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Braid equivalences in 3-manifolds with rational surgery description ☆

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A B S T R A C T

In this paper we provide algebraic mixed braid classification of links in any c.c.o. 3-manifold M obtained by rational surgery along a framed link in S^3. We do this by representing M by a closed framed braid in S^3 and links in M by closed mixed braids in S^3. We first prove an analogue of the Reidemeister theorem for links in M. We then give geometric formulations of the mixed braid equivalence using the L-moves and the braid band moves. Finally we formulate the algebraic braid equivalence in terms of the mixed braid groups $B_{m,n}$, using cabling and the parting and combing techniques for mixed braids. Our results set a homogeneous algebraic ground for studying links in 3-manifolds and in families of 3-manifolds using computational tools. We provide concrete formuli of the braid equivalences in lens spaces, in Seifert manifolds, in homology spheres obtained from the trefoil and in manifolds obtained from torus knots.

Our setting is appropriate for constructing Jones-type invariants for links in families of 3-manifolds via quotient algebras of the mixed braid groups $B_{m,n}$, as well as for studying skein modules of 3-manifolds, since they provide a controlled algebraic framework and much of the diagrammatic complexity has been absorbed into the proofs. Further, our moves can be used in a braid analogue of Rolfsen's rational calculus and potentially in computing Witten invariants.

0. Introduction

In the study of knots and links in 3-manifolds, such as handlebodies, knot complements, closed, connected, oriented (c.c.o.) 3-manifolds, as well as in the study of 3-manifolds themselves, it can prove very useful to

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take an approach via braids, as the use of braids provides more structure and more control on the topological equivalence moves. After the construction of the Jones polynomial for links in S^3, many mathematicians focused on expressing link isotopy in oriented 3-manifolds via appropriate braids, using different approaches, cf. for example [18,19,21,22,20,12,10,13,7].

In [12] braid equivalences have been obtained for isotopy of knots and links in knot complements and in c.c.o. 3-manifolds with integral surgery description. Integral surgery covers the generality, since every c.c.o. 3-manifold can be constructed via integral surgery along a framed link in S^3, the components of which may be assumed to be simple closed curves, giving rise to a closed framed pure braid. So, for a 3-manifold, say M, a surgery description via a closed framed braid $\widehat{B}$ in S^3 is fixed and we write $M = \chi(S^3, \widehat{B})$. Then, links in M can be represented unambiguously by *mixed links* in S^3 (see Fig. 1 and Fig. 3), that is, links in S^3 that contain $\widehat{B}$ as a fixed sublink. Mixed links are then represented by geometric *mixed braids* which contain B as a fixed subbraid. Link isotopy in M comprises isotopy in the complement $S^3 \backslash \widehat{B}$ together with the *band moves*, which come from the handle sliding moves in M according to the surgery description of M (see Fig. 2). Isotopy in M is then translated into mixed link equivalence. For obtaining the geometric mixed braid equivalences in M, the authors sharpened first the classic Markov theorem giving only one type of equivalence moves, the *L-moves* (see Fig. 3), which are geometric as well as algebraic. Then, it was proved that link isotopy in M is generated by the L-moves and the *braid band moves* (see Fig. 4). Further, in [13] the geometric statements were reformulated into algebraic language, via the cosets of the braid B in the *mixed braid groups* $B_{m,n}$ (see (1) and Fig. 5), introduced and studied in [10], and the techniques of parting and combing mixed braids (see Fig. 6). *Parting* a geometric mixed braid means to separate its strands into two sets: the strands of the fixed subbraid B and the 'moving strands' of the braid representing a link in M. *Combing* a parted mixed braid means to separate the braiding of the fixed subbraid B from the braiding of the moving strands (see Figs. 4 and 7). The above techniques have been also applied for obtaining mixed braid equivalences in knot complements and in handlebodies [13,7] (see also [20]).

Integral surgery is a special case of rational surgery. There are c.c.o. 3-manifolds which have simpler description when obtained from S^3 via rational surgery. There are even whole families of 3-manifolds described by rational surgery along the same link. Representative examples are the lens spaces $L(p, q)$: they are all obtained from the trivial knot with rational surgery description p/q, while with integral surgery description different, non-trivial links are needed, see for example [15]. Another important example comprise the homology spheres obtained by rational surgery $1/n$ along the trefoil knot: with integral surgery they would be described by more complicated knots (see [16]). Other known classes of 3-manifolds given by the same surgery description (with different surgery coefficients) comprise the Seifert manifolds ([17]) and manifolds obtained by surgery along torus knots ([14]). We note that a whole family of 3-manifolds described by different framings on the same link, in our setting is represented by the same cosets of the mixed braid groups $B_{m,n}$.

The purpose of this paper is to provide mixed braid equivalences, geometric as well as algebraic, for isotopy of oriented links in families of c.c.o. 3-manifolds obtained by rational surgery along framed links in S^3. A simpler surgery description of a c.c.o. 3-manifold M is expected to induce simpler algebraic expressions for the braid equivalence in M. As an example, compare [13, §4] with Section 7.1 in this paper for the case of lens spaces: in this paper the $\mathbb{Q}$-mixed braid equivalence is in the mixed braid groups $B_{1,n}$ and there is only one expression for the braid band moves, while in [13] there are many, according to the integer surgery coefficient of each strand of the surgery pure braid; on top of that combing is also needed. Further, in Section 7 we give the algebraic $\mathbb{Q}$-mixed braid equivalences for links in all four families of 3-manifolds mentioned above. In the paper we use the setting and the results of [12,13] and our results extend the results of [12,13] to rational surgery descriptions and to arbitrary framed braids. We first formulate the geometric $\mathbb{Q}$-mixed braid equivalence via the L-moves and the braid band moves and then we move gradually to the algebraic statement by introducing the notion of cabling and applying the parting and combing techniques of [13].

More precisely: let M be a c.c.o. 3-manifold obtained by rational surgery along a framed link $\widehat{B}$ in S^3. Note that the surgery braid B is not assumed to be a pure braid. Let s be a surgery component of B with surgery description p/q consisting of k strands, $s_1, \ldots, s_k$. When a *geometric* $\mathbb{Q}$*-braid band move* on s occurs, k sets of q new strands appear, each one running in parallel to a strand of s, and also a (p, q)-torus braid d' wraps around the last strand, s_k, p times, followed by a positive or negative crossing $c'_{\pm}$, see Figs. 17 (shaded region) and 14. These moves together with the L-moves lead to the *geometric* $\mathbb{Q}$*-mixed braid equivalence* in M (Theorem 7) (see also [12,19,22]). The $\mathbb{Q}$-braid band moves are clearly much more complicated than the $\mathbb{Z}$-braid band moves in [12]. However, a sharpened version of the Reidemeister theorem for links in M (Theorem 6; see also [12,18,21]), whereby only one type of band moves is used in the mixed link isotopy (see Fig. 8), makes the proof of Theorem 7 quite light.

In order to move toward an algebraic statement we adapt the techniques and results of [13] using the notion of a *q-strand cable*. A q-strand cable represents a set of q new strands arising from the performance of a geometric braid band move. So, we show first that standard parting of a q-strand cable is equivalent to standard parting of each strand of the cable one by one; in other words that parting and cabling commute. Treating now each one of the k q-strand cables as one thickened strand leads to the *parted* $\mathbb{Q}$*-mixed braid equivalence* (Theorem 8), assuming the corresponding result with integral surgery from [13]. We continue by finding algebraic expressions for the loopings of the cables around the fixed strands of B (Lemma 3). Then, after a parted $\mathbb{Q}$-braid band move is performed (Fig. 27a), we part locally the (p, q)-torus braid d', the crossing $c'_{\pm}$ and the loop generators a_j between the moving and the fixed strands obtaining their corresponding algebraic expressions (see Figs. 24, 25, 26). In this way we obtain the algebraic expression of an *algebraic* $\mathbb{Q}$*-braid band move*, which takes place on elements of the braid groups $B_{m,n}$ (see top part of Fig. 27b) and Definition 5(i)). Finally, we do combing through the fixed subbraid B and we show that combing and cabling commute (see Figs. 21 to 23). After the combing our parted mixed braids as well as the $\mathbb{Q}$-braid band moves get separated from the fixed subbraid B, having picked information from it. So, we obtain the *algebraic* $\mathbb{Q}$*-mixed braid equivalence* for links in M in terms of the mixed braid groups $B_{m,n}$ and this is our main result (Theorem 9). Further, in Section 7.1–Section 7.4 we apply Theorem 9 to give the concrete algebraic expressions for the $\mathbb{Q}$-mixed braid equivalences in the aforementioned families of 3-manifolds.

Our results set a homogeneous algebraic ground for studying links in families of 3-manifolds with the computational advantage. Indeed, as we discuss in Section 7.5, our setting is the right one for constructing *Jones type invariants* (such as analogues of the Jones polynomial and the 2-variable Jones or Homflypt polynomial) for links in 3-manifolds via appropriate quotient algebras of the mixed braid groups $B_{m,n}$ (such as analogues of the Temperley–Lieb algebras and the Iwahori–Hecke algebras) which support Markov traces. This topological motivation gives rise to new algebras worth studying. Then one can derive link invariants in the complement $S^{\setminus}\widehat{B}$, which then have to satisfy all possible band moves, for extending them to link invariants in the manifold $M = \chi(S^3, \widehat{B})$. Our results can be equally applied to the study of *skein modules* of c.c.o. 3-manifolds, using braid techniques (see Section 7.5). The advantage of the braid approach is that the algebraic mixed braid equivalences provide good control over the band moves, better than in the diagrammatic setting, and much of the diagrammatic complexity is absorbed into the proofs of the algebraic statements. We only need to consider one type of orientations patterns and the braid band moves are limited. A good example and the simplest one demonstrating the above is the case of the lens spaces $L(p, 1)$: in [11] a generic analogue of the Homflypt polynomial for links in the solid torus, ST, has been defined from the generalized Hecke algebras of type B via a Markov trace constructed on them. This invariant recovers the Homflypt skein module of ST. In order to extend this to an invariant of links in $L(p, 1)$ in [1,4] we solve an infinite system of equations resulting from the braid band moves and we show that it has a unique solution, which proves the freeness of the module. In [5] the same problem has been solved using the diagrammatic approach. As a consequence of the above, in [3] we try to compute the Homflypt skein module of the general case $L(p, q)$ using our results of Section 7.1. Finally, our $\mathbb{Q}$-braid band move

Fig. 1. A mixed link diagram.

can be used for providing a braid analogue of the Rational calculus, which is Rolfsen's analogue to the Kirby calculus for manifolds with rational surgery description [16], extending the braid approach to the Kirby calculus by Ko and Smolinsly [9] (see Section 7.6). Then, our results can potentially lead to a braid computational approach to the Witten invariants.

The paper is organized as follows. In Section 1 we recall the setting and the essential techniques and results from [12,10,13], such as the braid groups $B_{m,n}$ and the techniques of parting and combing. In Section 2 we prove the sharpened version of the Reidemeister theorem for knots and links in c.c.o. 3-manifolds with rational surgery description (Theorem 6). In Section 3 we derive the geometric $\mathbb{Q}$-mixed braid equivalence for links in such 3-manifolds (Theorem 7) and we introduce the cabling. In Section 4 we derive the parted $\mathbb{Q}$-mixed braid equivalence using the cabling and in Section 5 we show that combing and cabling commute. These lead to Section 6 where we give the algebraic $\mathbb{Q}$-mixed braid equivalence (Theorem 9). In Section 7.1–Section 7.4 the reader will find the application of Theorem 9 to the aforementioned families of 3-manifolds. In Section 7.5 we discuss applications to Jones-type invariants of links in 3-manifolds and to skein modules of 3-manifolds; finally, in Section 7.6 we discuss the potential application to formulating Rolfsen's Rational Calculus in terms of braids and to the computation of the Witten invariants.

1. Background results in the case of integral surgery description

In this section we recall from [12,10] and [13] the topological and algebraic setting, the techniques and the results that we will be using in this paper. For the rest of this section we fix a c.c.o. 3-manifold M, which is obtained by integral surgery on a framed link in S^3 given in the form of a closed braid $\widehat{B}$, and we denote $M = \chi_{\mathbb{Z}}(S^3, \widehat{B})$.

1.1. Mixed links and isotopy

Let K be an oriented link in M. Fixing $\widehat{B}$ pointwise on its projection plane, K can be represented unambiguously by a *mixed link* in S^3, denoted $\widehat{B} \cup K$, consisting of the *fixed part* $\widehat{B}$ and the *moving part* K that links with $\widehat{B}$. A *mixed link diagram* is a diagram $\widehat{B} \cup \widetilde{K}$ of $\widehat{B} \cup K$ on the plane of $\widehat{B}$, where this plane is equipped with the top-to-bottom direction of the braid B. See Fig. 1 for an example.

An isotopy of K in M can be then translated into a finite sequence of moves of the mixed link $\widehat{B} \bigcup K$ in S^3 as follows. As we know, surgery along $\widehat{B}$ is realized by taking first the complement $S^3 \backslash \widehat{B}$ and then attaching to it solid tori according to the surgery description. Thus, isotopy in M can be viewed as isotopy in $S^3 \backslash \widehat{B}$ together with the band moves in S^3, which are similar to the second Kirby move. A *band move* is a non-isotopy move in $S^3 \backslash \widehat{B}$ that reflects isotopy in M and is the band connected sum of a component, say s, of K with the specified (from the framing) parallel curve l of a surgery component, say c, of $\widehat{B}$. Note that l bounds a disc in M. There are two types of band moves according to the orientations of the component s of K and of the surgery curve c, as illustrated and exemplified in Fig. 2. In the α-type the orientation of s is opposite to the orientation of c (and of its parallel curve l), but after the performance of the move their orientations agree. In the β-type the orientation of s agrees initially with the orientation of c, but disagrees after the performance of the move. Note that the two types of band moves are related by a twist

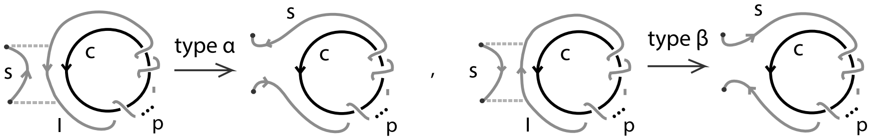

Fig. 2. The two types of $\mathbb{Z}$-band moves.

