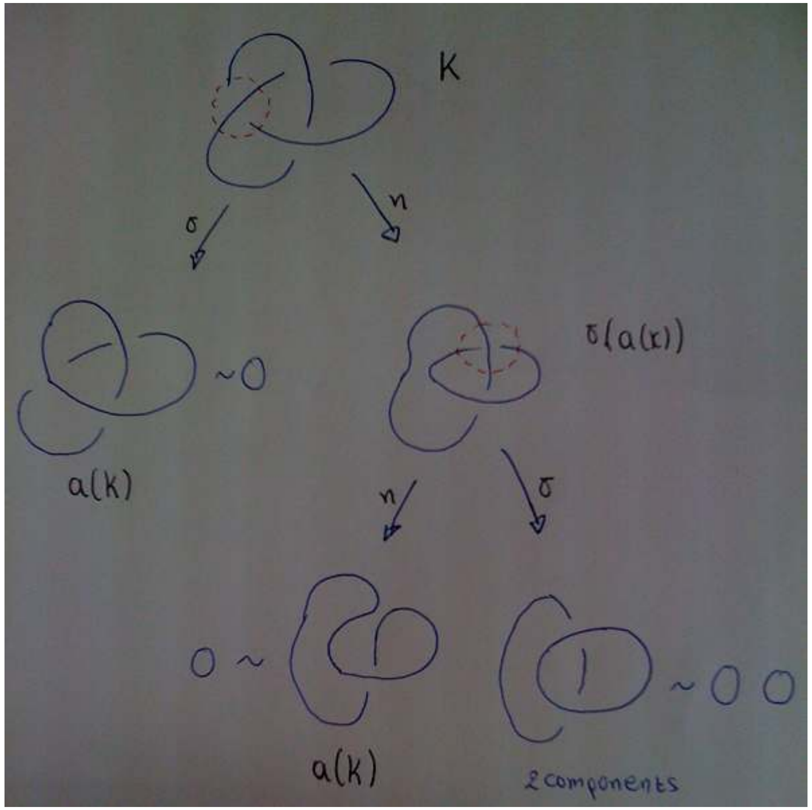
- *Ascending (descending) element L* of $\mathcal{Z}$:

  When traversing the components of L in their given order and from their base points in the direction specified by their orientation, every crossing is 1st encountered as an overcrossing (resp. undercrossing).



- Clearly: *L* ~ unlink

- *Standard ascending projection* of the unlink of *n* components (*α(K)*) is associated to a generic projection, *K*, of an oriented, ordered, based link of *n* components.



Inductive Hypothesis

- Assume that to each *K* in $\mathcal{L}_{n-1}$, there is associated an element $\mathcal{P}$(*K*) in $\mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$ which is independent of the choices of base points and the ordering of the components, is invariant under those Reidemeister moves which do not increase the number of crossings beyond *n-1*, and which satisfies:

- (I) $lK_{+}(l,m) + l^{-1}K_{-}(l,m) + mK_{0}(l,m) = 0$

- (II') $U^{c} = \mu^{c-1}$, where $U^{c} \in L_{n-1}$ denotes any ascending projection of c components and:

$$\mu = {}^{-\left(l + l^{-1}\right)}\!\big/_{m}$$

- The induction starts with zero crossing projections.

- Let $P(K(l,m)) = K(l,m) \in \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$

- Order the components of *K*: $c_1, \ldots, c_s$

- Label the crossings: $\{1, 2, \ldots, n\}$

- We need to show that:

A. $\mathscr{P}(K)$ is independent of:

(1) The crossing changes to achieve *α(K)* from *K.*
(2) The choice of base points.
(3) The choice of order of the components.

B. $\mathscr{P}(K)$ satisfies formula (I).

C. $\mathscr{P}(K)$ is invariant under *R-moves.*

Proposition 1

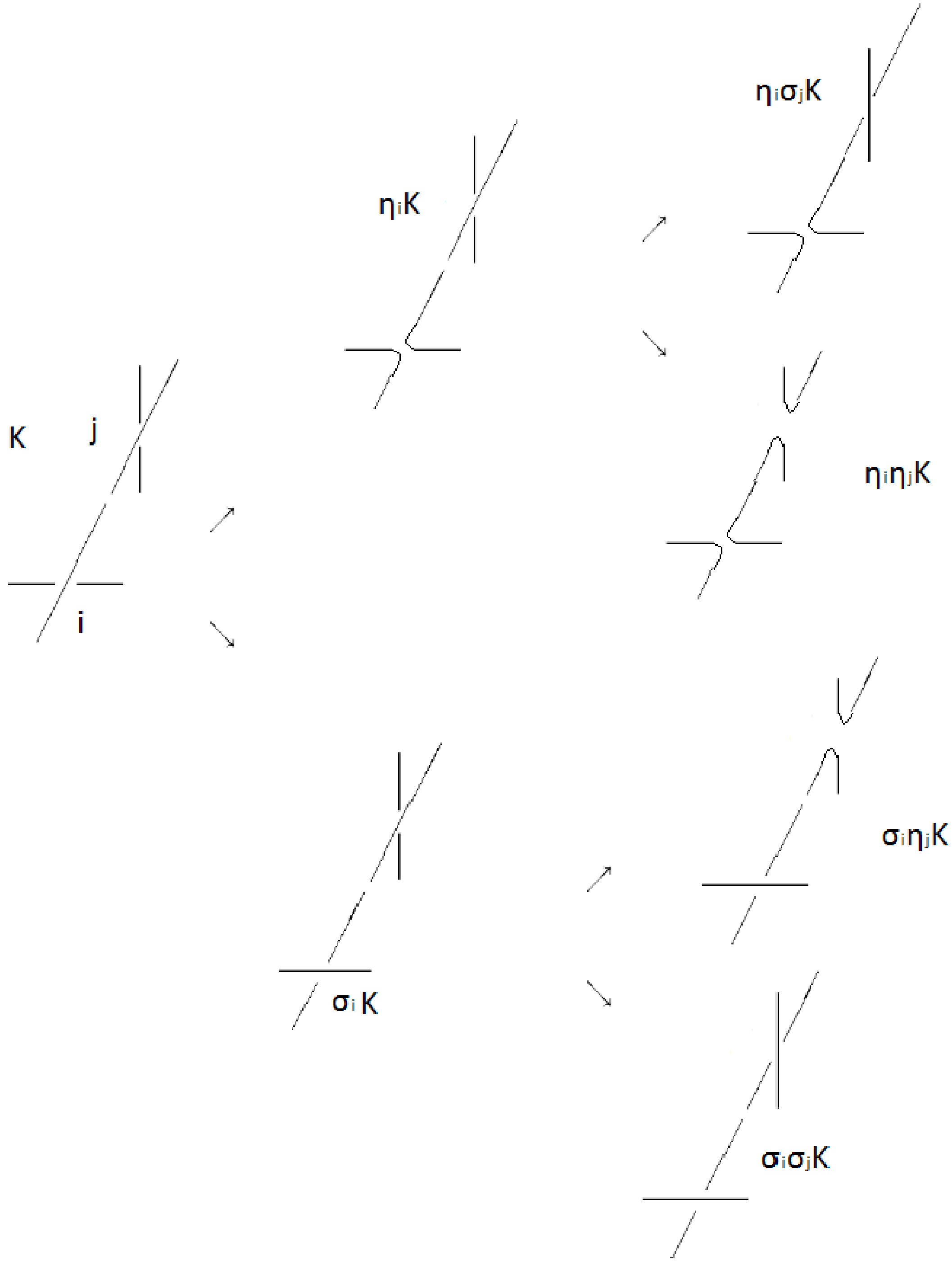
- Suppose *K* in $\mathcal{L}_n$. If the crossings of *K* differ from those of *α(K)* are changed in any sequence to achieve *α(K),* then the corresponding calculation (using (I) and (II’)) yields $\mathcal{P}$*(K).*

Proof

- Induction on the number of crossing differences between *K* and *α(K).*

- *It is only necessary to consider altering the sequence by interchanging the first two crossing switches which the algorithm requires at, say, the crossing labeled i and then at the crossing labeled j.*

- Let $\sigma_i K$ and $\eta_i K$ be the same as *K* except that the *i*-th crossing is switched in $\sigma_i K$ and nullified in $\eta_i K$.


K
j
i
$\eta_i K$
$\eta_i\sigma_j K$
$\eta_i\eta_j K$
$\sigma_i K$
$\sigma_i\eta_j K$
$\sigma_i\sigma_j K$

- *Base points and component order are chosen arbitrarily (induction) in* $\eta_i K$.

- Let ε_i be the sign of the *i*-th crossing in *K*.

- σ_i before σ_j :

$$K(l,m) = -l^{-2\varepsilon_i}\underline{(\sigma_i K)} - ml^{-\varepsilon_i}\left(\eta_i K\right) =$$

$$= -l^{-2\varepsilon_i}\left(-l^{-2\varepsilon_j}(\sigma_j \sigma_i K) - ml^{-\varepsilon_j}\left(\eta_j \sigma_i K\right)\right) - ml^{-\varepsilon_i}\left(\eta_i K\right) \Rightarrow$$

$$\boxed{K(l,m) = l^{-2\varepsilon_i - 2\varepsilon_j}(\sigma_j \sigma_i K) + ml^{-2\varepsilon_i - \varepsilon_j}\left(\eta_j \sigma_i K\right) - ml^{-\varepsilon_i}\left(\eta_i K\right)} \quad (1)$$

- σ_j before σ_i :

$$\boxed{K'(l,m) = l^{-2\varepsilon_i - 2\varepsilon_j}(\sigma_i \sigma_j K) + m l^{-2\varepsilon_j - \varepsilon_i}\left(\eta_i \sigma_j K\right) - m l^{-\varepsilon_j}\left(\eta_j K\right)} \quad (2)$$

- Since $\sigma_i \sigma_j = \sigma_j \sigma_i$, we have that :

$$l^{-2\varepsilon_i - 2\varepsilon_j}(\sigma_i \sigma_j K) = l^{-2\varepsilon_i - 2\varepsilon_j}(\sigma_j \sigma_i K)$$

- Inductive hypothesis for $\eta_i K, \eta_j K$:

$$\boxed{\left(\eta_i K\right) = -l^{-2\varepsilon_j}(\sigma_j \eta_i K) - m l^{-\varepsilon_j}\left(\eta_i K\right) - m l^{-\varepsilon_j}\left(\eta_j \eta_i K\right)} \quad (3)$$

$$\boxed{\left(\eta_j K\right) = -l^{-2\varepsilon_i}(\sigma_i \eta_j K) - m l^{-\varepsilon_i}\left(\eta_i K\right) - m l^{-\varepsilon_j}\left(\eta_i \eta_j K\right)} \quad (4)$$

$$(1)\overset{(3)}{\Rightarrow} K(l,m) = l^{-2\varepsilon_i-2\varepsilon_j}\left(\sigma_j\sigma_i K\right) + ml^{-2\varepsilon_i-\varepsilon_j}\left(\eta_j\sigma_i K\right) + ml^{-2\varepsilon_j-\varepsilon_i}\left(\sigma_j\eta_i K\right) + m^2 l^{-\varepsilon_i-\varepsilon_j}\left(\eta_j\eta_i K\right)$$

$$(2)\overset{(4)}{\Rightarrow} K'(l,m) = l^{-2\varepsilon_i-2\varepsilon_j}\left(\sigma_i\sigma_j K\right) + ml^{-2\varepsilon_j-\varepsilon_i}\left(\eta_i\sigma_j K\right) - ml^{-\varepsilon_j}\left(-l^{-2\varepsilon_i}\left(\sigma_i\eta_j K\right) - ml^{-\varepsilon_i}\left(\eta_i\eta_j K\right)\right)$$

