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A new basis for the Homflypt skein module of the solid torus ☆

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A B S T R A C T

In this paper we give a new basis, Λ, for the Homflypt skein module of the solid torus, $\mathcal{S}(\mathrm{ST})$, which topologically is compatible with the handle sliding moves and which was predicted by J.H. Przytycki. The basis Λ is different from the basis Λ', discovered independently by Hoste and Kidwell [1] and Turaev [2] with the use of diagrammatic methods, and also different from the basis of Morton and Aiston [3]. For finding the basis Λ we use the generalized Hecke algebra of type B, $\mathrm{H}_{1,n}$, which is generated by looping elements and braiding elements and which is isomorphic to the affine Hecke algebra of type A [4]. More precisely, we start with the well-known basis Λ' of $\mathcal{S}(\mathrm{ST})$ and an appropriate linear basis Σ_n of the algebra $\mathrm{H}_{1,n}$. We then convert elements in Λ' to sums of elements in Σ_n. Then, using conjugation and the stabilization moves, we convert these elements to sums of elements in Λ by managing gaps in the indices, by ordering the exponents of the looping elements and by eliminating braiding tails in the words. Further, we define total orderings on the sets Λ' and Λ and, using these orderings, we relate the two sets via a block diagonal matrix, where each block is an infinite lower triangular matrix with invertible elements in the diagonal. Using this matrix we prove linear independence of the set Λ, thus Λ is a basis for $\mathcal{S}(\mathrm{ST})$. $\mathcal{S}(\mathrm{ST})$ plays an important role in the study of Homflypt skein modules of arbitrary c.c.o. 3-manifolds, since every c.c.o. 3-manifold can be obtained by integral surgery along a framed link in S^3 with unknotted components. In particular, the new basis of $\mathcal{S}(\mathrm{ST})$ is appropriate for computing the Homflypt skein module of the lens spaces. In this paper we provide some basic algebraic tools for computing skein modules of c.c.o. 3-manifolds via algebraic means.

1. Introduction

Let M be an oriented 3-manifold, $R = \mathbb{Z}[u^{\pm 1}, z^{\pm 1}]$, $\mathcal{L}$ the set of all oriented links in M up to ambient isotopy in M and let S be the submodule of $R\mathcal{L}$ generated by the skein expressions $u^{-1}L_+ - uL_- - zL_0$,

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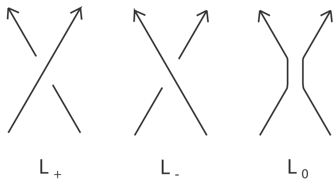

Fig. 1. The links L_+, L_-, L_0 locally.

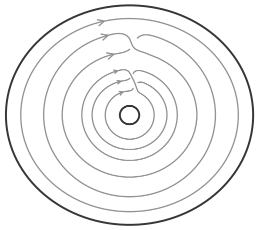

Fig. 2. A basic element of $\mathcal{S}(\mathrm{ST})$.

where L_+, L_- and L_0 are oriented links that have identical diagrams, except in one crossing, where they are as depicted in Fig. 1.

For convenience we allow the empty knot, $\emptyset$, and add the relation $u^{-1}\emptyset - u\emptyset = zT_1$, where T_1 denotes the trivial knot. Then the *Homflypt skein module* of M is defined to be:

$$\mathcal{S}\left(M\right) = \mathcal{S}\left(M; \mathbb{Z}\left[u^{\pm 1}, z^{\pm 1}\right], u^{-1}L_+ - uL_- - zL_0\right) = {}^{R\mathcal{L}}\!/_{S}.$$

Unlike the Kauffman bracket skein module, the Homflypt skein module of a 3-manifold, also known as *Conway skein module* and as *third skein module*, is very hard to compute (see [5] for the case of the product of a surface and the interval).

Let ST denote the solid torus. In [2,1] the Homflypt skein module of the solid torus has been computed using diagrammatic methods by means of the following theorem:

Theorem 1 *(Turaev, Kidwell–Hoste). The skein module $\mathcal{S}(\mathrm{ST})$ is a free, infinitely generated $\mathbb{Z}[u^{\pm 1}, z^{\pm 1}]$-module isomorphic to the symmetric tensor algebra $SR\widehat{\pi}^0$, where $\widehat{\pi}^0$ denotes the conjugacy classes of non-trivial elements of $\pi_1(\mathrm{ST})$.*

A basic element of $\mathcal{S}(\mathrm{ST})$ in the context of [2,1], is illustrated in Fig. 2. In the diagrammatic setting of [2] and [1], ST is considered as Annulus × Interval. The Homflypt skein module of ST is particularly important, because any closed, connected, oriented (c.c.o.) 3-manifold can be obtained by surgery along a framed link in S^3 with unknotted components.

A different basis of $\mathcal{S}(\mathrm{ST})$, known as Young idempotent basis, is based on the work of Morton and Aiston [3] and Blanchet [6].

In [4], $\mathcal{S}(\mathrm{ST})$ has been recovered using algebraic means. More precisely, the generalized Hecke algebra of type B, $\mathrm{H}_{1,n}(q)$, is introduced, which is related to the affine Hecke algebra of type A, $\widetilde{\mathrm{H}_n}(q)$ [4]. Then, a unique Markov trace is constructed on the algebras $\mathrm{H}_{1,n}(q)$ leading to an invariant for links in ST, the universal analogue of the Homflypt polynomial for ST. This trace gives distinct values on distinct elements of the [2,1]-basis of $\mathcal{S}(\mathrm{ST})$. The link isotopy in ST, which is taken into account in the definition of the skein module and which corresponds to conjugation and the stabilization moves on the braid level, is captured by the conjugation property and the Markov property of the trace, while the defining relation of the skein module is reflected into the quadratic relation of $\mathrm{H}_{1,n}(q)$. In the algebraic language of [4] the basis of $\mathcal{S}(\mathrm{ST})$, described in Theorem 1, is given in open braid form by the set Λ' in Eq. (4). Fig. 8 illustrates the basic

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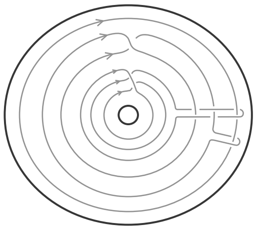

Fig. 3. An element of the new basis Λ.

element of Fig. 2 in braid notation. Note that in the setting of [4] ST is considered as the complement of the unknot (the bold curve in the figure). The looping elements $t'_i \in \mathrm{H}_{1,n}(q)$ in the monomials of Λ' are all conjugates, so they are consistent with the trace property and they enable the definition of the trace via simple inductive rules.

In this paper we give a new basis Λ for $\mathcal{S}(\mathrm{ST})$, which was predicted by J.H. Przytycki, using the algebraic methods developed in [4] (see Fig. 3). The motivation of this work is the computation of $\mathcal{S}\left(L(p,q)\right)$ via algebraic means. The new basic set is described in Eq. (1) in open braid form (see Fig. 9). The looping elements t_i are in the algebras $\mathrm{H}_{1,n}(q)$ and they are commuting. For a comparative illustration and for the defining formulas of the t_i's and the t'_i's the reader is referred to Fig. 7 and Eq. (3) respectively. Moreover, the t_i's are consistent with the handle sliding move or band move used in the link isotopy in $L(p,q)$, in the sense that a braid band move can be described naturally with the use of the t_i's (see for example [7] and references therein).

Our main result is the following:

Theorem 2. *The following set is a $\mathbb{Z}[q^{\pm 1}, z^{\pm 1}]$-basis for $\mathcal{S}(\mathrm{ST})$:*

$$\Lambda = \{t^{k_0} t_1^{k_1} \dots t_n^{k_n},\ k_i \in \mathbb{Z} \setminus \{0\},\ k_i \geq k_{i+1}\ \forall i,\ n \in \mathbb{N}\}. \tag{1}$$

Our method for proving Theorem 2 is the following:

- We define total orderings in the sets Λ' and Λ,
- we show that the two ordered sets are related via a lower triangular infinite matrix with invertible elements on the diagonal, and
- using this matrix, we show that the set Λ is linearly independent.