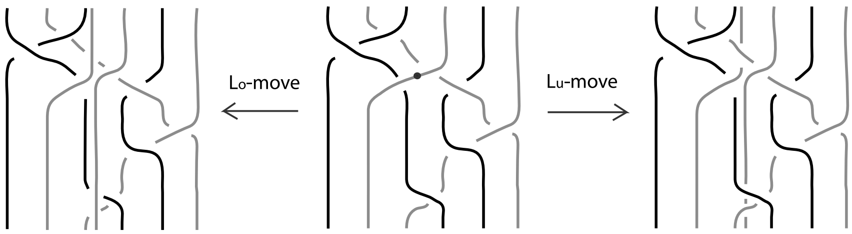

Fig. 3. A geometric mixed braid and the two types of L-moves.

of s (Reidemeister I move in $S^3\backslash\widehat{B}$). Further, in terms of mixed link diagrams, isotopy in $S^3\backslash\widehat{B}$ is realized in S^3 by the classical Reidemeister moves and planar moves for the moving part together with the *mixed Reidemeister moves*. These are the Reidemeister II and III moves involving the fixed and the moving part of the mixed link (cf. Definition 5.1 [12]). The above are summarized in the following:

Theorem 1 *(Reidemeister for $M = \chi_{\mathbb{Z}}(S^3, \widehat{B})$, Thm. 5.8 [12]). Two oriented links L_1, L_2 in M are isotopic if and only if any two corresponding mixed link diagrams of theirs, $\widehat{B} \cup \widetilde{L_1}$ and $\widehat{B} \cup \widetilde{L_2}$, differ by isotopy in $S^3\backslash\widehat{B}$ together with a finite sequence of the two types α and β of band moves.*

1.2. Geometric mixed braids and the L-moves

In order to translate isotopy of links in M into braid equivalence, we need to introduce the notion of a geometric mixed braid. A *geometric mixed braid* related to M and to a link K in M is an element of the group B_{m+n}, where m strands form the fixed *surgery braid* B representing M and n strands form the *moving subbraid* β representing the link K in M. For an illustration see the middle picture of Fig. 3. We further need the notions of the L-moves and the braid band moves ([12, Definitions 2.1 and 5.6]).

An *L-move* on a geometric mixed braid $B \bigcup \beta$, consists in cutting an arc of the moving subbraid β open and pulling the upper cutpoint downward and the lower upward, so as to create a new pair of braid strands with corresponding endpoints (on the vertical line of the cutpoint), and such that both strands cross entirely *over* or *under* with the rest of the braid. Stretching the new strands over will give rise to an L_o*-move* and under to an L_u*-move*. For an illustration see Fig. 3. Two geometric mixed braids shall be called *L-equivalent* if and only if they differ by a sequence of L-moves and braid isotopy. Note that an L-move does not touch the fixed subbraid B.

1.3. Geometric $\mathbb{Z}$-mixed braid equivalence

A *geometric $\mathbb{Z}$-braid band move* is a move between geometric mixed braids which is a band move between their closures. It starts with a little band oriented downward, which, before sliding along a surgery strand, gets one twist *positive* or *negative*. See Fig. 4 (a) and (b).

Remark 1. *(i)* In [12] it is shown that braid equivalence in S^3 as well as mixed braid equivalence in $S^3\backslash\widehat{B}$ are generated only by the L-moves. A concrete demonstration for conjugation can be found in [13] Fig. 14. *(ii)* A geometric $\mathbb{Z}$-braid band move may be always assumed, up to L-equivalence, to take place at the top part of a mixed braid and on the right of the specific surgery strand ([13] Lemma 5).

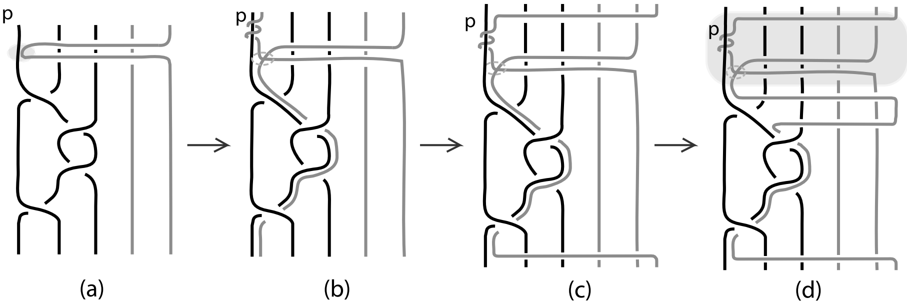

Fig. 4. A geometric $\mathbb{Z}$-braid band move (b), a parted $\mathbb{Z}$-b.b.m. (c) and an algebraic $\mathbb{Z}$-b.b.m. (shaded part in (d)).

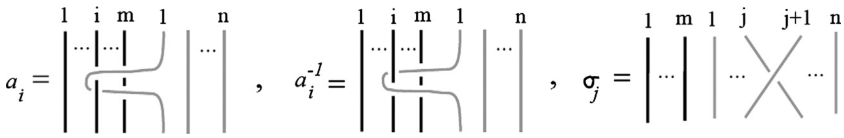

Fig. 5. The loop generators a_i, a_i^{-1} and the braiding generators σ_j of $B_{m,n}$.

So we have the following:

Theorem 2 *(Geometric braid equivalence for $M = \chi_{\mathbb{Z}}(S^3, \widehat{B})$, Theorem 5.10 [12]). Two oriented links in M are isotopic if and only if any two corresponding geometric mixed braids in S^3 differ by mixed braid isotopy, by the L-moves and by the geometric $\mathbb{Z}$-braid band moves.*

1.4. Algebraic mixed braids

We will pass now from the geometric mixed braid equivalence to an algebraic statement for links in M. An *algebraic mixed braid* is a mixed braid on $m + n$ strands such that the first m strands are fixed and form the identity braid on m strands and the next n strands are moving strands and represent a link in the manifold M. The set of all algebraic mixed braids on $m + n$ strands forms a subgroup of B_{m+n}, denoted $B_{m,n}$, called the *mixed braid group*. The mixed braid group $B_{m,n}$ has been introduced and studied in [10] and it is shown that it has the presentation:

$$B_{m,n} = \left\langle \begin{array}{c} a_1, \ldots, a_m, \\ \sigma_1, \ldots, \sigma_{n-1} \end{array} \left| \begin{array}{l} \sigma_k \sigma_j = \sigma_j \sigma_k, \quad |k-j| > 1 \\ \sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}, \quad 1 \leq k \leq n-1 \\ a_i \sigma_k = \sigma_k a_i, \quad k \geq 2, \ 1 \leq i \leq m \\ a_i \sigma_1 a_i \sigma_1 = \sigma_1 a_i \sigma_1 a_i, \quad 1 \leq i \leq m \\ a_i (\sigma_1 a_r \sigma_1^{-1}) = (\sigma_1 a_r \sigma_1^{-1}) a_i, \quad r < i \end{array} \right. \right\rangle, \tag{1}$$

where the *loop generators* a_i and the *braiding generators* σ_j are as illustrated in Fig. 5.

1.5. Parted $\mathbb{Z}$-mixed braid equivalence

In order to give an algebraic statement for braid equivalence in M, we first part the mixed braids and we translate the geometric L-equivalence of Theorem 2 to an equivalence of parted mixed braids (see [13]). *Parting* a geometric mixed braid $B \bigcup \beta$ on $m+n$ strands means to separate its endpoints into two different sets, the first m belonging to the subbraid B and the last n to β, and so that the resulting braids have isotopic closures. This can be realized by pulling each pair of corresponding moving strands to the right and *over* or *under* each strand of B that lies on their right. We start from the rightmost pair respecting the position of the endpoints. The result of parting is a *parted mixed braid*. If the strands are pulled always

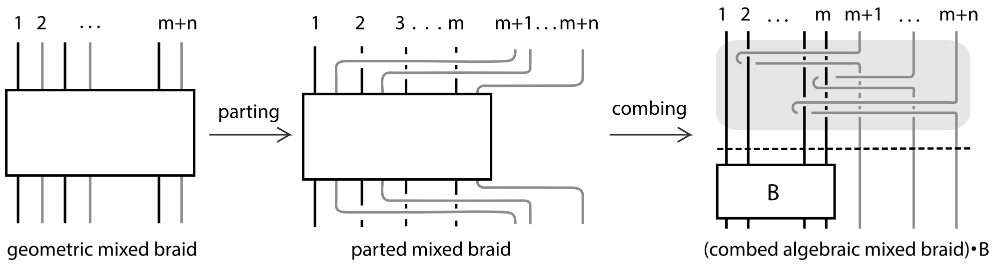

Fig. 6. Parting and combing a geometric mixed braid.

over the strands of B, then this parting is called *standard parting*. See the middle illustration of Fig. 6 for the standard parting of an abstract mixed braid.

Then, in order to restrict Theorem 2 to the set of all parted mixed braids related to M, we need the following moves between parted mixed braids. *Loop conjugation* of a parted mixed braid β is its concatenation by a loop from above and by its inverse from below, that is: $\beta \sim a_i^{\pm 1}\beta a_i^{\mp 1}$. As it turns out, two partings of a geometric mixed braid differ by loop conjugations (cf. Lemma 2 [13]). A *parted L-move* is an L-move between parted mixed braids so that if the endpoints of the new pair of strands appear before the last fixed strand, then we pull the two endpoints to the rightmost position of the moving part, over or under all strands, according to the type of the L-move. Further, a mixed braid with an L-move performed can be parted to a parted mixed braid with a parted L-move performed by making the parting consistent with the label of the L-move (cf. Lemma 3 [13]). A *parted $\mathbb{Z}$-braid band move* is a geometric $\mathbb{Z}$-braid band move between parted mixed braids, such that: it takes place at the top part of the braid, the little band *starts from the last strand* of the moving subbraid and it moves over each moving strand and each component of the surgery braid until it reaches from the right the specific component, and afterwards the move is followed by parting, whereby we pull the new pair of strands over all strands of the fixed and the moving subbraids (see Fig. 4(a) and (c)). Thus, performing a geometric $\mathbb{Z}$-braid band move on a mixed braid and then parting, the result is equivalent, up to L-moves and loop conjugation, to performing a parted $\mathbb{Z}$-braid band move (cf. Lemma 5 [13]).

Theorem 3 *(Parted version of braid equivalence for $M = \chi_{\mathbb{Z}}(S^3, \widehat{B})$, Theorem 3 [13]). Two oriented links in $M = \chi_{\mathbb{Z}}(S^3, \widehat{B})$ are isotopic if and only if any two corresponding parted mixed braids differ by a finite sequence of parted mixed braid isotopies, parted L-moves, loop conjugations and parted $\mathbb{Z}$-braid band moves.*

1.6. Algebraic $\mathbb{Z}$-mixed braid equivalence

In order to translate the parted mixed braid equivalence to an equivalence between algebraic mixed braids we comb the parted mixed braids. *Combing* a parted mixed braid means to separate the knotting and linking of the moving part away from the fixed subbraid using mixed braid isotopy. More precisely, let Σ_k denote the crossing between the kth and the $(k+1)$st strand of the fixed subbraid. Then, for all $j = 1, \ldots, n-1$ and $k = 1, \ldots, m-1$ we have: $\Sigma_k \sigma_j = \sigma_j \Sigma_k$. Thus, the only generating elements of the moving part that are affected by the combing are the loops a_i. This is illustrated in Fig. 7. In Lemma 6 [13] algebraic formuli are given for the effect of combing on the a_i's (see Lemma 2 below).

The effect of combing a parted mixed braid is to separate it into two distinct parts: the *algebraic part* at the top, which has all fixed strands forming the identity braid, so it is an element of some mixed braid group $B_{m,n}$, and which contains all the knotting and linking information of the link L in M; and the *coset part* at the bottom, which contains only the fixed subbraid B and an identity braid for the moving part (see rightmost illustration in Fig. 6). Let now $C_{m,n}$ denote the set of parted mixed braids on n moving strands with fixed subbraid B. Concatenating two elements of $C_{m,n}$ is not a closed operation since it alters

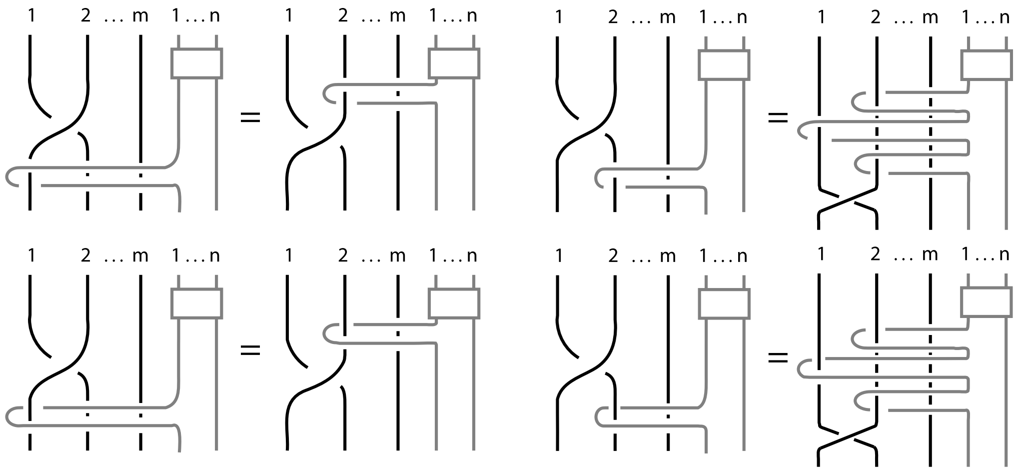

Fig. 7. Combing a parted mixed braid.

the braid description of the manifold. However, as a result of the combing, the set $C_{m,n}$ is a coset of $B_{m,n}$ in B_{m+n} characterized by the fixed subbraid B. Fore details on the above the reader is referred to [10].