- Since: $\sigma_i\sigma_j = \sigma_j\sigma_i \;,\; \eta_i\eta_j = \eta_j\eta_i$

  $\sigma_i\eta_j = \eta_j\sigma_i \;,\; \sigma_j\eta_i = \sigma_i\eta_j$

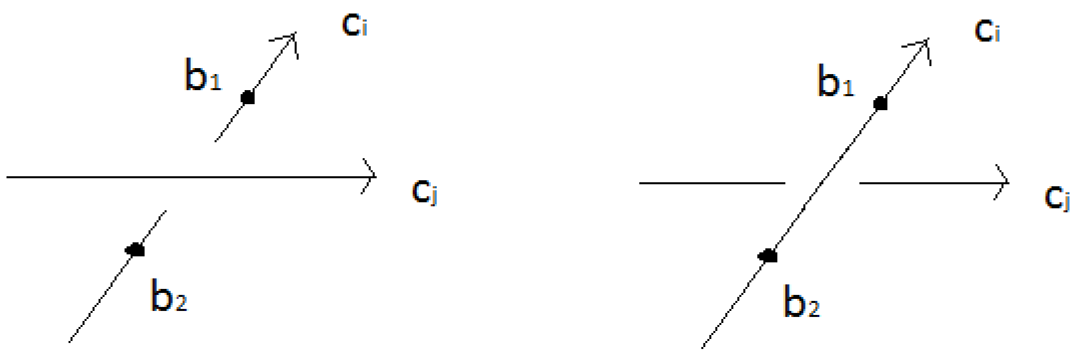
We have that : $K(l,m) = K'(l,m)$

□

Proposition 2

- $\mathscr{P}(K)$ is independent of the choice of base points.

Proof



- <u>Case (a):</u> $i \neq j$ (different components):

$$\alpha(K_1) = \alpha(K_2) \Rightarrow P(K_1) = P(K_2)$$

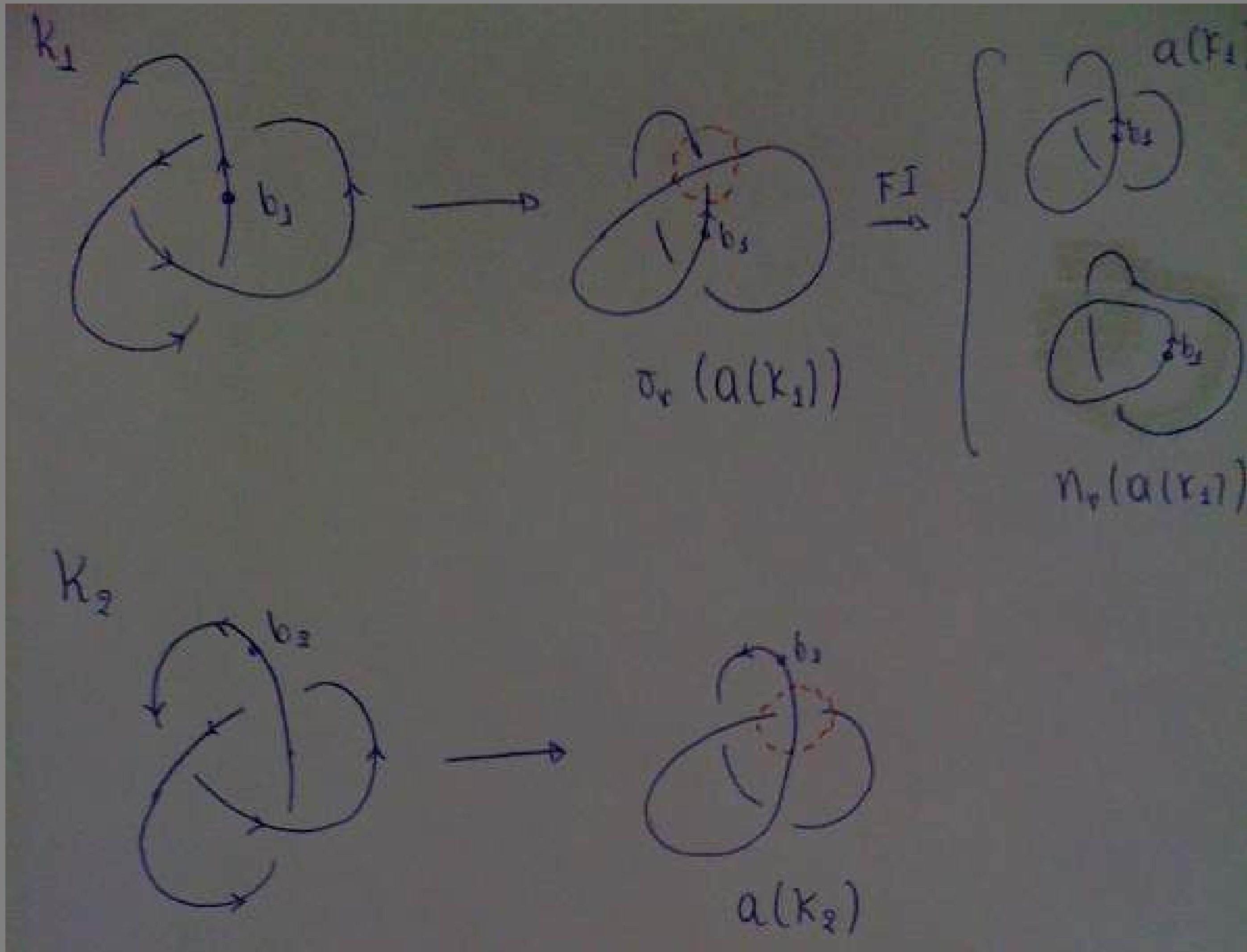
- <u>Case (b):</u> $i=j$ (same component):

~ $\alpha(K_1)$, $\alpha(K_2)$ differ only at the crossing under consideration, labeled r.

~ By Prop.1, $P(K_1)$ can be calculated by changing all the other relevant crossings first, giving $P(K_1)=f(P\sigma_r \alpha(K_1))$.

(f : linear function).

~ Similarly: $P(K_2)=f(P(\alpha(K_2)))$ (same function f)



K1
b1
σv (a(K1))
FI
a(K1
n_v(a(K1))
K2
b2
b2
a(K2)

By definition:

$$P\left(\alpha\left(K_1\right)\right)=\mu^{c-1},\ P\left(\alpha\left(K_2\right)\right)=\mu^{c-1},\ P\left(\eta_r\left(\alpha\left(K_1\right)\right)\right)=\mu^{c}$$

Further:

$$l^{\varepsilon}P\left(\sigma_r\alpha\left(K_1\right)\right)+l^{-\varepsilon}P\left(\alpha\left(K_1\right)\right)+mP\left(\eta_r\alpha\left(K_1\right)\right)=0\overset{\dots}{\Rightarrow}$$

$$\overset{\dots}{\Rightarrow}P\left(\sigma_r\alpha\left(K_1\right)\right)=\mu^{c-1}=P\left(\alpha\left(K_2\right)\right)\Rightarrow$$

$$\Rightarrow\boxed{P\left(K_1\right)=P\left(K_2\right)}$$

$\square$

Proposition 3

- $P / \mathcal{Z}_n$ satisfies formula I. - i.e. it does not lead to a contradiction –

Proof

Proposition 4

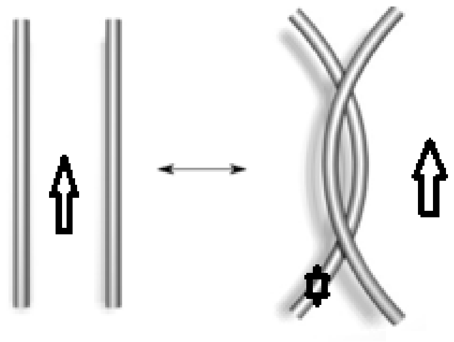
- $\mathscr{P}(K)$ is invariant under Reidemeister moves which do not increase the number of crossings beyond *n*.

Proof

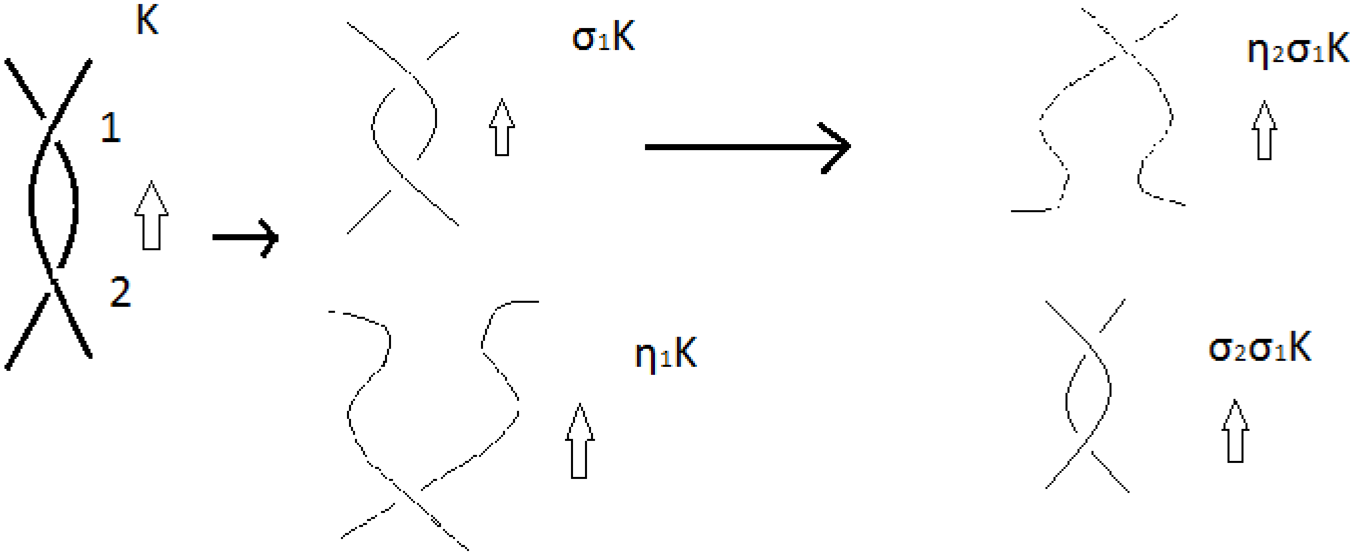
- *Type I:* Place the base point immediately before the crossing to be removed by one of the *R-I* moves and a base point on the corresponding arc with that crossing removed.

$$\alpha\left(K_1\right)=\alpha\left(K_2\right)\Rightarrow P\left(\alpha\left(K_1\right)\right)=P\left(\alpha\left(K_2\right)\right)$$

- <u>*Type II:*</u> If $i<j$ or $i=j$, we may place the base point so that no crossing switch occurs.

