A new basis for the Homflypt skein module of the solid torus

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**A B S T R A C T**

In this paper we give a new basis, $\Lambda$, for the Homflypt skein module of the solid torus, $\mathcal{S}(ST)$, which topologically is compatible with the handle sliding moves and which was predicted by J.H. Przytycki. The basis $\Lambda$ is different from the basis $\Lambda'$, discovered independently by Hoste and Kidwell [1] and Turaev [2] with the use of diagrammatic methods, and also different from the basis of Morton and Aiston [3]. For finding the basis $\Lambda$ we use the generalized Hecke algebra of type $B$, $H_{1,n}$, which is generated by looping elements and braiding elements and which is isomorphic to the affine Hecke algebra of type $A$ [4]. More precisely, we start with the well-known basis $\Lambda'$ of $\mathcal{S}(ST)$ and an appropriate linear basis $\Sigma_n$ of the algebra $H_{1,n}$. We then convert elements in $\Lambda'$ to sums of elements in $\Sigma_n$. Then, using conjugation and the stabilization moves, we convert these elements to sums of elements in $\Lambda$ by managing gaps in the indices, by ordering the exponents of the looping elements and by eliminating braiding tails in the words. Further, we define total orderings on the sets $\Lambda'$ and $\Lambda$ and, using these orderings, we relate the two sets via a block diagonal matrix, where each block is an infinite lower triangular matrix with invertible elements in the diagonal. Using this matrix we prove linear independence of the set $\Lambda$, thus $\Lambda$ is a basis for $\mathcal{S}(ST)$. $\mathcal{S}(ST)$ plays an important role in the study of Homflypt skein modules of arbitrary c.c.o. 3-manifolds, since every c.c.o. 3-manifold can be obtained by integral surgery along a framed link in $S^3$ with unknotted components. In particular, the new basis of $\mathcal{S}(ST)$ is appropriate for computing the Homflypt skein module of the lens spaces. In this paper we provide some basic algebraic tools for computing skein modules of c.c.o. 3-manifolds via algebraic means.

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1. Introduction

Let $M$ be an oriented 3-manifold, $R = \mathbb{Z}[u^{\pm 1}, z^{\pm 1}]$, $\mathcal{L}$ the set of all oriented links in $M$ up to ambient isotopy in $M$ and let $S$ be the submodule of $R\mathcal{L}$ generated by the skein expressions $u^{-1}L_+ - uL_- - zL_0$,

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where $L_+, L_-$ and $L_0$ are oriented links that have identical diagrams, except in one crossing, where they are as depicted in Fig. 1.

For convenience we allow the empty knot, $\emptyset$, and add the relation $u^{-1}\emptyset - u\emptyset = zT_1$, where $T_1$ denotes the trivial knot. Then the Homflypt skein module of $M$ is defined to be:

$$S(M) = S(M; \mathbb{Z}[u^{\pm 1}, z^{\pm 1}], u^{-1}L_+ - uL_- - zL_0) = \mathcal{R}/\mathcal{S}.$$ 

Unlike the Kauffman bracket skein module, the Homflypt skein module of a 3-manifold, also known as Conway skein module and as third skein module, is very hard to compute (see [5] for the case of the product of a surface and the interval).

Let ST denote the solid torus. In [2,1] the Homflypt skein module of the solid torus has been computed using diagrammatic methods by means of the following theorem:

Theorem 1 (Turaev, Kidwell–Hoste). The skein module $S(ST)$ is a free, infinitely generated $\mathbb{Z}[u^{\pm 1}, z^{\pm 1}]$-module isomorphic to the symmetric tensor algebra $SR\hat{\pi}^0$, where $\hat{\pi}^0$ denotes the conjugacy classes of non-trivial elements of $\pi_1(ST)$.

A basic element of $S(ST)$ in the context of [2,1], is illustrated in Fig. 2. In the diagrammatic setting of [2] and [1], ST is considered as Annulus $\times$ Interval. The Homflypt skein module of ST is particularly important, because any closed, connected, oriented (c.c.o.) 3-manifold can be obtained by surgery along a framed link in $S^3$ with unknotted components.

A different basis of $S(ST)$, known as Young idempotent basis, is based on the work of Morton and Aiston [3] and Blanchet [6].

In [4], $S(ST)$ has been recovered using algebraic means. More precisely, the generalized Hecke algebra of type $B$, $H_{1,n}(q)$, is introduced, which is related to the affine Hecke algebra of type $A$, $\tilde{H}_n(q)$ [4]. Then, a unique Markov trace is constructed on the algebras $H_{1,n}(q)$ leading to an invariant for links in ST, the universal analogue of the Homflypt polynomial for ST. This trace gives distinct values on distinct elements of the $[2,1]$-basis of $S(ST)$. The link isotopy in ST, which is taken into account in the definition of the skein module and which corresponds to conjugation and the stabilization moves on the braid level, is captured by the conjugation property and the Markov property of the trace, while the defining relation of the skein module is reflected into the quadratic relation of $H_{1,n}(q)$. In the algebraic language of [4] the basis of $S(ST)$, described in Theorem 1, is given in open braid form by the set $\Lambda'$ in Eq. (4). Fig. 8 illustrates the basic
element of Fig. 2 in braid notation. Note that in the setting of [4] ST is considered as the complement of the unknot (the bold curve in the figure). The looping elements \( t'_i \in H_{1,n}(q) \) in the monomials of \( \Lambda' \) are all conjugates, so they are consistent with the trace property and they enable the definition of the trace via simple inductive rules.

In this paper we give a new basis \( \Lambda \) for \( S(ST) \), which was predicted by J.H. Przytycki, using the algebraic methods developed in [4] (see Fig. 3). The motivation of this work is the computation of \( S(L(p,q)) \) via algebraic means. The new basic set is described in Eq. (1) in open braid form (see Fig. 9). The looping elements \( t_i \) are in the algebras \( H_{1,n}(q) \) and they are commuting. For a comparative illustration and for the defining formulas of the \( t_i \)'s and the \( t'_i \)'s the reader is referred to Fig. 7 and Eq. (3) respectively. Moreover, the \( t_i \)'s are consistent with the handle sliding move or band move used in the link isotopy in \( L(p,q) \), in the sense that a braid band move can be described naturally with the use of the \( t_i \)'s (see for example [7] and references therein).

Our main result is the following:

**Theorem 2.** The following set is a \( \mathbb{Z}[q^\pm 1, z^\pm 1] \)-basis for \( S(ST) \):

\[
\Lambda = \{ t^{k_0} t_1^{k_1} \cdots t_n^{k_n} \, ; \, k_i \in \mathbb{Z} \setminus \{0\} , \, k_i \geq k_{i+1} \, \forall i , \, n \in \mathbb{N} \}.
\]

(1)

Our method for proving Theorem 2 is the following:

- We define total orderings in the sets \( \Lambda' \) and \( \Lambda \),
- we show that the two ordered sets are related via a lower triangular infinite matrix with invertible elements on the diagonal, and
- using this matrix, we show that the set \( \Lambda \) is linearly independent.

More precisely, two analogous sets, \( \Sigma_n \) and \( \Sigma'_n \), are given in [4] as linear bases for the algebra \( H_{1,n}(q) \). See Theorem 4 in this paper. The set \( \bigcup_n \Sigma_n \) includes \( \Lambda \) as a proper subset and the set \( \bigcup_n \Sigma'_n \) includes \( \Lambda' \) as a proper subset. The sets \( \Sigma_n \) come directly from the works of S. Ariki and K. Koike, and M. Broué and G. Malle on the cyclotomic Hecke algebras of type B. See [4] and references therein. The second set \( \bigcup_n \Sigma'_n \) includes \( \Lambda' \) as a proper subset. The sets \( \Sigma'_n \) appear naturally in the structure of the braid groups of type B, \( B_{1,n} \); however, it is very complicated to show that they are indeed basic sets for the algebras \( H_{1,n}(q) \). The sets \( \Sigma_n \) play an intrinsic role in the proof of Theorem 2. Indeed, when trying to convert a monomial \( \lambda' \) in \( \Lambda' \) into a linear combination of elements in \( \Lambda \) we pass by elements of the sets \( \Sigma_n \). This means that in the converted expression of \( \lambda' \) we have monomials in the \( t_i \)'s with possible gaps in the indices and possible non-ordered exponents, followed by monomials in the braiding generators \( g_i \). So, in order to reach expressions in the set \( \Lambda \) we need:

- to manage the gaps in the indices of the \( t_i \)'s,
- to order the exponents of the \( t_i \)'s and
- to eliminate the braiding 'tails'.

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Fig. 4. A mixed link in $S^3$.

The paper is organized as follows. In Section 2 we recall the algebraic setting and the results needed from [4]. In Section 3 we define the orderings in the two sets $\Sigma_n$ and $\Sigma'_n$, which include the sets $\Lambda$ and $\Lambda'$ as subsets, and we prove that these sets are totally ordered. In Section 4 we prove a series of lemmas for converting elements in $\Lambda'$ to elements in the sets $\Sigma_n$. In Section 5 we convert elements in $\Sigma_n$ to elements in $\Lambda$ using conjugation and the stabilization moves. Finally, in Section 6 we prove that the sets $\Lambda'$ and $\Lambda$ are related through a lower triangular infinite matrix mentioned above and that the set $\Lambda$ is linearly independent. A computer program converting elements in $\Lambda'$ to elements in $\Sigma_n$ has been developed by K. Karvounis and will be soon available on http://www.math.ntua.gr/~sofia.

The algebraic techniques developed here will serve as basis for computing Homflypt skein modules of arbitrary c.c.o. 3-manifolds using the braid approach. The advantage of this approach is that we have an already developed homogeneous theory of braid structures and braid equivalences for links in c.c.o. 3-manifolds [8,9,7]. In fact, these algebraic techniques are used and developed further in [10] for knots and links in 3-manifolds represented by the 2-unlink.

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2. The algebraic settings

2.1. Mixed links in $S^3$

We now view ST as the complement of a solid torus in $S^3$. An oriented link $L$ in ST can be represented by an oriented mixed link in $S^3$ (see Fig. 4), that is a link in $S^3$ consisting of the unknotted fixed part $\hat{I}$ representing the complementary solid torus in $S^3$ and the moving part $L$ that links with $\hat{I}$.

A mixed link diagram is a diagram $\hat{I} \cup \hat{L}$ of $\hat{I} \cup L$ on the plane of $\hat{I}$, where this plane is equipped with the top-to-bottom direction of $I$.

Consider now an isotopy of an oriented link $L$ in ST. As the link moves in ST, its corresponding mixed link will change in $S^3$ by a sequence of moves that keep the oriented $\hat{I}$ pointwise fixed. This sequence of moves consists in isotopy in the $S^3$ and the mixed Reidemeister moves. In terms of diagrams we have the following result for isotopy in ST:

The mixed link equivalence in $S^3$ includes the classical Reidemeister moves and the mixed Reidemeister moves, which involve the fixed and the standard part of the mixed link, keeping $\hat{I}$ pointwise fixed.

2.2. Mixed braids in $S^3$

By the Alexander theorem for knots in solid torus, a mixed link diagram $\hat{I} \cup \hat{L}$ of $\hat{I} \cup L$ may be turned into a mixed braid $I \cup \beta$ with isotopic closure (see Fig. 5). This is a braid in $S^3$ where, without loss of generality, its first strand represents $\hat{I}$, the fixed part, and the other strands, $\beta$, represent the moving part $L$. The subbraid $\beta$ shall be called the moving part of $I \cup \beta$. 
Fig. 5. The closure of a mixed braid to a mixed link.

Fig. 6. The generators of $B_{1,n}$.

The sets of braids related to the ST form groups, which are in fact the Artin braid groups of type B, denoted $B_{1,n}$, with presentation:

$$B_{1,n} = \left< t, \sigma_1, \ldots, \sigma_{n-1} \middle| \begin{array}{l}
\sigma_1 t \sigma_1 t = t \sigma_1 t \sigma_1 \\
\sigma_1 t = \sigma_1 t, \quad i > 1 \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n-2 \\
\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1
\end{array} \right>,$$

where the generators $\sigma_i$ and $t$ are illustrated in Fig. 6.

Isotopy in ST is translated on the level of mixed braids by means of the following theorem.

**Theorem 3.** (See [11, Theorem 3].) Let $L_1$, $L_2$ be two oriented links in ST and let $I \cup \beta_1$, $I \cup \beta_2$ be two corresponding mixed braids in $S^3$. Then $L_1$ is isotopic to $L_2$ in ST if and only if $I \cup \beta_1$ is equivalent to $I \cup \beta_2$ in $\bigcup_{n=1}^{\infty} B_{1,n}$ by the following moves:

(i) **Conjugation:** $\alpha \sim \beta^{-1} \alpha \beta$, if $\alpha, \beta \in B_{1,n}$.

(ii) **Stabilization moves:** $\alpha \sim \alpha \sigma_n^{\pm 1} \in B_{1,n+1}$, if $\alpha \in B_{1,n}$.