More precisely, two analogous sets, Σ_n and Σ'_n, are given in [4] as linear bases for the algebra $\mathrm{H}_{1,n}(q)$. See Theorem 4 in this paper. The set $\bigcup_n \Sigma_n$ includes Λ as a proper subset and the set $\bigcup_n \Sigma'_n$ includes Λ' as a proper subset. The sets Σ_n come directly from the works of S. Ariki and K. Koike, and M. Brouè and G. Malle on the cyclotomic Hecke algebras of type B. See [4] and references therein. The second set $\bigcup_n \Sigma'_n$ includes Λ' as a proper subset. The sets Σ'_n appear naturally in the structure of the braid groups of type B, $B_{1,n}$; however, it is very complicated to show that they are indeed basic sets for the algebras $\mathrm{H}_{1,n}(q)$. The sets Σ_n play an intrinsic role in the proof of Theorem 2. Indeed, when trying to convert a monomial λ' in Λ' into a linear combination of elements in Λ we pass by elements of the sets Σ_n. This means that in the converted expression of λ' we have monomials in the t_i's with possible gaps in the indices and possible non-ordered exponents, followed by monomials in the braiding generators g_i. So, in order to reach expressions in the set Λ we need:

- to manage the gaps in the indices of the t_i's,
- to order the exponents of the t_i's and
- to eliminate the braiding 'tails'.

Fig. 4. A mixed link in S^3.

The paper is organized as follows. In Section 2 we recall the algebraic setting and the results needed from [4]. In Section 3 we define the orderings in the two sets Σ_n and Σ'_n, which include the sets Λ and Λ' as subsets, and we prove that these sets are totally ordered. In Section 4 we prove a series of lemmas for converting elements in Λ' to elements in the sets Σ_n. In Section 5 we convert elements in Σ_n to elements in Λ using conjugation and the stabilization moves. Finally, in Section 6 we prove that the sets Λ' and Λ are related through a lower triangular infinite matrix mentioned above and that the set Λ is linearly independent. A computer program converting elements in Λ' to elements in Σ_n has been developed by K. Karvounis and will be soon available on http://www.math.ntua.gr/~sofia.

The algebraic techniques developed here will serve as basis for computing Homflypt skein modules of arbitrary c.c.o. 3-manifolds using the braid approach. The advantage of this approach is that we have an already developed homogeneous theory of braid structures and braid equivalences for links in c.c.o. 3-manifolds [8,9,7]. In fact, these algebraic techniques are used and developed further in [10] for knots and links in 3-manifolds represented by the 2-unlink.

The authors wish to express their gratitude for some critical comments of the Referee, which led to the improvement of the presentation of the paper.

2. The algebraic settings

2.1. Mixed links in S^3

We now view ST as the complement of a solid torus in S^3. An oriented link L in ST can be represented by an oriented *mixed link* in S^3 (see Fig. 4), that is a link in S^3 consisting of the unknotted fixed part $\widehat{I}$ representing the complementary solid torus in S^3 and the moving part L that links with $\widehat{I}$.

A *mixed link diagram* is a diagram $\widehat{I} \cup \widetilde{L}$ of $\widehat{I} \cup L$ on the plane of $\widehat{I}$, where this plane is equipped with the top-to-bottom direction of I.

Consider now an isotopy of an oriented link L in ST. As the link moves in ST, its corresponding mixed link will change in S^3 by a sequence of moves that keep the oriented $\widehat{I}$ pointwise fixed. This sequence of moves consists in isotopy in the S^3 and the *mixed Reidemeister moves*. In terms of diagrams we have the following result for isotopy in ST:

The mixed link equivalence in S^3 includes the classical Reidemeister moves and the mixed Reidemeister moves, which involve the fixed and the standard part of the mixed link, keeping $\widehat{I}$ pointwise fixed.

2.2. Mixed braids in S^3

By the Alexander theorem for knots in solid torus, a mixed link diagram $\widehat{I} \cup \widetilde{L}$ of $\widehat{I} \cup L$ may be turned into a *mixed braid* $I \cup \beta$ with isotopic closure (see Fig. 5). This is a braid in S^3 where, without loss of generality, its first strand represents $\widehat{I}$, the fixed part, and the other strands, β, represent the moving part L. The subbraid β shall be called the *moving part* of $I \cup \beta$.

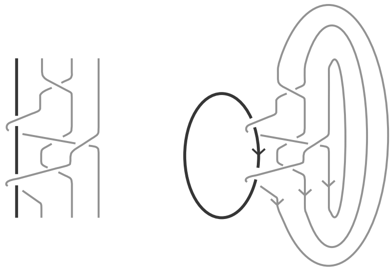

Fig. 5. The closure of a mixed braid to a mixed link.

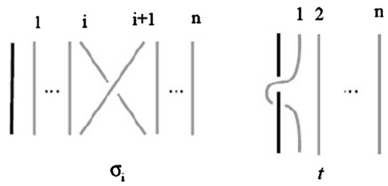

Fig. 6. The generators of $B_{1,n}$.

The sets of braids related to the ST form groups, which are in fact the Artin braid groups of type B, denoted $B_{1,n}$, with presentation:

$$B_{1,n} = \left\langle \; t, \sigma_1, \ldots, \sigma_{n-1} \quad \left| \begin{array}{l} \sigma_1 t \sigma_1 t = t \sigma_1 t \sigma_1 \\ t\sigma_i = \sigma_i t, \quad i > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n-2 \\ \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1 \end{array} \right. \right\rangle,$$

where the generators σ_i and t are illustrated in Fig. 6.

Isotopy in ST is translated on the level of mixed braids by means of the following theorem.

Theorem 3. *(See [11, Theorem 3].) Let L_1, L_2 be two oriented links in* ST *and let $I \cup \beta_1$, $I \cup \beta_2$ be two corresponding mixed braids in S^3. Then L_1 is isotopic to L_2 in* ST *if and only if $I \cup \beta_1$ is equivalent to $I \cup \beta_2$ in $\bigcup_{n=1}^{\infty} B_{1,n}$ by the following moves:*

(i)	*Conjugation:*	$\alpha \sim \beta^{-1}\alpha\beta,$	*if* $\alpha, \beta \in B_{1,n}$.
(ii)	*Stabilization moves:*	$\alpha \sim \alpha\sigma_n^{\pm 1} \in B_{1,n+1},$	*if* $\alpha \in B_{1,n}$.

2.3. The generalized Iwahori–Hecke algebra of type B

It is well known that $B_{1,n}$ is the Artin group of the Coxeter group of type B, which is related to the Hecke algebra of type B, $\mathrm{H}_n(q,Q)$ and to the cyclotomic Hecke algebras of type B. In [4] it has been established that all these algebras form a tower of B-type algebras and are related to the knot theory of ST. The basic one is $\mathrm{H}_n(q,Q)$, a presentation of which is obtained from the presentation of the Artin group $B_{1,n}$ by adding the quadratic relations

$$g_i^2 = (q-1)g_i + q \tag{2}$$

and the relation $t^2 = (Q-1)\,t + Q$, where $q, Q \in \mathbb{C}\backslash\{0\}$ are seen as fixed variables. The middle B-type algebras are the cyclotomic Hecke algebras of type B, $\mathrm{H}_n(q,d)$, whose presentations are obtained by the quadratic relation (2) and $t^d = (t-u_1)(t-u_2)\ldots(t-u_d)$. The topmost Hecke-like algebra in the tower is

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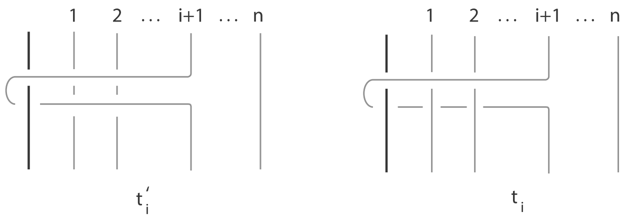

Fig. 7. The elements t'_i and t_i.

the *generalized Iwahori–Hecke algebra of type B*, $\mathrm{H}_{1,n}(q)$, which, as observed by T. tom Dieck, is related to the affine Hecke algebra of type A, $\widetilde{\mathrm{H}}_n(q)$ (cf. [4]). The algebra $\mathrm{H}_{1,n}(q)$ has the following presentation:

$$\mathrm{H}_{1,n}(q) = \left\langle \; t, g_1, \ldots, g_{n-1} \;\left|\; \begin{array}{l} g_1tg_1t = tg_1tg_1 \\ tg_i = g_it, \quad i > 1 \\ g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}, \quad 1 \le i \le n-2 \\ g_ig_j = g_jg_i, \quad |i-j| > 1 \\ {g_i}^2 = (q-1)g_i + q, \quad i = 1, \ldots, n-1 \end{array} \right. \right\rangle.$$

That is:

$$\mathrm{H}_{1,n}(q) = \frac{\mathbb{Z}\left[q^{\pm 1}\right] B_{1,n}}{\langle \sigma_i^2 - (q-1)\,\sigma_i - q \rangle}.$$

Note that in $\mathrm{H}_{1,n}(q)$ the generator t satisfies no polynomial relation, making the algebra $\mathrm{H}_{1,n}(q)$ infinite dimensional. Also that in [4] the algebra $\mathrm{H}_{1,n}(q)$ is denoted as $\mathrm{H}_n(q,\infty)$.