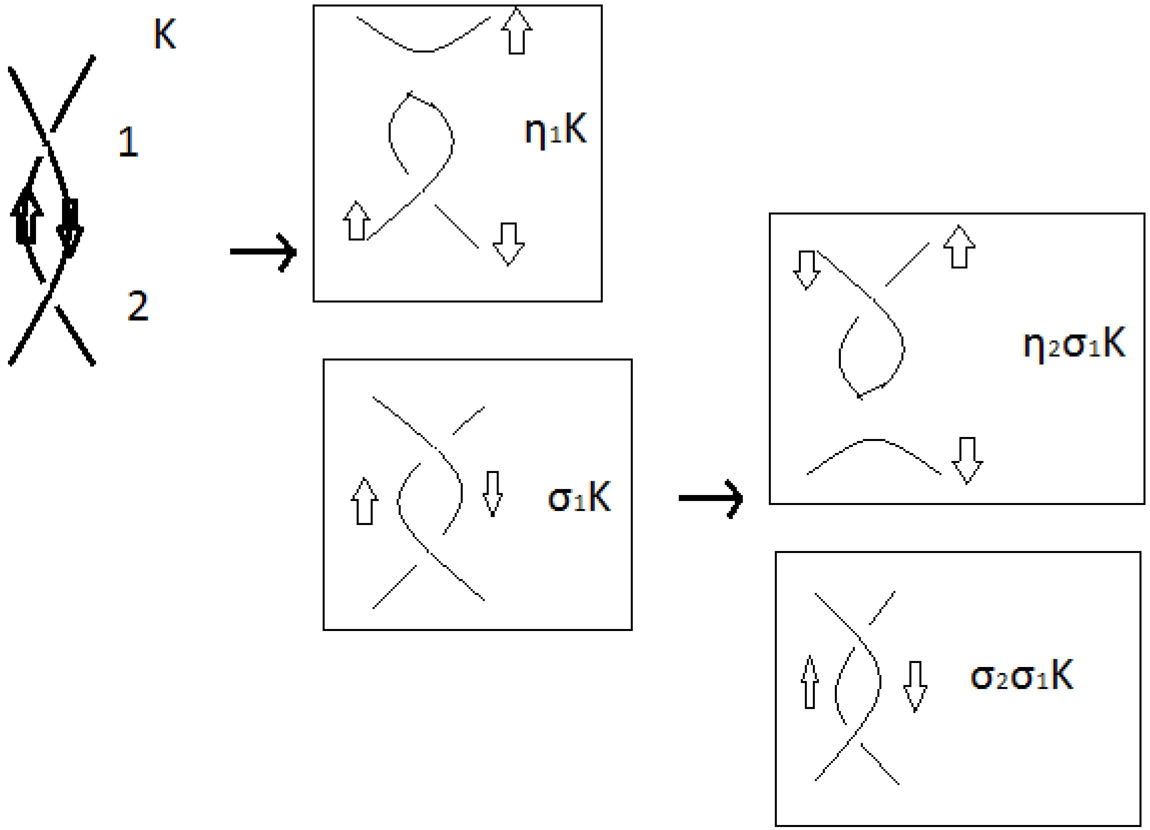


- If $j<i$ and $\varepsilon=1$:



$$
\begin{aligned}
K &= -l^{-2}\left(\sigma_1 K\right) - ml^{-1}\left(\eta_1 K\right) = \\
&= -l^{-2}\left(-l^{2}\left(\sigma_2\sigma_1 K\right) - ml\left(\eta_2\sigma_1 K\right)\right) - ml^{-1}\left(\eta_1 K\right) = \\
&= \left(\sigma_2\sigma_1 K\right) + ml^{-1}\left(\left(\eta_2\sigma_1 K\right) - \left(\eta_1 K\right)\right) = \left(\sigma_2\sigma_1 K\right)
\end{aligned}
$$

- If $j<i$ and $\varepsilon=-1$:

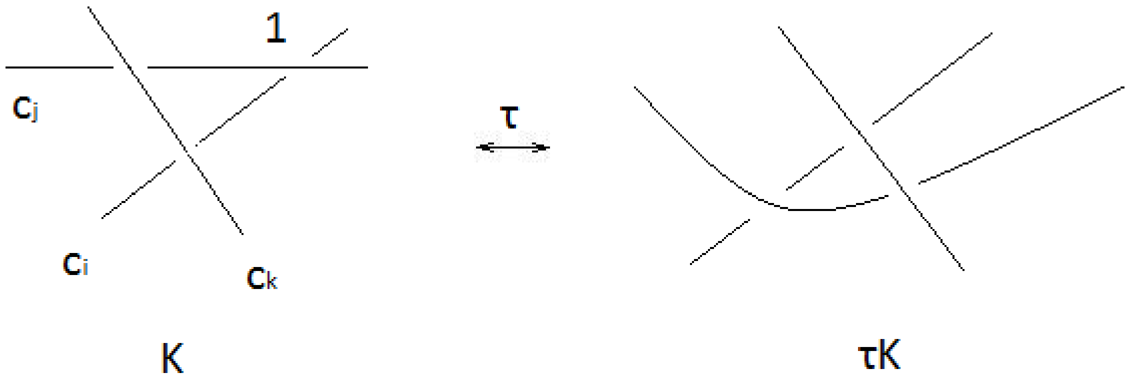


$$K = -l^2\left(\sigma_1 K\right) - ml\left(\eta_1 K\right) =$$

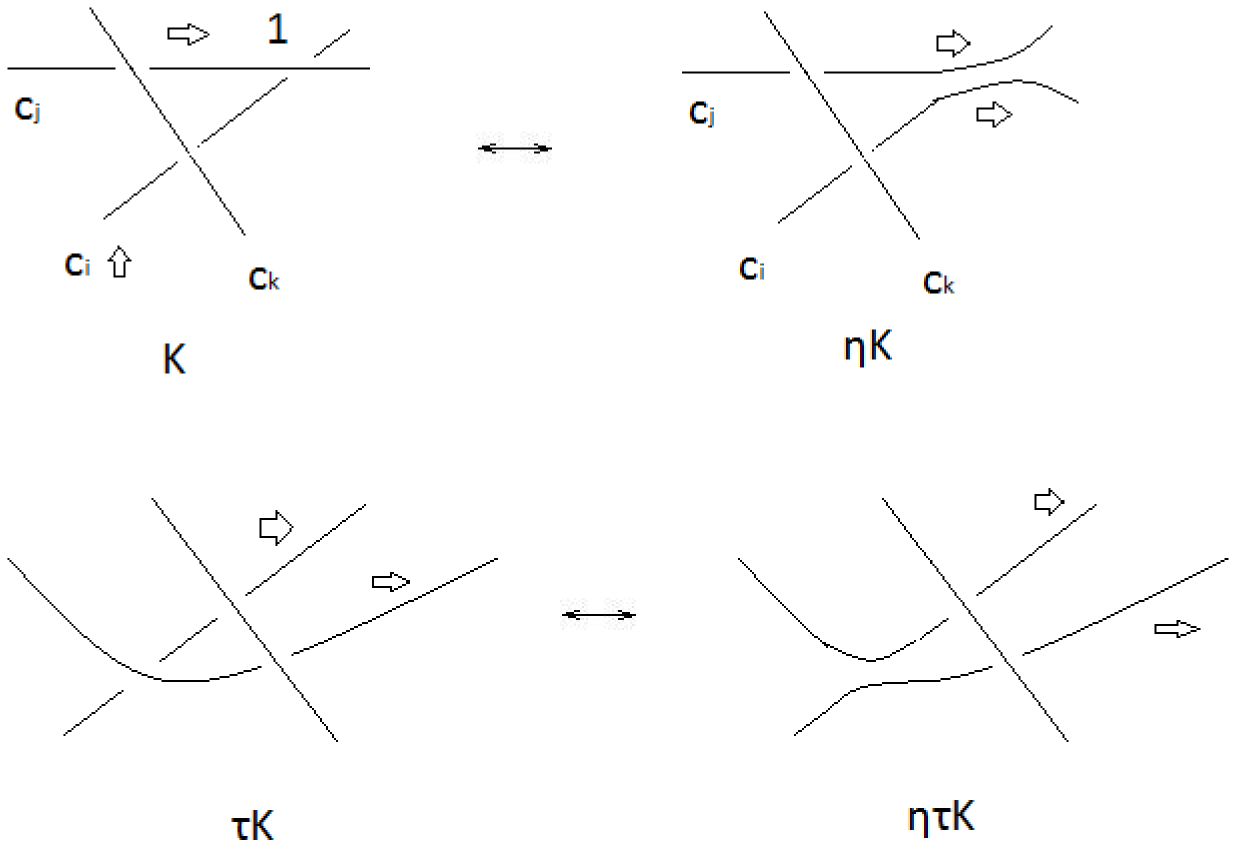
$$= -l^2\left(-l^{-2}\left(\sigma_2\sigma_1 K\right) - ml^{-1}\left(\eta_2\sigma_1 K\right)\right) - ml\left(\eta_1 K\right) =$$

$$= \left(\sigma_2\sigma_1 K\right) + ml\left(\left(\eta_2\sigma_1 K\right) - \left(\eta_1 K\right)\right) = \left(\sigma_2\sigma_1 K\right)$$