2.3. The generalized Iwahori–Hecke algebra of type B

It is well known that $B_{1,n}$ is the Artin group of the Coxeter group of type B, which is related to the Hecke algebra of type B, $H_n(q,Q)$ and to the cyclotomic Hecke algebras of type B. In [4] it has been established that all these algebras form a tower of B-type algebras and are related to the knot theory of ST. The basic one is $H_n(q,Q)$, a presentation of which is obtained from the presentation of the Artin group $B_{1,n}$ by adding the quadratic relations

$$g_i^2 = (q-1)g_i + q \quad (2)$$

and the relation $t^2 = (Q-1)t + Q$, where $q, Q \in \mathbb{C}\setminus\{0\}$ are seen as fixed variables. The middle B-type algebras are the cyclotomic Hecke algebras of type B, $H_n(q,d)$, whose presentations are obtained by the quadratic relation (2) and $t^d = (t - u_1)(t - u_2) \cdots (t - u_d)$. The topmost Hecke-like algebra in the tower is
the generalized Iwahori–Hecke algebra of type B, \( H_{1,n}(q) \), which, as observed by T. tom Dieck, is related to the affine Hecke algebra of type A, \( H_n(q) \) (cf. [4]). The algebra \( H_{1,n}(q) \) has the following presentation:

\[
H_{1,n}(q) = \left\{ t, g_1, \ldots, g_{n-1} \mid \begin{align*}
g_1 t g_1 &= t g_1 t g_1 \\
t g_i &= g t, \quad i > 1 \\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1}, \quad 1 \leq i \leq n - 2 \\
g_i g_j &= g_j g_i, \quad |i - j| > 1 \\
g_i^2 &= (q - 1) g_i + q, \quad i = 1, \ldots, n - 1
\end{align*} \right\}.
\]

That is:

\[
H_{1,n}(q) = \frac{\mathbb{Z}[q^{\pm 1}]}{<\sigma_1^2 - (q - 1) \sigma_1 - q>}. \tag{3}
\]

Note that in \( H_{1,n}(q) \) the generator \( t \) satisfies no polynomial relation, making the algebra \( H_{1,n}(q) \) infinite dimensional. Also that in [4] the algebra \( H_{1,n}(q) \) is denoted as \( H_n(q, \infty) \).

In [12] V.F.R. Jones gives the following linear basis for the Iwahori–Hecke algebra of type A, \( H_n(q) \):

\[
S = \left\{ (g_{i_1} g_{i_1 - 1} \cdots g_{i_1 - k_1}) (g_{i_2} g_{i_2 - 1} \cdots g_{i_2 - k_2}) \cdots (g_{i_p} g_{i_p - 1} \cdots g_{i_p - k_p}) \right\},
\]

for \( 1 \leq i_1 < \ldots < i_p \leq n - 1 \).

The basis \( S \) yields directly an inductive basis for \( H_n(q) \), which is used in the construction of the Ocneanu trace, leading to the Homflypt or 2-variable Jones polynomial.

In \( H_{1,n}(q) \) we define the elements:

\[
t_i := g_i g_{i-1} \cdots g_1 t g_1 \cdots g_{i-1} g_i \quad \text{and} \quad t'_i := g_i g_{i-1} \cdots g_1 t g_1^{-1} \cdots g_{i-1} g_i^{-1}, \quad \tag{3}
\]

as illustrated in Fig. 7.

In [4] the following result has been proved.

**Theorem 4.** (See [4, Proposition 1, Theorem 1].) The following sets form linear bases for \( H_{1,n}(q) \):

\[
\begin{align*}
(i) & \quad \Sigma_n = \left\{ t^{k_1} t_1^{k_1} \cdots t^{k_r} t_r^{k_r} \cdot \sigma, \text{ where } 1 \leq i_1 < \ldots < i_r \leq n - 1 \right\}, \\
(ii) & \quad \Sigma'_n = \left\{ t'^{k_1} t'_1^{k_1} \cdots t'^{k_r} t'_r^{k_r} \cdot \sigma, \text{ where } 1 \leq i_1 < \ldots < i_r \leq n \right\},
\end{align*}
\]

where \( k_1, \ldots, k_r \in \mathbb{Z} \) and \( \sigma \) is a basic element in \( H_n(q) \).

**Remark 1.**

(i) The indices of the \( t'_i \)'s in the set \( \Sigma'_n \) are ordered but are not necessarily consecutive, neither do they need to start from \( t \).

(ii) A more straightforward proof that the sets \( \Sigma'_n \) form bases for \( H_{1,n}(q) \) can be found in [13].
In [4] the basis $\Sigma_n$ is used for constructing a Markov trace on $\bigcup_{n=1}^{\infty} H_{1,n}(q)$.

**Theorem 5.** (See [4, Theorem 6].) Given $z$, $s_k$, with $k \in \mathbb{Z}$ specified elements in $R = \mathbb{Z}[q^{\pm 1}]$, there exists a unique linear Markov trace function

$$\text{tr} : \bigcup_{n=1}^{\infty} H_{1,n}(q) \rightarrow R(z,s_k), \ k \in \mathbb{Z}$$

determined by the rules:

1. $\text{tr}(ab) = \text{tr}(ba)$ for $a, b \in H_{1,n}(q)$
2. $\text{tr}(1) = 1$ for all $H_{1,n}(q)$
3. $\text{tr}(a g_n) = z \text{tr}(a)$ for $a \in H_{1,n}(q)$
4. $\text{tr}(a t_n^k) = s_k \text{tr}(a)$ for $a \in H_{1,n}(q)$, $k \in \mathbb{Z}$.

Note that the use of the looping elements $t'_i$ enables the trace $\text{tr}$ to be defined by just extending the three rules of the Ocneanu trace on the algebras $H_n(q)$ [12] by rule (4). Using $\text{tr}$ Lambropoulou constructed a universal Homflypt-type invariant for oriented links in ST. Namely, let $\mathcal{L}$ denote the set of oriented links in ST. Then:

**Theorem 6.** (See [4, Definition 1].) The function $X : \mathcal{L} \rightarrow R(z,s_k)$

$$X_{\hat{\alpha}} = \left[ \frac{1 - \lambda q}{\sqrt{\lambda} (1 - q)} \right]^{n-1} \left( \sqrt{\lambda} \right)^{e \text{tr} (\pi (\alpha))},$$

where $\lambda := \frac{z^{1+q} q}{q z}$, $\alpha \in B_{1,n}$ is a word in the $\sigma_i$’s and $t'_i$’s, $\hat{\alpha}$ is the closure of $\alpha$, $e$ is the exponent sum of the $\sigma_i$’s in $\alpha$, and $\pi$ the canonical map of $B_{1,n}$ in $H_{1,n}(q)$, such that $t \mapsto t$ and $\sigma_i \mapsto g_i$, is an invariant of oriented links in ST.

The invariant $X$ satisfies a skein relation [4]. Theorems 4, 5 and 6 hold also for the algebras $H_n(q, Q)$ and $H_n(q, d)$, giving rise to all possible Homflypt-type invariants for knots in ST. For the case of the Hecke algebra of type B, $H_n(q, Q)$, see also [11] and [14].

2.4. The basis of $\mathcal{S}(\text{ST})$ in algebraic terms

Let us now see how $\mathcal{S}(\text{ST})$ is described in the above algebraic language. We note first that an element $\alpha$ in the basis of $\mathcal{S}(\text{ST})$ described in Theorem 1 when ST is considered as Annullus $\times$ Interval, can be illustrated equivalently as a mixed link in $S^3$ when ST is viewed as the complement of a solid torus in $S^3$. So we correspond the element $\alpha$ to the minimal mixed braid representation, which has decreasing order of twists around the fixed strand. Fig. 8 illustrates an example of this correspondence. Denoting

$$\Lambda' = \{ t^{k_0} t_1^{k_1} t_2^{k_2} \ldots t_n^{k_n}, \ k_i \in \mathbb{Z} \setminus \{0\}, \ k_i \geq k_{i+1}, \forall i, \ n \in \mathbb{N} \},$$

we have that $\Lambda'$ is a subset of $\bigcup_n H_{1,n}$. In particular $\Lambda'$ is a subset of $\bigcup_n \Sigma'_n$.

Applying the inductive trace rules to a word $w$ in $\bigcup_n \Sigma'_n$ will eventually give rise to linear combinations of monomials in $R(z,s_k)$. In particular, for an element of $\Lambda'$ we have:

$$\text{tr}(t^{k_0} t_1^{k_1} t_2^{k_2} \ldots t_{n-1}^{k_{n-1}}) = s_{k_{n-1}} \ldots s_{k_1} s_{k_0}.$$
Moreover, ignoring \( t \) before \( \sum_{i=0}^{\nu} \mu_i \), we will need the notion of the index of a word in \( \Lambda' \) or in \( \Lambda \).

**Definition 1.** The *index* of a word \( w \) in \( \Lambda' \) or in \( \Lambda \), denoted \( \text{ind}(w) \), is defined to be the highest index of the \( t'_i \)'s, resp. the \( t_i \)'s, in \( w \). Similarly, the *index* of an element in \( \Sigma'_n \) or in \( \Sigma_n \) is defined in the same way by ignoring possible gaps in the indices of the looping generators and by ignoring the braiding part in \( H_n(q) \).

Moreover, the index of a monomial in \( H_n(q) \) is equal to 0.

For example, \( \text{ind}(t'^{k_0}t'_{i_1}^{k_1} \cdots t'_{i_n}^{k_n}) = \text{ind}(t^{u_0} \cdots t^{u_n}) = n \).

**Definition 2.** We define the following *ordering* in the sets \( \Sigma'_n \).

Let \( w = t^{k_1}_i t^{k_2}_{i_2} \cdots t^{k_{\nu}}_{i_{\nu}} \) and \( \sigma = t^{\lambda_1}_{j_1} t^{\lambda_2}_{j_2} \cdots t^{\lambda_{\nu}}_{j_{\nu}} \), where \( k_i, \lambda_s \in \mathbb{Z} \), for all \( t, s \). Then:

(a) If \( \sum_{i=0}^{\nu} k_i < \sum_{i=0}^{\nu} \lambda_i \), then \( w < \sigma \).

(b) If \( \sum_{i=0}^{\nu} k_i = \sum_{i=0}^{\nu} \lambda_i \), then:

(i) if \( \text{ind}(w) < \text{ind}(\sigma) \), then \( w < \sigma \),

(ii) if \( \text{ind}(w) = \text{ind}(\sigma) \), then:

(a) if \( i_1 = j_1, i_2 = j_2, \ldots, i_{s-1} = j_{s-1}, i_s < j_s \), then \( w > \sigma \),

(b) if \( i_t = j_t \) \( \forall t \) and \( k_{\mu} = \lambda_{\mu}, k_{\mu-1} = \lambda_{\mu-1}, \ldots, k_{i+1} = \lambda_{i+1}, |k_i| < |\lambda_i| \), then \( w < \sigma \),

(c) if \( i_t = j_t \) \( \forall t \) and \( k_{\mu} = \lambda_{\mu}, k_{\mu-1} = \lambda_{\mu-1}, \ldots, k_{i+1} = \lambda_{i+1}, |k_i| = |\lambda_i| \) and \( k_i > \lambda_i \), then \( w < \sigma \),

(d) if \( i_t = j_t \) \( \forall t \) and \( k_i = \lambda_i, \forall i \), then \( w = \sigma \).
(c) In the general case where \( w = t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m} \cdot \beta_1 \) and \( \sigma = t_1^{j_1} t_2^{j_2} \cdots t_m^{j_m} \cdot \beta_2 \), where \( \beta_1, \beta_2 \in \text{H}_n(q) \), the ordering is defined in the same way by ignoring the braiding parts \( \beta_1, \beta_2 \).

The same ordering is defined on the set \( \Lambda' \) by ignoring the braiding parts. Moreover, the same ordering is defined on the sets \( \Sigma_n \) and \( \Lambda \), where the \( t_i^{j_i} \)'s are replaced by the corresponding \( t_i \)'s.

**Proposition 1.** The set \( \Sigma_n' \) equipped with the ordering given in Definition 2, is a totally ordered set.

**Proof.** In order to show that the set \( \Sigma_n' \) is a totally ordered set when equipped with the ordering given in Definition 2, we need to show that the ordering relation is antisymmetric, transitive and total. We only show that the ordering relation is transitive. Antisymmetric property follows similarly. Totality follows from Definition 2 since all possible cases have been considered. Let \( w, \sigma, v \in \Sigma_n \) such that:

\[
\begin{align*}
    w &= t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m} \cdot \beta_1, \\
    \sigma &= t_1^{j_1} t_2^{j_2} \cdots t_m^{j_m} \cdot \beta_2, \\
    v &= t_1^{p_1} t_2^{p_2} \cdots t_m^{p_m} \cdot \beta_3,
\end{align*}
\]

where \( \beta_1, \beta_2, \beta_3 \in \text{H}_n(q) \) and let \( w < \sigma \) and \( \sigma < v \). Since \( w < \sigma \), one of the following holds:

(a) Either \( \sum_{i=1}^m k_i < \sum_{i=1}^n \lambda_i \) and since \( \sigma < v \), we have that \( \sum_{i=1}^n \lambda_i \leq \sum_{i=1}^p \mu_i \) and so \( \sum_{i=1}^m k_i < \sum_{i=1}^p \mu_i \). Thus \( w < v \).

(b) Either \( \sum_{i=1}^m k_i = \sum_{i=1}^n \lambda_i \) and \( \text{ind}(w) = m < n = \text{ind}(\sigma) \). Then, since \( \sigma < v \) we have that either \( \sum_{i=1}^n \lambda_i < \sum_{i=1}^p \mu_i \) (same as in case (a)) or \( \sum_{i=1}^n \lambda_i = \sum_{i=1}^p \mu_i \) and \( \text{ind}(\sigma) \leq p = \text{ind}(v) \). Thus, \( \text{ind}(w) = m < p = \text{ind}(v) \) and so we conclude that \( w < v \).