In [12] V.F.R. Jones gives the following linear basis for the Iwahori–Hecke algebra of type A, $\mathrm{H}_n(q)$:

$$S = \left\{ (g_{i_1}g_{i_1-1}\ldots g_{i_1-k_1})(g_{i_2}g_{i_2-1}\ldots g_{i_2-k_2})\ldots(g_{i_p}g_{i_p-1}\ldots g_{i_p-k_p}) \right\},$$

for $1 \le i_1 < \ldots < i_p \le n-1$.

The basis S yields directly an inductive basis for $\mathrm{H}_n(q)$, which is used in the construction of the Ocneanu trace, leading to the Homflypt or 2-variable Jones polynomial.

In $\mathrm{H}_{1,n}(q)$ we define the elements:

$$t_i := g_ig_{i-1}\ldots g_1tg_1\ldots g_{i-1}g_i \text{ and } t'_i := g_ig_{i-1}\ldots g_1tg_1^{-1}\ldots g_{i-1}^{-1}g_i^{-1}, \tag{3}$$

as illustrated in Fig. 7.

In [4] the following result has been proved.

Theorem 4. *(See [4, Proposition 1, Theorem 1].) The following sets form linear bases for* $\mathrm{H}_{1,n}(q)$*:*

$$\begin{array}{lll} (i) & \Sigma_n & = \{t_{i_1}^{k_1}t_{i_2}^{k_2}\ldots t_{i_r}^{k_r}\cdot\sigma, \ \text{where } 1 \le i_1 < \ldots < i_r \le n-1\}, \\ (ii) & \Sigma'_n & = \{{t'_{i_1}}^{k_1}{t'_{i_2}}^{k_2}\ldots {t'_{i_r}}^{k_r}\cdot\sigma, \ \text{where } 1 \le i_1 < \ldots < i_r \le n\}, \end{array}$$

where $k_1, \ldots, k_r \in \mathbb{Z}$ *and* σ *is a basic element in* $\mathrm{H}_n(q)$.

Remark 1.

(i) The indices of the t'_i's in the set Σ'_n are ordered but are not necessarily consecutive, neither do they need to start from t.
(ii) A more straightforward proof that the sets Σ'_n form bases for $\mathrm{H}_{1,n}(q)$ can be found in [13].

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In [4] the basis Σ'_n is used for constructing a Markov trace on $\bigcup_{n=1}^{\infty} \mathrm{H}_{1,n}(q)$.

Theorem 5. *(See [4, Theorem 6].) Given z, s_k, with $k \in \mathbb{Z}$ specified elements in $R = \mathbb{Z}\left[q^{\pm 1}\right]$, there exists a unique linear Markov trace function*

$$\mathrm{tr} : \bigcup_{n=1}^{\infty} \mathrm{H}_{1,n}(q) \to R\left(z, s_k\right), k \in \mathbb{Z}$$

determined by the rules:

$$\begin{array}{llll} (1) & \mathrm{tr}(ab) & = \mathrm{tr}(ba) & \textit{for } a, b \in \mathrm{H}_{1,n}(q) \\ (2) & \mathrm{tr}(1) & = 1 & \textit{for all } \mathrm{H}_{1,n}(q) \\ (3) & \mathrm{tr}(ag_n) & = z\,\mathrm{tr}(a) & \textit{for } a \in \mathrm{H}_{1,n}(q) \\ (4) & \mathrm{tr}(a{t'_n}^{k}) & = s_k\,\mathrm{tr}(a) & \textit{for } a \in \mathrm{H}_{1,n}(q),\ k \in \mathbb{Z}. \end{array}$$

Note that the use of the looping elements t'_i enables the trace tr to be defined by just extending the three rules of the Ocneanu trace on the algebras $\mathrm{H}_n(q)$ [12] by rule (4). Using tr Lambropoulou constructed a universal Homflypt-type invariant for oriented links in ST. Namely, let $\mathcal{L}$ denote the set of oriented links in ST. Then:

Theorem 6. *(See [4, Definition 1].) The function $X : \mathcal{L} \to R(z, s_k)$*

$$X_{\widehat{\alpha}} = \left[-\frac{1-\lambda q}{\sqrt{\lambda}\,(1-q)}\right]^{n-1} \left(\sqrt{\lambda}\right)^{e} \mathrm{tr}\left(\pi\left(\alpha\right)\right),$$

where $\lambda := \frac{z+1-q}{qz}$, $\alpha \in B_{1,n}$ is a word in the σ_i's and t'_i's, $\widehat{\alpha}$ is the closure of α, e is the exponent sum of the σ_i's in α, and π the canonical map of $B_{1,n}$ in $\mathrm{H}_{1,n}(q)$, such that $t \mapsto t$ and $\sigma_i \mapsto g_i$, is an invariant of oriented links in ST.

The invariant X satisfies a skein relation [4]. Theorems 4, 5 and 6 hold also for the algebras $\mathrm{H}_n(q,Q)$ and $\mathrm{H}_n(q,d)$, giving rise to all possible Homflypt-type invariants for knots in ST. For the case of the Hecke algebra of type B, $\mathrm{H}_n(q,Q)$, see also [11] and [14].

2.4. The basis of $\mathcal{S}(\mathrm{ST})$ in algebraic terms

Let us now see how $\mathcal{S}(\mathrm{ST})$ is described in the above algebraic language. We note first that an element α in the basis of $\mathcal{S}(\mathrm{ST})$ described in Theorem 1 when ST is considered as Annulus × Interval, can be illustrated equivalently as a mixed link in S^3 when ST is viewed as the complement of a solid torus in S^3. So we correspond the element α to the minimal mixed braid representation, which has decreasing order of twists around the fixed strand. Fig. 8 illustrates an example of this correspondence. Denoting

$$\Lambda' = \{t^{k_0}{t'_1}^{k_1}{t'_2}^{k_2}\ldots{t'_n}^{k_n},\ k_i \in \mathbb{Z}\setminus\{0\},\ k_i \geq k_{i+1}\ \forall i,\ n \in \mathbb{N}\}, \tag{4}$$

we have that Λ' is a subset of $\bigcup_n \mathrm{H}_{1,n}$. In particular Λ' is a subset of $\bigcup_n \Sigma'_n$.

Applying the inductive trace rules to a word w in $\bigcup_n \Sigma'_n$ will eventually give rise to linear combinations of monomials in $R(z, s_k)$. In particular, for an element of Λ' we have:

$$\mathrm{tr}(t^{k_0}{t'_1}^{k_1}\ldots{t'_{n-1}}^{k_{n-1}}) = s_{k_{n-1}}\ldots s_{k_1}s_{k_0}.$$

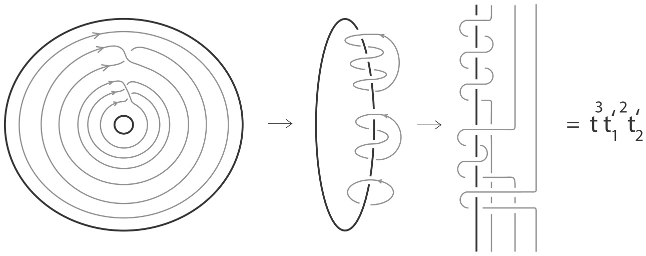

Fig. 8. An element in Λ'.