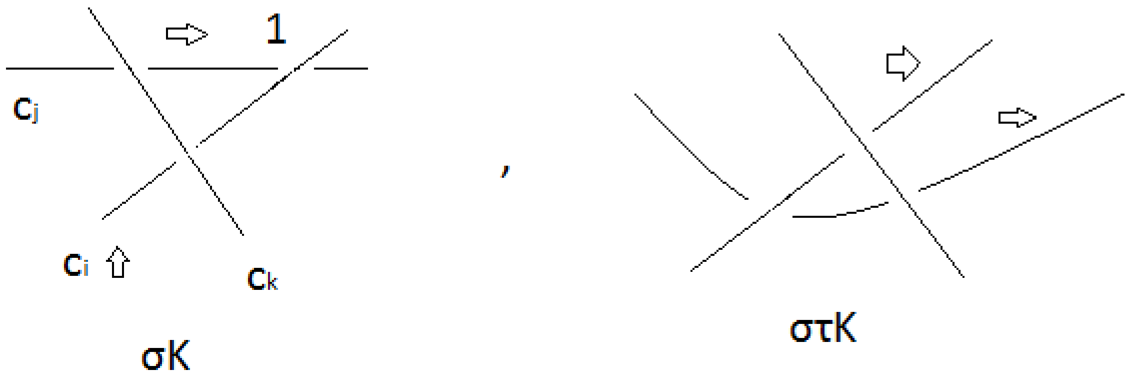
- ***<u>Type III:</u>***

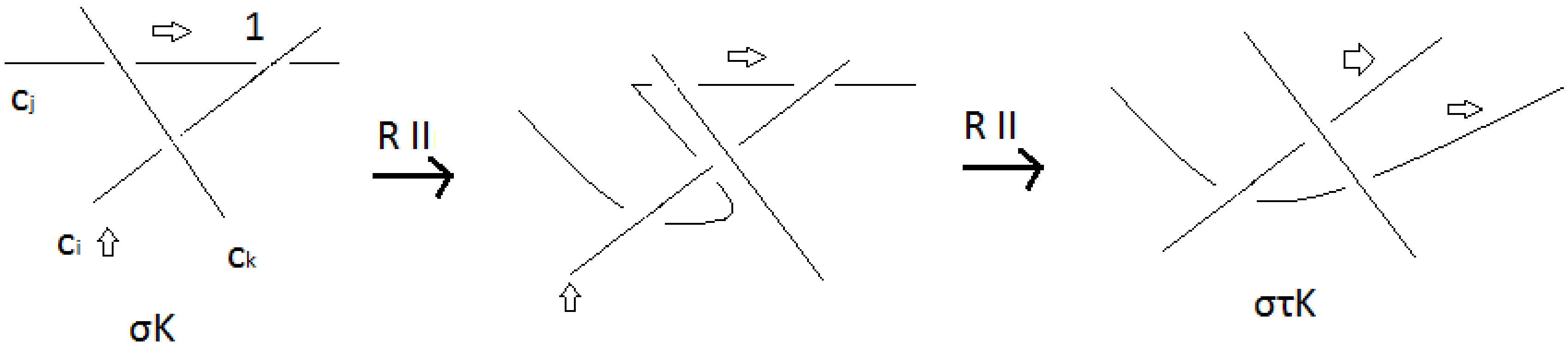


$$K = -l^{-2\varepsilon}\left(\sigma K\right) - ml^{-\varepsilon}\left(\eta K\right)$$
$$\tau K = -l^{-2\varepsilon}\left(\sigma\tau K\right) - ml^{-\varepsilon}\left(\eta\tau K\right)$$



$$(\eta K) = (\eta \tau K)$$

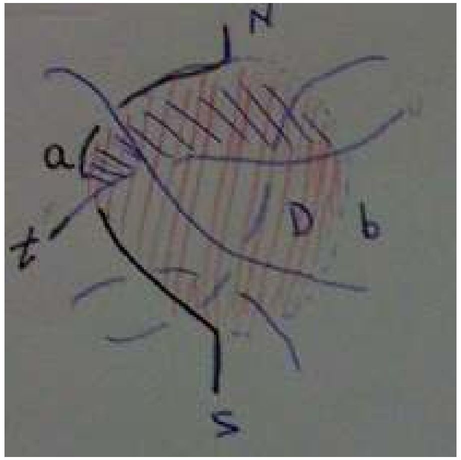




$$\left(\sigma K\right) = \left(\sigma\tau K\right)$$

□

- Suppose that K is a non-standard ascending element of $\mathcal{L}_n$. Let D be a disk in the projection plane s.t. $D \cap K$ is the union of an arc α in ∂D and a finite number of arcs (to be called *transversals*) properly embedded in D.
- Suppose that no base point is in D, that each transversal crosses α in one point and that no pair of transversals cross in more than one point.
- Let b be the closure of $\partial D - \alpha$ and let $\widehat{K}$ be the result of substituting b for α in K.



Proposition 5

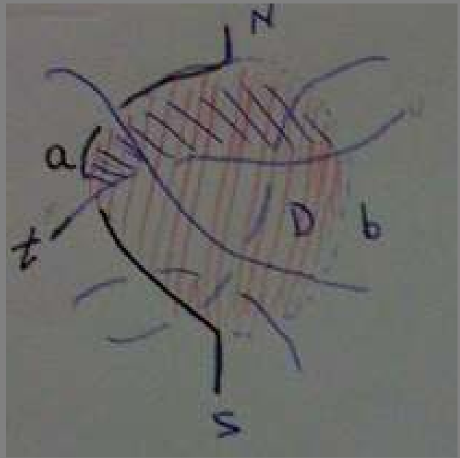
- $\mathscr{P}(K) = \mathscr{P}\left(\widehat{K}\right)$.

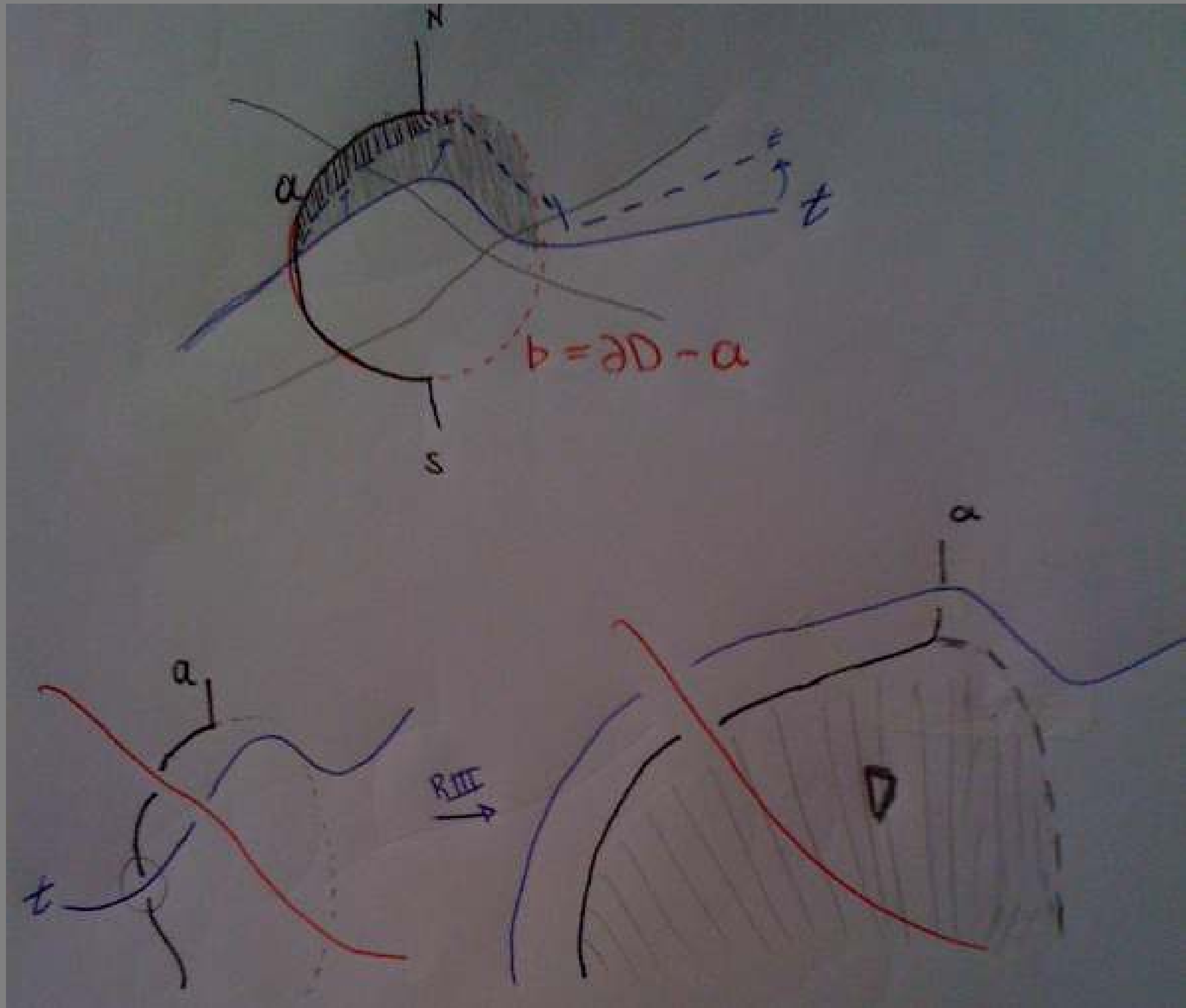
Proof

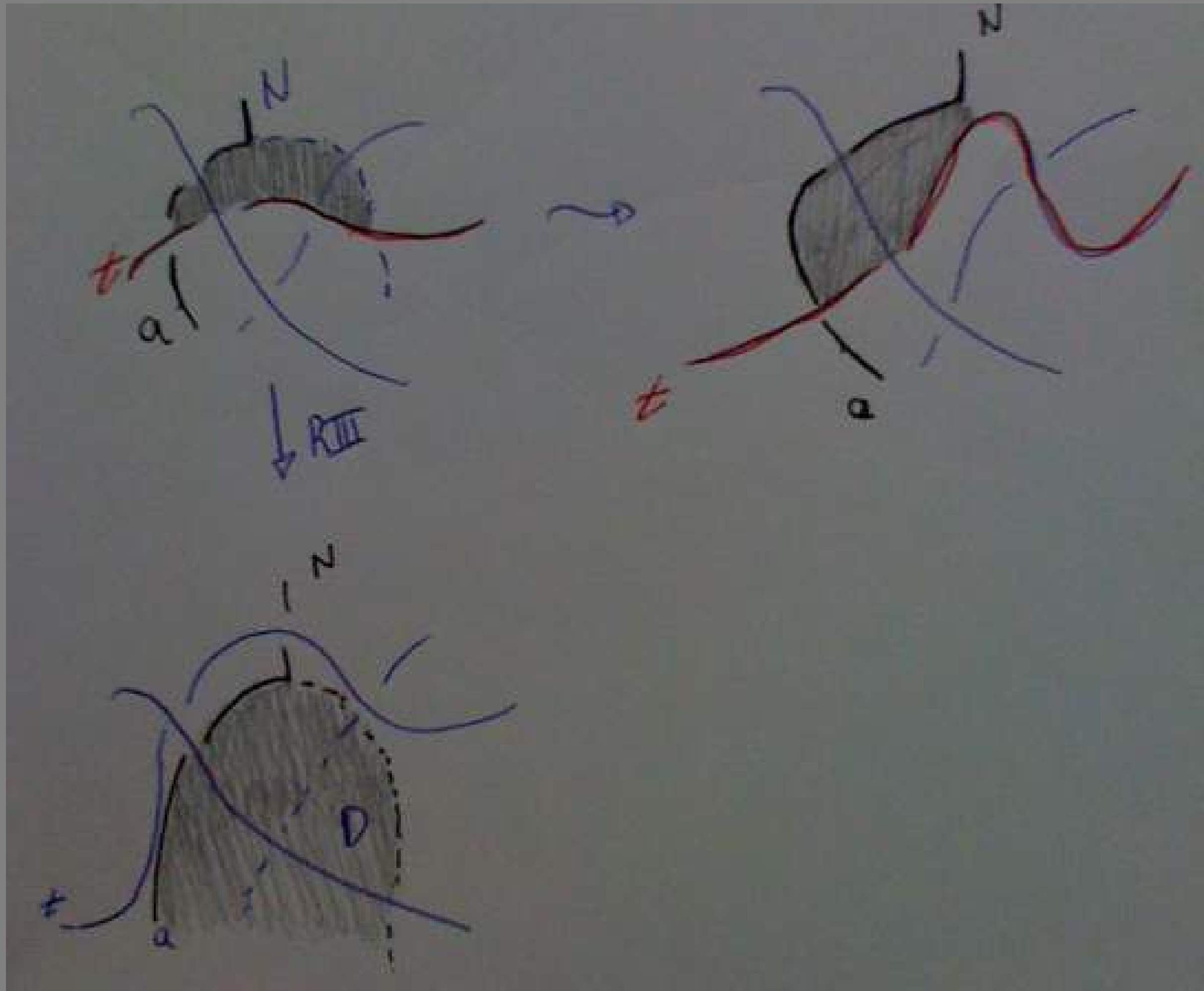
Induction on the number, *u,* of transversals.