(c) Either \( \sum_{i=1}^m k_i = \sum_{i=1}^n \lambda_i \), \( \text{ind}(w) = \text{ind}(\sigma) \) and \( t_1 = j_1, \ldots, t_{i-1} = j_{s-1}, t_s > j_s \). Then, since \( \sigma < v \), we have that either:

- \( \sum_{i=1}^n \lambda_i < \sum_{i=1}^p \mu_i \), same as in case (a), or
- \( \sum_{i=1}^n \lambda_i = \sum_{i=1}^p \mu_i \) and \( \text{ind}(\sigma) < \text{ind}(v) \), same as in case (b), or
- \( \text{ind}(\sigma) = \text{ind}(v) \) and \( j_1 = \varphi_1, \ldots, j_p > \varphi_p \). Then:
  - \( i \) if \( p = s \) we have that \( t_s > j_s > \varphi_s \) and we conclude that \( w < v \),
  - \( ii \) if \( p < s \) we have that \( t_p = j_p > \varphi_p \) and thus \( w < v \) and if \( s < p \) we have that \( t_s > j_s = \varphi_s \) and so \( w < v \).

(d) Either \( \sum_{i=1}^m k_i = \sum_{i=1}^n \lambda_i \), \( \text{ind}(w) = \text{ind}(\sigma) \) and \( k_n = \lambda_n, \ldots, |k_q| < |\lambda_q| \). Then, since \( \sigma < v \), we have that either:

- \( \sum_{i=1}^n \lambda_i < \sum_{i=1}^p \mu_i \), same as in case (a), or
- \( \sum_{i=1}^n \lambda_i = \sum_{i=1}^p \mu_i \) and \( \text{ind}(\sigma) < \text{ind}(v) \), same as in case (b), or
- \( \text{ind}(\sigma) = \text{ind}(v) \) and \( j_1 = \varphi_1, \ldots, j_q > \varphi_q \), same as in case (c), or
- \( j_n = \varphi_n \), for all \( n \) and \( \mu_n = \lambda_n, \ldots, \mu_{c+1} = \lambda_{c+1}, |\mu_c| \geq |\lambda_c| \) for some \( c \), then:
  - \( i \) if \( |\mu_c| > |\lambda_c| \), then:
    - \( i \) if \( c > q \) then \( |k_c| = |\lambda_c| < |\mu_c| \) and thus \( w < v \).
    - \( ii \) if \( c < q \) then \( |k_q| < |\lambda_q| = |\mu_q| \) and thus \( w < v \).
    - \( iii \) if \( c = q \) then \( |k_q| < |\lambda_q| < |\mu_q| \) and thus \( w < v \).

(2) If \( |\mu_c| = |\lambda_c| \), such that \( \mu_c < \lambda_c \), then:

- \( i \) if \( c > q \) then \( |k_c| = |\lambda_c| = |\mu_c| \) and \( k_c = \lambda_c > \mu_c \). Thus \( w < v \).
  - \( ii \) if \( c \leq q \) then \( |k_q| < |\lambda_q| = |\mu_q| \) and thus \( w < v \).

(c) Either \( \sum_{i=1}^m k_i = \sum_{i=1}^n \lambda_i \), \( \text{ind}(w) = \text{ind}(\sigma) \) and \( k_n = \lambda_n, \ldots, |k_q| = |\lambda_q| \), such that \( k_q > \lambda_q \). Then, since \( \sigma < v \), we have that either:
\[ \sum_{i=1}^{n} \lambda_i < \sum_{i=1}^{n} \mu_i, \text{ same as in case (a), or} \]
\[ \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i \text{ and } \text{ind}(\sigma) < \text{ind}(v), \text{ same as in case (b), or} \]
\[ \text{ind}(\sigma) = \text{ind}(v) \text{ and } j_1 = \varphi_1, \ldots, j_q > \varphi_q, \text{ same as in case (c), or} \]
\[ j_n = \varphi_n, \text{ for all } n \text{ and } \mu_n = \lambda_n, \ldots, \mu_{c+1} = \lambda_{c+1}, |\mu_c| \geq |\lambda_c| \text{ for some } c, \text{ then:} \]

1. If \(|\mu_c| > |\lambda_c|\), then:
   (i) If \(c > q\) then \(|k_c| = |\lambda_c| < |\mu_c|\), thus \(w < v\).
   (ii) If \(c \leq q\) then \(|k_q| = |\lambda_q| = |\mu_q|\) and \(k_q > \lambda_q = \mu_q\), thus \(w < v\).

2. If \(|\mu_c| = |\lambda_c|\) such that \(\lambda_c > \mu_c\), then:
   (i) If \(c > q\) then \(|k_c| = |\lambda_c| = |\mu_c|\) and \(k_c = \lambda_c > \mu_c\), thus \(w < v\).
   (ii) If \(c < q\) then \(|k_q| = |\lambda_q| = |\mu_q|\) and \(k_q > \lambda_q = \mu_q\), thus \(w < v\).
   (iii) If \(c = q\), then \(|k_q| = |\lambda_q| = |\mu_q|\) and \(k_q > \lambda_q > \mu_q\), thus \(w < v\).

So, we conclude that the ordering relation is transitive. \(\square\)

**Remark 3.** Proposition 1 also holds for the sets \(\Sigma_n, \Lambda'\) and \(\Lambda\).

**Definition 3.** We define the subset of level \(k\), \(\Lambda_k\), of \(\Lambda\) to be the set

\[ \Lambda_k := \{t_{k_1}^{k_1} \cdots t_{k_m}^{k_m} \mid \sum_{i=0}^{m} k_i = k, \ k_i \in \mathbb{Z} \setminus \{0\}, \ k_i \geq k_{i+1} \ \forall i\} \]

and similarly, the subset of level \(k\) of \(\Lambda'\) to be

\[ \Lambda'_k := \{t_{k_1}^{k_1'} \cdots t_{k_m}^{k_m'} \mid \sum_{i=0}^{m} k_i = k, \ k_i \in \mathbb{Z} \setminus \{0\}, \ k_i \geq k_{i+1} \ \forall i\}. \]

**Remark 4.** Let \(w \in \Lambda_k\) be a monomial containing gaps in the indices and \(u \in \Lambda_k\) a monomial with consecutive indices such that \(\text{ind}(w) = \text{ind}(u)\). Then, it follows from Definition 2 that \(w < u\).

**Proposition 2.** The sets \(\Lambda_k\) are totally ordered and well-ordered for all \(k\).

**Proof.** Since \(\Lambda_k \subseteq \Lambda\), \(\forall k\), \(\Lambda_k\) inherits the property of being a totally ordered set from \(\Lambda\). Moreover, \(t^k\) is the minimum element of \(\Lambda_k\) and so \(\Lambda_k\) is a well-ordered set. \(\square\)

We also introduce the notion of *homologous words* as follows:

**Definition 4.** We shall say that two words \(w' \in \Lambda'\) and \(w \in \Lambda\) are *homologous*, denoted \(w' \sim w\), if \(w\) is obtained from \(w'\) by turning \(t'_i\) into \(t_i\) for all \(i\).

With the above notion the proof of Theorem 2 is based on the following idea: Every element \(w' \in \Lambda'\) can be expressed as linear combinations of monomials \(w_i \in \Lambda\) with coefficients in \(\mathbb{C}\), such that:

1. \(\exists j\) such that \(w_j \sim w'\),
2. \(w_j < w_i\), for all \(i \neq j\),
3. the coefficient of \(w_j\) is an invertible element in \(\mathbb{C}\).
4. From \( \Lambda' \) to \( \Sigma_n \)

In this section we prove a series of lemmas relating elements of the two different basic sets \( \Sigma_n, \Sigma'_n \) of \( H_{1,n}(q) \). In the proofs we underline expressions which are crucial for the next step. Since \( \Lambda' \) is a subset of \( \Sigma'_n \), all lemmas proved here apply also to \( \Lambda' \) and will be used in the context of the bases of \( S(ST) \).

4.1. Some useful lemmas in \( H_{1,n}(q) \)

We will need the following results from [4]. The first lemma gives some basic relations of the braiding generators.

**Lemma 1.** (See [4, Lemma 1].) For \( \epsilon \in \{ \pm 1 \} \) the following hold in \( H_{1,n}(q) \):

\[
\begin{align*}
(i) \quad & g_i^m = (q^{m-1} - q^{m-2} + \cdots + (-1)^{m-1}) g_i + (q^{m-1} - q^{m-2} + \cdots + (-1)^{m-2} q) \\
& g_i^{-m} = (q^{-m} - q^{-m+1} + \cdots + (-1)^{-m} q^{-1}) g_i + \\
& \quad + (q^{-m} - q^{-m+1} + \cdots + (-1)^{-m} q^{-1} + (1)^{-m}) \\
(ii) \quad & g_i^\epsilon (g_k^{\pm 1}_{k-1} g_j^{\pm 1}_{j-1} \cdots g_{j-1}^{\pm 1}) = (g_k^{\pm 1}_{k-1} g_j^{\pm 1}_{j-1} \cdots g_{j-1}^{\pm 1}) g_{i+1}^\epsilon, \text{ for } k > i \geq j, \\
& g_i^\epsilon (g_j^{\pm 1}_{j+1} g_{j+1}^{\pm 1} \cdots g_{k-1}^{\pm 1}) = (g_j^{\pm 1}_{j+1} g_{j+1}^{\pm 1} \cdots g_{k-1}^{\pm 1}) g_{i-1}^\epsilon, \text{ for } k \geq i > j,
\end{align*}
\]

where the sign of the \( \pm 1 \) exponent is the same for all generators.

\[
\begin{align*}
(iii) \quad & g_i g_{i-1} \cdots g_{j+1} g_{j+1} \cdots g_i = g_j g_{j+1} \cdots g_i - g_{i-1} g_i - g_{j+1} g_j \\
& g_i g_{i-1} \cdots g_{j+1} g_{j+1} \cdots g_i = g_j g_{j+1} \cdots g_i - g_{i-1} g_i - g_{j+1} g_j^{-1} \\
(iv) \quad & g_i^\epsilon \cdots g_{n-1}^\epsilon g_n^{2^\epsilon} g_{n-1}^\epsilon \cdots g_i^\epsilon = \sum_{r=0}^{n-1} (q^\epsilon - 1)^r q^{r} (g_i^\epsilon \cdots g_{n-r}^\epsilon \cdots g_i^\epsilon),
\end{align*}
\]

where \( \epsilon_r = 1 \) if \( r \leq n - i \) and \( \epsilon_{n-i+1} = 0 \). Similarly,

\[
\begin{align*}
(v) \quad & g_i^\epsilon \cdots g_{2^\epsilon} g_1^{2^\epsilon} g_2^\epsilon \cdots g_i^\epsilon = \sum_{r=0}^{i} (q^\epsilon - 1)^r q^{r} (g_i^\epsilon \cdots g_{r+2^\epsilon} g_{r+1}^\epsilon g_{r+2}^\epsilon \cdots g_i^\epsilon),
\end{align*}
\]

where \( \epsilon_r = 1 \) if \( r \leq i - 1 \) and \( \epsilon_i = 0 \).

The next lemma comprises relations between the braiding generators and the looping generator \( t \).

**Lemma 2.** (Cf. [4, Lemmas 1, 4, 5].) For \( \epsilon \in \{ \pm 1 \}, i, k \in \mathbb{N} \) and \( \lambda \in \mathbb{Z} \) the following hold in \( H_{1,n}(q) \):

\[
\begin{align*}
(i) \quad & t^\lambda g_1 g_1 = g_1 t^\lambda g_1 \\
(ii) \quad & t^\epsilon g_i^t e^k g_i^\epsilon = g_i^\epsilon e^k g_i^t e^\epsilon + (q^\epsilon - 1) t^\epsilon g_i^t e^k + (1 - q^\epsilon) t^\epsilon g_i^t e^\epsilon \\
& t^{-\epsilon} g_i^t e^k g_i^\epsilon = g_i^\epsilon e^k g_i^t e^\epsilon + (q^\epsilon - 1) t^{-\epsilon} g_i^t e^k + (1 - q^\epsilon) t^{-\epsilon} g_i^t e^\epsilon \\
(iii) \quad & t^{\epsilon} g_i^t e^k g_i^\epsilon = g_i^\epsilon e^k g_i^t e^\epsilon + (q^\epsilon - 1) \sum_{j=1}^{i} t^{\epsilon} g_i^t e^{(k+i-j)} + \\
& + (1 - q^\epsilon) \sum_{j=1}^{i} t^{\epsilon} g_i^t e^{(k+i-j)} \\
& t^{-\epsilon} g_i^t e^k g_i^\epsilon = g_i^\epsilon e^k g_i^t e^\epsilon + (q^\epsilon - 1) \sum_{j=1}^{i} t^{-\epsilon} g_i^t e^{(k+i-j)} + \\
& + (1 - q^\epsilon) \sum_{j=1}^{i} t^{-\epsilon} g_i^t e^{(k+i-j)}
\end{align*}
\]

The next lemma gives the interactions of the braiding generators and the loopings \( t_i \)'s and \( t_i'^\lambda \)'s.
Lemma 3. (See [4, Lemmas 1 and 2].) The following relations hold in $H_{1,n}(q)$:

(i) $g_it_k^e = t_k^e g_i$ for $k > i$, $k < i - 1$

(ii) $g_it_i = qt_{i-1} g_i + (q - 1)t_i$

(iii) $g_it_{i-1} = q^{1-t_i} g_i + (q - 1)t_{i-1} = t_{i-1} g_i - 1$

(iv) $g_it_{i-1}^e = q^{1-t_i} g_i + (q - 1)t_{i-1}^e + (1 - q)t_{i-1}^e$

(v) $t_i^k = g_i ... g_it_k g_{i-1} ... g_{i-1}$ for $k \in \mathbb{Z}$.

Using now Lemmas 1, 2 and 3 we prove the following relations, which we will use for converting elements in $A'$ to elements in $\Sigma_n$. Note that whenever a generator is overlined, this means that the specific generator is omitted from the word.