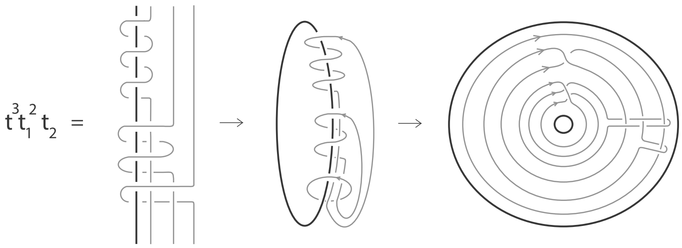

Fig. 9. An element of Λ.

Further, the elements of Λ' are in bijective correspondence with decreasing n-tuples of integers, $(k_0, k_1, \dots, k_{n-1})$, $n \in \mathbb{N}$, and these are in bijective correspondence with monomials in $s_{k_0}, s_{k_1}, \dots, s_{k_{n-1}}$. (See Fig. 8.)

Remark 2. The invariant X recovers the Homflypt skein module of ST since it gives different values for different elements of Λ' by rule (4) of the trace.

3. An ordering in the sets Λ and Λ'

In this section we define an ordering relation in the sets Σ'_n and Σ_n, which include Λ' and Λ as subsets. Before that, we will need the notion of the index of a word in Λ' or in Λ.

Definition 1. The *index* of a word w in Λ' or in Λ, denoted $ind(w)$, is defined to be the highest index of the t_i''s, resp. of the t_i's, in w. Similarly, the *index* of an element in Σ'_n or in Σ_n is defined in the same way by ignoring possible gaps in the indices of the looping generators and by ignoring the braiding part in $\mathrm{H}_n(q)$. Moreover, the index of a monomial in $\mathrm{H}_n(q)$ is equal to 0.

For example, $ind({t'}^{k_0} {t_1'}^{k_1} \dots {t_n'}^{k_n}) = ind(t^{u_0} \dots t_n^{u_n}) = n$.

Definition 2. We define the following *ordering* in the sets Σ'_n.

Let $w = {t'_{i_1}}^{k_1} {t'_{i_2}}^{k_2} \dots {t'_{i_\mu}}^{k_\mu}$ and $\sigma = {t'_{j_1}}^{\lambda_1} {t'_{j_2}}^{\lambda_2} \dots {t'_{j_\nu}}^{\lambda_\nu}$, where $k_t, \lambda_s \in \mathbb{Z}$, for all t, s. Then:

(a) If $\sum_{i=0}^{\mu} k_i < \sum_{i=0}^{\nu} \lambda_i$, then $w < \sigma$.
(b) If $\sum_{i=0}^{\mu} k_i = \sum_{i=0}^{\nu} \lambda_i$, then:
  (i) if $ind(w) < ind(\sigma)$, then $w < \sigma$,
  (ii) if $ind(w) = ind(\sigma)$, then:
    (α) if $i_1 = j_1$, $i_2 = j_2, \dots, i_{s-1} = j_{s-1}$, $i_s < j_s$, then $w > \sigma$,
    (β) if $i_t = j_t\ \forall t$ and $k_\mu = \lambda_\mu$, $k_{\mu-1} = \lambda_{\mu-1}, \dots, k_{i+1} = \lambda_{i+1}$, $|k_i| < |\lambda_i|$, then $w < \sigma$,
    (γ) if $i_t = j_t\ \forall t$ and $k_\mu = \lambda_\mu$, $k_{\mu-1} = \lambda_{\mu-1}, \dots, k_{i+1} = \lambda_{i+1}$, $|k_i| = |\lambda_i|$ and $k_i > \lambda_i$, then $w < \sigma$,
    (δ) if $i_t = j_t\ \forall t$ and $k_i = \lambda_i$, $\forall i$, then $w = \sigma$.

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(c) In the general case where $w = {t'_{i_1}}^{k_1} {t'_{i_2}}^{k_2} \ldots {t'_{i_\mu}}^{k_\mu} \cdot \beta_1$ and $\sigma = {t'_{j_1}}^{\lambda_1} {t'_{j_2}}^{\lambda_2} \ldots {t'_{j_\nu}}^{\lambda_\nu} \cdot \beta_2$, where $\beta_1, \beta_2 \in \mathrm{H}_n(q)$, the ordering is defined in the same way by ignoring the braiding parts β_1, β_2.

The same ordering is defined on the set Λ' by ignoring the braiding parts. Moreover, the same ordering is defined on the sets Σ_n and Λ, where the t'_i's are replaced by the corresponding t_i's.

Proposition 1. *The set Σ'_n equipped with the ordering given in Definition 2, is a totally ordered set.*

Proof. In order to show that the set Σ'_n is a totally ordered set when equipped with the ordering given in Definition 2, we need to show that the ordering relation is antisymmetric, transitive and total. We only show that the ordering relation is transitive. Antisymmetric property follows similarly. Totality follows from Definition 2 since all possible cases have been considered. Let $w, \sigma, v \in \Sigma_n$ such that:

$$\begin{aligned} w &= {t'_{i_1}}^{k_1} {t'_{i_2}}^{k_2} \ldots {t'_{i_m}}^{k_m} \cdot \beta_1, \\ \sigma &= {t'_{j_1}}^{\lambda_1} {t'_{j_2}}^{\lambda_2} \ldots {t'_{j_n}}^{\lambda_n} \cdot \beta_2, \\ v &= {t'_{\phi_1}}^{\mu_1} {t'_{\phi_2}}^{\mu_2} \ldots {t'_{\phi_p}}^{\mu_p} \cdot \beta_3, \end{aligned}$$

where $\beta_1, \beta_2, \beta_3 \in \mathrm{H}_n(q)$ and let $w < \sigma$ and $\sigma < v$. Since $w < \sigma$, one of the following holds:

(a) Either $\sum_{i=1}^m k_i < \sum_{i=1}^n \lambda_i$ and since $\sigma < v$, we have that $\sum_{i=1}^n \lambda_i \leq \sum_{i=1}^p \mu_i$ and so $\sum_{i=1}^m k_i < \sum_{i=1}^p \mu_i$. Thus $w < v$.

(b) Either $\sum_{i=1}^m k_i = \sum_{i=1}^n \lambda_i$ and $ind(w) = m < n = ind(\sigma)$. Then, since $\sigma < v$ we have that either $\sum_{i=1}^n \lambda_i < \sum_{i=1}^p \mu_i$ (same as in case (a)) or $\sum_{i=1}^n \lambda_i = \sum_{i=1}^p \mu_i$ and $ind(\sigma) \leq p = ind(v)$. Thus, $ind(w) = m < p = ind(v)$ and so we conclude that $w < v$.

(c) Either $\sum_{i=1}^m k_i = \sum_{i=1}^n \lambda_i$, $ind(w) = ind(\sigma)$ and $i_1 = j_1, \ldots, i_{s-1} = j_{s-1}$, $i_s > j_s$. Then, since $\sigma < v$, we have that either:
- $\sum_{i=1}^n \lambda_i < \sum_{i=1}^p \mu_i$, same as in case (a), or
- $\sum_{i=1}^n \lambda_i = \sum_{i=1}^p \mu_i$ and $ind(\sigma) < ind(v)$, same as in case (b), or
- $ind(\sigma) = ind(v)$ and $j_1 = \varphi_1, \ldots, j_p > \varphi_p$. Then:
  - (i) if $p = s$ we have that $i_s > j_s > \varphi_s$ and we conclude that $w < v$,
  - (ii) if $p < s$ we have that $i_p = j_p > \varphi_p$ and thus $w < v$ and if $s < p$ we have that $i_s > j_s = \varphi_s$ and so $w < v$.