~ The case *u=0* is trivial.

~ Suppose that the proposition is true for *(u-1)* transversals.


N
a
t
D
b
S

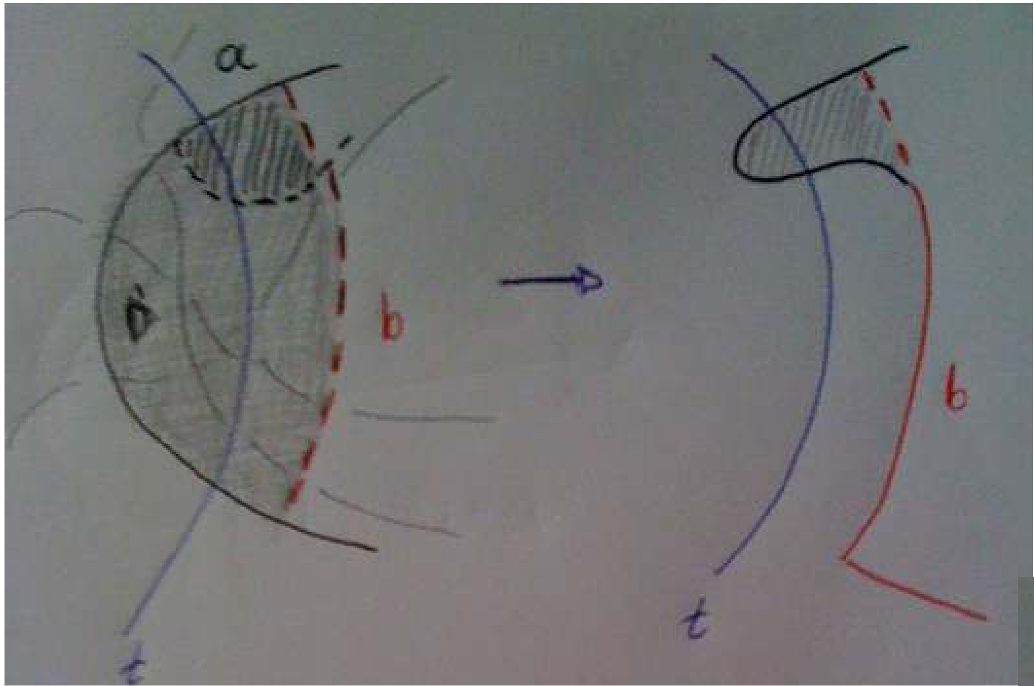

N
a
t
b = ∂D − a
S
a
t
RIII
a
D


N
t
a
RIII
N
t
a
N
D
t
a

Corollary 5.1

Suppose that *K* and *D* are as in Prop.5. Suppose, furthermore, that the transversals now have the properties that no two cross at more than one point, one transversal, denoted *t*, crosses *α* at two points, and each other transversal crosses each of *α,t* and *b* at one point. If, as before, $\widehat{K}$ is the result of replacing *α* with *b*, then $\mathscr{P}(K) = \mathscr{P}\left(\widehat{K}\right)$.

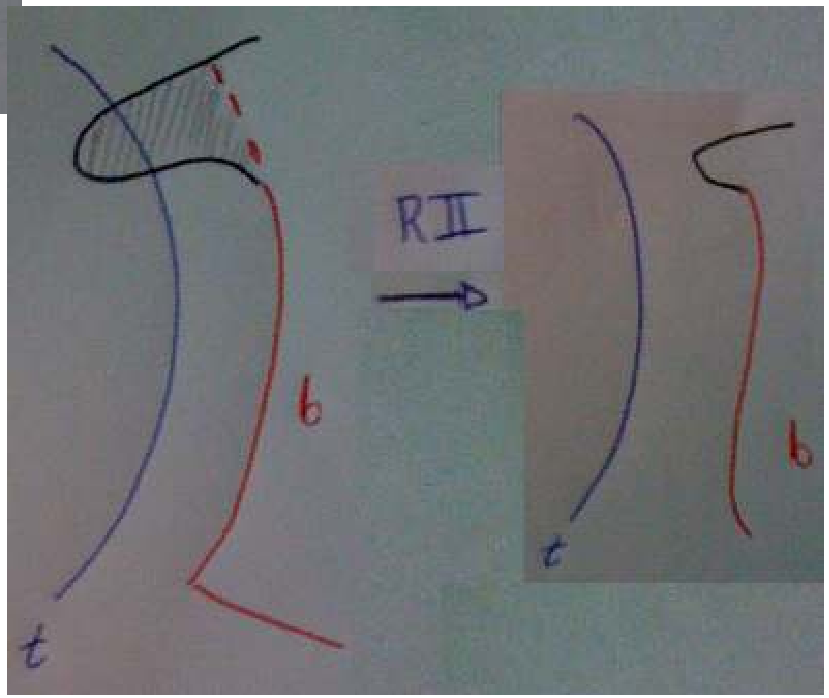
Proof



←

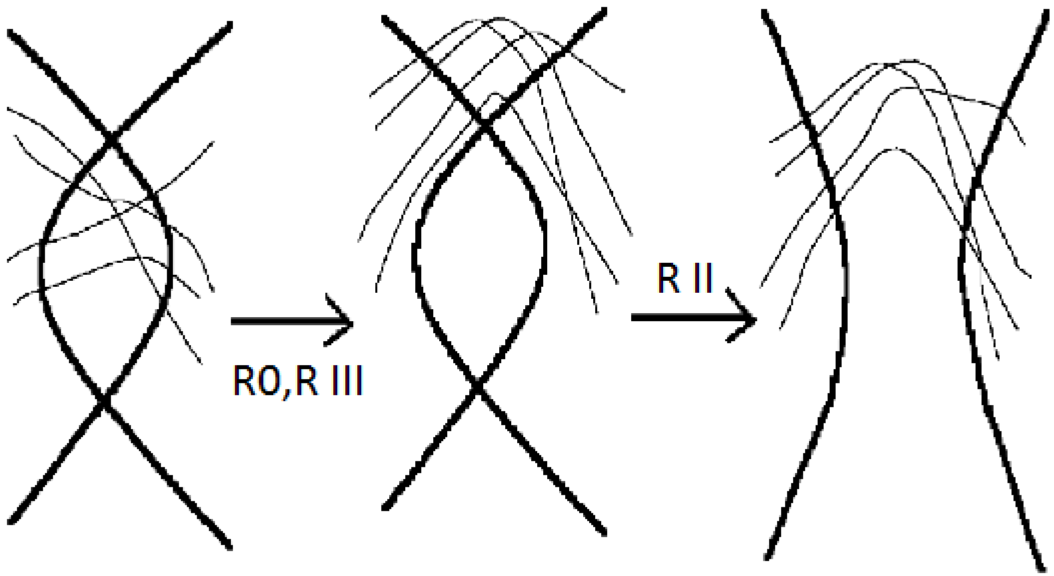
- Apply Prop.5 to the case of $D' \subset D$.

- Apply Prop.4. →



□

Remark : Przytycki-Traczyk Approach



Proposition 6

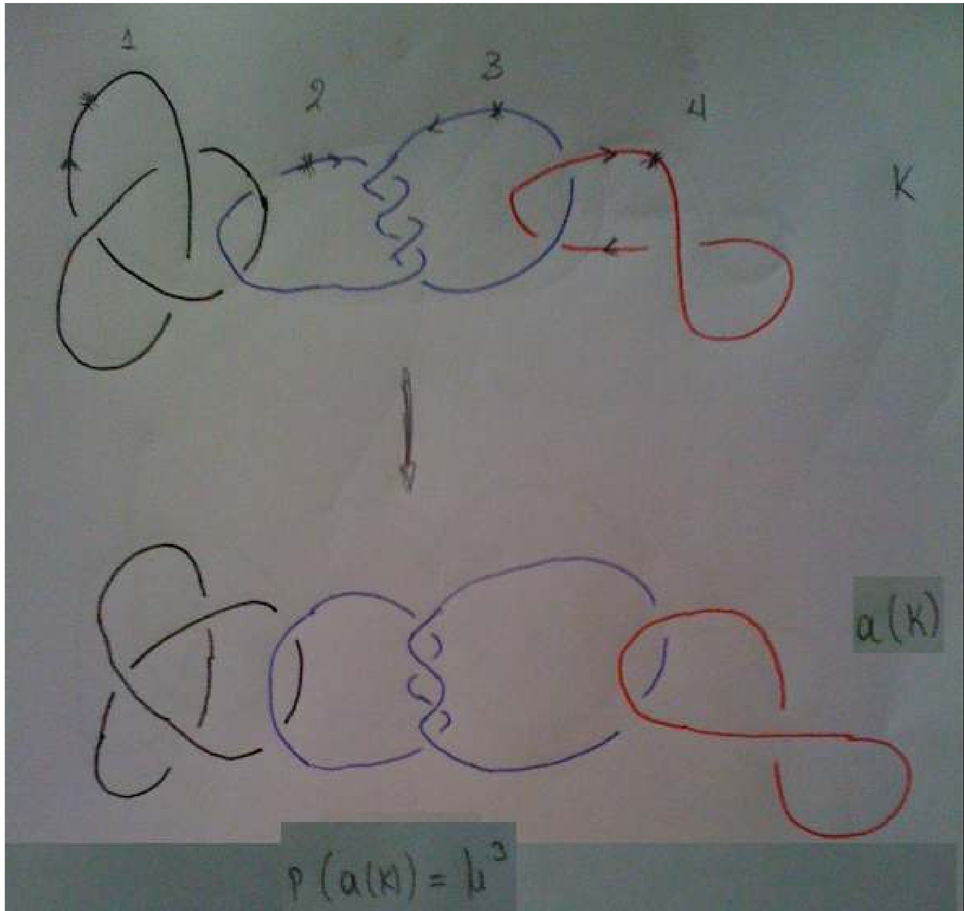
- The polynomial for *K* in $\mathcal{L}_n$ is independent of the choice of order of the components.