Lemma 4. The following relations hold in $H_{1,n}(q)$ for $k \in \mathbb{N}$:

(i) $g_{m+1}t_m^k = q^{-(k-1)}t_m^{k}g_{m+1}^{-1} + \sum_{j=1}^{k-1} q^{-(k-1-j)}(q - 1)t_{m+j}^{-1}$ (see Fig. 10),

(ii) $g_{m+1}^{-1}t_m^{-k} = q^{-(k-1)}t_m^{-k}g_{m+1}^{-1} + \sum_{j=1}^{k-1} q^{-(k-1-j)}(q - 1)t_{m+j}^{-1}$.

Proof. We prove relation (i) by induction on $k$. Relation (ii) follows similarly. For $k = 1$ we have that $g_{m+1}t_m = t_{m+1}g_{m+1}$, which holds from Lemma 3(i). Suppose that the relation holds for $k - 1$. Then, for $k$ we have:

Lemma 5. In $H_{1,n}(q)$ the following relations hold:

(i) For the expression $A = (g_r g_{r-1} ... g_{r-s}) \cdot t_k$ the following hold for the different values of $k \in \mathbb{N}$:

1. $A = t_k (g_r ... g_{r-s})$ for $k > r$ or $k < r - s - 1$
2. $A = t_r (g_r^{-1} ... g_{r-s})$ for $k = r - s - 1$
3. $A = qt_{r-1} (g_r ... g_{r-s}) + (q - 1)t_{r} (g_{r-1} ... g_{r-s})$ for $k = r$
4. $A = qt_{r-s-1} (g_r ... g_{r-s}) + (q - 1)t_{r} (g_{r-1} ... g_{r-s+1})$ for $k = r - s$
5. $A = t_{m-1} (g_r ... g_{r-s}) + (q - 1)t_{r} (g_r^{-1} ... g_{m+1}) (g_m ... g_{r-s})$

for $k = m \in \{r - s + 1, ..., r - 1\}$. 

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(ii) For the expression \( A = (g_r g_{r-1} \ldots g_{r-s}) \cdot t_k^{-1} \) the following hold for the different values of \( k \in \mathbb{N} \):

1. \( A = t_{k-1} (g_r \ldots g_{r-s}) \) for \( k > r \) or \( k < r-s-1 \)
2. \( A = t_{r-s}^{-1} (g_r \ldots g_{r-s+1} g_{r-s}) \) for \( k = r-s \) (see Fig. 11)
3. \( A = t_{m-1}^{-1} (g_r g_{r-1} \ldots g_{m+1} g_m \ldots g_{k-s}) \) for \( k = m \in \{r-s+1, \ldots, r\} \)
4. \( A = q^{s+1} t_r^{-1} (g_r \ldots g_{r-s}) + (q-1) \sum_{j=1}^{s+1} q^{s-j+1} t_r^{-1} \cdot (g_r \ldots g_{r-j+2} g_{r-j} \ldots g_{r-s}) \) for \( k = r-s-1 \).

**Proof.** We only prove relation (ii) for \( k = r-s-1 \) by induction on \( s \) (case (4)). All other relations follow from Lemma 3(i).

For \( s = 1 \) we have:

\[
g_r g_{r-1} t_{r-2} = g_r[q t_{r-1} g_r + (q-1) t_{r-2}] = g_r q t_{r-1} g_r + (q-1) g_r t_{r-2} = q t_r^{-1} g_r + (q-1) t_r^{-1} g_r + (q-1) t_{r-2} g_r = q^2 t_r^{-1} (g_r g_r) + (q-1) [q t_{r-1}^{-1} g_r + q^2 t_{r-2} g_r] ,
\]

and so the relation holds for \( s = 1 \). Suppose that the relation holds for \( s = n \). We will show that it holds for \( s = n + 1 \). Indeed we have:

\[
\begin{align*}
(g_r \ldots g_{r-n-1}) t_{r-n-2}^{-1} &= (g_r \ldots g_{r-n}) (g_r \ldots g_{r-n-1}) t_{r-n-2}^{-1} \\
(g_r \ldots g_{r-n}) (q t_{r-n-1} g_{r-n-1} + (q-1) t_{r-n-2}) &= q (g_r \ldots g_{r-n} t_{r-n-1}^{-1} g_{r-n} + (q-1) g_r \ldots g_{r-n}) t_{r-n-2}^{-1} \text{ ind. step} \\
&= q^{n+2} t_r^{-1} (g_r \ldots g_{r-n-1}) + \\
&+ (q-1) \sum_{j=1}^{n+1} q^{n-j+2} t_{r-j}^{-1} (g_r \ldots g_{r-j+2} g_{r-j} \ldots g_{r-n-1}) + \\
&+ (q-1) t_{r-n-2}^{-1} (g_r \ldots g_{r-n}) = q^{n+2} t_r^{-1} (g_r \ldots g_{r-n-1}) + \\
&+ (q-1) \sum_{j=1}^{n+1} q^{n-j+2} t_{r-j}^{-1} (g_r \ldots g_{r-j+2} g_{r-j} \ldots g_{r-n-1}) .
\end{align*}
\]

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Before proceeding with the next lemma we introduce the notion of length of \( w \in H_n(q) \). For convenience we set \( \delta_{k,r} := g_k g_{k-1} \ldots g_{r+1} g_r \) for \( k > r \) and by convention we set \( \delta_{k,k} := g_k \).

**Definition 5.** We define the *length* of \( \delta_{k,r} \in H_n(q) \) to be the number of braiding generators, that is, \( l(\delta_{k,r}) := k - r + 1 \) and since every element of the Iwahori–Hecke algebra of type A can be written as \( \prod_{i=1}^{n-1} \delta_{k_i,r_i} \), so that \( k_j < k_{j+1} \, \forall \, j \), we define the length of an element \( w \in H_n(q) \) as:

\[
l(w) := \sum_{i=1}^{n-1} l_i(\delta_{k_i,r_i}) = \sum_{i=1}^{n-1} k_i - r_i + 1.
\]

Note that \( l(g_k) = l(\delta_{k,k}) = k - k + 1 = 1 \).

**Lemma 6.** For \( k > r \) the following relations hold in \( H_{1,n}(q) \):

\[
t_k \delta_{k,r} = \sum_{i=0}^{k-r} q^i(q - 1) \delta_{k,k-i,r} t_{k-i} + q^{l(\delta_{k,r})} \delta_{k,r} t_{r-1},
\]

where \( \delta_{k,k-i,r} := g_k g_{k-1} \ldots g_{k-i+1} g_{k-i} \ldots g_r := g_k \ldots \overline{g_{i-1}} \ldots g_r \).

**Proof.** We prove relations by induction on \( k \). For \( k = 1 \) we have that \( t_1 g_1 = (q-1) t_1 + q g_1 t \), which holds. Suppose that the relation holds for \( (k-1) \), then for \( k \) we have:

\[
t_k \delta_{k,r} = t_k g_k \cdot \delta_{k-1,r} = (q - 1) t_k \delta_{k-1,r} + q g_k t_{k-1} \delta_{k-1,r} = \frac{(q - 1) \delta_{k-1,r} - t_k g_k \delta_{k-1,r}}{1 - q} + q^{l(\delta_{k-1,r})+1} g_k \delta_{k-1,r} t_{r-1} = \sum_{i=0}^{k-r} q^i(q - 1) \delta_{k,k-i,r} t_{k-i} + q^{l(\delta_{k,r})} \delta_{k,r} t_{r-1}.
\]

**Lemma 7.** In \( H_{1,n}(q) \) the following relations hold:

(i) For the expression \( A = (g_r g_{r+1} \ldots g_{r+s}) \cdot t_k \) the following hold for the different values of \( k \in \mathbb{N} \):

\[
(1) \quad A = t_k \, (g_r \ldots g_{r+s}) \quad \text{for} \quad k \geq r + s + 1 \quad \text{or} \quad k < r - 1
\]

\[
(2) \quad A = t_{k+1} \, (g_r \ldots g_{k+1} g_{k+2} \ldots g_{r+s})
\]

\[
\quad \quad \text{for} \quad r - 1 \leq k < r + s
\]

\[
(3) \quad A = (q - 1) \sum_{i=0}^{r+s} q^{r+s-i} t_i \, (g_r \ldots \overline{g_i} \ldots g_{r+s}) + q^{s+1} t_{r-1} \, (g_r \ldots g_{r+s})
\]

\[\text{for} \quad k = r + s\]

(ii) For the expression \( A = (g_r g_{r+1} \ldots g_{r+s}) \cdot t_k^{-1} \) the following hold for the different values of \( k \in \mathbb{N} \):

\[
(1) \quad A = t_k^{-1} \, (g_r g_{r+1} \ldots g_{r+s}) \quad \text{for} \quad k \geq r + s + 1 \quad \text{or} \quad k < r - 1
\]

\[
(2) \quad A = q \, t_{k+1}^{-1} \, (g_r \ldots g_{r+s}) + (q - 1) \, t_{r-1}^{-1} \, (g_r \ldots g_{k+1} g_{k+2} \ldots g_{r+s})
\]

\[\quad \quad \text{for} \quad r - 1 \leq k < r + s
\]

\[
(3) \quad A = t_{r-1}^{-1} \, (g_r \ldots \overline{g_{r+s}}) \quad \text{for} \quad k = r + s
\]

**Proof.** We prove relation (i) for \( r + s = k \) by induction on \( k \) (case (3)). All other relations follow from Lemmas 1 and 3.
For \( k = 1 \) we have: \( g_1 t_1 = \frac{q^2}{q} g_1 t_1 = qt g_1 + (q - 1) t_1 \). Suppose that the relation holds for \( k = n \). Then, for \( k = n + 1 \) we have that:

\[
\frac{g_r \cdots g_{n+1} t_{n+1}}{q^n} = (q - 1)^n \frac{g_r \cdots g_n t_n}{q^n} + (q - 1)(g_r \cdots g_n t_{n+1}) + q^{-n+1} \sum_{i=1}^{n} q^{-n+i} t_i (g_r \cdots g_{n+i+1} t_{n+i+1} + (q - 1) t_{n+i+1} (g_r \cdots g_{n+i+1}) =
\]

\[
= (q - 1)^n \frac{g_r \cdots g_n t_{n+1}}{q^n} + q^{-n+1} \sum_{i=1}^{n} q^{-n+i} t_i (g_r \cdots g_{n+i+1} t_{n+i+1} + (q - 1) t_{n+i+1} (g_r \cdots g_{n+i+1}). \quad \Box
\]

Lemma 8. The following relations hold in \( H_{1,n}(q) \) for \( k \in \mathbb{N} 

(ii) \( \frac{(g_r \cdots g_{n+1} t_{n+1})}{q^n} = (g_r \cdots g_{n+1} t_{n+1}) + q^{-n+1} \sum_{i=1}^{n} q^{-n+i} t_i (g_r \cdots g_{n+i+1} t_{n+i+1} + (q - 1) t_{n+i+1} (g_r \cdots g_{n+i+1})). \quad \Box
\]

Proof. We prove relation (i) by induction on \( i \). All other relations follow similarly. For \( i = 1 \) we have:

\[
\frac{g_1 t}{q^2} = g_1 t \frac{g_1 t_1}{q} = g_1 t_1 \frac{g_1 t_1}{q} = (q - 1)t g_1 + qt. \]

Suppose that the relation holds for \( i = n \). Then, for \( i = n + 1 \) we have:

\[
\frac{(g_1 \cdots g_n t_1 \cdots g_1 t_1)}{q^n} = (q - 1)^n \frac{(g_1 \cdots g_n t_1 \cdots g_1 t_1)}{q^n} + q^{-n+1} \sum_{i=1}^{n} q^{-n+i} t_i (g_1 \cdots g_n t_1 + (q - 1) t_1 (g_1 \cdots g_n) =
\]

\[
= (q - 1)^n \frac{(g_1 \cdots g_n t_1 \cdots g_1 t_1)}{q^n} + q^{-n+1} \sum_{i=1}^{n} q^{-n+i} t_i (g_1 \cdots g_n t_1 + (q - 1) t_1 (g_1 \cdots g_n). \quad \Box
\]

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4.2. Converting elements in \( \Lambda' \) to elements in \( \Sigma_n \)

We are now in the position to prove a set of relations converting monomials of \( t'_i \)'s to expressions containing the \( t_i \)'s. In [13] we provide lemmas converting monomials of \( t_i \)'s to monomials of \( t'_i \)'s in the context of giving a simple proof that the sets \( \Sigma_n' \) form bases of \( H_{1,n}(q) \).

**Lemma 9.** The following relations hold in \( H_{1,n}(q) \) for \( k \in \mathbb{N} \):

\[
(i) \quad t'_1^{-k} = q^k t_1^{-k} + \sum_{j=1}^{k} q^{k-j}(q-1) t_j^{-1} t'_1^{-1} \cdot g_1^{-1}, \\
(ii) \quad t'_1^{k} = q^{-k} t_1^{k} + \sum_{j=1}^{k} q^{-(k-j)}(q-1) t_j^{1} t'_1^{1-j} \cdot g_1^{-1}.
\]

**Proof.** We prove relation (i) by induction on \( k \). Relation (ii) follows similarly. For \( k = 1 \) we have: \( t'_1^{-1} = g_1 t_1^{-1} g_1^{-1} = q g_1^{-1} t_1^{-1} g_1^{-1} + (q-1) t_1^{-1} g_1^{-1} = q t_1^{-1} + (q-1) t_1^{-1} g_1^{-1} \).