(d) Either $\sum_{i=1}^m k_i = \sum_{i=1}^n \lambda_i$, $ind(w) = ind(\sigma)$ and $k_n = \lambda_n, \ldots, |k_q| < |\lambda_q|$. Then, since $\sigma < v$, we have that either:
- $\sum_{i=1}^n \lambda_i < \sum_{i=1}^p \mu_i$, same as in case (a), or
- $\sum_{i=1}^n \lambda_i = \sum_{i=1}^p \mu_i$ and $ind(\sigma) < ind(v)$, same as in case (b), or
- $ind(\sigma) = ind(v)$ and $j_1 = \varphi_1, \ldots, j_q > \varphi_q$, same as in case (c), or
- $j_n = \varphi_n$, for all n and $\mu_n = \lambda_n, \ldots, \mu_{c+1} = \lambda_{c+1}$, $|\mu_c| \geq |\lambda_c|$ for some c, then:
  - (1) If $|\mu_c| > |\lambda_c|$, then:
    - (i) If $c > q$ then $|k_c| = |\lambda_c| < |\mu_c|$ and thus $w < v$.
    - (ii) If $c < q$ then $|k_q| < |\lambda_q| = |\mu_q|$ and thus $w < v$.
    - (iii) If $c = q$ then $|k_q| < |\lambda_q| < |\mu_q|$ and thus $w < v$.
  - (2) If $|\mu_c| = |\lambda_c|$, such that $\mu_c < \lambda_c$, then:
    - (i) If $c > q$ then $|k_c| = |\lambda_c| = |\mu_c|$ and $k_c = \lambda_c > \mu_c$. Thus $w < v$.
    - (ii) If $c \leq q$ then $|k_q| < |\lambda_q| = |\mu_q|$ and thus $w < v$.

(e) Either $\sum_{i=1}^m k_i = \sum_{i=1}^n \lambda_i$, $ind(w) = ind(\sigma)$ and $k_n = \lambda_n, \ldots, |k_q| = |\lambda_q|$, such that $k_q > \lambda_q$. Then, since $\sigma < v$, we have that either:

- $\sum_{i=1}^{n}\lambda_i < \sum_{i=1}^{p}\mu_i$, same as in case (a), or
- $\sum_{i=1}^{n}\lambda_i = \sum_{i=1}^{p}\mu_i$ and $ind(\sigma) < ind(v)$, same as in case (b), or
- $ind(\sigma) = ind(v)$ and $j_1 = \varphi_1, \ldots, j_q > \varphi_q$, same as in case (c), or
- $j_n = \varphi_n$, for all n and $\mu_n = \lambda_n, \ldots, \mu_{c+1} = \lambda_{c+1}$, $|\mu_c| \geq |\lambda_c|$ for some c, then:
  (1) If $|\mu_c| > |\lambda_c|$, then:
    (i) If $c > q$ then $|k_c| = |\lambda_c| < |\mu_c|$, thus $w < v$.
    (ii) If $c \leq q$ then $|k_q| = |\lambda_q| = |\mu_q|$ and $k_q > \lambda_q = \mu_q$, thus $w < v$.
  (2) If $|\mu_c| = |\lambda_c|$ such that $\lambda_c > \mu_c$, then:
    (i) If $c > q$ then $|k_c| = |\lambda_c| = |\mu_c|$ and $k_c = \lambda_c > \mu_c$, thus $w < v$.
    (ii) If $c < q$ then $|k_q| = |\lambda_q| = |\mu_q|$ and $k_q > \lambda_q = \mu_q$, thus $w < v$.
    (iii) If $c = q$, then $|k_q| = |\lambda_q| = |\mu_q|$ and $k_q > \lambda_q > \mu_q$, thus $w < v$.

So, we conclude that the ordering relation is transitive. □

Remark 3. Proposition 1 also holds for the sets Σ_n, Λ' and Λ.

Definition 3. We define the subset of *level* k, Λ_k, of Λ to be the set

$$\Lambda_k := \{t^{k_0}t_1^{k_1}\ldots t_m^{k_m} | \sum_{i=0}^{m} k_i = k,\ k_i \in \mathbb{Z}\setminus\{0\},\ k_i \geq k_{i+1}\ \forall i\}$$

and similarly, the subset of *level* k of Λ' to be

$$\Lambda'_k := \{t^{k_0}{t'_1}^{k_1}\ldots {t'_m}^{k_m} | \sum_{i=0}^{m} k_i = k,\ k_i \in \mathbb{Z}\setminus\{0\},\ k_i \geq k_{i+1}\ \forall i\}.$$

Remark 4. Let $w \in \Lambda_k$ be a monomial containing gaps in the indices and $u \in \Lambda_k$ a monomial with consecutive indices such that $ind(w) = ind(u)$. Then, it follows from Definition 2 that $w < u$.

Proposition 2. *The sets Λ_k are totally ordered and well-ordered for all k.*

Proof. Since $\Lambda_k \subseteq \Lambda$, $\forall k$, Λ_k inherits the property of being a totally ordered set from Λ. Moreover, t^k is the minimum element of Λ_k and so Λ_k is a well-ordered set. □

We also introduce the notion of *homologous words* as follows:

Definition 4. We shall say that two words $w' \in \Lambda'$ and $w \in \Lambda$ are *homologous*, denoted $w' \sim w$, if w is obtained from w' by turning t'_i into t_i for all i.

With the above notion the proof of Theorem 2 is based on the following idea: Every element $w' \in \Lambda'$ can be expressed as linear combinations of monomials $w_i \in \Lambda$ with coefficients in $\mathbb{C}$, such that:

(i) $\exists\ j$ such that $w_j \sim w'$,
(ii) $w_j < w_i$, for all $i \neq j$,
(iii) the coefficient of w_j is an invertible element in $\mathbb{C}$.

4. From Λ' to Σ_n

In this section we prove a series of lemmas relating elements of the two different basic sets Σ_n, Σ'_n of $\mathrm{H}_{1,n}(q)$. In the proofs we underline expressions which are crucial for the next step. Since Λ' is a subset of Σ'_n, all lemmas proved here apply also to Λ' and will be used in the context of the bases of $\mathcal{S}(\mathrm{ST})$.

4.1. Some useful lemmas in $\mathrm{H}_{1,n}(q)$

We will need the following results from [4]. The first lemma gives some basic relations of the braiding generators.

Lemma 1. *(See [4, Lemma 1].) For $\epsilon \in \{\pm 1\}$ the following hold in $\mathrm{H}_{1,n}(q)$:*

$$\begin{array}{lrcl}
(i) & g_i^m & = & \left(q^{m-1} - q^{m-2} + \ldots + (-1)^{m-1}\right) g_i + \left(q^{m-1} - q^{m-2} + \cdots + (-1)^{m-2} q\right) \\
 & g_i^{-m} & = & \left(q^{-m} - q^{1-m} + \ldots + (-1)^{m-1} q^{-1}\right) g_i + \\
 & & + & \left(q^{-m} - q^{1-m} + \cdots + (-1)^{m-1} q^{-1} + (-1)^m\right) \\
(ii) & g_i{}^{\epsilon}(g_k{}^{\pm 1} g_{k-1}^{\pm 1} \ldots g_j{}^{\pm 1}) & = & (g_k{}^{\pm 1} g_{k-1}^{\pm 1} \ldots g_j{}^{\pm 1}) g_{i+1}{}^{\epsilon}, \quad \text{for } k > i \geq j, \\
 & g_i{}^{\epsilon}(g_j{}^{\pm 1} g_{j+1}^{\pm 1} \ldots g_k{}^{\pm 1}) & = & (g_j{}^{\pm 1} g_{j+1}^{\pm 1} \ldots g_k{}^{\pm 1}) g_{i-1}{}^{\epsilon}, \quad \text{for } k \geq i > j,
\end{array}$$

where the sign of the ± 1 exponent is the same for all generators.