Translating the parted braid equivalence into an equivalence between algebraic mixed braids (after combing), gives rise to an algebraic statement of Theorem 3. Since loop conjugation does not take into account the combing of the loop through the fixed subbraid, we need the notion of *combed loop conjugation*. A combed loop conjugation is a move between algebraic mixed braids and is the result of a loop conjugation on a combed mixed braid, followed by combing; so it can be described algebraically as: $\beta \sim \alpha_i^{\mp 1}\beta\rho_i^{\pm 1}$ for $\beta, a_i, \rho_i \in B_{m,n}$, where ρ_i is the combing of the loop a_i through the fixed subbraid B. We also define an *algebraic L-move* to be an L-move between algebraic mixed braids. An algebraic L-move has the following algebraic expression for an L_o-move and an L_u-move respectively:

$$\begin{aligned} \alpha &= \alpha_1\alpha_2 \overset{L_o}{\sim} \sigma_i^{-1}\ldots\sigma_n^{-1}\alpha_1'\sigma_{i-1}^{-1}\ldots\sigma_{n-1}^{-1}\sigma_n^{\pm 1}\sigma_{n-1}\ldots\sigma_i\alpha_2'\sigma_n\ldots\sigma_i \\ \alpha &= \alpha_1\alpha_2 \overset{L_u}{\sim} \sigma_i\ldots\sigma_n\alpha_1'\sigma_{i-1}\ldots\sigma_{n-1}\sigma_n^{\pm 1}\sigma_{n-1}^{-1}\ldots\sigma_i^{-1}\alpha_2'\sigma_n^{-1}\ldots\sigma_i^{-1} \end{aligned} \tag{2}$$

where α_1, α_2 are elements of $B_{m,n}$ and α_1', $\alpha_2' \in B_{m,n+1}$ are obtained from α_1, α_2 by replacing each σ_j by σ_{j+1} for $j = i, \ldots, n-1$.

Finally, we define *M-conjugation* of an algebraic mixed braid to be its conjugation by a crossing σ_j (or by σ_j^{-1}). An *M-move* is defined to be the insertion of a crossing $\sigma_n^{\pm 1}$ on the right hand side of an algebraic mixed braid. Note that M-conjugation, the M-moves and the algebraic L-moves commute with combing. We finally need to understand how a parted $\mathbb{Z}$-braid band move is combed through B.

Definition 1 *(Definition 7 [13]).* An *algebraic $\mathbb{Z}$-braid band move* is defined to be a parted braid band move between algebraic mixed braids (see top part of Fig. 4(d)). Setting:

$$\lambda_{n-1,1} := \sigma_{n-1}\ldots\sigma_1 \quad \text{and} \quad t_{k,n} := \sigma_n\ldots\sigma_1 a_k \sigma_1^{-1}\ldots\sigma_n^{-1},$$

an algebraic band move has the following algebraic expression:

$$\beta_1\beta_2 \ \sim \ \beta_1'\, t_{k,n}^{p_k}\, \sigma_n^{\pm 1}\, \beta_2',$$

where $\beta_1, \beta_2 \in B_{m,n}$ and $\beta_1', \beta_2' \in B_{m,n+1}$ are the words β_1, β_2 respectively with the substitutions:

$$\begin{array}{l} a_k^{\pm 1} \longleftrightarrow [(\lambda_{n-1,1}^{-1}\sigma_n^2\lambda_{n-1,1})\, a_k]^{\pm 1} \\ a_i^{\pm 1} \longleftrightarrow (\lambda_{n-1,1}^{-1}\sigma_n^2\lambda_{n-1,1})\, a_i^{\pm 1}\, (\lambda_{n-1,1}^{-1}\sigma_n^2\lambda_{n-1,1}^{-1}), \quad \text{if } i<k \\ a_i^{\pm 1} \longleftrightarrow a_i^{\pm 1}, \quad \text{if } i>k. \end{array}$$

Further, a *combed algebraic $\mathbb{Z}$-braid band move* is a move between algebraic mixed braids and is defined to be a parted $\mathbb{Z}$-braid band move that has been combed through B. So it is the composition of an algebraic $\mathbb{Z}$-braid band move with the combing of the parallel strand, and it has the following algebraic expression:

$$\beta_1\beta_2 \;\sim\; \beta_1'\, t_{k,n}^{p_k}\, \sigma_n^{\pm 1}\, \beta_2'\, r_k,$$

where r_k is the combing of the parted parallel strand to the kth surgery strand through B.

The group $B_{m,n}$ embeds naturally into the group $B_{m,n+1}$. We shall denote

$$B_{m,\infty} := \bigcup_{n=1}^{\infty} B_{m,n} \quad \text{and similarly} \quad C_{m,\infty} = \bigcup_{n=1}^{\infty} C_{m,n}.$$

Recalling now Remark 1(i) we are in the position to give the algebraic mixed braid equivalence for M.

Theorem 4 *(Algebraic mixed braid equivalence for $M = \chi_{\mathbb{Z}}(S^3, \widehat{B})$, Theorem 5 [13]). Two oriented links in $M = \chi_{\mathbb{Z}}(S^3, \widehat{B})$ are isotopic if and only if any two corresponding algebraic mixed braid representatives in $B_{m,\infty}$ differ by a finite sequence of the following moves:*

(1) *M-moves: $\beta_1\beta_2 \sim \beta_1\sigma_n^{\pm 1}\beta_2$, for $\beta_1, \beta_2 \in B_{m,n}$,*
(2) *M-conjugation: $\beta \sim \sigma_j^{\pm 1}\beta\sigma_j^{\mp 1}$, for $\beta, \sigma_j \in B_{m,n}$,*
(3) *Combed loop conjugation: $\beta \sim a_i^{\mp 1}\beta\rho_i^{\pm 1}$, for $\beta, a_i, \rho_i \in B_{m,n}$, where ρ_i is the combing of the loop a_i through B,*
(4) *Combed algebraic $\mathbb{Z}$-braid band moves: For every $k = 1, \dots, m$ we have:*

$$\beta_1\beta_2 \;\sim\; \beta_1'\, t_{k,n}^{p_k}\, \sigma_n^{\pm 1}\, \beta_2'\, r_k,$$

where $\beta_1, \beta_2 \in B_{m,n}$ and $\beta_1', \beta_2' \in B_{m,n+1}$ are as in Definition 1 and where r_k is the combing of the parted parallel strand to the kth surgery strand through B. Equivalently, by the same moves as above, where (1) and (2) are replaced by: (1′) algebraic L-moves.

2. The Reidemeister Theorem for links in 3-manifolds

From now on M will denote a c.c.o. 3-manifold obtained from S^3 by rational surgery, that is surgery along a framed link $\widehat{B}$ with rational coefficients, denoted $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$. Let L be an oriented link in M. By the discussion in Section 1.1, isotopy in M is translated into isotopy in $S^3\backslash\widehat{B}$ together with the two types, α and β, of band moves for mixed links in S^3. The band moves in this case are described as follows. Let c be a component of $\widehat{B}$ with framing p/q. The specified parallel curve l of c is a (p, q)-torus knot on the boundary of a tubular neighborhood of c which, by construction, bounds a disc in M. Then, a $\mathbb{Q}$-*band move* along c is the connected sum of a component of L with the (p, q)-torus knot l and there are two types, α and β, according to the orientations. The two types of band moves are illustrated in Fig. 8, where c is a trefoil knot with 2/3 surgery coefficient and where “band move” is shortened to “b.m.”. Clearly, Theorem 1 applies also to $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$. Namely:

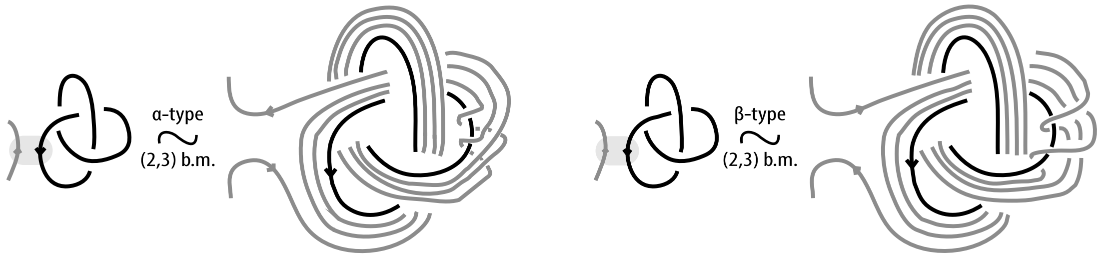

Fig. 8. The two types of $\mathbb{Q}$-band moves.

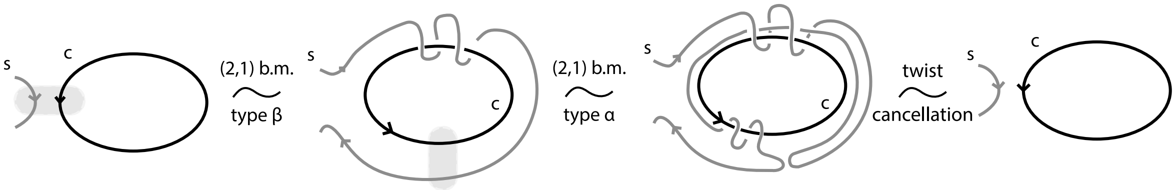

Fig. 9. A type-β band move follows from a type-α band move in the case of integral surgery coefficient.

Theorem 5 *(Reidemeister for $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$ with two types of band moves). Two oriented links L_1, L_2 in M are isotopic if and only if any two corresponding mixed link diagrams of theirs, $\widehat{B} \cup \widetilde{L_1}$ and $\widehat{B} \cup \widetilde{L_2}$, differ by isotopy in $S^3 \backslash \widehat{B}$ together with a finite sequence of the two types α and β of band moves.*

In this section we sharpen Theorem 5. More precisely, we show that only one of the two types of band moves is necessary in order to describe isotopy for links in M. The proof is based on a known contrivance, which was used in the proof of Theorem 5.10 [12] (Theorem 2) for establishing the sufficiency of the geometric braid band moves in the mixed braid equivalence for the case of integral surgery (see Fig. 9). Theorem 6 simplifies the proof of Theorem 7.

Theorem 6 *(Reidemeister for $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$ with one type of band moves). Two oriented links L_1, L_2 in M are isotopic if and only if any two corresponding mixed link diagrams of theirs, $\widehat{B} \bigcup L_1$ and $\widehat{B} \bigcup L_2$, differ by a finite sequence of the band moves of type α (or equivalently of type β) and isotopy in $S^3 \backslash \widehat{B}$.*

Proof. Let L be an oriented link in M. By Theorem 5, it suffices to show that a band move of type β can be obtained from a band move of type α and isotopy in the knot complement. We will first demonstrate the proof for an unknotted surgery component c with integral coefficient p. (Note that integral surgery description can be considered as a special case of rational surgery description.) We shall follow the steps of the proof in Fig. 9 where $p = 2$. We start with performing a band move of type β using a component s of the link L. In Fig. 9 we see the two twists of the band move wrapping around the surgery curve c in the righthand sense. Then, using an arc of the same link component s, we perform a second band move of type α. This will take place within a thinner tubular neighborhood than the first band move. So, the two twists of the second band move, which also wrap around c in the righthand sense, commute with the two twists of the first band move. We arrange all $2p$ twists in pairs as follows. We pass one twist from the second band move (the closest) through all twists of the first band move, see Fig. 10. Since all twists follow the righthand sense, the two innermost twists coming from the second and the first band move, create a little band which can be eliminated using isotopy in the knot complement of c. This is the cancellation of the first pair of the $2p$ twists. Repeating the same procedure we cancel all p pairs and we end up with the component s of the link L as it was in the initial position before the band moves.

For the more general case of rational surgery along any knot c we follow the same idea. More precisely, we perform a $\mathbb{Q}$-band move of type β along c and we obtain an outer (p, q)-torus knot. Then, we perform

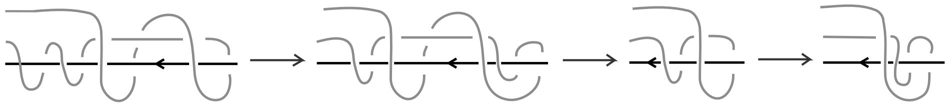

Fig. 10. Twist cancellation.

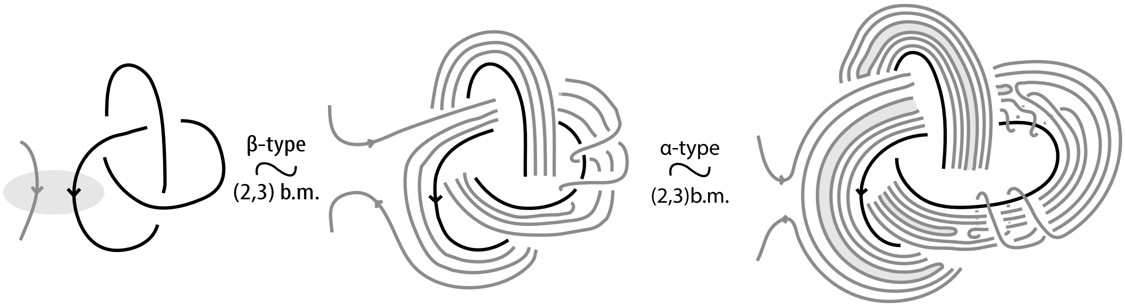

Fig. 11. A band move of type β followed by a band move of type α.

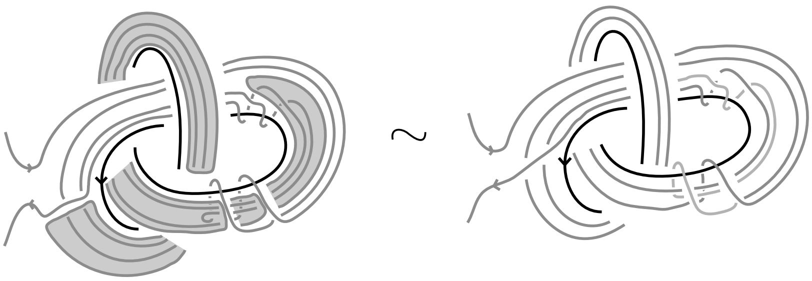

Fig. 12. Band with boundary two parallel arcs of opposite orientations.