Suppose that the relation holds for \( k - 1 \). Then, for \( k \) we have:

\[
t'_1^{-k} = t'_1^{-(k-1)} t_1^{-1} \quad \text{ind. step} \quad q^{k-1} t_1^{-(k-1)} t_1^{-1} + \\
+ \sum_{j=1}^{k-1} q^{k-j-1}(q-1) t_j^{-1} t'_1^{-1} \cdot g_1^{-1} = \\
= q^k t_1^{-k} + q^{-k} t_1^{-1} t'_1^{-1} \cdot g_1^{-1} + \sum_{j=1}^{k-1} q^{k-j}(q-1) t_j^{1} t'_1^{1-j} \cdot t_1^{-1} g_1^{-1} = \\
= q^k t_1^{-k} + q^{-k}(q-1) t_1^{-1} t'_1^{-1} \cdot g_1^{-1} + \\
+ \sum_{j=1}^{k-1} q^{-j}(q-1) t_1^{-1} t'_1^{j} \cdot g_1^{-1} = \\
= q^k t_1^{-k} + \sum_{j=1}^{k} q^{k-j}(q-1) t_j^{-1} t'_1^{j} \cdot g_1^{-1}.
\]

**Lemma 10.** The following relations hold in \( H_{1,n}(q) \) for \( k \in \mathbb{N} \):

\[
t'_k^{-1} = q^k t_k^{-1} + (q-1) \sum_{i=0}^{k-1} q^i t_i^{-1} ( g_k g_{k-1} \cdots g_{i+2} g_{i+1}^{-1} \cdots g_{k-1}^{-1} g_k^{-1} ).
\]

**Proof.** We prove the relations by induction on \( k \). For \( k = 1 \) we have:

\[
t'_1^{-1} = g_1 t_1^{-1} g_1^{-1} = q g_1^{-1} t_1^{-1} g_1^{-1} + (q-1) t_1^{-1} g_1^{-1} = q t_1^{-1} + (q-1) t_1^{-1} g_1^{-1}.
\]

Suppose that the relations hold for \( k = n \). Then, for \( k = n+1 \) we have that:

\[
t'_{n+1}^{-1} = g_{n+1} t_{n+1}^{-1} g_{n+1}^{-1} \quad \text{ind. step} \quad g_{n+1} t_{n+1}^{-1} g_{n+1}^{-1} = \\
= q^n g_{n+1} [q^{-n} t_n^{-1} g_n^{-1} + (q-1) \sum_{i=0}^{n-1} q^i t_i^{-1} (g_n \cdots g_{i+2} g_{i+1}^{-1} \cdots g_{n-1}^{-1}) g_{n+1}^{-1} ] = \\
= q^n [q^{-n} t_n^{-1} g_n^{-1} + (q-1) \sum_{i=0}^{n-1} q^i t_i^{-1} g_{n+1}^{-1} + (q-1) \sum_{i=0}^{n-1} q^i t_i^{-1}] = \\
= q^n t_{n+1}^{-1} + (q-1) t_{n+1}^{-1} g_{n+1}^{-1} + (q-1) \sum_{i=0}^{n-1} q^i t_i^{-1}. \\
\]

**Lemma 11.** The following relations hold in \( H_{1,n}(q) \) for \( k \in \mathbb{Z} \setminus \{0\} \):

\[
t'_m^{k} = q^{-mk} t_m^{k} + \sum_{i} f_i(q) t_m^{k} w_i + \sum_{i} g_i(q) t_{\lambda_i}^{\lambda_i} \cdots t_{\lambda_m}^{\lambda_m} u_i,
\]

where \( w_i, u_i \in H_{m+1}(q) \), \( \forall i, \sum_{i=0}^{m} \lambda_i = k \) and \( \lambda_i \geq 0 \), \( \forall i, k > 0 \) and \( \lambda_i \leq 0, \forall i, k < 0 \).
Proof. We prove relations by induction on \( m \). The case \( m = 1 \) is Lemma 9. Suppose now that the relations hold for \( m - 1 \). Then, for \( m \) we have:

\[
\begin{align*}
\lambda' & \in T' \\
\Lambda & \ni T \sim T' \\
\Lambda & \ni T_1 \sim T' \\
\sum_n & \ni T, \forall i \\
\sum_u & \ni H_{1,n}(q), \forall i
\end{align*}
\]

\[
\begin{array}{c}
\text{Fig. 13. Illustrating Theorem 7.}
\end{array}
\]

Using now Lemma 11 we have that every element \( u \in \Lambda' \) can be expressed to linear combinations of elements \( v_i \in \Sigma_n \), where \( \exists j : v_j \sim u \). More precisely:

**Theorem 7.** The following relations hold in \( H_{1,n}(q) \) for \( k \in \mathbb{Z} \):

\[
t^{k_0}t_1^{k_1} \cdots t_m^{k_m} = q^{-\sum_{n=1}^m nk_n}t^{k_0}t_1^{k_1} \cdots t_m^{k_m} + \sum_i f_i(q) t^{k_0}t_1^{k_1} \cdots t_m^{k_m} w_i + \sum_j g_j(q) \tau_j \cdot u_j,
\]

where \( w_i, u_j \in H_{m+1}(q), \forall i, \tau_j \in \Sigma_n \), such that \( \tau_j \neq t^{k_0}t_1^{k_1} \cdots t_m^{k_m}, \forall j \). (See Fig. 13.)

Proof. We prove relations by induction on \( m \). Let \( k_1 \in \mathbb{N} \), then for \( m = 1 \) we have:

\[
t^{k_0}t_1^{k_1} = q^{-k_1}t^{k_0}t_1^{k_1} + \sum_{j=1}^{k_1} q^{-(k_1-j)}(q^{-1} - 1)t_{k_0+j-1}t_1^{k_1-j} g_1^{-1} =
\]

\[
q^{-k_1}t^{k_0}t_1^{k_1} + q^{-k_1}(q^{-1} - 1)t_{k_0}t_1^{k_1} g_1^{-1} + \sum_{j=2}^{k_1} q^{-(k_1-j)}(q^{-1} - 1)t_{k_0+j-1}t_1^{k_1-j} g_1^{-1}.
\]

On the right hand side we obtain a term which is the homologous word of \( t^{k_0}t_1^{k_1} \) with scalar \( q^{-k_1} \in \mathbb{C} \), the homologous word again followed by \( g_1^{-2} \in H_2(q) \) and with scalar \( q^{-(k_1-1)}(q^{-1} - 1) \in \mathbb{C} \) and the terms \( t_{k_0+j-1}t_1^{k_1-j} \), which are of less order than the homologous word \( t^{k_0}t_1^{k_1} \), since \( k_1 > k_1 + 1 - j \), for all \( j \in \{2, 3, \ldots, k_1\} \). So the statement holds for \( m = 1 \) and \( k_1 \in \mathbb{N} \). The case \( m = 1 \) and \( k_1 \in \mathbb{Z}\setminus\mathbb{N} \) is similar.

Suppose now that the relations hold for \( m - 1 \). Then, for \( m \) we have:

\[
t^{k_0}t_1^{k_1} \cdots t_m^{k_m} \text{ ind. step } q^{-\sum_{m=1}^{m-1} nk_n} \cdots t^{k_0} \cdots t_{m-1} \cdot t_m^{k_m} + \\
+ \sum_i f_i(q) t^{k_0}t_1^{k_1} \cdots t_{m-1}^{k_{m-1}} w_i \cdot t_m^{k_m} + \\
+ \sum_j g_j(q) \tau_j \cdot u_j \cdot t_m^{k_m}.
\]

Now, since \( w_i, u_i \in H_m(q), \forall i \) we have that \( w_it^{k_m} = t_m^{k_m}w_i \) and \( u_it^{k_m} = t_m^{k_m}u_i, \forall i \). Applying now Lemma 11 to \( t_m^{k_m} \) we obtain the requested relation. \( \Box \).
Example 1. We convert the monomial $tt_1 t_2^{-2} \in \Lambda$ to linear combination of elements in $\Sigma_n$. We have that:

\[
\begin{align*}
t_1' &= q^{-1}t_1 + (q^{-1} - 1)t_1g_1^{-1} \quad \text{(Lemma 9)}, \\
t_2'^{-2} &= q^4 t_2^{-2} + q^3(q - 1)t_1^{-1}t_2^{-1}g_2^{-1} + q^2(q - 1)t_2^{-1}g_2g_1^{-1}g_2^{-1} + q^2(q - 1)t_1^{-2}g_2^{-1} + q(q - 1)t_1^{-1}g_2^{-1} + (q - 1)t_2^{-2}g_2g_1^{-1}g_2^{-1} \quad \text{(Lemma 10)},
\end{align*}
\]
and so:

\[
\begin{align*}
tt_1 t_2'^{-2} &= q^3 \cdot tt_1 t_2'^{-2} + q^4(q^{-1} - 1) \cdot tt_1 t_2'^{-2} \cdot g_1^{-1} + 1 \cdot tt_1 t_2'^{-2} + t_1^{-1} \cdot \left( (q - 1)(q^2 - q + 1) \cdot g_2^{-1} - (q - 1) \cdot g_1 g_2^{-1} g_2^{-1} \right) + \\
&+ tt_2'^{-1} \cdot \left( q^2(q - 1) \cdot g_2^{-1} + q(q - 1)^3 \cdot g_2^{-1} - q(q - 1) \cdot g_2 g_1^{-1} g_2^{-1} \right) + \\
&+ t_1^{-1} \cdot (q(q - 1) \cdot g_2 g_1^{-1} g_2^{-1} - q(q - 1)^2 \cdot g_1 g_2^{-1} g_2^{-1} + q(q - 1)) \cdot g_2^{-1} + \\
&+ t^{-1} \cdot \left( (q - 1) \cdot g_2 g_1^{-1} g_2^{-1} - q(q - 1)^2 \cdot g_1^{-1} g_2^{-1} \right)
\end{align*}
\]

where $u = (q - 1)^2 g_1^{-1} g_2^{-1} - (q - 1)^3 g_1^{-2} g_2^{-1} - q^{-1}(q - 1)^3 g_2 g_1^{-1} g_2^{-1} + q^{-1}(q - 1)^3 g_2^{-1}$.

We obtain the homologous word $w = tt_1 t_2'^{-2}$, the homologous word again followed by the braiding generator $g_1^{-1}$ and terms in $\Sigma_n$ of less order than $w$, since either their index is less that $\text{ind}(w)$ (the terms $tt_1'^{-1}$, 1 and $t^{-1}t_{1}$), either they contain gaps in the indices (the terms $tt_2'^{-2}$ and $tt_1'^{-1}$).

5. From $\Sigma_n$ to $\Lambda$

In order to prove Theorem 2 we need to show that the set $\Lambda$ is a spanning set of $\mathcal{S}(ST)$ and also that it is linearly independent. In this section we show that every element in $\Lambda$ can be expressed in terms of elements in the set $\Lambda$. Linear independence of the set $\Lambda$ is shown in the next section.

Before proceeding we need to discuss the following situation. According to Lemma 9, for a word $w' = t^k t_1^{-\lambda} \in \Lambda'$, where $k, \lambda \in \mathbb{N}$ and $k < \lambda$ we have that:

\[
w' = t^k t_1^{-\lambda} = t^{k-1} t_1^{-\lambda+1} \alpha_1 + t^{k-2} t_1^{-\lambda+2} \alpha_2 + \ldots + \\
+ t^0 t_1^{-\lambda+k} \alpha_k + t^{-1} t_1^{-\lambda+k+1} \alpha_{k+1} + \ldots + t^{-\lambda+k} \alpha_{\lambda},
\]

where $\alpha_i \in H_n(q), \; \forall i$. We observe that in this particular case, in the right hand side there are terms which do not belong to the set $\Lambda$. These are the terms of the form $t^q t_1^p$, where $p > q$ and the term $t_1^n$. So these elements cannot be compared with the highest order term $w \sim w'$. The point now is that these terms are elements in the basis $\Sigma_n$ on the Hecke algebra level, but, when we are working in $\mathcal{S}(ST)$, such elements must be considered up to conjugation by any braiding generator and up to stabilization moves. Topologically, conjugation corresponds to closing the braiding part of a mixed braid. Conjugating $t_1$ by $g_1^{-1}$ we obtain $tg_1^2$ (view Fig. 14) and similarly conjugating $t_1^m$ by $g_1^{-1}$ we obtain $tg_1^m$ and $tt_2^m$. Then, applying Lemma 3 we obtain the expression $\sum_{k=1}^{m-1} t^k t_1^{m-k} v_k$, where $v_k \in H_n(q)$, for all $k$, that is, we obtain now elements with consecutive indices but not necessarily with ordered exponents.
We shall first deal with elements where the looping generators do not have consecutive indices, and then with elements where the exponents are not in decreasing order. For the expressions that we obtain after appropriate conjugations we shall use the notation $\widehat{\cdot}$.

5.1. Managing the gaps

We will call gaps in monomials of the $t_i$’s, gaps occurring in the indices and size of the gap $k_i^{j_1}k_j$ the number $s_{i,j} = j - i \in \mathbb{N}$.

**Lemma 12.** For $k_0, k_1 \ldots k_i \in \mathbb{Z}$, $\epsilon = 1$ or $\epsilon = -1$ and $s_{i,j} > 1$ the following relation holds in $H_{1,n}(q)$:

$$t_{k_0}t_{k_1}^{i_1} \ldots t_{k_{i-1}}^{i_{i-1}} t_{i}^{i_i} \cdot t_j^{j} = t_{k_0}t_{k_1}^{i_1} \ldots t_{k_{i-1}}^{i_{i-1}} t_{i}^{i_i} \cdot t_{i+1}^{i_{i+1}} (g_{j+2} \ldots g_j) (g_{j+2} \ldots g_j) \ldots g_1) + g_{j+2} \ldots g_j) \ldots g_1).$$

**Proof.** We have that $t_j^j = (g_j^2 \ldots g_j^{j+2}) t_{i+1}^{i_{i+1}} (g_{j+2} \ldots g_j^j) \ldots g_1) + g_{j+2} \ldots g_j) \ldots g_1)$ and so:

$$t_{k_0}t_{k_1}^{i_1} \ldots t_{k_{i-1}}^{i_{i-1}} t_{i}^{i_i} \cdot t_j^{j} = t_{k_0}t_{k_1}^{i_1} \ldots t_{k_{i-1}}^{i_{i-1}} t_{i}^{i_i} (g_j^2 \ldots g_j^{j+2}) t_{i+1}^{i_{i+1}} (g_{j+2} \ldots g_j^j) \ldots g_1) + g_{j+2} \ldots g_j) \ldots g_1).$$

In order to pass to a general way for managing gaps in monomials of $t_i$’s we first deal with gaps of size one. For this we have the following.