$$\begin{array}{lrcl}
(iii) & g_i g_{i-1} \ldots g_{j+1} g_j g_{j+1} \ldots g_i & = & g_j g_{j+1} \ldots g_{i-1} g_i g_{i-1} \ldots g_{j+1} g_j \\
 & g_i{}^{-1} g_{i-1}^{-1} \ldots g_{j+1}^{-1} g_j{}^{\epsilon} g_{j+1} \ldots g_i & = & g_j g_{j+1} \ldots g_{i-1} g_i{}^{\epsilon} g_{i-1}^{-1} \ldots g_{j+1}^{-1} g_j{}^{-1} \\
(iv) & g_i{}^{\epsilon} \ldots g_{n-1}{}^{\epsilon} g_n{}^{2\epsilon} g_{n-1}{}^{\epsilon} \ldots g_i{}^{\epsilon} & = & \sum_{r=0}^{n-i+1} (q^{\epsilon} - 1)^{\epsilon_r} q^{\epsilon r} \left(g_i{}^{\epsilon} \ldots g_{n-r}{}^{\epsilon} \ldots g_i{}^{\epsilon}\right),
\end{array}$$

where $\epsilon_r = 1$ if $r \leq n - i$ and $\epsilon_{n-i+1} = 0$. Similarly,

$$(v) \quad g_i{}^{\epsilon} \ldots g_2{}^{\epsilon} g_1{}^{2\epsilon} g_2{}^{\epsilon} \ldots g_i{}^{\epsilon} = \sum_{r=0}^{i} (q^{\epsilon} - 1)^{\epsilon_r} q^{\epsilon r} \left(g_i{}^{\epsilon} \ldots g_{r+2}{}^{\epsilon} g_{r+1}{}^{\epsilon} g_{r+2}{}^{\epsilon} \ldots g_i{}^{\epsilon}\right),$$

where $\epsilon_r = 1$ if $r \leq i - 1$ and $\epsilon_i = 0$.

The next lemma comprises relations between the braiding generators and the looping generator t.

Lemma 2. *(Cf. [4, Lemmas 1, 4, 5].) For $\epsilon \in \{\pm 1\}$, $i, k \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$ the following hold in $\mathrm{H}_{1,n}(q)$:*

$$\begin{array}{lrl}
(i) & t^{\lambda} g_1 t g_1 & = g_1 t g_1 t^{\lambda} \\
(ii) & t^{\epsilon} g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} & = g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} t^{\epsilon} + (q^{\epsilon} - 1) t^{\epsilon} g_1{}^{\epsilon} t^{\epsilon k} + (1 - q^{\epsilon}) t^{\epsilon k} g_1{}^{\epsilon} t^{\epsilon} \\
 & t^{-\epsilon} g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} & = g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} t^{-\epsilon} + (q^{\epsilon} - 1) t^{\epsilon(k-1)} g_1{}^{\epsilon} + (1 - q^{\epsilon}) g_1{}^{\epsilon} t^{\epsilon(k-1)} \\
(iii) & t^{\epsilon i} g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} & = g_1^{\epsilon} t^{\epsilon k} g_1^{\epsilon} t^{\epsilon i} + (q^{\epsilon} - 1) \sum_{j=1}^{i} t^{\epsilon j} g_1^{\epsilon} t^{\epsilon(k+i-j)} + \\
 & & + (1 - q^{\epsilon}) \sum_{j=0}^{i-1} t^{\epsilon(k+j)} g_1^{\epsilon} t^{\epsilon(i-j)} \\
 & t^{-\epsilon i} g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} & = g_1{}^{\epsilon} t^{\epsilon k} g_1{}^{\epsilon} t^{-\epsilon i} + (q^{\epsilon} - 1) \sum_{j=1}^{i} t^{\epsilon(k-j)} g_1^{\epsilon} t^{-\epsilon(i-j)} + \\
 & & + (1 - q^{\epsilon}) \sum_{j=1}^{i} t^{\epsilon(i-j)} g_1^{\epsilon} t^{\epsilon(k-j)}
\end{array}$$

The next lemma gives the interactions of the braiding generators and the loopings t_i's and t'_i's.

Lemma 3. *(See [4, Lemmas 1 and 2].) The following relations hold in* $\mathrm{H}_{1,n}(q)$*:*

$$\begin{array}{lll}
(i) & g_i{t_k}^{\epsilon} & = {t_k}^{\epsilon} g_i \quad \text{for } k>i,\ k<i-1 \\
 & g_i t_i & = q t_{i-1} g_i + (q-1) t_i \\
 & g_i t_{i-1} & = q^{-1} t_i g_i + (q^{-1}-1) t_i \ = \ t_i g_i^{-1} \\
 & g_i t_{i-1}^{-1} & = q {t_i}^{-1} g_i + (q-1) {t_{i-1}}^{-1} \\
 & g_i t_i^{-1} & = q^{-1} {t_{i-1}}^{-1} g_i + (q^{-1}-1) {t_{i-1}}^{-1} \ = \ t_{i-1}^{-1} g_i^{-1} \\
(ii) & t_n^k g_n & = (q-1)\sum_{j=0}^{k-1} q^j t_{n-1}^j t_n^{k-j} + q^k g_n t_{n-1}^k, \ \text{if } k\in\mathbb{N} \\
 & t_n^k g_n & = (1-q)\sum_{j=0}^{k-1} q^j t_{n-1}^j t_n^{k-j} + q^k g_n t_{n-1}^k, \ \text{if } k\in\mathbb{Z}-\mathbb{N} \\
(iii) & {t_i}^k {t_j}^{\lambda} & = {t_j}^{\lambda} {t_i}^k \quad \text{for } i\neq j \quad \text{and } k,\lambda\in\mathbb{Z} \\
(iv) & g_i {t'_k}^{\epsilon} & = {t'_k}^{\epsilon} g_i \quad \text{for } k>i,\ \ k<i-1 \\
 & g_i {t'_i}^{\epsilon} & = {t'_{i-1}}^{\epsilon} g_i + (q-1){t'_i}^{\epsilon} + (1-q){t'_{i-1}}^{\epsilon} \\
 & g_i {t'_{i-1}}^{\epsilon} & = {t'_i}^{\epsilon} g_i \\
(v) & {t'_i}^k & = g_i \dots g_1 t^k {g_1}^{-1} \dots {g_i}^{-1} \quad \text{for } k\in\mathbb{Z}.
\end{array}$$

Using now Lemmas 1, 2 and 3 we prove the following relations, which we will use for converting elements in Λ' to elements in Σ_n. Note that whenever a generator is overlined, this means that the specific generator is omitted from the word.

Lemma 4. *The following relations hold in* $\mathrm{H}_{1,n}(q)$ *for* $k\in\mathbb{N}$*:*

$$\begin{array}{lll}
(i) & g_{m+1} t_m^k & = q^{-(k-1)} t_{m+1}^k g_{m+1}^{-1} \ + \ \sum_{j=1}^{k-1} q^{-(k-1-j)} (q^{-1}-1) t_m^j t_{m+1}^{k-j} \quad (\text{see Fig. 10}), \\
(ii) & g_{m+1}^{-1} t_m^{-k} & = q^{(k-1)} t_{m+1}^{-k} g_{m+1} \ + \ \sum_{j=1}^{k-1} q^{(k-1-j)} (q-1) t_m^{-j} t_{m+1}^{-(k-j)}.
\end{array}$$

Proof. We prove relation (i) by induction on k. Relation (ii) follows similarly. For $k=1$ we have that $g_{m+1}t_m = t_{m+1}g_{m+1}^{-1}$, which holds from Lemma 3(i). Suppose that the relation holds for $k-1$. Then, for k we have:

$$\begin{aligned}
g_{m+1} t_m^k &= g_{m+1} t_m^{k-1} t_m \overset{\text{ind.}}{\underset{\text{step}}{=}} q^{-(k-2)} t_{m+1}^{k-1} \underline{g_{m+1}^{-1}} t_m \ + \\
&+ \sum_{j=1}^{k-2} q^{-(k-2-j)} (q^{-1}-1) t_m^j t_{m+1}^{k-1-j} t_m = \\
&= q^{1-k} g_{m+1} t_m + q^{2-k} (q^{-1}-1) t_m t_{m+1}^{k-1} + \sum_{j=1}^{k-2} q^{-(k-2-j)} (q^{-1}-1) t_m^{j+1} t_{m+1}^{k-1-j} \\
&= q^{-(k-1)} t_{m+1} g_{m+1}^{-1} \ + \ \sum_{j=1}^{k} q^{-(k-1-j)} (q^{-1}-1) t_m^j t_{m+1}^{k-j}. \qquad \square
\end{aligned}$$