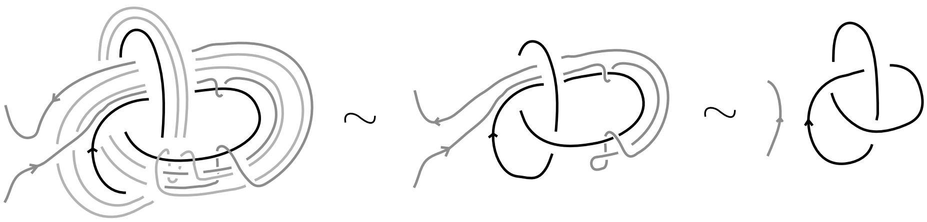

Fig. 13. Retracting the band along the surgery component.

a $\mathbb{Q}$-band move of type α along c and we obtain an inner (p, q)-torus knot. In Fig. 11 we illustrate this for the case where $p = 2$, $q = 3$ and c a trefoil knot.

Without loss of generality (by isotopy in the complement of c), the second band move is performed on the innermost arc of the q arcs parallel to c, creating q new parallel arcs even closer to c. After the second $\mathbb{Q}$-band move is performed, the outer arc of the q new arcs and the inner arc of the q arcs coming from the first band move of type α form a band (see shaded area in Fig. 11). Then, using isotopy in the complement of c, we eliminate this band by pulling it along c. This will result in the elimination of $p - q$ pairs of parallel arcs to c. In our example, this is done in Fig. 12.

As in the case of integral surgery the twists coming from the two band moves commute. Arranging these $2p$ twists pairwise, they cancel out by the fact that all twists have the same handiness, but opposite orientation. In the end, s is left as in its initial position.

So, a $\mathbb{Q}$-band move of type β can be performed using a $\mathbb{Q}$-band move of type α and isotopy in the complement of the surgery component c (see Fig. 13). The proof of Theorem 6 is now concluded. □

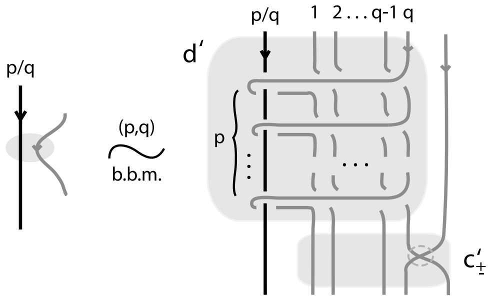

Fig. 14. A $\mathbb{Q}$-braid band move locally.

3. Geometric $\mathbb{Q}$-mixed braid equivalence

In this section we extend Theorem 2 to manifolds with rational surgery description, that is $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$, using the sharpened Reidemeister theorem for M (Theorem 6). We first need the following.

Definition 2. A *geometric* $\mathbb{Q}$*-braid band move* is a move between geometric mixed braids which is a $\mathbb{Q}$-band move of type α between their closures. It starts with a little band (an arc of the moving subbraid) close to a surgery strand with surgery coefficient p/q. The little band gets first one twist *positive* or *negative*, which shall be denoted as $c'_{\pm}$ and then is replaced by q strands that run in parallel to all strands of the same surgery component and link only with that surgery strand, wrapping around it p times and, thus, forming a (p, q)-torus knot. See Fig. 14 for local and Fig. 17 (shaded area) for global illustration. This braided (p, q)-torus knot is denoted as d'. A geometric $\mathbb{Q}$-braid band move with a positive (resp. negative) twist shall be called a positive geometric $\mathbb{Q}$-braid band move (resp. negative geometric $\mathbb{Q}$-braid band move).

By Remark 1(ii) a $\mathbb{Q}$-braid band move may be assumed to take place at the top part of a mixed braid and all strands from a $\mathbb{Q}$-braid band move may be assumed to lie on the righthand side of the surgery strands. We shall now prove the following.

Theorem 7 *(Geometric braid equivalence for $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$). Two oriented links in M are isotopic if and only if any two corresponding geometric mixed braids in S^3 differ by mixed braid isotopy, by L-moves that do not touch the fixed subbraid B and by the geometric $\mathbb{Q}$-braid band moves.*

Proof. The proof is completely analogous to and is based on the proof of Theorem 5.10 [12] (Theorem 2). Let K_1 and K_2 be two isotopic oriented links in M. By Theorem 6, the corresponding mixed links $\widehat{B} \bigcup K_1$ and $\widehat{B} \bigcup K_2$ differ by isotopy in the complement of $\widehat{B}$ and $\mathbb{Q}$-band moves of type α. Note that, by Theorem 6 we do not need to consider band moves of type β. By Theorem 5.10 [12], isotopy in the complement of $\widehat{B}$ translates into geometric braid isotopy and the L-moves. Let now $\widehat{B} \bigcup K_1$ and $\widehat{B} \bigcup K_2$ differ by a $\mathbb{Q}$-band move of type α (recall Fig. 8). Let $\widehat{B} \bigcup \widetilde{K_1}$ and $\widehat{B} \bigcup \widetilde{K_2}$ be two mixed link diagrams of the mixed links $\widehat{B} \bigcup K_1$ and $\widehat{B} \bigcup K_2$ which differ only by the places illustrated in Fig. 15. As in [12], by the braiding algorithm given therein, the diagrams $\widehat{B} \bigcup \widetilde{K_1}$ and $\widehat{B} \bigcup \widetilde{K_2}$ may be assumed braided everywhere except for the places where the $\mathbb{Q}$-band move is performed.

We now braid the up-arc in Fig. 15(b) and obtain a geometric mixed braid $\widehat{B} \bigcup b_1$ corresponding to the diagram $\widehat{B} \bigcup \widetilde{K_1}$ (see Fig. 15(a)). Note that Fig. 15(c) is already in braided form and let $B \bigcup b_2$ denote the geometric mixed braid corresponding to the diagram $\widehat{B} \bigcup \widetilde{K_2}$.

We would like to show that the two mixed braids $B \bigcup b_1$ and $B \bigcup b_2$ differ by the moves given in the statement of the theorem.

We perform a Reidemeister I move on $\widehat{B} \bigcup \widetilde{K_1}$ with a *negative* crossing and obtain the diagram $\widehat{B} \bigcup \widetilde{K'_1}$. Then, the corresponding mixed braids, $B \bigcup b_1$ and $B \bigcup b'_1$, differ by mixed braid isotopy and L-moves

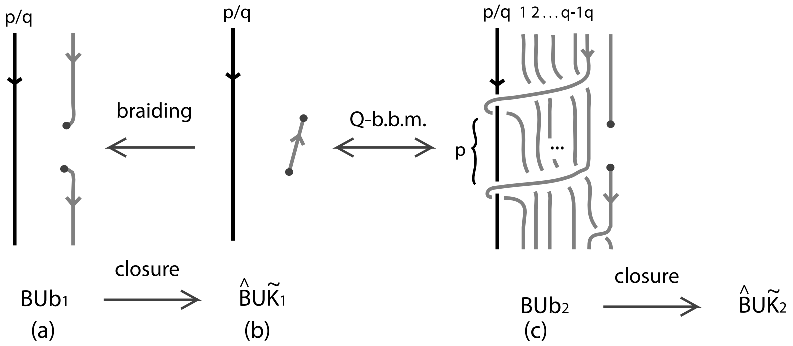

Fig. 15. A type α band move and its braiding (locally).

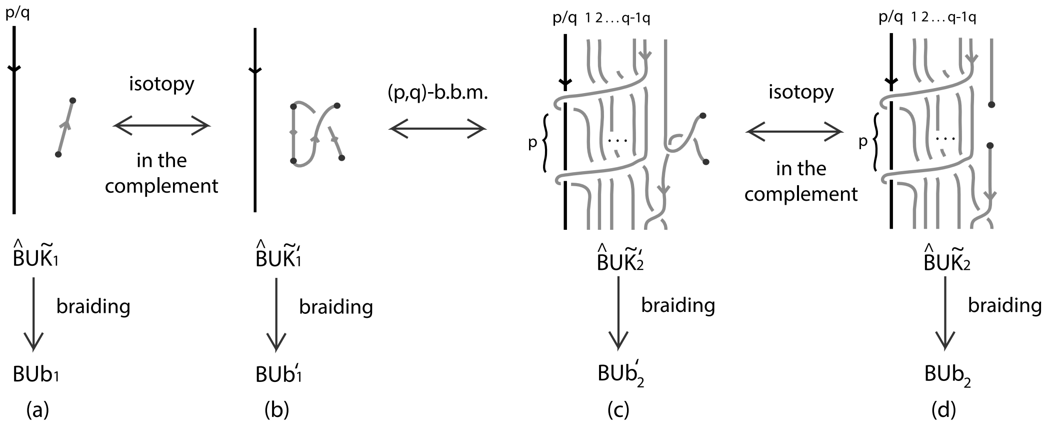

Fig. 16. The steps of the proof of Theorem 7.

(see Fig. 16(a) and (b)). We then perform a *positive* $\mathbb{Q}$-braid band move on $B\bigcup b_1'$ and obtain the mixed braid $B\bigcup b_2'$. In the closure of $B\bigcup b_2'$ we unbraid and re-introduce the two up-arcs illustrated in Fig. 16(b), obtaining a diagram $\widehat{B}\bigcup\widetilde{K_2'}$ with the formation of a Reidemeister II move. Performing this move on $\widehat{B}\bigcup\widetilde{K_2'}$ we obtain the diagram $\widehat{B}\bigcup\widetilde{K_2}$, which is already in braided form and its corresponding mixed braid is $B\bigcup b_2$ (see Fig. 16(c) and (d)). So, the mixed braids $B\bigcup b_2'$ and $B\bigcup b_2$ differ by mixed braid isotopy and L-moves. Therefore, we showed that the braids $B\bigcup b_1$ and $B\bigcup b_2$ in Fig. 15(a) and (c) differ by mixed braid isotopy, L-moves and a braid band move. This concludes the proof. □

3.1. Introducing cabling

In order to translate the geometric mixed braid equivalence to an equivalence of algebraic mixed braids we follow the strategy in [13]. Namely, we apply to the geometric mixed braids first parting and then combing. What makes things more complicated in the case of rational surgery description is that the surgery braid B is in general not a pure braid and when we apply a $\mathbb{Q}$-braid band move on a mixed braid, the little band that approaches the surgery strand is replaced by q strands that run in parallel to all strands of the same surgery component. In order to proceed we need the notion of a q-strand cable.

Definition 3. We define a *q-strand cable* to be a set of q parallel strands coming from a $\mathbb{Q}$-braid band move and following one strand of the specified surgery component.

Treating the new strands coming from the braid band move as cables running in parallel to the strands of a surgery component, that is, treating each cable as one thickened strand, we may adopt and apply results from [13].

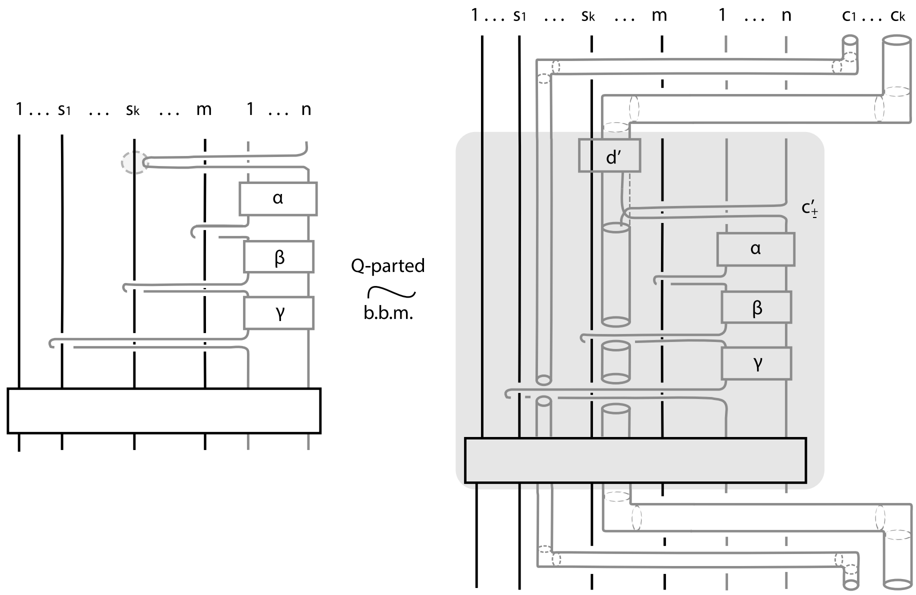

Fig. 17. A parted $\mathbb{Q}$-braid band move using cables.

4. Parted $\mathbb{Q}$-mixed braid equivalence

Let $B \bigcup \beta$ be a geometric mixed braid and suppose that a $\mathbb{Q}$-braid band move is performed on it. We part $B \bigcup \beta$ following the exact procedure as in [13]. More precisely, we have the following.

Lemma 1. *Cabling and standard parting commute. That is, standard parting of a mixed braid with a $\mathbb{Q}$-braid band move performed and then cabling, is the same as cabling first the set of new strands and then standard parting.*

Proof. Let $B \bigcup \beta$ be a geometric mixed braid on $m + n$ strands and let a $\mathbb{Q}$-braid band move be performed on a surgery component s of B. Let also $s_1, \ldots, s_k \in \{1, \ldots, m\}$ be the numbers of the strands of the surgery component s and let $c_1, \ldots, c_k$ denote the q-strand cables corresponding to $s_1, \ldots, s_k$. On the one hand, after the $\mathbb{Q}$-braid band move is performed and before any cablings occur, we part the geometric mixed braid following the procedure of the standard parting as described in Section 1.3 (recall middle illustration of Fig. 6). On the other hand we cable first each set of q-strands resulting from the $\mathbb{Q}$-braid band move and then we part the geometric mixed braid with the standard parting, treating each cable as one (thickened) strand. Since both cabling and parting a geometric mixed braid respect the position of the endpoints of each pair of corresponding moving strands, it follows that cabling and parting commutes. $\square$

Recall from Section 1.3 that a geometric L-move can be turned to a parted L-move. In order to give the analogue of Theorem 3 in the case of rational surgery we also need to introduce the following adaptation of a parted $\mathbb{Z}$-braid band move.