**Lemma 13.** For $k \in \mathbb{N}$, $\epsilon = 1$ or $\epsilon = -1$ and $\alpha \in H_{1,n}(q)$ the following relations hold:

$$t_{i}^{t_{k}} \cdot \alpha \equiv \sum_{u=1}^{k-1} q^{(u-1)}(q^u - 1)t_{i+1}^{u}t_{i}^{(k-u)}(\alpha g_i^u) + q^{(k-u)}t_{i+1}^{u}(g_i^u \alpha g_i^u).$$

**Proof.** We prove the relations by induction on $k$. For $k = 1$ we have $t_{i}^{t_{1}} \cdot \alpha \equiv g_{i}t_{i+1}^{1}g_{i} \cdot \alpha \equiv t_{i+1}^{1}g_{i} \cdot \alpha \cdot g_{i}^1$. Suppose that the assumption holds for $k - 1 > 1$. Then for $k$ we have:

$$t_{i}^{t_{k}} \cdot \alpha \equiv t_{i}^{(k-1)}(t_{i+1}^{t_{1}} \cdot \alpha) = \beta \quad \text{ind. step}$$

$$\equiv \sum_{u=1}^{k-2} q^{(u-1)}(q^u - 1)t_{i+1}^{u+1}t_{i}^{(k-u)}(\beta g_i^u) + q^{(k-u)}t_{i+1}^{u+1}(g_i^u \beta g_i^u) = \beta = t_{i}^{t_{k}} \cdot \alpha \equiv \sum_{u=1}^{k-2} q^{(u-1)}(q^u - 1)t_{i+1}^{u+1}t_{i}^{(k-u)}(\alpha g_i^u) + q^{(k-u)}t_{i+1}^{u+1}(g_i^u \alpha g_i^u) = \sum_{u=1}^{k-2} q^{(u-1)}(q^u - 1)t_{i+1}^{u+1}t_{i}^{(k-u)}(\alpha g_i^u) + q^{(k-u)}t_{i+1}^{u+1}(g_i^u \alpha g_i^u) = \sum_{u=1}^{k-1} q^{(u-1)}(q^u - 1)t_{i+1}^{u+1}t_{i}^{(k-u)}(\alpha g_i^u) + q^{(k-u)}t_{i+1}^{u+1}(g_i^u \alpha g_i^u).$$

We now introduce the following notation.

**Notation 1.** We set $\tau_{i,i+m} := t_{i}^{k_{i}}t_{i+1}^{k_{i+1}} \ldots t_{i+m}^{k_{i+m}}$, where $m \in \mathbb{N}$ and $k_j \neq 0$ for all $j$ and $m 

$$\delta_{i,j} := \begin{cases} g_i g_{i+1} \ldots g_j g_{j-1} & \text{if } i < j \\
 g_i g_{i-1} \ldots g_{j+1} g_j & \text{if } i > j \end{cases} \quad \delta_{i,k,j} := \begin{cases} g_i g_{i+1} \ldots g_k g_{k+1} \ldots g_j g_{j-1} & \text{if } i < j \\
 g_i g_{i-1} \ldots g_{k+1} g_k \ldots g_{j+1} g_j & \text{if } i > j \end{cases}$$

We also set $u_{i,j}$ an element in $H_{j+1}(q)$ where the minimum index in $w$ is $i$.

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Using now the notation introduced above, we apply Lemma 13 $s_{i,j}$-times to 1-gap monomials of the form $\tau_{0,i}^{k_{i,j}} \cdot t_{j}^{k_{j}}$ and we obtain monomials with no gaps in the indices, followed by words in $H_{n}(q)$.

Example 2. For $s_{i,j} > 1$ and $\alpha \in H_{n}(q)$ we have:

\[
\begin{align*}
(i) & \quad \tau_{0,i}^{k_{i,j}} \cdot t_{j} \cdot \alpha \equiv \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta, \\
(ii) & \quad \tau_{0,i}^{k_{i,j}} \cdot t_{j}^{2} \cdot \alpha \equiv \left( q^{j-(i+1)+2} \right) \frac{\frac{k_{i,j}}{t_{i+1}^{2}} \cdot \tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2}^{2}}{\delta_{j,i+2}} \cdot \gamma + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2} \cdot t_{i+3} \cdot \mu}{\delta_{j,i+2}}, \\
(iii) & \quad \tau_{0,i}^{k_{i,j}} \cdot t_{j}^{3} \cdot \alpha \equiv \left( q^{j-(i+3)+1} \right) \frac{q^{j-(i+2)+2} \cdot \tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2}^{2}}{\delta_{j,i+2}} \cdot \gamma + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2} \cdot t_{i+3} \cdot \mu}{\delta_{j,i+2}},
\end{align*}
\]

where $\beta = \left( q^{j-(i+3)+1} \right) \frac{q^{j-(i+2)+2} \cdot \tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2}^{2}}{\delta_{j,i+2}} \cdot \gamma + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2} \cdot t_{i+3} \cdot \mu}{\delta_{j,i+2}}$.

Applying Lemma 13 to the one gap word $\tau_{0,i}^{k_{i,j}} \cdot t_{j}^{k_{j}}$, where $k_{j} \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in H_{n}(q)$ we obtain:

\[
\tau_{0,i}^{k_{i,j}} \cdot t_{j}^{k_{j}} \cong \sum_{\lambda, i} \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1}^{\lambda_{i+1}} \cdots t_{j}^{k_{j}} \cdot \alpha'}{\delta_{j,i+2}} \text{ if } k_{j} < s_{i,j},
\]

where $\alpha', \beta', \in H_{n}(q), \sum_{\mu=1}^{i} \lambda_{\mu} = k_{j}, \lambda_{\mu} \geq 0, \forall \mu$ and if $\lambda_{u} = 0$, then $\lambda_{v} = 0, \forall v \geq u$.

More precisely:

Lemma 14. For the 1-gap word $A = \tau_{0,i}^{k_{i,j}} \cdot t_{j}^{k_{j}} \cdot \alpha$, where $\alpha \in H_{n}(q)$ we have:

\[
\begin{align*}
(i) & \quad \text{If } |k_{j}| < s_{i,j}, \text{ then: } A \equiv \left( q^{j-(i+1)+2} \right) \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2}^{2}}{\delta_{j,i+2}} \cdot \gamma + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2} \cdot t_{i+3} \cdot \mu}{\delta_{j,i+2}}, \\
(ii) & \quad \text{If } |k_{j}| \geq s_{i,j}, \text{ then: } A \equiv \left( q^{j-(i+3)+1} \right) \frac{q^{j-(i+2)+2} \cdot \tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2}^{2}}{\delta_{j,i+2}} \cdot \gamma + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2} \cdot t_{i+3} \cdot \mu}{\delta_{j,i+2}},
\end{align*}
\]

where $\beta$ and $\beta'$ are of the form $w_{i+1,j} \in H_{j+1}(q), \sum_{k_{j}} \frac{f(q,z) \tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta'$, means a sum of elements in $\Sigma_{n}$, such that in each one of them, the sum of the exponents of the looping generators $t_{i+1}, \ldots, t_{i+k_{j}}$ is equal to $k_{j}$, and such that $|k_{i+1,j}| < |k_{j}|$. Moreover, if $k_{j} = 0$, for some index $\mu$, then $k_{\mu} = 0$ for all $s > \mu$.

Proof. We prove the relations by induction on $k_{j}$. Let $0 < k_{j} < j - i$.

For $k_{j} = 1$ we have $A \equiv \left( q^{j-(i+1)+2} \right) \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2}^{2}}{\delta_{j,i+2}} \cdot \gamma + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2} \cdot t_{i+3} \cdot \mu}{\delta_{j,i+2}}$ (Lemma 12). Suppose that the relation holds for $k_{j} - 1 > 1$. Then for $k_{j}$ we have:

\[
A = \tau_{0,i}^{k_{i,j}} \cdot t_{j}^{k_{j}-1} \cdot (t_{j} \alpha) \cong \underbrace{A}_{\text{ind. step}} + \sum_{\substack{k_{i,j}+1 \leq j \leq k_{i,j}+1 \cdot \beta \cdot \gamma \cdot \mu}} \cdot \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot \delta_{i+2,j}}{\delta_{j,i+2}} \cdot \beta + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2}^{2}}{\delta_{j,i+2}} \cdot \gamma + \frac{\tau_{0,i}^{k_{i,j}} \cdot t_{i+1} \cdot t_{i+2} \cdot t_{i+3} \cdot \mu}{\delta_{j,i+2}}.
\]
We now consider $B$ and $C$ separately and apply Lemma 4 to both expressions:

$$B^{(L,4)} = \left[ q^{k_j-2} \right]^{j-(i+1)} t_{i+1}^{k_{j+1}} \sum_{t_{i+1}^{j+1}} \delta_{i+2,j} \alpha \delta_{j,i+2} + \tau_{0,i}^{k_{j+1}} \cdot t_{i+1}^{j+1} + \tau_{0,i}^{k_{j+1}} \cdot t_{i+1}^{j+1} \cdot \delta_{i+2,j} \alpha \delta_{j,i+2}.$$  

We now do conjugation on the $(j - (i + 3))$-one gap words that occur and since $t_k \beta = t_{i+3,k} \beta \delta_{k,i+3}$ we obtain:

$$B \equiv \left[ q^{k_j-1} \right]^{j-(i+1)} t_{0,i}^{k_{j+1}} \cdot \tau_{0,i}^{k_{j+1}} \cdot \delta_{i+2,j} \alpha \delta_{j,i+2} + \tau_{0,i}^{k_{j+1}} \cdot t_{i+1}^{j+1} \cdot \sum_{t_{i+1}^{j+1}} f(q) \delta_{i+3,k} \alpha \delta_{j,i+3} + \tau_{0,i}^{k_{j+1}} \cdot t_{i+1}^{j+1} \cdot \delta_{i+2,j} \alpha \delta_{j,i+2}.$$  

where $\beta_1 \in H_{j+1}(q)$.

Moreover, $C = \sum_{r_1} f(q) \tau_{0,i}^{k_{j+1}} \cdot \tau_{t_{i+1},i+1}^{k_{j+3}} \cdot \beta t_j \beta'$ and since $\beta = w_{t_{i+1},i+1} \cdot \beta_2$, we have that: $\beta \cdot t_j \equiv \beta' \cdot t_j$ (L, 4)

$\sum_{s=i+1}^j t_s \cdot \gamma_s$, where $\gamma_s \in H_{j+1}(q)$ and so: $C \equiv \sum_{r_1} f(q) \tau_{0,i}^{k_{j+1}} \cdot \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \beta_2$, where $\beta_2 \in H_{j+1}(q)$.  

This concludes the proof. \qed

We now pass to the general case of one-gap words.

**Proposition 3.** For the 1-gap word $B = \tau_{0,i}^{k_{j+1}} \cdot \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \alpha$, where $\alpha \in H_n(q)$ we have:

$$B \equiv \prod_{s=0}^m (q^{k_{j+1}-1})^{j-(i+1)} \cdot \tau_{0,i}^{k_{j+1}} \cdot \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \sum_{s=0}^m (\delta_{i+m+2-s,j+s} \cdot \alpha \cdot \prod_{s=0}^m (\delta_{i+m+2-s,j+s} \cdot \alpha'))$$

where $\alpha' \in H_n(q)$, $\sum u_{1,m} = k_j$ such that $u_1 < k_j$ and if $u_{\mu} = 0$, then $u_s = 0$, $\forall s > \mu$.

**Proof.** The proof follows from Lemma 14. The idea is to apply Lemma 14 on the expression $\tau_{0,i}^{k_{j+1}} \cdot t_{i+1}^{k_{j+1}} \cdot \rho_1$, where $\rho_1 = \tau_{t_{i+1},i+1}^{k_{j+1}}$ and obtain the terms $\tau_{0,i}^{k_{j+1}} \cdot t_{i+1}^{k_{j+1}} \cdot \rho_2$ and $\tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \rho_2$ and follow the same procedure until there is no gap in the word. \qed

We are now ready to deal with the general case, that is, words with more than one gap in the indices of the generators.