Lemma 5. *In* $\mathrm{H}_{1,n}(q)$ *the following relations hold:*

(i) For the expression $A = (g_r g_{r-1} \dots g_{r-s}) \cdot t_k$ *the following hold for the different values of* $k\in\mathbb{N}$*:*

$$\begin{array}{ll}
(1) & A = t_k\,(g_r \dots g_{r-s}) \quad \text{for } k>r \text{ or } k<r-s-1 \\
(2) & A = t_r\,\left(g_r^{-1} \dots g_{r-s}^{-1}\right) \quad \text{for } k=r-s-1 \\
(3) & A = q t_{r-1}\,(g_r \dots g_{r-s}) + (q-1) t_r (g_{r-1} \dots g_{r-s}) \quad \text{for } k=r \\
(4) & A = q t_{r-s-1}\,(g_r \dots g_{r-s}) + (q-1) t_r\,\left(g_r^{-1} \dots g_{r-s+1}^{-1}\right) \ \text{for } k=r-s \\
(5) & A = t_{m-1}\,(g_r \dots g_{r-s}) + (q-1) t_r\,\left(g_r^{-1} \dots g_{m+1}^{-1}\right) (g_{m-1} \dots g_{r-s}) \\
 & \qquad \text{for } k=m\in\{r-s+1,\dots,r-1\}.
\end{array}$$

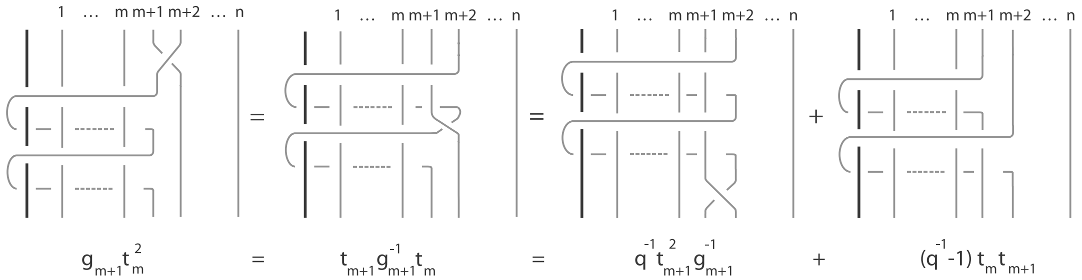

Fig. 10. Illustrating Lemma 4(i) for $k = 2$.

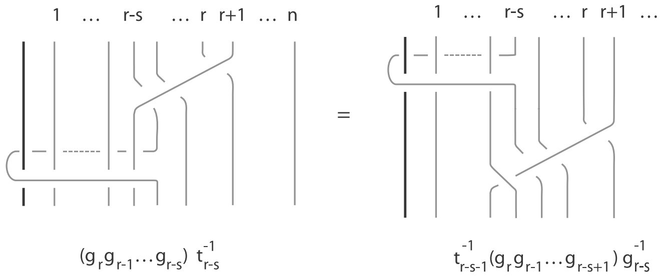

Fig. 11. Illustrating Lemma 5(ii) for $k = r - s$.

(ii) For the expression $A = (g_r g_{r-1} \ldots g_{r-s}) \cdot t_k^{-1}$ the following hold for the different values of $k \in \mathbb{N}$:

$$\begin{array}{lll}
(1) & A = t_k^{-1}\,(g_r \ldots g_{r-s}) & \textit{for } k > r \text{ or } k < r - s - 1 \\
(2) & A = t_{r-s-1}^{-1}\left(g_r \ldots g_{r-s+1} g_{r-s}^{-1}\right) & \textit{for } k = r - s \quad (\textit{see Fig. 11}) \\
(3) & A = t_{m-1}^{-1}\left(g_r g_{r-1} \ldots g_{m+1} g_m^{-1} g_{m-1} \ldots g_{k-s}\right) & \\
 & \quad \textit{for } k = m \in \{r - s + 1, \ldots, r\} & \\
(4) & A = q^{s+1} t_r^{-1}\,(g_r \ldots g_{r-s}) + (q-1) \sum_{j=1}^{s+1} q^{s-j+1} t_{r-j}^{-1} \cdot & \\
 & \quad \cdot\,(g_r \ldots g_{r-j+2} g_{r-j} \ldots g_{r-s}) \quad \textit{for } k = r - s - 1. &
\end{array}$$

Proof. We only prove relation (ii) for $k = r - s - 1$ by induction on s (case (4)). All other relations follow from Lemma 3(i).

For $s = 1$ we have:

$$\begin{aligned}
g_r \underline{g_{r-1} t_{r-2}^{-1}} &= g_r[q t_{r-1}^{-1} g_{r-1} + (q-1) t_{r-2}^{-1}] \;=\; q \underline{g_r t_{r-1}^{-1}} g_{r-1} + (q-1) g_r t_{r-2}^{-1} \\
&= q[q t_r^{-1} g_r + (q-1) t_{r-1}^{-1}] g_{r-1} + (q-1) t_{r-2}^{-1} g_r \\
&= q^2 t_r^{-1} (g_r g_{r-1}) + (q-1)\left[q t_{r-1}^{-1} g_{r-1} \;+\; q^0 t_{r-2}^{-1} g_r\right],
\end{aligned}$$

and so the relation holds for $s = 1$. Suppose that the relation holds for $s = n$. We will show that it holds for $s = n + 1$. Indeed we have:

$$\begin{aligned}
& (g_r \ldots g_{r-n-1}) t_{r-n-2}^{-1} = (g_r \ldots g_{r-n}) (\underline{g_{r-n-1} t_{r-n-2}^{-1}}) = \\
& (g_r \ldots g_{r-n}) \left[q t_{r-n-1}^{-1} g_{r-n-1} + (q-1) t_{r-n-2}^{-1}\right] = \\
& = q(\underline{g_r \ldots g_{r-n} t_{r-n-1}^{-1}}) g_{r-n-1} + (q-1)(g_r \ldots g_{r-n}) t_{r-n-2}^{-1} \overset{\text{ind. step}}{=} \\
& = q^{n+2} t_r^{-1} (g_r \ldots g_{r-n-1}) + \\
& + (q-1) \sum_{j=1}^{n+1} q^{n-j+2} t_{r-j}^{-1} (g_r \ldots g_{r-j+2} g_{r-j} \ldots g_{r-n-1}) + \\
& + (q-1) t_{r-n-2}^{-1} (g_r \ldots g_{r-n}) \;=\; q^{n+2} t_r^{-1} (g_r \ldots g_{r-n-1}) + \\
& + (q-1) \sum_{j=1}^{n+2} q^{(n+1)-j+1} t_{r-j}^{-1} (g_r \ldots g_{r-j+2} g_{r-j} \ldots g_{r-n-1}). \qquad \square
\end{aligned}$$

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Before proceeding with the next lemma we introduce the notion of length of $w \in \mathrm{H}_n(q)$. For convenience we set $\delta_{k,r} := g_k g_{k-1} \dots g_{r+1} g_r$ for $k > r$ and by convention we set $\delta_{k,k} := g_k$.

Definition 5. We define the *length* of $\delta_{k,r} \in \mathrm{H}_n(q)$ to be the number of braiding generators, that is, $l(\delta_{k,r}) := k - r + 1$ and since every element of the Iwahori–Hecke algebra of type A can be written as $\prod_{i=1}^{n-1} \delta_{k_i, r_i}$ so that $k_j < k_{j+1}\ \forall j$, we define the *length* of an element $w \in \mathrm{H}_n(q)$ as:

$$l(w) := \sum_{i=1}^{n-1} l_i(\delta_{k_i,r_i}) = \sum_{i=1}^{n-1} k_i - r_i + 1.$$

Note that $l(g_k) = l(\delta_{k,k}) = k - k + 1 = 1$.