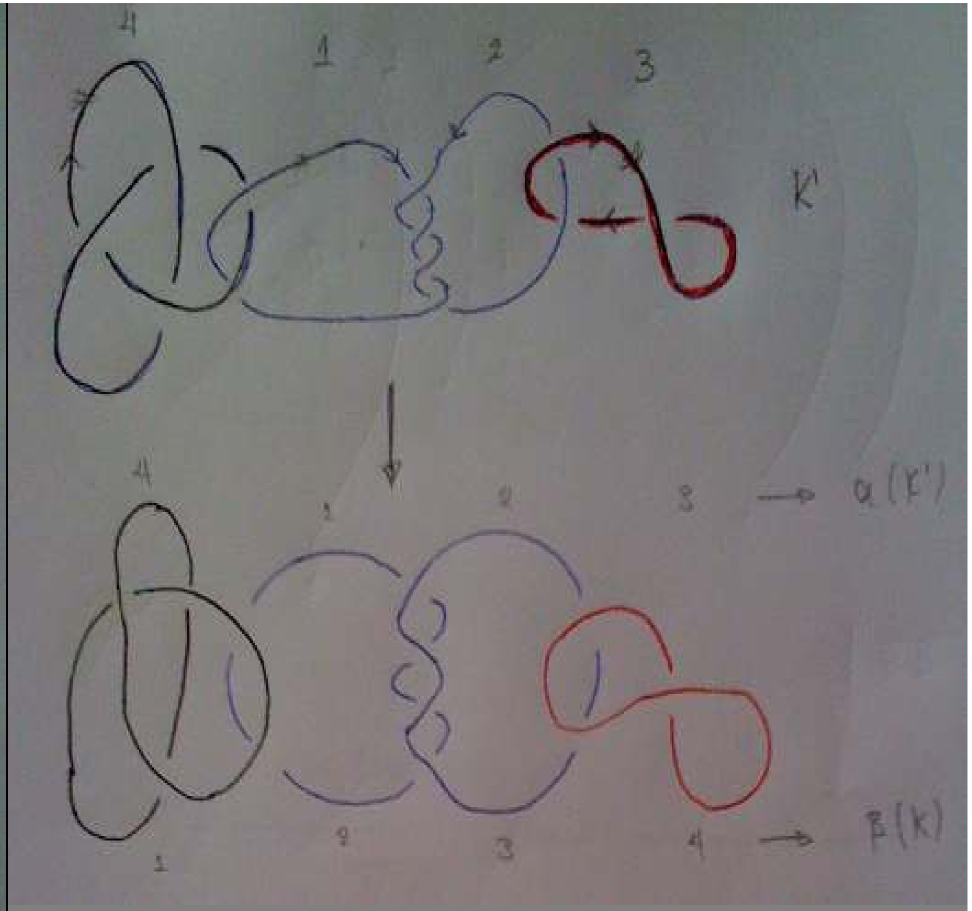
Proof

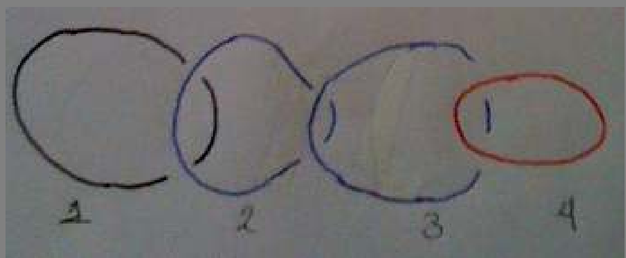
$$K \overset{alg.}{\rightarrow} \alpha(K) \Rightarrow P(\alpha(K)) = \mu^{c-1}$$

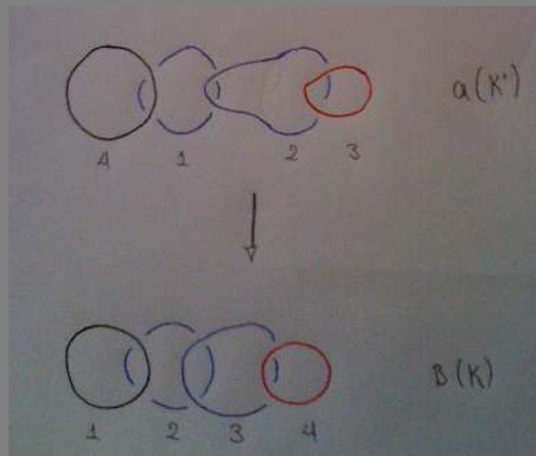
$$K' \underset{dif.\ order}{\overset{same\ link}{\rightarrow}} \alpha(K') \underset{order}{\overset{give\ original}{\rightarrow}} \beta(K) \Rightarrow P(\beta(K)) \overset{induction}{=} \mu^{c-1}$$

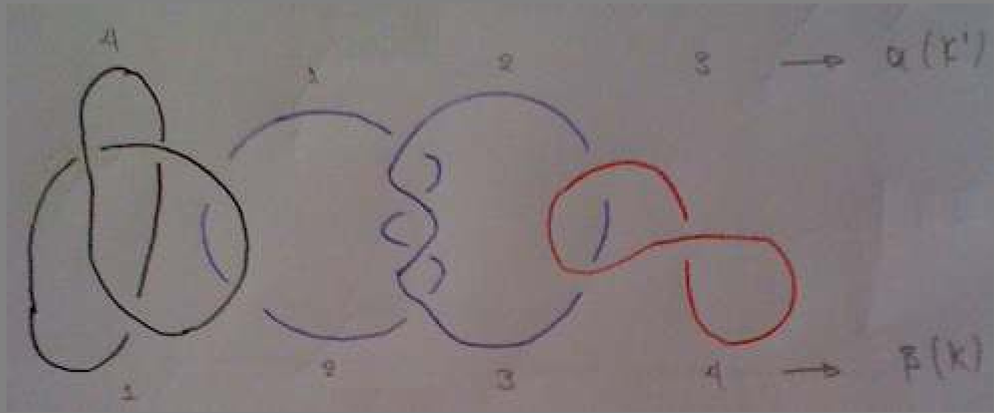
$$\alpha(K) \overset{F.I}{\rightarrow} \beta(K) \overset{F.I}{\rightarrow} K$$

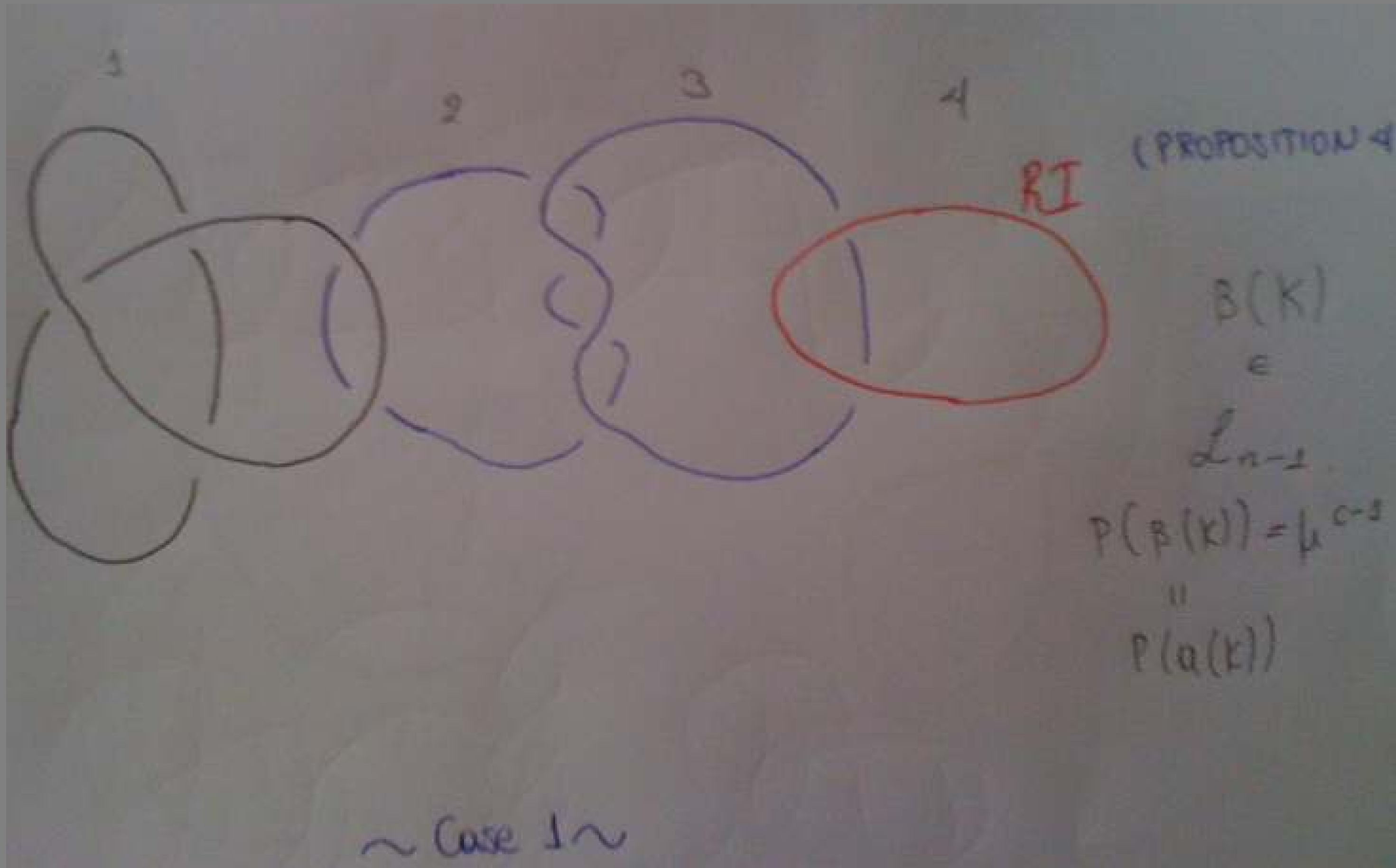

1
2
3
4
K
a(K)
p (a(K)) = μ^3

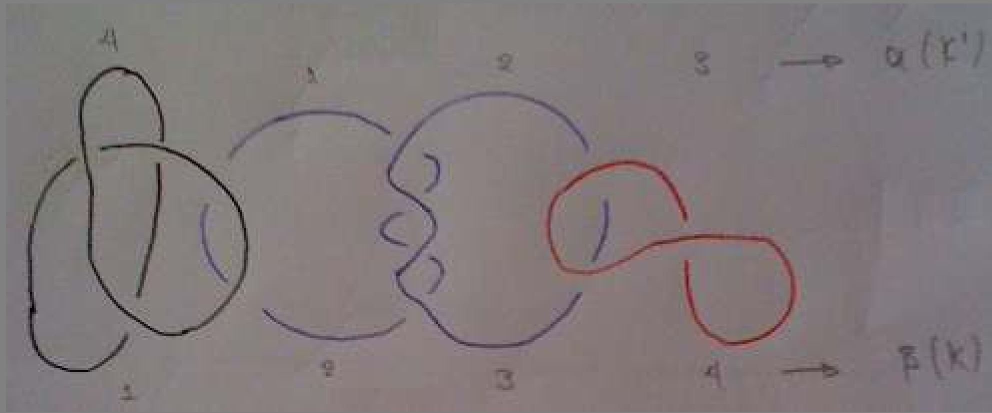

4
1
2
3
K'
4 1 2 3 → a(K')
1 2 3 4 → β(K)