**Theorem 8.** For the $\phi$-gap words:

$$C = \tau_{0,i}^{k_{j+1}} \cdot \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \cdots \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \alpha,$$

where $k_i \in \mathbb{Z} \setminus \{0\}$ for all $i, \alpha \in H_n(q), \sigma_j, \mu_j \in \mathbb{N}$, such that $s_1 > 1$ and $s_j > s_{j-1} + \mu_{j-1}$ for all $j$ we have:

$$C \equiv \prod_{p=1}^\phi (q^{k_{i+j}-1})^{s_j-j-\sum_{p=1}^{j-1} \mu_p} \cdot \tau_{0,i}^{k_{j+1}} \cdot \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \cdots \tau_{t_{i+1},i+1}^{k_{j+1}} \cdot \alpha \cdot \left( \prod_{p=1}^\phi \alpha_p \right) + \sum_{v} f_v(q) \tau_{0,i}^{k_{j+1}} \cdot w_v,$$

where
\\((i)\ \alpha_j = \prod_{i=0}^{\phi} \delta_{i+j+1+\sum k=1 \mu_k-\lambda_j, \ i+s_j+\mu_j-\lambda_j}, \ j = \{1,2,\ldots, \phi\}\),
\\((ii)\ \alpha_j' = \prod_{i=0}^{\phi} \delta_{i+j+1+\sum k=1 \mu_k+\lambda_j, \ i+s_j+\lambda_j}, \ j = \{1,2,\ldots, \phi\}\),
\\((iii)\ \sum_{0<i+\phi+\sum p=1 \mu_p} = \prod_{i=1}^{\phi} \tau_{i+j+1+\sum k=1 \mu_k+\lambda_j+\lambda_j+\sum p=1 \mu_p},
\\((iv)\ \tau_{0,v} < \sum_{0<i+\phi+\sum p=1 \mu_p} = \sum_{0<i+\phi+\sum p=1 \mu_p} = \sum_{0<i+\phi+\sum p=1 \mu_p},
\\(v)\ w_v of the form \ w_{i+2,i+s_j+\phi} = H_{i+s_j+\phi+1}(q), \ for all v,
\\(vi)\ the scalars f_v(q) are expressions of q \in \mathbb{C} for all v.\)

**Proof.** We prove the relations by induction on the number of gaps. For the 1-gap word \ \tau_{0,0}^{\mu_j} \cdot \tau_{i+s_j+\phi}^{\mu_j} \cdot \alpha, \ where \ \alpha \in H_n(q), \ have:

\[ A = \left[ \prod_{\lambda=0}^{\mu} (q^{k_{i+s_j+\phi}-1} s^{1}-1) \cdot \tau_{0,0}^{\mu_j} \cdot \tau_{i+s_j+\phi}^{\mu_j} \cdot \alpha \cdot \sum_{v} f_v(q) \cdot \tau_{0,v}^{\mu} \cdot \cdot w_v, \right. \]

which holds from **Proposition 3**.

Suppose that the relation holds for \ (\phi - 1) \cdot gaps. Then for a \ \phi \cdot gap word we have:

\[ \sum_{v} f_v(q) \cdot \tau_{0,v}^{\mu} \cdot \cdot w_v. \]

**Example 3.** For the 2-gap word \ t_{1}^{k_{0,1}}t_{3}^{k_{3,1}}t_{5}^{k_{5,1}}t_{6}^{k_{6,1}} \in \Sigma_n \ we have:

\[ t_{1}^{k_{0,1}}t_{3}^{k_{3,1}}t_{5}^{k_{5,1}}t_{6}^{k_{6,1}} = t_{1}^{k_{0,1}}g_{1}t_{3}^{k_{3,1}}g_{3}t_{5}^{k_{5,1}}g_{5}t_{6}^{k_{6,1}}g_{3} = t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

\[ = t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

\[ = g_{3}g_{5}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = g_{3}g_{5}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

\[ = t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{3}g_{5}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

\[ = q^{2}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{3}g_{5}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

\[ = q^{2}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{3}g_{5}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

\[ = q^{2}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{3}g_{5}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

\[ = q^{2}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{3}g_{5}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

\[ = q^{2}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{3}g_{5}t_{1}^{k_{0,1}}t_{2}^{k_{2,1}}t_{3}^{k_{3,1}}t_{4}^{k_{4,1}}t_{6}^{k_{6,1}}g_{2} = \]

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\[ + q(q - 1)t^{k_0}t_i^1t_2t_3g_3g_4^2g_5g_6g_7g_8 + (q - 1)t^{k_0}t_i^1t_2t_3g_5g_4^2g_6^1. \]

\[ \cdot (g_4^2g_5g_6g_7) = q^2g_6^1g_5^1q^{k_0}t_i^1t_2t_3t_4^2g_5g_6^1g_7g_8 + \]

\[ + q(q - 1)g_6^1t^{k_0}t_i^1t_2t_3t_4^1g_5g_6^1g_7g_8 + \]

\[ + (q - 1)g_5t^{k_0}t_i^1t_2t_3t_4^1g_5g_6^1g_7g_8 \]

\[ \cong q^2t^{k_0}t_i^1t_2t_3t_4^1g_5g_6^1g_7g_8 + \]

\[ + q(q - 1)t^{k_0}t_i^1t_2t_3t_4^1g_5g_6^1g_7g_8 + (q - 1)t^{k_0}t_i^1t_2t_3t_4^1g_5g_6^1g_7g_8. \]

5.2. Ordering the exponents

We now deal with elements in \( \Sigma_n \), where the looping generators have consecutive indices but their exponents are not in decreasing order. More precisely, we will show that these elements can be expressed as sums of elements in the \( \bigcup_n H_n(q) \)-module \( \Lambda \), namely, as sums of elements in \( \Lambda \) followed by a braiding tail.

We will need the following lemma.

**Lemma 15.** The following relations hold in \( H_{1,n}(q) \) for \( \lambda \in \mathbb{N} \):

\[ t_i^k \cdot t_{i+1}^{k+\lambda} = \sum_j t_i^{u_j}t_{i+1}^{v_j} \cdot w_j, \]

where \( u_j + v_j = 2k + \lambda \), \( u_j \geq v_j \) and \( w_j \in H_n(q), \ \forall j \).

**Proof.** We have that

\[ t_i^k \cdot t_{i+1}^{k+\lambda} = t_i^k \cdot t_{i+1}^{k+\lambda} \overset{\text{L.13}}{=} \]

\[ = t_i^k \cdot t_{i+1}^{k+\lambda} \cdot \left( q^{\lambda-1}g_{i+1}t_i^1g_{i+1} + \sum_{j=0}^{\lambda-2} q^j(q - 1)t_{i+1}^{j+1}t_i^{\lambda-j} \right) = \]

\[ = q^{\lambda-1}t_i^k \cdot t_{i+1}^{k+\lambda} \cdot g_{i+1}t_i^1g_{i+1} + \sum_{j=0}^{\lambda-2} q^j(q - 1)t_{i+1}^{k+j+1}t_i^{\lambda-1-j}. \]

We obtained the term \( t_i^k \cdot t_{i+1}^{k+\lambda} \cdot g_{i+1}t_i^1g_{i+1} \), terms where the exponent of \( t_i \) is greater than the exponent of \( t_{i+1} \) and terms of the form \( t_i^{p_1}t_{i+1}^{p_2} \), where \( k < p_1 < p_2 < k + \lambda \). We apply Lemma 13 on the terms of the last form and repeat the same procedure until there are only elements of the form \( t_{i+1}^{n_1}t_i^{n_2} \), \( u_1 > u_2 \) left in each sum. Note that each time Lemma 13 is performed, a term of the form \( t_i^{m_1} \cdot t_{i+1}^{m_2} \cdot g_{i+1}t_i^1g_{i+1} \) appears.

For these elements we have:

\[ t_{i+1}^{m_1} \cdot t_i^{m_2} \cdot g_{i+1}t_i^1g_{i+1} \overset{\text{L.3}}{=} t_i^{m_1} \cdot \left( q(q - 1)^{m_1} \sum_{j=0}^{m_1-1} q^j t_{i+1}^{m_1-j} + q^{m_1}g_{i+1}t_i^{m_1} \right) \cdot t_i^{m_2}g_{i+1} \]

\[ = (q - 1)^{m_1} \sum_{j=0}^{m_1-1} q^j t_{i+1}^{m_1+m_2+j}g_{i+1}t_i^{m_1+j} + q^{m_1}t_i^{m_1} \cdot g_{i+1}t_i^{m_1+m_2} \cdot g_{i+1}. \]
We have obtained now elements where the exponent of $t_i$ is greater than the exponent of $t_{i+1}$ and the term
\[
t^m_i \cdot g_{i+1} t^{m_1+m_2} \cdot g_{i+1} = t^m_i \cdot g_{i+1} t^{m_1+m_2},
\]
and this concludes the proof. □

**Remark 5.** Let $\tau_{0,m}^k \in \Sigma_n$ such that $k < k_{i+1}$. Applying Lemma 15 on $\tau_{0,m}^k$ we obtain a sum of elements $\tau_j \in \Sigma_n$, such that $\tau_j < \tau$, $\forall j$, since the exponent of the generator $t_{i+1}$ in $\tau_j$ is less than $k_{i+1}$ for all $j$ (see Definition 2).

**Example 4.** Consider the element $tt^2_1 t^2_2 \in \Sigma_n$ and apply Lemma 15 on the first “bad” exponent occurring in the word, starting from right to left.
\[
\begin{align*}
&tt^2_1 t^2_2 = f_1(q) \cdot tt^3_1 t^2_2 \cdot w_1 + f_2(q) \cdot tt^4_1 t^2_2 \cdot w_2. \\
\end{align*}
\]
The terms obtained are still in $\Sigma_n$ but they have one “bad” exponent less. We apply Lemma 15 again and obtain:
\[
\begin{align*}
&tt^3_1 t^2_2 = f_3(q) \cdot t^3 t^3_1 t^2_2 \cdot w_3 + f_4(q) \cdot t^2 t^2_2 t^2 \cdot w_4 \\
&tt^4_1 t^2_2 = f_5(q) \cdot t^4 t^3_1 t^2_2 \cdot w_5 + f_6(q) \cdot t^3 t^2_2 t^2 \cdot w_6
\end{align*}
\]
All terms obtained now are in the $\bigcup_n H_n(q)$-module $\Lambda$ except from the element $t^3 t^2_1 t^2_2$. We apply Lemma 15 again and obtain:
\[
\begin{align*}
&t^3 t^2_1 t^2_2 = f_7(q) \cdot t^3 t^2_2 t^2 \cdot w_7.
\end{align*}
\]
So:
\[
\begin{align*}
&tt^2_1 t^2_2 = g_1(q) \cdot t^3 t^3_1 t^2_2 \cdot u_1 + g_2(q) \cdot t^2 t^2_2 t^2 \cdot u_2 + g_3(q) \cdot t^4 t^3_1 t^2_2 \cdot u_3
\end{align*}
\]
where $u_1, \ldots, u_5 \in H_n(q)$ and $g_1(q), \ldots, g_5(q) \in \mathbb{C}$.

**Theorem 9.** Applying conjugation on an element in $\Sigma_n$ we have that:
\[
\tau_{0,m}^k \cdot w = \sum_j \tau_{0,j}^l \cdot w_j,
\]
where $\tau_{0,j}^l \in \Lambda$ and $w, w_j \in H_n(q)$, $\forall j$.

**Proof.** We prove the statement by induction on the order of $\tau_{0,m}^k \cdot w \in \Sigma_n$, where order of an element in $\Sigma_n$ denotes the position of this element in $\Sigma_n$ with respect to total-ordering.

The base of the induction is Lemma 15 for $i = 0$. Suppose that the relation holds for all $\tau_j \cdot u_j \in \Sigma_n$ of less order than $\tau_{0,m}^k \cdot w$. Then, for $\tau_{0,m}^k \cdot w$ we have:

Let $k_0 > k_1 > \ldots > k_i < k_{i+1}$. Applying Lemma 15 on $\tau_{0,m}^k \cdot w$ we obtain:
\[
\begin{align*}
&\tau_{0,m}^k \cdot w := k_{0}^{k_0} k_1^{k_1} \ldots k_{i+1}^{k_{i+1}} \ldots k_{m}^{k_{m}} \cdot w = \sum_j t_{0}^{u_j} t_1^{v_j} \ldots t_{i+1}^{u_{i+1}} \ldots t_{m}^{u_{m}} \cdot w_j,
\end{align*}
\]
where $u_j > v_j < k_{i+1}$, $\forall j$, that is, a sum of lower order terms than $\tau_{0,m}^k \cdot w$ (see Remark 5). So, by the induction hypothesis, the relation holds. □
5.3. Eliminating the tails

So far we have seen how to convert elements in the basis \( \Lambda' \) to sums of elements in \( \Sigma_n \) and then, using conjugation, how these elements are expressed as sums of elements in the \( \bigcup_n H_n(q) \)-module \( \Lambda \). We will show now that using conjugation and stabilization moves all these elements of the \( \bigcup_n H_n(q) \)-module \( \Lambda \) are expressed to sums of elements in the set \( \Lambda \) with scalars in the field \( \mathbb{C} \). We will use the symbol \( \simeq \) when a stabilization move is performed and \( \tilde{\simeq} \) when both stabilization moves and conjugation are performed.

Let us consider a generic word in \( H_{1,n+1}(q) \). This is of the form \( \tau_{0,n}^{k_0} \cdot w_{n+1} \), where \( w_{n+1} \in H_{n+1}(q) \). Without loss of generality we consider the exponent of the braiding generator with the highest index to be \((-1)\) when the exponent of the corresponding loop generator is in \( \mathbb{N} \) and \((+1)\) when the exponent of the corresponding loop generator is in \( \mathbb{Z}\setminus\mathbb{N} \). We then apply Lemmas 3 and 4 in order to interact \( t_n^{k_n} \) with \( g_n^{\pm 1} \) and obtain words of the following form:

\[
\begin{align*}
(1) & \quad \tau_{0,p}^{\lambda_{0,p}} \cdot v, \quad \text{where } \tau_{0,p}^{\lambda_{0,p}} < \tau_{0,n}^{k_0} \text{ and } v \in H_{n+1}(q) \text{ of any length, or} \\
(2) & \quad \tau_{0,q}^{\lambda_{0,q}} \cdot u, \quad \text{where } \tau_{0,q}^{\lambda_{0,q}} < \tau_{0,n}^{k_0} \text{ and } u \in H_n(q) \text{ such that } l(u) < l(w).
\end{align*}
\]

In the first case we obtain monomials of \( t_i \)'s of less order than the initial monomial, followed by a word in \( H_{n+1}(q) \) of any length. After at most \((k_n+1)\)-interactions of \( t_n \) with \( g_n \), the exponent of \( t_n \) will become zero and so by applying a stabilization move we obtain monomials of \( t_i \)'s of less index, and thus of less order (Definition 2), followed by a word in \( H_n(q) \).