Definition 4. A *parted $\mathbb{Q}$-braid band move* is defined to be a geometric $\mathbb{Q}$-braid band move between parted mixed braids, such that it takes place at the top part of the braid and on the right of the rightmost strand, s_k, of the specific surgery component, s, consisting of the strands $s_1, \ldots, s_k$. Moreover, the little band starts from the last strand of the moving subbraid and it moves over each moving strand and each component of the surgery braid, until it reaches the last strand of s, and then is followed by parting of the resulting mixed braid, as illustrated in Fig. 17.

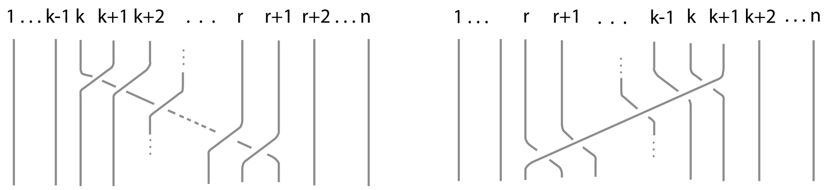

Fig. 18. The elements $\lambda_{k,r}$.

Then Theorem 7 restricts to the following.

Theorem 8 *(Parted version of braid equivalence for $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$). Two oriented links in $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$ are isotopic if and only if any two corresponding parted mixed braids in $C_{m,\infty}$ differ by a finite sequence of parted L-moves, loop conjugations and parted $\mathbb{Q}$-braid band moves.*

Proof. By Lemma 1 the cables resulting from a geometric $\mathbb{Q}$-braid band move are treated as one strand, so we can apply Theorem 3. Moreover, by Lemma 9 in [13] a geometric $\mathbb{Q}$-braid band move may be always assumed, up to L-equivalence, to take place on the right of the rightmost strand of the specific surgery component. □

5. Combing and cabling

In order to translate Theorem 8 into an algebraic equivalence between elements of $B_{m,\infty}$ we need the following lemmas.

Lemma 2 *(Combing Lemma, Lemma 6 [13]). The crossings Σ_k, $k = 1, \ldots, m-1$ of the fixed subbraid B, and the loops a_i, for $i = 1, \ldots, m$, satisfy the following* 'combing' relations*:*

$$\begin{array}{lll}
\Sigma_k a_k^{\pm 1} & = a_{k+1}^{\pm 1}\Sigma_k & \\
\Sigma_k a_{k+1}^{\pm 1} & = a_{k+1}^{-1} a_k^{\pm 1} a_{k+1}\Sigma_k & \\
\Sigma_k a_i^{\pm 1} & = a_i^{\pm 1}\Sigma_k & \textit{if} \ \ i \neq k, k+1 \\
\Sigma_k^{-1} a_k^{\pm 1} & = a_k a_{k+1}^{\pm 1} a_k^{-1}\Sigma_k^{-1} & \\
\Sigma_k^{-1} a_{k+1}^{\pm 1} & = a_k^{\pm 1}\Sigma_k^{-1} & \\
\Sigma_k^{-1} a_i^{\pm 1} & = a_i^{\pm 1}\Sigma_k^{-1} & \textit{if} \ \ i \neq k, k+1.
\end{array}$$

Notation: We set $\lambda_{k,r} := \sigma_k\sigma_{k+1}\ldots\sigma_{r-1}\sigma_r$, for $k < r$ and $\lambda_{k,r} := \sigma_k\sigma_{k-1}\ldots\sigma_{r+1}\sigma_r$, for $r < k$. We note that $\lambda_{i,i} := \sigma_i$. Also, by convention we set $\lambda_{0,i} = \lambda_{i,0} := 1$ (see Fig. 18).

Then we have the following:

Lemma 3. *A positive looping between a q-strand cable and the jth fixed strand of the fixed subbraid B has the algebraic expressions:*

$$\prod_{i=0}^{q-1} \lambda_{i,1} a_j \lambda_{i,1}^{-1} = \prod_{i=0}^{q-1} \lambda_{1,(q-1)-i}^{-1} a_j \lambda_{1,(q-1)-i} \ ,$$

while a negative looping has the algebraic expressions:

$$\prod_{i=0}^{q-1} \lambda_{1,i}^{-1} a_j^{-1} \lambda_{1,i} = \prod_{i=0}^{q-1} \lambda_{(q-1)-i,1} a_j^{-1} \lambda_{(q-1)-i,1}^{-1} \ .$$

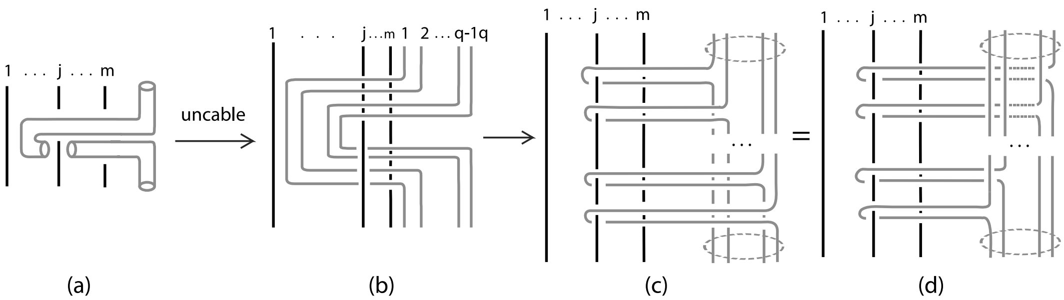

Fig. 19. A positive looping between a cable and a fixed strand.

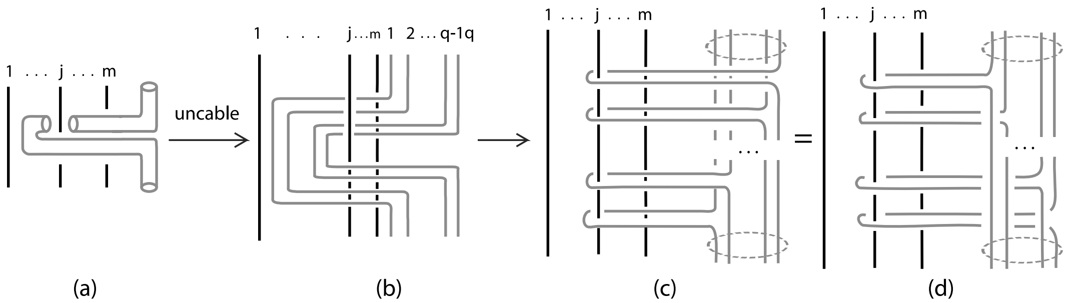

Fig. 20. A negative looping between a cable and a fixed strand.

Proof. We start with Fig. 19(a) where a positive looping between a q-strand cable and a fixed stand of the mixed braid is shown. In Fig. 19(b) the cable is replaced by the q strands according to Definition 3. Then, using mixed braid isotopy, we end up with Fig. 19(c), whereby we can read directly the algebraic expression $\prod_{i=0}^{q-1} \lambda_{i,1} a_j \lambda_{i,1}^{-1}$. The second algebraic expression comes from the illustration of Fig. 19(d). Similarly, in Fig. 20 we illustrate the case where a negative looping between a q-strand cable and a fixed strand of the mixed braid occurs. □

Lemma 4. *Cabling and combing commute. That is, treating a q-strand cable as a thickened moving strand and combing it through the fixed subbraid B, the result is equivalent to combing one by one each strand of the cable.*

Proof. According to the Combing Lemma we have to consider all cases between looping and crossings of the subbraid B. We will only examine the four cases illustrated in Fig. 7 as representative cases. All others are completely analogous. The first case is illustrated in Fig. 21, where a positive looping between the cable and the kth fixed strand of B is being considered and the crossing of the fixed strands is positive. For a negative looping the proof is similar.

We now consider the case illustrated in Fig. 22, where a positive looping between the cable and the $(k+1)$th fixed strand of B is being considered, and the crossing in B is positive. We shall prove this case by induction on the number of strands that belong to the cable, since, as we can see from Fig. 22, the resulting algebraic expressions are not directly comparable.

The case where the cable consists of one strand is trivial. For a two-strand cable, combing the cable first and then uncabling (see top part of Fig. 22) results in the algebraic expression:

$$\alpha_2^{-1}\ (\sigma_1^{-1}\alpha_2^{-1}\sigma_1)\ \alpha_1\ (\sigma_1^{-1}\alpha_1\sigma_1)\ \alpha_2\ (\sigma_1^{-1}\alpha_2\sigma_1),$$

while uncabling first and then combing (bottom part of Fig. 22) results in the algebraic expression:

$$(\alpha_2^{-1}\alpha_1\alpha_2)\ (\sigma_1^{-1}\alpha_2^{-1}\alpha_1\alpha_2\sigma_1).$$

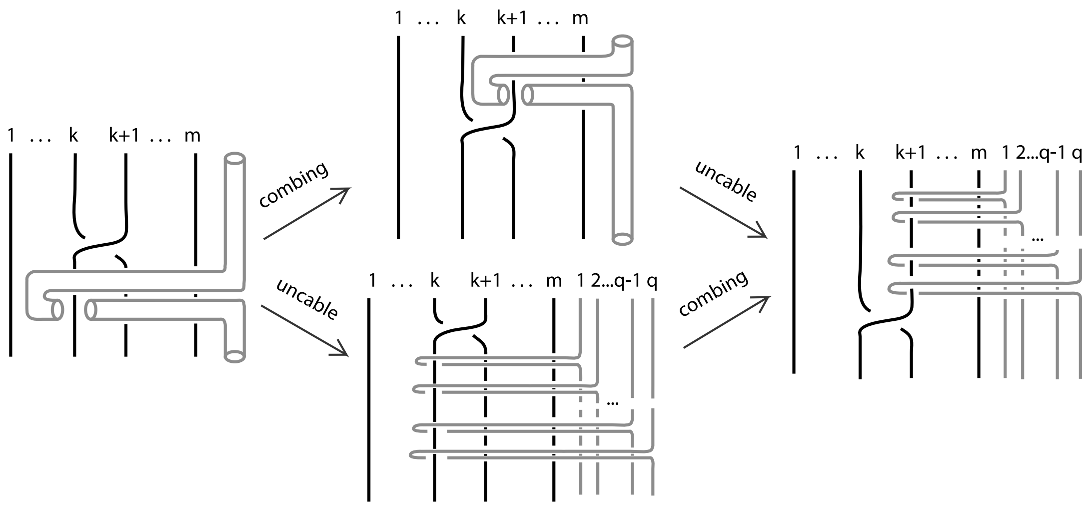

Fig. 21. Combing and cabling commute: Proof of Case 1.

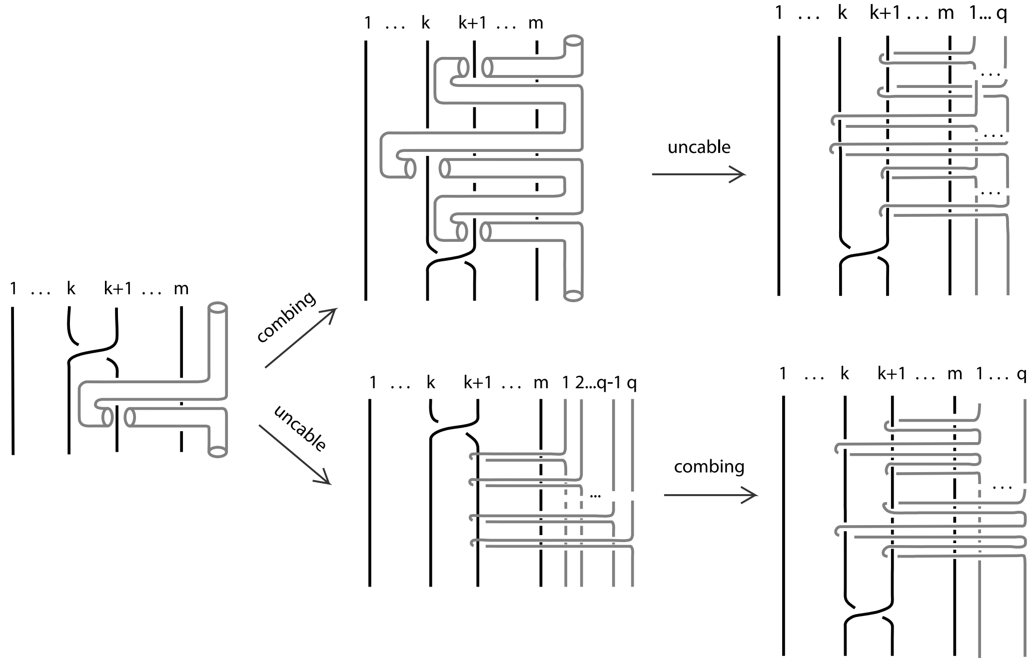

Fig. 22. Combing and cabling commute: Case 2.

We show below that these algebraic expressions are equal, whereby we have underlined expressions which are crucial for the next step. Indeed:

$$\begin{aligned}
\underline{\alpha_2^{-1}}(\sigma_1^{-1}\alpha_2^{-1}\sigma_1)\alpha_1(\sigma_1\alpha_1\sigma_1^{-1})\alpha_2(\sigma_1\underline{\alpha_2\sigma_1^{-1}}) &= (\underline{\alpha_2^{-1}}\alpha_1\alpha_2)(\sigma_1\alpha_2^{-1}\alpha_1\underline{\alpha_2\sigma_1^{-1}}) && \Leftrightarrow\\
(\sigma_1^{-1}\alpha_2^{-1}\sigma_1)\alpha_1(\underline{\sigma_1\alpha_1\sigma_1^{-1})\alpha_2}(\sigma_1) &= (\alpha_1\alpha_2)(\sigma_1\alpha_2^{-1}\alpha_1) && \Leftrightarrow\\
\sigma_1^{-1}\alpha_2^{-1}\sigma_1\alpha_1\alpha_2\sigma_1\underline{\alpha_1\sigma_1^{-1}\sigma_1} &= \alpha_1\alpha_2\sigma_1\alpha_2^{-1}\underline{\alpha_1} && \Leftrightarrow\\
\sigma_1^{-1}\underline{\alpha_2^{-1}\sigma_1\alpha_1(\sigma_1^{-1}}\sigma_1)\alpha_2\sigma_1 &= \alpha_1\alpha_2\sigma_1\alpha_2^{-1} && \Leftrightarrow\\
\underline{\sigma_1^{-1}\sigma_1\alpha_1}\sigma_1^{-1}\alpha_2^{-1}\sigma_1\alpha_2\sigma_1 &= \underline{\alpha_1}\alpha_2\sigma_1\alpha_2^{-1} && \Leftrightarrow\\
\underline{\sigma_1^{-1}\alpha_2^{-1}}\sigma_1\alpha_2\sigma_1 &= \alpha_2\sigma_1\underline{\alpha_2^{-1}} && \Leftrightarrow\\
\sigma_1\alpha_2\sigma_1\alpha_2 &= \alpha_2\sigma_1\alpha_2\sigma_1 &&
\end{aligned}$$

We ended up with one of the defining relations of the mixed braid group $B_{m,n}$, recall (1).