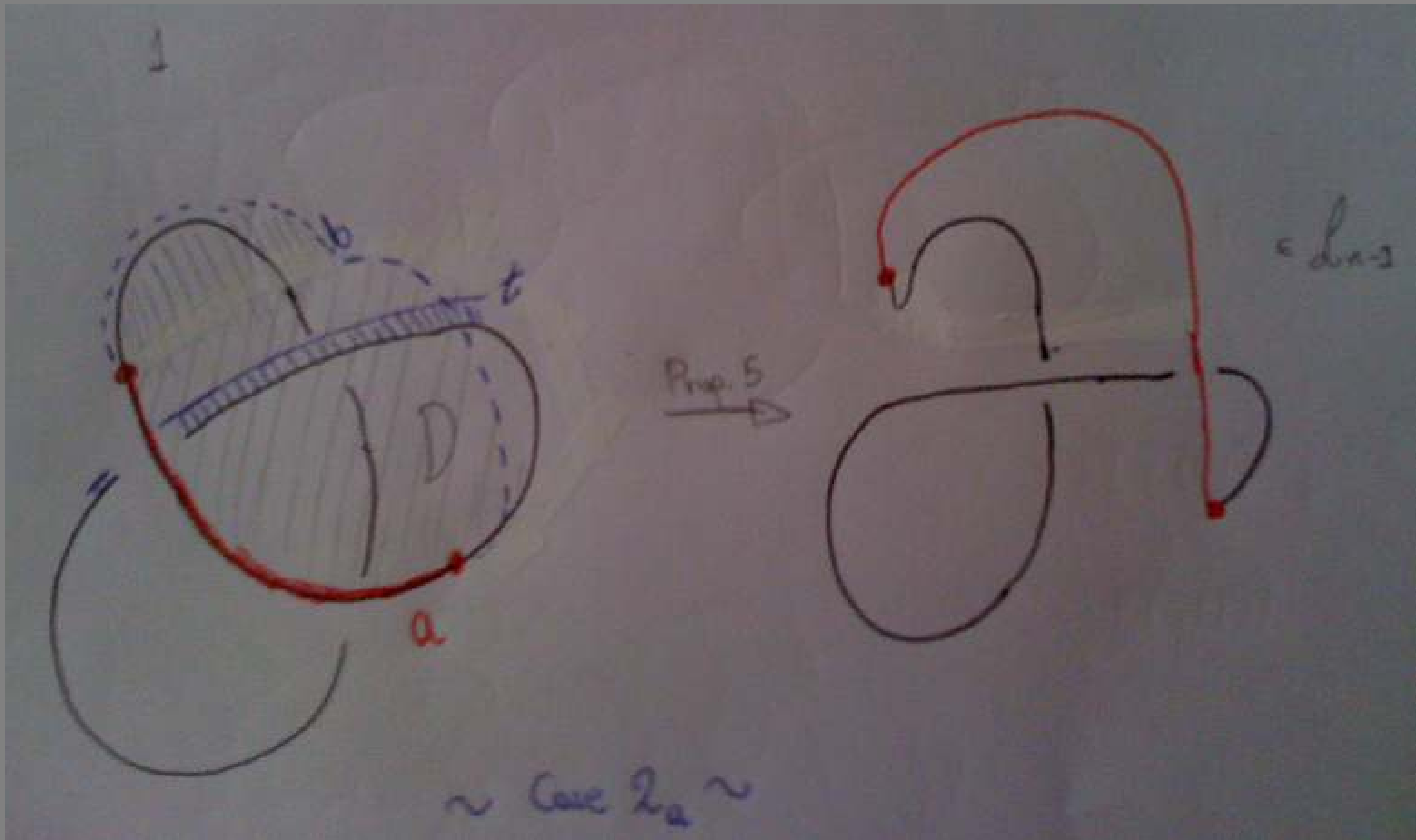

1
2
3
4

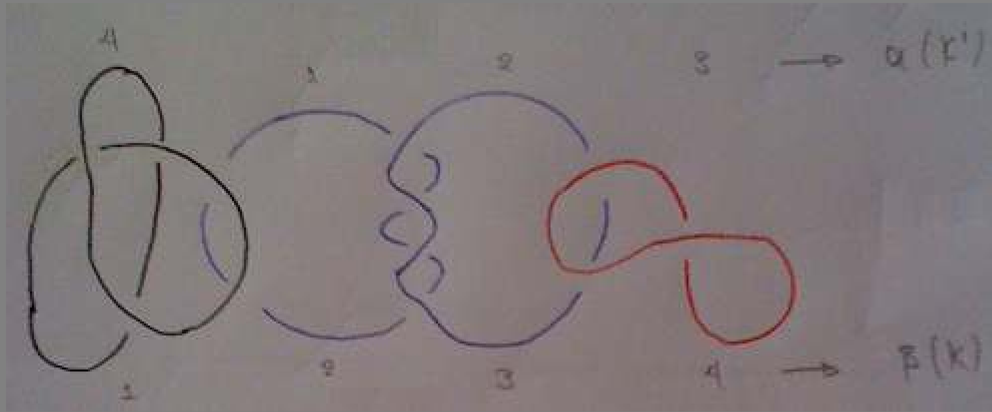

a(K')
4 1 2 3
β(K)
1 2 3 4

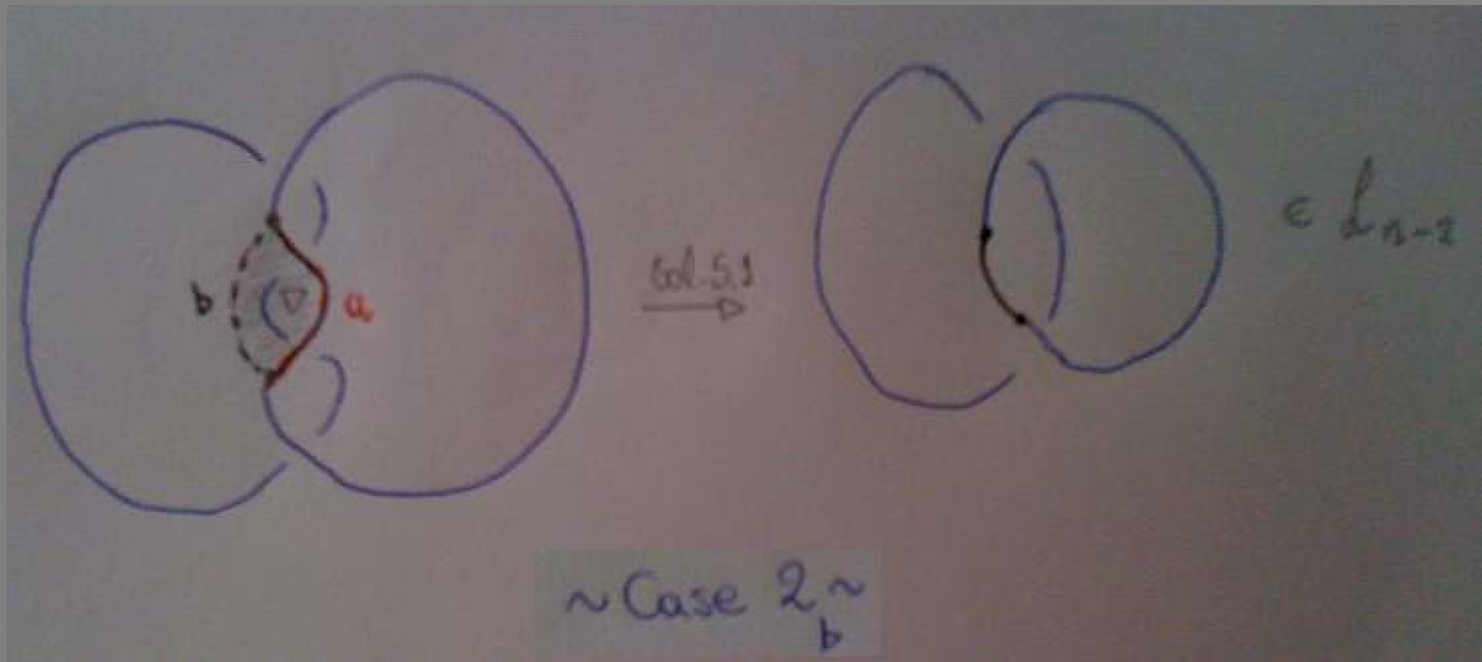

4
1
2
3 → α(K')
1
2
3
4 → β(K)


1
2
3
4
(PROPOSITION 4
RI
β(K)
∈
$\mathcal{L}_{n-1}$
P(β(K)) = μ^{c-1}
||
P(α(K))
~ Case 1 ~


4
1
2
3
→ α(K')
1
2
3
4
→ β(K)


1
b
t
D
a
Prop. 5
~ Case 2a ~


4
1
2
3
α(K')
1
2
3
4
β(K)


b
a
Col. 5.1
∈ L
~Case 2~
b

□

The knot quandle and the Conway algebra

Przytycki – Traczyk approach

Introduction

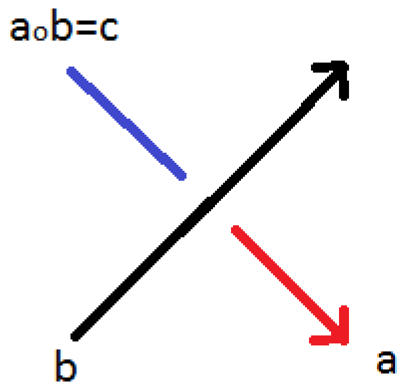
- Let Γ be an arbitrary finite set; all elements of Γ are to be called *colors*. Suppose the set Γ is equipped with a binary operation $\alpha : \Gamma \times \Gamma \rightarrow \Gamma$.

- Denote : $a \circ b = \alpha(a, b)$

- *Proper coloring* of a diagram *D* of an oriented link *K*:

  Associate some color with each arc of *D* s.t. for each overcrossing arc (that has color *b*), undercrossing arc lying on the left hand (color *a*) and undercrossing lying on the right (color *c*), the following relation holds:

$$a \circ b = c$$



Conditions for $\circ$, s.t. the number of proper colorings is invariant under *R*-moves:

- 1. *RI* : $a \circ a = a, \ \forall a \in \Gamma$
- 2. *RII* : $x \circ a = b \Rightarrow x \in \Gamma, \ \forall a, b \in \Gamma$
- 3. *RIII* : $(a \circ b) \circ c = (a \circ c) \circ (b \circ c), \ \forall a, b, c \in \Gamma$

Quandle

- Each set with an operation o satisfying the 1-2-3 properties is called a *quandle*.

- Proposition: The number of proper colorings by elements of any quandle is a link invariant.

Remark:

- In any quandle, the reverse operation for $\circ$ is denoted by $/$ and

$$(x \circ a) = b \Leftrightarrow x = b / a$$

- Each quandle with operation $\circ$ is a quandle with respect to the operation $/$.

- Let A be an alphabet consisting of letters. A *word* in the alphabet A is an arbitrary finite sequence of elements of A and symbols o and $/$.

- The set of *admissible* words, $D(A)$:

1. For all α in A, the word consisting of only the letter α is admissible.
2. If W_1 , W_2 are admissible, then $(W_1)\text{o}(W_2)$ and $(W_1) / (W_2)$ are admissible.
3. There are no other admissible words except for those obtained by rules 1 and 2.

- Let *R* be a set of relations, i.e. identities of type $r_a=s_a$, where r_a,s_a in *D(A)* and *a* runs over some set *X* of indices.