In the second case, we have monomials of \( t_i \)'s of less order than the initial monomial followed by words \( u \in H_n(q) \) such that \( l(u) < l(w) \). We interact the generator with the maximum index of \( u, g_n \), with the corresponding loop generator until the exponent of \( t_m \) becomes zero. A gap in the indices of the monomials of the \( t_i \)'s occurs and we apply Theorem 8. This leads to monomials of \( t_i \)'s of less order followed by words of the braiding generators of any length. We then apply stabilization moves and repeat the same procedure until the braiding ‘tails’ are eliminated.

**Theorem 10.** Applying conjugation and stabilization moves on a word in the \( \bigcup_{\infty} H_n(q) \)-module, \( \Lambda \) we have that:

\[
\tau_{0,m}^{k_0} \cdot w_{n} \simeq \sum_j f_j(q, z) \cdot \tau_{0,u_j}^{v_{0,u_j}},
\]

such that \( \sum v_{0,u_j} = \sum k_{0,m} \) and \( \tau_{0,u_j}^{v_{0,u_j}} < \tau_{0,m}^{k_0} \), for all \( j \).

The logic for the induction hypothesis is explained above. We shall now proceed with the proof of the theorem.

**Proof of Theorem 10.** We prove the statement by double induction on the length of \( w_{n} \in H_n(q) \) and on the order of \( \tau_{0,m}^{k_0} \in \Lambda \), where order of \( \tau_{0,m}^{k_0} \) denotes the position of \( \tau_{0,m}^{k_0} \) in \( \Lambda \) with respect to total-ordering.

For \( l(w) = 0 \), that is for \( w = e \) we have that \( \tau_{0,m}^{k_0} \simeq \tau_{0,m}^{k_0} \) and there’s nothing to show. Moreover, the minimal element in the set \( \Lambda \) is \( t_k \) and for any word \( w \in H_n(q) \) we have that \( t_k \cdot w \simeq f(q, z) \cdot t_k \), by the quadratic relation and stabilization moves.

Suppose that the relation holds for all \( \tau_{0,p}^{l_0} \cdot w' \), where \( \tau_{0,p}^{l_0} \leq \tau_{0,m}^{k_0} \) and \( l(w') = l \), and for all \( \tau_{0,q}^{l_0} \cdot w \), where \( \tau_{0,q}^{l_0} < \tau_{0,m}^{k_0} \) and \( l(w) = l + 1 \). We will show that it holds for \( \tau_{0,m}^{k_0} \cdot w \). Let the exponent of \( t_r, k_r \in \mathbb{N} \) and let \( w \in H_{r+1}(q) \). Then, \( w \) can be written as \( w' \cdot g_{r}^{-1} \cdot \delta_{r-1,d} \), where \( w' \in H_{r}(q) \) and \( d < r \). We have that:
We have that \( \left( \frac{\tau_{j_0, r_1, r_2}}{\tau_{0, r_1, r_2}} \right)^m \cdot \tau_{j_0, r_1, r_2}^{1-n} \cdot \tau_{0, r_1, r_2}^{1-n} = \left( \frac{\tau_{j_0, r_1, r_2}}{\tau_{0, r_1, r_2}} \right)^m \cdot \tau_{j_0, r_1, r_2}^{1-n} \cdot \tau_{0, r_1, r_2}^{1-n} \), for all \( j \in \{ 1, 2, \ldots, r - 1 - d \} \) and \( l \left( \omega \cdot \delta_{r_1, r_2} \right) = l \) and \( l \left( \frac{\tau_{j_0, r_1, r_2}}{\tau_{0, r_1, r_2}} \right)^m \cdot \tau_{j_0, r_1, r_2}^{1-n} \cdot \tau_{0, r_1, r_2}^{1-n} \). So, by the induction hypothesis, the relation holds. \( \square \)

**Example 5.** In this example we demonstrate how to eliminate the braiding ‘tail’ in a word in \( \Sigma_n \).

\[
t^{-1}t_2t_1^{-1}g_1^{-1} = t^{-1}t_1^{-1}t_2^{-1}g_1^{-1} = t^{-1}t_1^{-1}t_2^{-1}g_1^{-1} = t_1^{-1}t_2^{-1}g_1 = (q - 1)t_1^{-1}t_2^{-1}g_1 = q^{-1}(q - 1)g_1g_2 = (q - 1)g_2^{-1}g_2^{-1} + q^{-1}(q - 1)g_2^{-1}g_2^{-1}.
\]

We have that:

\[
g_2^{-1}g_1^{-1} = q^{-2}g_1g_2g_1 + q^{-1}(q - 1)g_2g_1 + q^{-1}(q - 1)g_1g_2 + (q - 1)^2g_2,
\]

\[
g_2^{-1}g_2^{-1} = q^{-2}(q - 1)g_1g_2g_1 - (q - 1)^2g_2g_1 - (q - 1)g_1g_2 + (q - 1)(q - 1)g_2^{-1} + q(q - 1)g_2^{-1} + 1,
\]

and so

\[
(q - 1)t_1^{-1}g_2^{-1}g_2^{-1} \simeq (q - 1) + q^{-1}(q - 1)^3 \cdot t_1^{-1} - q^{-3}(q - 1)^3z \cdot 1 + + 3q^{-3}(q - 1)^3z \cdot 1 - q^{-1}(q - 1)^2z \cdot 1 - q^{-3}(q - 1)^5 \cdot 1,
\]

\[
q^{-1}g_2^{-1}g_1^{-1} \simeq z \cdot t_1^{-1} + q^{-1}(q - 1)^3z \cdot 1 + 2(q - 1)^2z \cdot 1 + + q(q - 1)^3 \cdot 1.
\]

**6. The basis \( \Lambda \) of \( \mathcal{S}(ST) \)**

In this section we show that the set \( \Lambda \) is linearly independent. This is done in two steps:

- We first relate the two sets \( \Lambda \) and \( \Lambda' \) via an infinite lower triangular matrix with invertible elements in the diagonal.
- Then, using the matrix mentioned above, we prove that the set \( \Lambda \) is linearly independent.

**6.1. The infinite matrix**

With the orderings given in **Definition 2** we shall show that the infinite matrix converting elements of the basis \( \Lambda' \) to elements of the set \( \Lambda \) is a block diagonal matrix, where each block is an infinite lower triangular matrix with invertible elements in the diagonal. Note that applying conjugation and stabilization moves on an element of some \( \Lambda_k \) followed by a braiding part won’t alter the sum of the exponents of the loop generators and thus, the resulted terms will belong to the set of the same level \( \Lambda_k \). Fixing the level \( k \) of a subset of \( \Lambda' \), the proof of **Theorem 2** is equivalent to proving the following claims:
\[ N' \ni \tau' \quad \stackrel{\text{Thm.7}}{\Rightarrow} \quad q^A \cdot \tau' + \sum_i f_i(q,z) \cdot \tau \cdot w_i + \sum_i g_i(q,z) \cdot \tau \cdot u_i \]

(1) A monomial \( w' \in \Lambda'_k \subseteq \Lambda' \) can be expressed as linear combinations of elements in \( \Lambda_k \subseteq \Lambda \), \( v_i \), followed by monomials in \( H_n(q) \), with scalars in \( \mathbb{C} \) such that \( \exists j : v_j = w \sim w' \).

(2) Applying conjugation and stabilization moves on all \( v_i \)'s results in obtaining elements in \( \Lambda_k \), \( u_i \)'s, such that \( u_i < v_i \) for all \( i \).

(3) The coefficient of \( w \) is an invertible element in \( \mathbb{C} \).

(4) \( \Lambda_k \ni w < u \in \Lambda_{k+1} \).

Indeed we have the following: Let \( w' \in \Lambda'_k \subseteq \Lambda' \). Then, by Theorem 7 the monomial \( w' \) is expressed as a sum of elements in \( \Sigma_n \), where the only term that isn’t followed by a braiding part is the homologous monomial \( w \in \Lambda_k \subseteq \Lambda \). Other terms in the sum involve lower order terms than \( w \) (with possible gaps in the indices and possible non-ordered exponents) followed by a braiding part and words of the form \( w \cdot \beta \), where \( \beta \in H_n(q) \). Then, by Theorem 8 elements in \( \Sigma_n \) are expressed to linear combinations of elements in \( \Sigma_n \) with no gaps in the indices of the looping generators (regularizing elements with gaps) and obtaining words which are of less order than the initial word \( w \). Then, by Theorem 9 we express these elements to linear combinations of elements in the \( H_n(q) \)-module \( \Lambda \), again of less order than \( w \). In Theorem 10 all elements that are followed by a braiding part are expressed as sums of monomials in \( t_i \)'s with coefficients in \( \mathbb{C} \). It is essential to mention that when applying Theorem 10 to a word of the form \( w \cdot \beta \) one obtains monomials in \( t_i \)'s that are less ordered than \( w \). Some of these monomials in \( t_i \)'s are in \( \Lambda \) and some have their exponents in non-decreasing order, but all monomials are of less order than \( w \). We apply again Theorem 9 on these monomials \( \tau \) that don’t belong in the set \( \Lambda \) and obtain words of less order than \( \tau \), followed by a braiding part. We only consider now the monomials not in \( \Lambda \) and perform Theorem 9. We obtain elements in the \( H_n(q) \)-module \( \Lambda \) of less order than the initial monomials, followed by a braiding part. Eventually this procedure stops at the lower order term of \( \Lambda_k \), \( t^k \). So we have obtained elements in \( \Lambda \) of lower order terms than the initial element \( w \), and thus, we obtain a lower triangular matrix with entries in the diagonal of the form \( q^{-A} \) (see Theorem 7), which are invertible elements in \( \mathbb{C} \). The fourth claim follows directly from Definition 2. (See Fig. 15.)

If we denote as \([\Lambda_k]\) the block matrix converting elements in \( \Lambda'_k \) to elements in \( \Lambda_k \) for some \( k \), then the change of basis matrix will be of the form:
6.2. Linear independence of $\Lambda$

**Theorem 11.** The set $\Lambda$ is linearly independent.

**Proof.** Consider an arbitrary subset of $\Lambda$ with finite many elements $\tau_1, \tau_2, \ldots, \tau_k$. Without loss of generality we consider $\tau_1 < \tau_2 < \ldots < \tau_k$ according to Definition 2. We convert now each element $\tau_i \in \Lambda$ to linear combination of elements in $\Lambda'$ according to the infinite matrix. We have that

$$\tau_i \simeq A_i \tau_i' + \sum_j A_j \tau_j',$$

where $\tau_i' \sim \tau_i$, $A_i \in \mathbb{C} \setminus \{0\}$, $\tau_i' < t_i'$ and $A_j \in \mathbb{C}$, $\forall j$.

So, we have that:

$$\begin{align*}
\tau_1 & \simeq A_1 \tau_1' + \sum_{j} A_{1j} \tau_{1j}' \\
\tau_2 & \simeq A_2 \tau_2' + \sum_{j} A_{2j} \tau_{2j}' \\
& \vdots \\
\tau_{k-1} & \simeq A_{k-1} \tau_{k-1}' + \sum_{j} A_{(k-1)j} \tau_{(k-1)j}' \\
\tau_k & \simeq A_k \tau_k' + \sum_{j} A_{kj} \tau_{kj}'
\end{align*}$$

Note that each $\tau_i'$ can occur as an element in the sum $\sum_{j} A_{pj} \tau_{pj}'$ for $p > i$. We consider now the equation

$$\sum_{i=1}^{k} \lambda_i \cdot \tau_i = 0, \quad \lambda_i \in \mathbb{C}, \quad \forall i$$

and we show that this holds only when $\lambda_i = 0$, $\forall i$. Indeed, we have:

$$\sum_{i=1}^{k} \lambda_i \cdot \tau_i = 0 \iff \lambda_k A_k \tau_k' + \sum_{i=1}^{k} \lambda_i A_{ij} \tau_{ij}' = 0,$$

where $\tau_k' > \tau_{ij}', \forall i, j$. So we conclude that $\lambda_k = 0$. Using the same argument we have that:

$$\sum_{i=1}^{k} \lambda_i \cdot \tau_i = 0 \iff \sum_{i=1}^{k-1} \lambda_i \cdot \tau_i = 0 \iff \lambda_{k-1} A_{k-1} \tau_{k-1}' + \sum_{i=1}^{k-1} \lambda_i A_{ij} \tau_{ij}' = 0,$$

where $\tau_{k-1} > \tau_{ij}', \forall i, j$. So, $\lambda_{k-1} = 0$. Retrospectively we get:

$$\sum_{i=1}^{k} \lambda_i \cdot \tau_i = 0 \iff \lambda_i = 0, \quad \forall i,$$

and so an arbitrary finite subset of $\Lambda$ is linearly independent. Thus, the set $\Lambda$ is linearly independent. □
6.3. The proof of the main result

By Theorems 7, 8, 9 and 10 the set $\Lambda$ is a spanning set of $S(\text{ST})$. By Theorem 11 the set $\Lambda$ is also linearly independent. Thus, it forms a basis for $S(\text{ST})$ and the proof of Theorem 2 is now concluded.

7. Conclusions

In this paper we gave a new basis $\Lambda$ for $S(\text{ST})$, different from the Turaev–Hoste–Kidwell basis and the Morton–Aiston basis. The new basis is appropriate for describing the handle sliding moves, whilst the old basis $\Lambda'$ is consistent with the trace rules [4]. In a sequel paper we use the bases $\Lambda'$ and $\Lambda$ of $S(\text{ST})$ and the change of basis matrix in order to compute the Homflypt skein module of the lens spaces $L(p, 1)$.

References