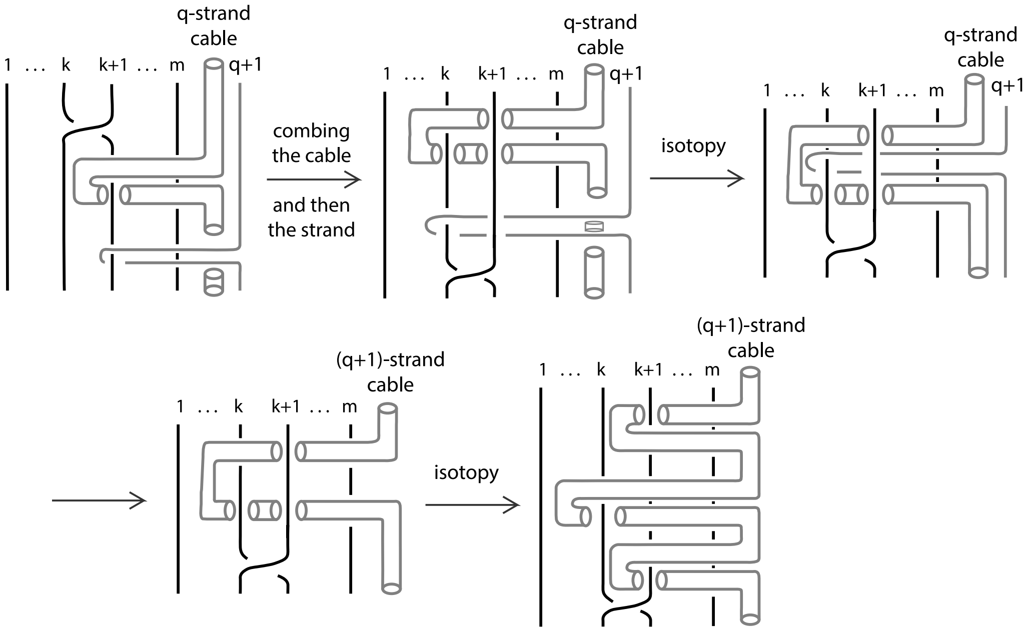

Fig. 23. Combing and cabling commute: Proof of Case 2.

We now consider a $(q+1)$-strand cable and we let the first q strands form a q-strand subcable. We first comb the q-strand cable and then the $(q+1)$st strand and the result follows by applying the case of a 2-strand cable and the induction hypothesis for the q-strand cable (see Fig. 23). □

6. Algebraic $\mathbb{Q}$-mixed braid equivalence

Let now $B \bigcup \beta$ be a parted mixed braid and let a parted $\mathbb{Q}$-braid band move be performed on the last strand, s_k, of a surgery component consisting of the strands $s_1, \ldots, s_k$. Recall Fig. 17. In order to give an algebraic expression for the parted $\mathbb{Q}$-braid band move, we part locally the subbraids d' and $c'_{\pm}$ and the loop generators a_i, $i = 1, \ldots, m$, and we use mixed braid isotopy in order to transform d' into d and $c'_{\pm}$ into $c_{\pm}$. See Figs. 24, 25, 26. Then, d has the algebraic expression:

$$d \ = \ [\ \lambda_{n+kq-1,n+(k-1)q+1}\ \lambda^{-1}_{n+1,n+(k-1)q}\ \lambda_{n,1}\ a_{s_k}\ \lambda^{-1}_{n,1}\ \lambda^{-1}_{n+1,n+(k-1)q}\ ]^p \tag{3}$$

and $c_{\pm}$ has the algebraic expression:

$$c_{\pm} \ = \ \lambda_{n,n+kq-2}\ \sigma^{\pm 1}_{n+kq-1}\ \lambda^{-1}_{n,n+kq-2}. \tag{4}$$

We are now in the position to give the definition of an algebraic $\mathbb{Q}$-braid band move.

Definition 5. (i) An *algebraic $\mathbb{Q}$-braid band move* is defined to be a parted $\mathbb{Q}$-braid band move between elements of $B_{n,\infty}$ and it has the following algebraic expression:

$$\beta \sim d\ c_{\pm}\ \beta',$$

where β' is the algebraic mixed braid β with the substitutions:

$$\begin{aligned} {a_i}^{\pm 1} &\longleftrightarrow {a_i}^{\pm 1}, \quad \text{for } i > s_k, \\ {a_i}^{\pm 1} &\longleftrightarrow \lambda^{-1}_{n-1,1}\lambda_{n,n+kq-1}\lambda_{n+kq-1,1}\ {a_i}^{\pm 1} \\ &\qquad \lambda^{-1}_{n-1,1}\lambda^{-1}_{n+kq-1,n}\lambda^{-1}_{n,n+kq-1}\lambda_{n-1,1}, \ \text{for } i < s_1, \end{aligned}$$

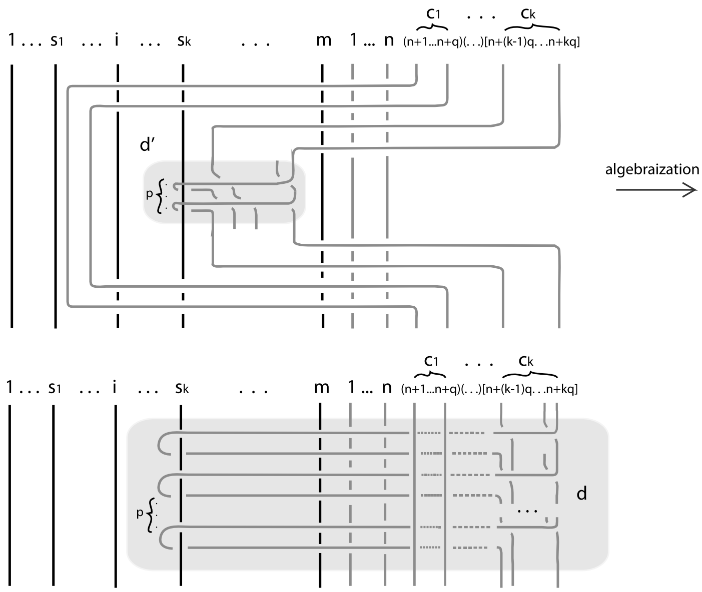

Fig. 24. Algebraization of the (p, q)-torus braid d' to d after a $\mathbb{Q}$-braid band move is performed.

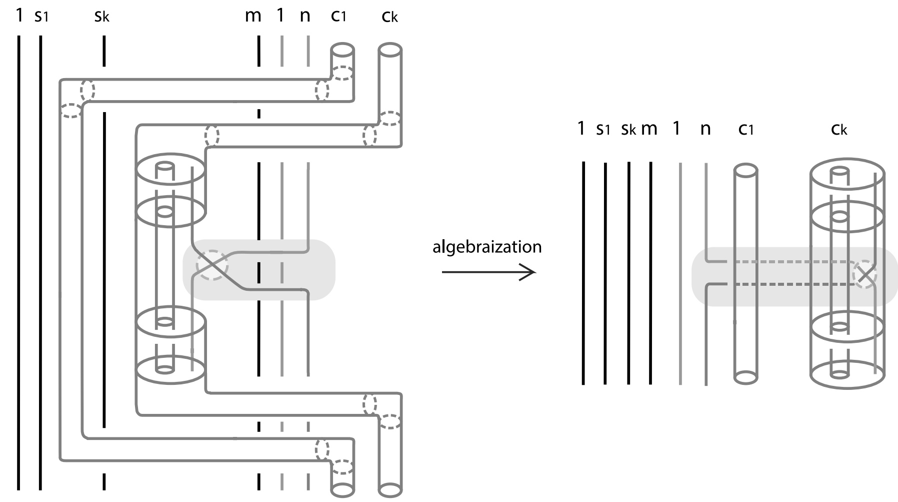

Fig. 25. Algebraization of the crossing part $c'_{\pm}$ to $c_{\pm}$ after a $\mathbb{Q}$-braid band move is performed.

$$
\begin{aligned}
a_{s_j} \;\longleftrightarrow\;& \lambda^{-1}_{n-1,1}\lambda_{n,n+kq-1}\lambda_{n+kq-1,n+(j-1)q}\lambda^{-1}_{n,n+(j-1)q-1}\lambda_{n-1,1}\; a_{s_j}\\
& \lambda^{-1}_{n-1,1}\lambda_{n,n+jq-1}\lambda^{-1}_{n+kq-1,n+jq}\lambda^{-1}_{n,n+kq-1}\lambda_{n-1,1}\\
a^{-1}_{s_j} \;\longleftrightarrow\;& \lambda^{-1}_{n-1,1}\lambda_{n,n+kq-1}\lambda_{n+kq-1,n+jq}\lambda^{-1}_{n,n+jq-1}\lambda_{n-1,1}\; a^{-1}_{s_j}\\
& \lambda^{-1}_{n-1,1}\lambda_{n,n+(j-1)q-1}\lambda^{-1}_{n+kq-1,n+(j-1)q}\lambda^{-1}_{n+kq-1,n}\lambda_{n-1,1},\\
& \text{for } s_j \in \{s_1, \ldots, s_k\},\\
a^{\pm 1}_{j} \;\longleftrightarrow\;& \lambda^{-1}_{n-1,1}\lambda_{n,n+kq-1}\lambda_{n+kq-1,n+(r-1)q}\lambda^{-1}_{n,n+(r-1)q-1}\lambda_{n-1,1}a^{\pm 1}_{j}\\
& \lambda^{-1}_{n-1,1}\lambda_{n,n+(r-1)q-1}\lambda^{-1}_{n+kq-1,n+(r-1)q}\lambda^{-1}_{n,n+kq-1}\lambda_{n-1,1}, \quad \text{for } s_{r-1} < j < s_r.
\end{aligned}
$$

(ii) A *combed algebraic $\mathbb{Q}$-braid band move* is a move between algebraic mixed braids and is defined to be a parted $\mathbb{Q}$-braid band move that has been combed through B. Moreover, it has the following algebraic expression:

$$\beta \sim d\; c_{\pm}\; \beta'\; comb_B(c_1, \ldots, c_k),$$

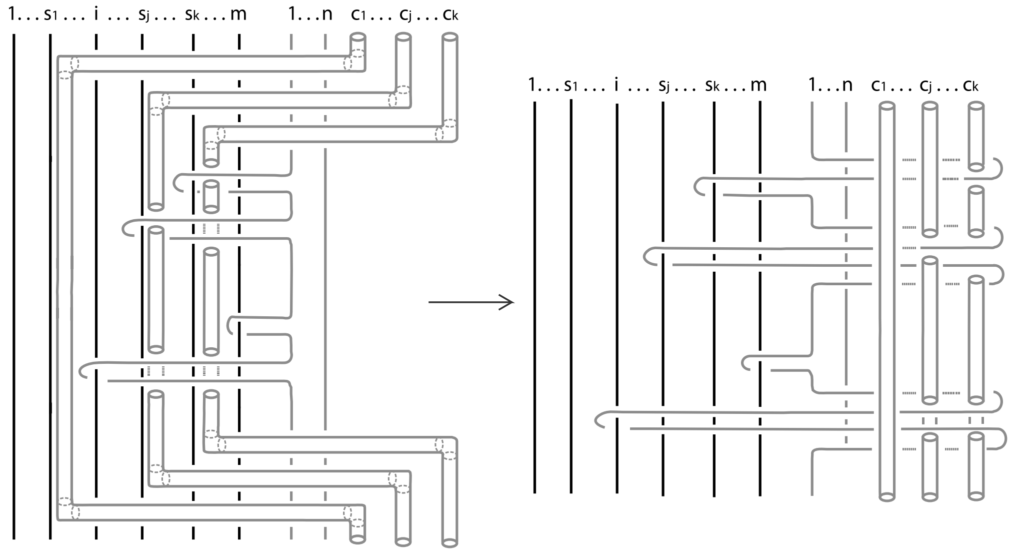

Fig. 26. Algebraization of the loop generators a_j after a $\mathbb{Q}$-braid band move is performed.

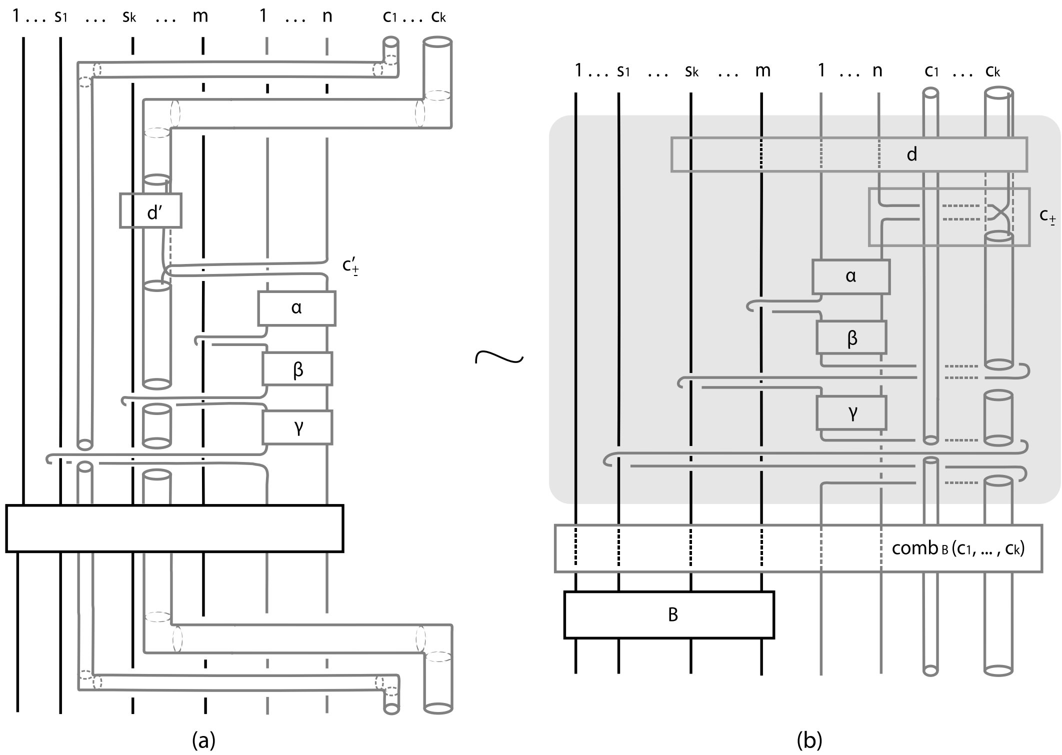

Fig. 27. Combing a parted $\mathbb{Q}$-braid band move results in an algebraic $\mathbb{Q}$-braid band move followed by its combing.

where $comb_B(c_1, \ldots, c_k)$ is the combing of the parted q-strand cables $c_1, \ldots, c_k$ through the surgery braid B (see Fig. 27).