- Equivalence relation for *D(A)*:

$W_1 \equiv W_2$ iff there exists a finite chain of transformations starting from W_1 and finishing at W_2 according to the rules:

$$1.\ x \circ x \Leftrightarrow x$$

$$2.\ (x \circ y)/y \Leftrightarrow x$$

$$3.\ (x/y) \circ y \Leftrightarrow x$$

$$4.\ (x \circ y) \circ z \Leftrightarrow (x \circ z) \circ (y \circ z)$$

$$5.\ r_i \Leftrightarrow s_i$$

- <u>Remark</u>:

The set of equivalent classes $\Gamma<A \mid R>$ is a quandle with respect to $\circ$.

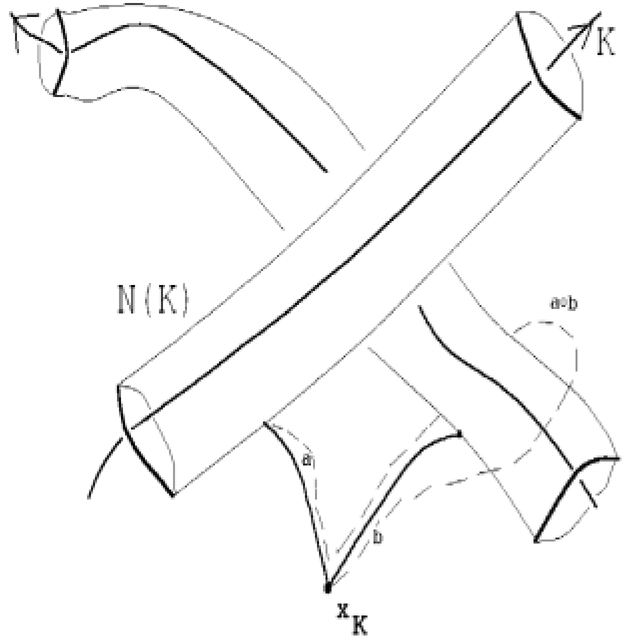
Geometric Description

- Let *K* be an oriented knot in $\mathbb{R}^3$ and let *N(K)* be its tubular neighborhood.
- Let $E(K)=\overline{\left(\mathbb{R}^3 \setminus N(K)\right)}$ be the complement to this neighborhood.
- Fix a base point x_K on *E(K).*

- Denote by *Γ$_K$* the set of *homotopy classes* of paths in the space *E(K)* with fixed initial point at x_K and end point on $\partial N(K)$.

- Let m_b be the oriented meridian hooking an arc *b*.

- Define $a \circ b = \left[b m_b b^{-1} a \right]$.



- Let as fix a point *x* outside the tubular neighborhood. With each element *γ* on the quandle Γ_K (path from *x* to $\partial E(K)$) we associate the $\gamma m \gamma^{-1}$, where *m* is the meridian at the point *x*.

Remark:

The fundamental group can be constructed by the quandle:

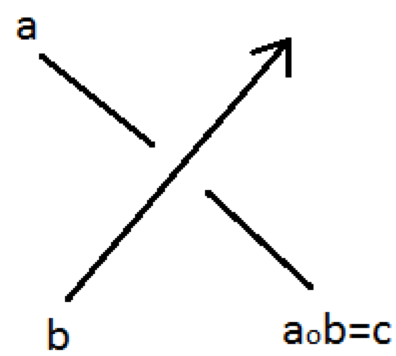
All meridians play the role of generators for the fundamental group.

All relations $a \circ b = c$ have to be replaced with $bab^{-1} = c$.

Algebraic Description

- Let *D* be a diagram of an oriented knot *K*. Denote the set of arcs of *D* by A_D.

- Let *P* be a crossing incident to two undercrossing arcs *a* and *c* and an overcrossing arc *b*. Then $a \circ b = c$, where *a* is lying on the left hand with respect to *b* and *c* is lying on the right hand with respect to *b*.



- Denote the set of all relations for all crossings by R_D and consider the quandle $\Gamma<A_D \mid R_D>$, defined by generators A_D and relations R_D.

Theorem Quandles Γ_K and $\Gamma<A_D \mid R_D>$ are isomorphic.

Remark:

The quandle corresponding to a knot is a complete knot invariant.

However, it is difficult to recognize quandles by their presentation.

It is possible though to *simplify* this invariant making it "*weaker*" but more recognizable..

The *Conway algebra* and polynomial invariants

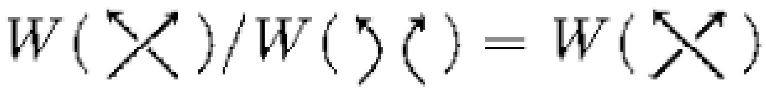
- Let *A* be the algebra with two binary operations $\circ$ and */*, such that the following properties hold:

$$\left(a \circ b\right) / b = a$$

$$\left(a / b\right) \circ b = a$$

- Denote by α_n the element of *A* corresponding to the *n*-component trivial link.

- Let us also require the following algebraic equation for any *Conway triple*:



- *W* is a map from the set of all links to the algebraic object to be constructed.

Definition An algebra *A* with tow operations ∘ and / (reverse to each other) and a fixed sequence α_n of elements is called a *Conway algebra* if the following conditions hold:

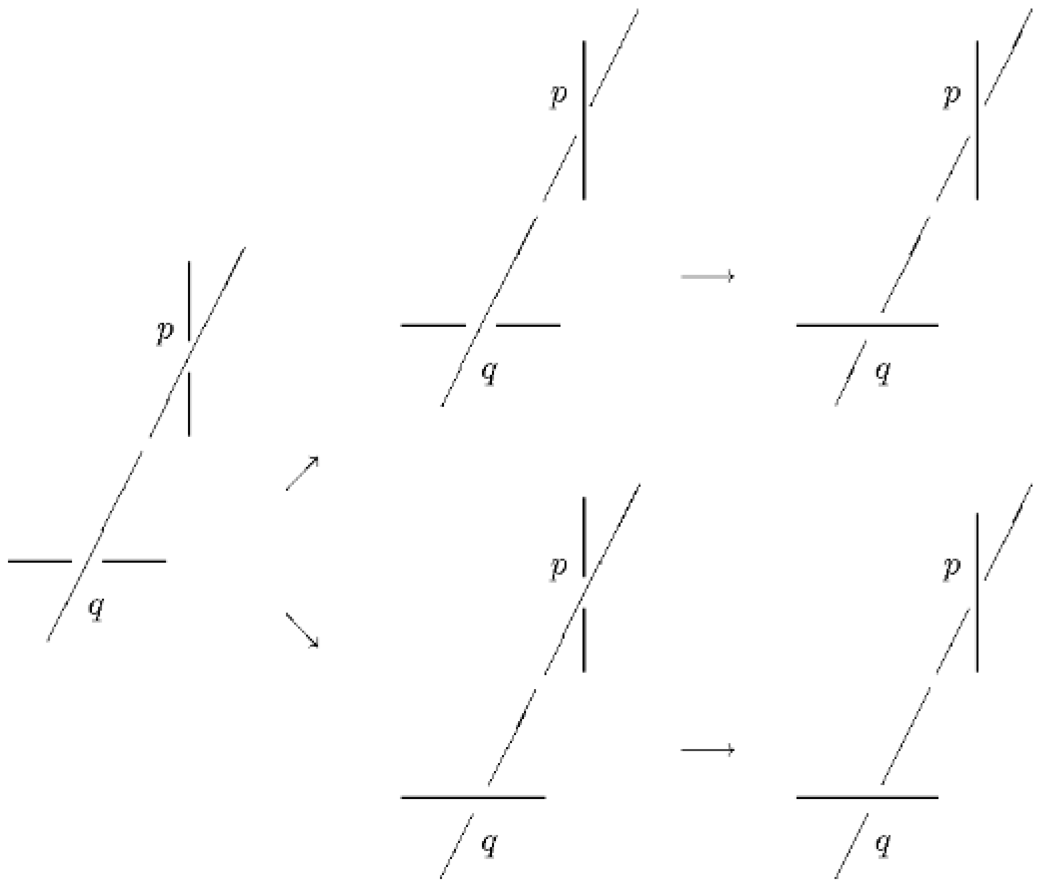
$$a_n = a_n \circ a_{n+1} \qquad (2)$$

$$a_n = a_n / a_{n+1} \qquad (3)$$

$$(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d) \qquad (4)$$

$$(a / b) / (c / d) = (a / c) / (b / d) \qquad (5)$$

$$(a / b) \circ (c / d) = (a \circ c) / (b \circ d) \qquad (6)$$



Two ways of resolving two crossings

- If *p* and *q* are positive crossings:

- First *p* and then *q*:

$$W\left(L_{++}\right)=W\left(L_{-+}\right)\circ W\left(L_{0+}\right)=\left(W\left(L_{--}\right)\circ W\left(L_{-0}\right)\right)\circ\left(W\left(L_{0-}\right)\circ W\left(L_{00}\right)\right)$$

- First *q* and then *p*:

$$W\left(L_{++}\right)=W\left(L_{+-}\right)\circ W\left(L_{+0}\right)=\left(W\left(L_{--}\right)\circ W\left(L_{0-}\right)\right)\circ\left(W\left(L_{-0}\right)\circ W\left(L_{00}\right)\right)$$

- So: $(a\circ b)\circ(c\circ d)=(a\circ c)\circ(b\circ d)\quad(4)$

$$a=W\left(L_{--}\right),\ \mathrm{b}=W\left(L_{0-}\right),\ \mathrm{c}=W\left(L_{-0}\right),\ \mathrm{d}=W\left(L_{00}\right)$$

Theorem For each Conway algebra, there exists a unique function $W(L)$ on link diagrams that has value α_n on the n-component unlink diagrams and satisfies (1).

Proof

1. Construction of *W*. (algorithm for descending diagrams)
2. *W* satisfies (1).
3. Independent of the choice of base points.
4. Invariant under *R*-moves.
5. Independent of the choice of order of components.

Family of Conway algebras

- Let *A* be an arbitrary commutative ring with the unit element, α_1, and α,β be some invertible elements of *A*.

- Let us define $\circ$ and $/$ as follows:

$$x \circ y = \alpha x + \beta y \qquad (7)$$

$$x / y = \alpha^{-1} x - \alpha^{-1} \beta y \qquad (8)$$

$$\alpha_n = [\beta^{-1}(1-\alpha)]^{n-1} \alpha_1, \quad n \geq 1 \qquad (9)$$

- Then the following proposition holds:

Proposition For any choice of invertible elements α, β and element α_1, the ring A endowed with operations $\circ$, $/$ defined above and with elements α_n (9), is a Conway algebra.

HOMFLYPT Let A be the integer coefficient Laurent polynomial ring of the variables l, m. Let $\alpha = -m/l$, $\beta = 1/l$ and $\alpha_1 = 1$.

Then $W(L)$ coincides with the *HOMFLYPT* polynomial.

CONWAY Let *A* be the ring of polynomials of variable *x* with integer coefficient. Let $\alpha=1$, $\beta=x$ and $\alpha_1=1$.

Then *W(L)* coincides with the *Conway* polynomial.

JONES Let *A* be the ring of Laurent polynomials in $\sqrt{q}$.

Let $a = q^2$, $\beta = \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)$ and $\alpha_1=1$.

Then *W(L)* coincides with the *Jones* polynomial.

□