We are, finally, in the position to state the following main result of the paper.

Theorem 9 *(Algebraic mixed braid equivalence for $M = \chi_{\mathbb{Q}}(S^3, \widehat{B})$). Let $s_1, \ldots, s_k$ be the numbers of the strands of a surgery component s and let $c_1, \ldots, c_k$ be the corresponding q-strand cables arising from a*

$\mathbb{Q}$-braid band move performed on s. Then, two oriented links in M are isotopic if and only if any two corresponding algebraic mixed braid representatives in $B_{m,\infty}$ differ by a finite sequence of the following moves:

(i) *M-moves: $\beta_1\beta_2 \sim \beta_1\sigma_n^{\pm 1}\beta_2$, for $\beta_1, \beta_2 \in B_{m,n}$,*
(ii) *M-conjugation: $\beta \sim \sigma_j^{\mp 1}\beta\sigma_j^{\pm 1}$, for $\beta, \sigma_j \in B_{m,n}$,*
(iii) *Combed loop conjugation: $\beta \sim \alpha_i^{\mp 1}\beta\ \rho_i^{\pm 1}$, for $\beta \in B_{m,n}$, where ρ_i is the combing of the loop α_i through B,*
(iv) *Combed algebraic braid band moves: $\beta\ \sim\ d\, c_{\pm}\ \beta'\ comb_B(c_1, \ldots, c_k)$, where the algebraic expressions of d and $c_{\pm}$ are as in Eqs. (3) and (4) respectively, β' is β with the substitutions of the loop generators as in Definition 5 and $comb_B(c_1, \ldots, c_k)$ is the combing of the resulting q-strand cables $c_1, \ldots, c_k$ through the fixed subbraid B. Equivalently, by the same moves as above, where (i) and (ii) are replaced by algebraic L-moves (see algebraic expressions in Eqs. (2)).*

Proof. The arguments for passing from parted braid equivalence (Theorem 8) to algebraic braid equivalence are the same as in those in the proof of the transition from Theorem 3 to Theorem 4 in the case of integral surgery. The only part we need to analyze in detail is the algebraization of a parted $\mathbb{Q}$-braid band move. Namely, we will show that the following diagram commutes.

$$\begin{array}{ccc}
C_{m,n} \ni B\bigcup\beta & \xrightarrow{\text{Parted Q-b.b.m.}} & B\bigcup\beta' \in C_{m,n+kq} \\
\| & & \| \\
comb_B(\beta) & & comb_B(\beta') \\
\downarrow & & \downarrow \\
B_{m,n} \ni alg_B(\beta) & \xrightarrow{\text{Algebraic Q-b.b.m.}} & alg_B(\beta') \in B_{m+n+kq}
\end{array}$$

In words, we start with a parted mixed braid $B\bigcup\beta \in C_{m,n}$ and we perform on it a parted $\mathbb{Q}$-braid band move (Definition 4) obtaining a parted mixed braid $B\bigcup\beta' \in C_{m,n+kq}$, where k is the number of strands forming the surgery component. We then comb both parted mixed braids obtaining $comb_B(\beta)$ and $comb_B(\beta')$ respectively. We will show that the corresponding algebraic parts, $alg_B(\beta) \in B_{m,n}$ and $alg_B(\beta') \in B_{m,n+kq}$ differ by the algebraic braid equivalence given in the statement of the theorem. We apply Lemma 8 in [13], where the q strands of a braid band move are placed in the cable and the cable is treated as one strand. More precisely, we note that the parted $\mathbb{Q}$-braid band move takes place at the top of the braid, so it forms an algebraic $\mathbb{Q}$-braid band move. We now comb away β to the top of B and on the other side we comb away β'. Since the q-strands cable of the parted $\mathbb{Q}$-braid band move lie very close to the surgery strands, this ensures that the loops $\alpha_j^{\pm 1}$ around any strand of the k strands of the specific surgery components get combed in the same way before and after the $\mathbb{Q}$-braid band move. So, having combed away β we are left at the bottom with the identity moving braid on the one hand, and with the combing of all cables of the braid band move on the other hand, which is precisely what we denote $comb_B()$. Finally, by Lemma 4, combing and cabling commute. Thus, the Theorem is proved. □

7. Applications

In this section we give the braid equivalences for knots in specific families of 3-manifolds that play a very important role in 3-dimensional topology, such as the lens spaces $L(p, q)$, homology spheres and Seifert manifolds. It is worth mentioning, in general, that any framed link gives rise to a whole family of 3-manifolds

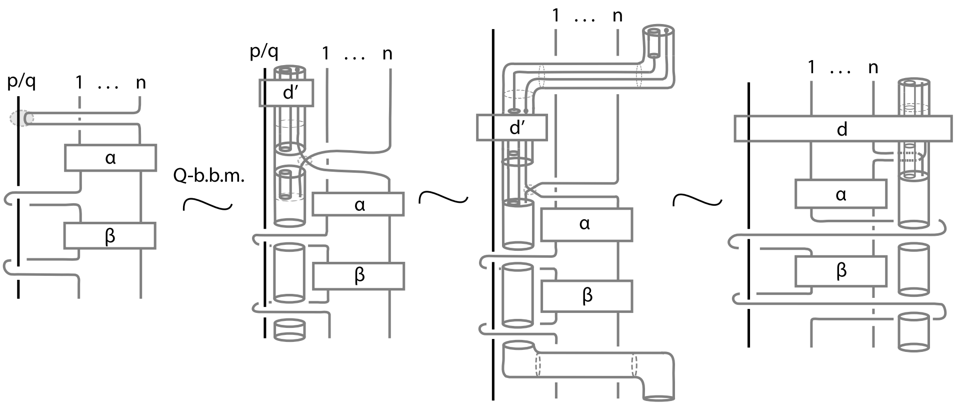

Fig. 28. A $\mathbb{Q}$-braid band move in $L(p,q)$ and its algebraization.

obtained from different rational surgeries along the link. This approach sets the ground for a homogeneous treatment for studying the knot theory of 3-manifolds, for example the skein modules of oriented 3-manifolds with or without boundary.

7.1. Lens spaces $L(p,q)$

It is known that the lens spaces $L(p,q)$ can be obtained by surgery on the unknot with surgery coefficient p/q. So, the fixed braid $\widehat{B}$ that represents $L(p,q)$ is the identity braid of one single strand and thus, no combing is needed (see Fig. 28). We have the following (compare with [13, §4]):
Two oriented links in $L(p,q)$ are isotopic if and only if any two corresponding algebraic mixed braids in $B_{1,\infty}$ differ by a finite sequence of the moves given in Theorem 9, where in particular:
(iv) Algebraic braid band moves: For $\beta \in B_{1,n}$ we have: $\beta \sim d\, c_{\pm}\, \beta'$, where:

$$d = [\lambda_{n+q-1,1}\ a_1\ \lambda^{-1}_{1,n+q-1}]^p,\ c_{\pm} = \lambda_{n,n+q-1}\ \sigma^{\pm 1}_{n+q-1}\ \lambda^{-1}_{n,n+q-1},$$

and where $\beta' \in B_{1,n+q}$ is the word β with the substitutions:

$$a_1 \longleftrightarrow (\lambda^{-1}_{n-1,1}\ \lambda_{n,n+q-1}\ \lambda_{n+q-1,1})\ a_1, \text{ and } {a_1}^{-1} \longleftrightarrow {a_1}^{-1}\ (\lambda^{-1}_{n+q-1,1}\ \lambda^{-1}_{n,n+q-1}\ \lambda_{n-1,1}).$$

7.2. Homology spheres

It is known that a Dehn surgery on a knot yields a homology sphere exactly when the surgery coefficient is the reciprocal of an integer (see [15], p. 262). For example, surgery on the right-handed trefoil, with surgery coefficient -1 yields the Poincare Manifold also known as dodecahedral space (for the algebraic braid equivalence in this case see [13, §4]). In this subsection we give the algebraic braid equivalence for knots in a homology sphere M obtained from S^3 by surgery on the trefoil knot with rational surgery coefficient $1/q$, where $q \in \mathbb{Z}$. As explained in [16] if one used integral surgery description, one would need a different knot for each q.

Two oriented links in M are isotopic if and only if any two corresponding algebraic mixed braids in $B_{2,\infty}$ differ by a finite sequence of the moves given in Theorem 9, where in particular:

(iv) Combed algebraic braid band moves: $\beta \sim d\ c_{\pm}\ \beta'\ comb_B(c_1, c_2)$, where: $\beta \in B_{2,n}$,

$$d = (\lambda_{n+2q-1,n+q+1}\ \lambda^{-1}_{n+1,n+q}\ \lambda_{n,1})\ a_2\ (\lambda^{-1}_{n,1}\ \lambda_{n+1,n+q}),$$

$$c_{\pm} = \lambda_{n,n+2q-1}\ \sigma^{\pm 1}_{n+2q-1}\ \lambda^{-1}_{n,n+2q-1},$$

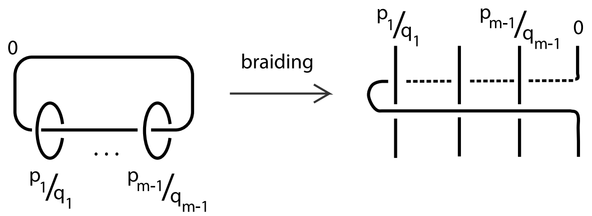

Fig. 29. Surgery description of a Seifert manifold.

β' is the word β with the substitutions:

$$
\begin{aligned}
a_1 &\longleftrightarrow (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+2q-1}\ \lambda_{n+2q-1,n+q}\ \lambda_{n,n+q-1}^{-1}\ \lambda_{n-1,1})a_1,\\
a_1^{-1} &\longleftrightarrow a_1^{-1}\ (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+q-1}\ \lambda_{n+2q-1,n+q}^{-1}\ \lambda_{n,n+2q-1}^{-1}\ \lambda_{n-1,1}),\\
a_2 &\longleftrightarrow (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+2q-1}\ \lambda_{n+2q-1,1})\ a_2\\
&\quad\ (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+q-1}\ \lambda_{n+2q-1,n+q}^{-1}\ \lambda_{n,n+2q-1}^{-1}\ \lambda_{n-1,1}),\\
a_2^{-1} &\longleftrightarrow (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+2q-1}\ \lambda_{n+2q-1,n+q}\ \lambda_{n,n+q-1}^{-1}\ \lambda_{n-1,1})\ a_2^{-1}\\
&\quad\ (\lambda_{n+2q-1,1}^{-1}\ \lambda_{n,n+2q-1}^{-1}\ \lambda_{n-1,1}),
\end{aligned}
$$

and $comb_B(c_1, c_2)$ is the combing of the q-strand cables (c_1 and c_2) through the fixed braid:

$$
\begin{aligned}
comb_B(c_1,c_2) = {} & \prod_{i=0}^{q-1} \lambda_{n+i,1}\ a_2\ \lambda_{n+i,1}^{-1}\ \prod_{i=0}^{q-1} \lambda_{n+2q-1-i,1}\ a_2^{-1}\ \lambda_{n+2q-1-i,1}^{-1}\\
& \prod_{i=0}^{q-1} \lambda_{n+q+i,1}\ a_1\ \lambda_{n+q+i,1}^{-1}\ \lambda_{n+q,1}\ a_2\ \lambda_{n,1}^{-1}\ \lambda_{n+1,n+q}\\
& \prod_{i=1}^{q-1} \lambda_{n+q+i,1}\ a_2\ \lambda_{n,1}^{-1}\ \lambda_{n+1,n+q}\ \lambda_{n+q+i,n+q+1}^{-1}\\
& \prod_{i=0}^{q-1} \lambda_{n+q-1-i,1}\ a_2\ \lambda_{n+q-1-i,1}^{-1}\ \prod_{i=0}^{q-1} \lambda_{n+i,1}\ a_1\ \lambda_{n+i,1}^{-1}\\
& \prod_{i=0}^{q-1} \lambda_{n+i,1}\ a_2\ \lambda_{n+i,1}^{-1}\ \prod_{i=0}^{q-1} \lambda_{n+i,1}\ a_1\ \lambda_{n+i,1}^{-1}\\
& \prod_{i=0}^{q-1} \lambda_{n+i,1}\ a_2\ \lambda_{n+i,1}^{-1}\ \prod_{i=0}^{q-1} \lambda_{n+q+i,n+1+i}.
\end{aligned}
$$

7.3. Seifert manifolds

It is known that a Seifert manifold $M((p_1, q_1), \ldots, (p_{m-1}, q_{m-1}))$ has a rational surgery description as shown in Fig. 29 (see [17], p. 33).

Two oriented links in a Seifert manifold $M((p_1, q_1), \ldots, (p_{m-1}, q_{m-1}))$ are isotopic if and only if any two corresponding algebraic mixed braids differ by a finite sequence of the moves given in Theorem 9, where in particular:

(iv) Combed algebraic braid band moves: For $\beta \in B_{m,n}$ we distinguish the cases:

- *If a $\mathbb{Q}$-braid band move is performed on the jth strand of the fixed braid with rational coefficient p/q (see Fig. 30) then: $\beta \sim d\ c_{\pm}\ \beta'\ comb_B(c_j)$, where $comb_B(c_j)$ is the combing of the c_j cable through B,*

$$d = [\lambda_{n+q-1,1}\ \alpha_i\ \lambda_{n-1,1}^{-1}]^p \quad \text{and} \quad c_{\pm} = \lambda_{n,n+q-1}\ \sigma_{n+q-1}^{-1}\ \lambda_{n,n+q-1}^{-1},$$

and where β' is β with the substitutions:

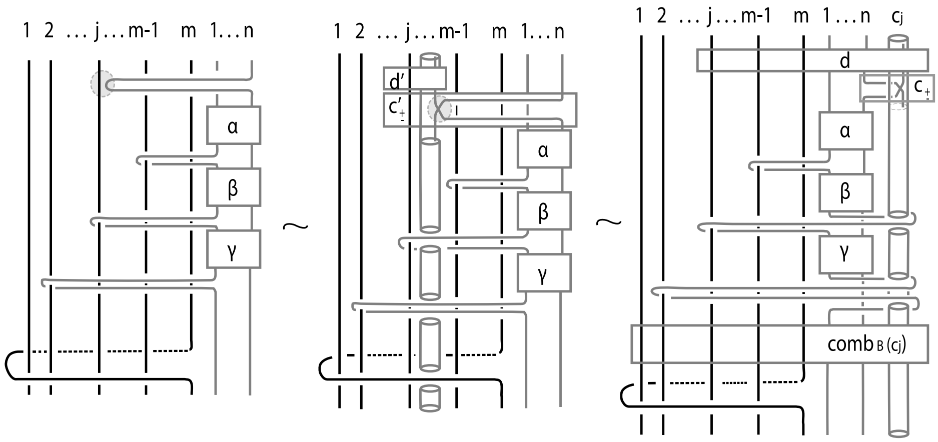

Fig. 30. A $\mathbb{Q}$-braid band move in a Seifert manifold and its algebraic expression.

$$
\begin{aligned}
&\text{For } i > j : a_i^{\pm 1} \longleftrightarrow a_i^{\pm 1},\\
&\text{For } i < j : a_i^{\pm 1} \longleftrightarrow \lambda_{n-1,1}^{-1}\ \lambda_{n,n+q-1}\ \lambda_{n+q-1,1}\ a_i^{\pm 1}\lambda_{n+q-1,1}^{-1}\ \lambda_{n,n+q-1}^{-1}\ \lambda_{n-1,1},\\
&\text{For } i = j : a_j \longleftrightarrow \lambda_{n-1,1}^{-1}\ \lambda_{n,n+q-1}\ \lambda_{n+q-1,1}\ a_j, \text{ and}\\
&\qquad\qquad a_j^{-1} \longleftrightarrow a_j^{-1}\lambda_{n+q-1,1}^{-1}\ \lambda_{n,n+q-1}^{-1}\ \lambda_{n-1,1}.
\end{aligned}
$$

- *If a $\mathbb{Q}$-braid band move is performed on the last strand of the fixed braid with surgery coefficient* 0*, then: $\beta \sim \sigma_n^{\pm 1}\ \beta'$, where β' is β with the substitutions:*

$$
\begin{aligned}
a_j^{\pm 1} &\longleftrightarrow \lambda_{n-1,1}^{-1}\ \sigma_n^2\ \lambda_{n-1,1}a_j^{\pm 1}\ \lambda_{n,1}^{-1}\ \sigma_n^{-1}\ \lambda_{n-1,1}, \text{ for } j = 1, \ldots, m-1,\\
a_m &\longleftrightarrow \lambda_{n-1,1}^{-1}\ \sigma_n^2\ \lambda_{n-1,1}\ a_m,\\
a_m^{-1} &\longleftrightarrow a_m^{-1}\ \lambda_{n-1,1}^{-1}\ \sigma_n^{-2}\ \lambda_{n-1,1}.
\end{aligned}
$$

7.4. Rational surgery along a torus knot

It is well-known that a manifold M obtained by rational surgery from S^3 along an (m, r)-torus knot with rational coefficient p/q is either the lens space $L(|q|, pr^2)$, or the connected sum of two lens spaces $L(m, r)\sharp L(r, m)$, or a Seifert manifold (for more details the reader is referred to [14]). For links in M we have:

Two oriented links in M are isotopic if and only if any two corresponding algebraic mixed braids differ by a finite sequence of the moves given in Theorem 9, where in particular: (iv) Combed algebraic braid band moves: For $\beta \in B_{m,n}$ we have:

$$\beta \sim d\ c_{\pm}\ \beta'\ comb_B(c_1, \ldots, c_m),$$

where

$$
\begin{aligned}
d &= [\ \lambda_{n+mq-1,n+(m-1)q+1}\ \lambda_{n,n+(m-1)q}^{-1}\ \lambda_{n-1,1}\ \alpha_j\ \lambda_{n-1,1}^{-1}\ \lambda_{n,n+(m-1)q}\ ]^p,\\
c_{\pm} &= \lambda_{n,n+mq-2}\ \sigma_{n+mq-1}^{\pm 1}\ \lambda_{n,n+mq-2}^{-1},
\end{aligned}
$$

$comb_B(c_1, \ldots, c_m)$ is the combing through the fixed braid of the parted moving cables parallel to the surgery strands and β' is the word β with the substitutions:

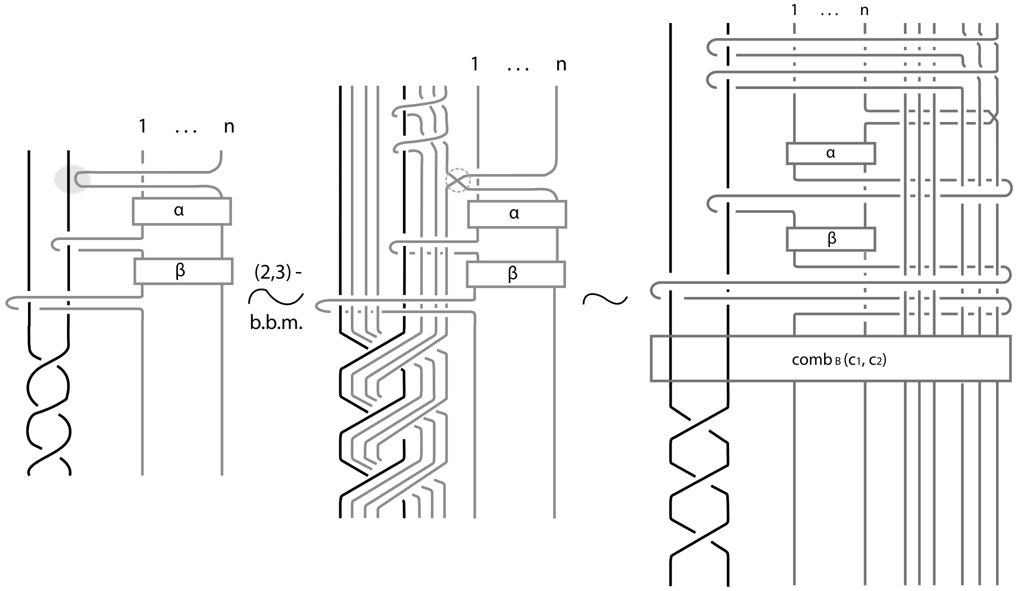

Fig. 31. Turning the geometric $(2, 3)$-braid band move into a combed algebraic $(2, 3)$-braid band move.

$$
\begin{aligned}
a_j \;&\longleftrightarrow\; (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+mq-1}\ \lambda_{n+mq-1,n+(j-1)q}\ \lambda_{n,n+(j-1)q-1}^{-1}\ \lambda_{n-1,1})\ a_j \\
&\qquad (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+jq-1}\ \lambda_{n+mq-1,n+jq}^{-1}\ \lambda_{n,n+mq-1}^{-1}\ \lambda_{n-1,1}), \\
a_j^{-1} \;&\longleftrightarrow\; (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+mq-1}\ \lambda_{n+mq-1,n+jq}\ \lambda_{n,n+jq-1}^{-1}\ \lambda_{n-1,1})\ a_j^{-1} \\
&\qquad (\lambda_{n-1,1}^{-1}\ \lambda_{n,n+(j-1)q-1}\ \lambda_{n+mq-1,n+(j-1)q}^{-1}\ \lambda_{n,n+mq-1}^{-1}\ \lambda_{n-1,1}),\ \text{for } j \in \{1, \ldots, m\}.
\end{aligned}
$$

In Fig. 31 we illustrate an example where the (m, r)-torus knot is the $(2, 3)$-torus knot, $p = 2$ and $q = 3$ (see Proposition 3.1 in [14] for details about the manifold obtained).

7.5. Jones-type invariants and skein modules of 3-manifolds

Our braiding approach is particularly useful for constructing Jones-type invariants and for computing skein modules of 3-manifolds. Jones-type invariants (such as analogues of the Jones polynomial and the 2-variable Jones or Homflypt polynomial) for links in 3-manifolds can be constructed via Markov traces on appropriate quotient algebras (such as analogues of the Temperley-Lieb algebras and the Iwahori-Hecke algebras) of the related mixed braid groups $B_{m,n}$, which support Markov traces. This topological motivation gives rise to many new algebras worth studying. From the Markov trace rules one can obtain link invariants in the complement $S^3 \setminus \widehat{B}$. These invariants can be then extended to link invariants in the manifold $M = \chi(S^3, \widehat{B})$ by forcing them to satisfy all possible band moves. Now, these are more limited if one uses the braiding setting and our Theorem 9. A good example and the simplest one demonstrating the above is the case of the lens spaces $L(p, 1)$: in [11] the most generic analogue of the Homflypt polynomial, X, for links in the solid torus ST has been derived from the generalized Hecke algebras of type B via a unique Markov trace constructed on them. Hence, X is appropriate for extending the results to the lens spaces $L(p, q)$, since the combinatorial setting is the same as for ST, only the braid equivalence includes the $\mathbb{Q}$-braid band move, which reflects the surgery description of $L(p, q)$. For the case of $L(p, 1)$, in order to extend X to an invariant of links in $L(p, 1)$ in [1,4] we solve an infinite system of equations resulting from the braid band moves and we show that it has a unique solution. Namely we force:

$$X_{\widehat{\alpha}} = X_{bbm(\widehat{\alpha})},$$

for all $\alpha \in \bigcup_\infty B_{1,n}$ and for all possible slidings of α. The above equations have particularly simple formulations with the use of a new basis Λ for the Homflypt skein module of ST, that we give in [1,2]. These handle sliding equations are very controlled in the algebraic setting, because they can be performed only on the first moving strand. Further, the infinite system of these equations splits into finite self-contained subsystems. In [3] we use Section 7.1 on the general case of $L(p,q)$ where we have to solve equations of the invariant X which derive by attaching anywhere on a link a 2-handle along a (p,q)-curve. Further, in [8] the authors are working on connected sums of two lens spaces, constructing the appropriate quotient algebras of the mixed braid groups $B_{2,n}$ and a Markov trace on these algebras.

Our results can be also applied to the study of skein modules of c.c.o. 3-manifolds, using braid techniques. A *skein module* of a 3-manifold, characterized by a given property, is equivalent to finding all possible knot invariants in the 3-manifold characterized by the same property. We are particularly interested in Homflypt skein modules of 3-manifolds, although our approach can be also used for computing other skein modules of 3-manifolds such as Kauffman bracket skein modules. We note that the computation of a Homflypt skein module of a 3-manifold M with the use of diagrammatic methods is very complicated. The advantage of the algebraic setting is that it gives more control over the band moves than the diagrammatic approach and much of the diagrammatic complexity is absorbed into the proofs of the algebraic statements. We only need to consider one type of orientations patterns and the braid band moves are limited. To draw the analogy in the simplest situation: in [11] the Homflypt skein module of the solid torus $S(\mathrm{ST})$ ([23,6]) has been recovered from the invariant X mentioned above. $S(\mathrm{ST})$ is related to $S(L(p,q))$. The unique solution of the infinite system of the sliding equations satisfied by X reflects the freeness of $S(L(p,1))$. In [3] we work on computing $S(L(p,q))$. As a consequence of the above, in [3] we work on computing $S(L(p,q))$ in the general case using our results of Section 7.1.

7.6. Application to the equivalence of 3*-manifolds*

In [9] the authors prove a braid version of the Kirby calculus, namely an equivalence relation between framed braids that represent homeomorphic 3-manifolds. As mentioned in the introduction, although every c.c.o. 3-manifold can be obtained by integral surgery along a link L in S^3, it is sometimes more convenient to consider rational surgery description for a c.c.o. 3-manifold. Rolfsen [16] extended the Kirby calculus to rational surgery coefficients, giving rise to the *Rational calculus* and introducing a handle sliding move called *Rolfsen twist*. It would be useful to extend the result in [9] and derive the braid analogue of the Rolfsen calculus. The braid analogue of the Rolfsen twist is precisely the $\mathbb{Q}$-braid band move (Definition 5). The difference here is that there are no fixed and moving strands in the setting; all braids involved are surgery braids. Moreover, when applying a $\mathbb{Q}$-braid band move along a component, the framings of the strands involved, will change, as shown in [16]. The braid moves reflecting framed link isotopy in S^3 as well as the blow up move are the same as in [9]. The difficulty in carrying through the braid analogue for the Rational calculus lies in the following: Since Kirby calculus as well as Rational calculus are applied to non-oriented links in S^3, and since the orientation of a link L is crucial in order to obtain its braid representation, one has to consider additionally how the change of orientation of any component of L would alter the surgery braid. For the case of integral surgery, as shown in [9], one may unknot the component that the change of orientation will occur, by applying Fenn–Rourke moves, then change the orientation of the component, and finally undo all Fenn–Rourke moves applied before. The result is a link L', that differs from L by a change of orientation of one component. For the case of rational surgery, this is a very complicated problem and will be the subject of future research.

Combining the above with the Kauffman bracket skein module of a 3-manifold, our results could potentially lead to a uniform algebraic approach to the Witten invariants.

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