Lemma 6. *For $k > r$ the following relations hold in $\mathrm{H}_{1,n}(q)$:*

$$t_k \delta_{k,r} = \sum_{i=0}^{k-r} q^i (q-1) \delta_{k,\overline{k-i},r} t_{k-i} + q^{l(\delta_{k,r})} \delta_{k,r} t_{r-1},$$

where $\delta_{k,\overline{k-i},r} := g_k g_{k-1} \dots g_{k-i+1} g_{k-i-1} \dots g_r := g_k \dots \overline{g_{k-i}} \dots g_r$.

Proof. We prove relations by induction on k. For $k = 1$ we have that $t_1 g_1 = (q-1)t_1 + q g_1 t$, which holds. Suppose that the relation holds for $(k-1)$, then for k we have:

$$\begin{aligned} t_k \delta_{k,r} &= \underline{t_k g_k} \cdot \delta_{k-1,r} = (q-1) t_k \delta_{k-1,r} + q g_k \underline{t_{k-1} \delta_{k-1,r}} = \\ &= (q-1)\delta_{k-1,r} t_k + q g_k \textstyle\sum_{i=0}^{k-1-r} q^i (q-1) \delta_{k-1,\overline{k-1-i},r} t_{k-1-i} + \\ &+ q^{l(\delta_{k-1,r})+1} g_k \delta_{k-1,r} t_{r-1} = \\ &= \textstyle\sum_{i=0}^{k-r} q^i (q-1) \delta_{k,\overline{k-1-i},r} t_{k-1-i} + q^{l(\delta_{k,r})} \delta_{k,r} t_{r-1}. \qquad \square \end{aligned}$$

Lemma 7. *In $\mathrm{H}_{1,n}(q)$ the following relations hold:*

(i) For the expression $A = (g_r g_{r+1} \dots g_{r+s}) \cdot t_k$ the following hold for the different values of $k \in \mathbb{N}$:

$$\begin{aligned} &(1) \quad A = t_k \left(g_r \dots g_{r+s}\right) \quad \textit{for } k \geq r+s+1 \text{ or } k < r-1 \\ &(2) \quad A = t_{k+1} \left(g_r \dots g_k g_{k+1}^{-1} g_{k+2} \dots g_{r+s}\right) \\ &\qquad\quad \textit{for } r-1 \leq k < r+s \\ &(3) \quad A = (q-1) \textstyle\sum_{i=r}^{r+s} q^{r+s-i} t_i \left(g_r \dots \overline{g_i} \dots g_{r+s}\right) + q^{s+1} t_{r-1} \left(g_r \dots g_{r+s}\right) \\ &\qquad\quad \textit{for } k = r+s \end{aligned}$$

(ii) For the expression $A = (g_r g_{r+1} \dots g_{r+s}) \cdot t_k^{-1}$ the following hold for the different values of $k \in \mathbb{N}$:

$$\begin{aligned} &(1) \quad A = t_k^{-1} \left(g_r g_{r+1} \dots g_{r+s}\right) \quad \textit{for } k \geq r+s+1 \text{ or } k < r-1 \\ &(2) \quad A = q\ t_{k+1}^{-1} \left(g_r \dots g_{r+s}\right) + (q-1)\ t_{r-1}^{-1} \left(g_r^{-1} \dots g_k^{-1} g_{k+2} \dots g_{r+s}\right) \\ &\qquad\quad \textit{for } r-1 \leq k < r+s \\ &(3) \quad A = t_{r-1}^{-1} \left(g_r^{-1} \dots g_{r+s}^{-1}\right) \quad \textit{for } k = r+s \end{aligned}$$

Proof. We prove relation (i) for $r + s = k$ by induction on k (case (3)). All other relations follow from Lemmas 1 and 3.

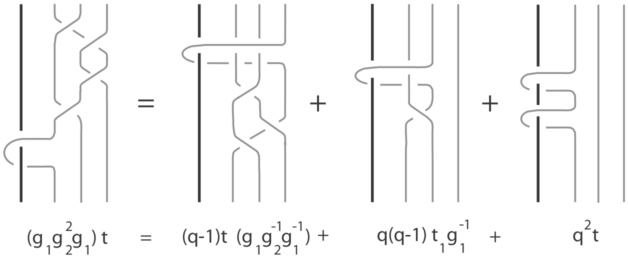

Fig. 12. Illustrating Lemma 8(i) for $i = 2$.

For $k = 1$ we have: $g_1 t_1 = \underline{g_1^2} t g_1 = q t g_1 + (q-1) t_1$. Suppose that the relation holds for $k = n$. Then, for $k = n + 1$ we have that:

$$
\begin{aligned}
& g_r \dots \underline{g_{n+1} t_{n+1}} = q(\underline{g_r \dots g_n t_n}) g_{n+1} \ + \ (q-1)\underline{(g_r \dots g_n) t_{n+1}} \overset{\text{ind. step}}{=} \\
& = q\left[(q-1)\textstyle\sum_{i=r}^{n} q^{n-i} t_i (g_r \dots \overline{g_i} \dots g_n) + q^{n-r+1} t_{r-1}(g_r \dots g_n)\right] g_{n+1} + \\
& + \ (q-1) t_{n+1}(g_r \dots g_n) \ = \\
& = \left((q-1)\textstyle\sum_{i=r}^{n} q^{n-i+1} t_i (g_r \dots \overline{g_i} \dots g_n g_{n+1}) + (q-1) t_{n+1}(g_r \dots g_n)\right) + \\
& + \ q^{n+1-r+1} t_{r-1}(g_r \dots g_n g_{n+1}) \ = \\
& = (q-1)\textstyle\sum_{i=r}^{n+1} q^{n+1-i} t_i (g_r \dots \overline{g_i} \dots g_{n+1}) + q^{n+1-r+1} t_{r-1}(g_r \dots g_{n+1}). \qquad \square
\end{aligned}
$$

Lemma 8. *The following relations hold in* $\mathrm{H}_{1,n}(q)$ *for* $k \in \mathbb{N}$*:*

$$
\begin{aligned}
(i) \quad & \left(g_1 \dots g_{i-1} g_i^2 g_{i-1} \dots g_1\right) \cdot t \ = \\
& (q-1)\textstyle\sum_{k=1}^{i} q^{i-k} t_k \left(g_1 \dots g_{k-1} g_k^{-1} g_{k-1}^{-1} \dots g_1^{-1}\right) + q^i t \quad (\textit{see Fig. 12}) \\
(ii) \quad & \left(g_1^{-1} \dots g_{i-1}^{-1} g_i^{-2} g_{i-1}^{-1} \dots g_1^{-1}\right) \cdot t^{-1} \ = \\
& (q^{-1}-1)\textstyle\sum_{k=1}^{i} q^{-(i-k)} t_k^{-1} \left(g_1^{-1} \dots g_{k-1}^{-1} g_k g_{k-1} \dots g_1\right) + q^{-i}\ t^{-1} \\
(iii) \quad & \left(g_k^{-1} \dots g_2^{-1} g_1^{-2} g_2^{-1} \dots g_k^{-1}\right) \cdot t_k \ = \\
& (q^{-1}-1)\textstyle\sum_{i=1}^{k-1} q^{-k} t_i \left(g_k^{-1} \dots g_{i+2}^{-1} g_{i+1} g_{i+2} \dots g_k\right) + q^{-k} t_k \\
(iv) \quad & \left(g_k^{-1} \dots g_2^{-1} g_1^{-2} g_2^{-1} \dots g_k^{-1}\right) \cdot t_k^{-1} \ = \\
& t^{-1} q^{-k} (q^{-1}-1) g_k^{-1} \dots g_1^{-1} \dots g_k^{-1} \ + \\
& + \textstyle\sum_{i=0}^{k-1} t_i^{-1} q^{-k+i} (q^{-1}-1) g_k^{-1} \dots g_1^{-2} \dots g_i^{-1} g_{i+2}^{-1} \dots g_k^{-1} \ + \\
& + t_k^{-1}\left[\textstyle\sum_{i=2}^{k} q^{-k+i} (q^{-1}-1)^2 g_{i-1}^{-1} \dots g_2^{-1} g_1^{-2} g_2^{-1} \dots g_{i-1}^{-1} \ + \right. \\
& \left. + \ q^{-(k+1)}(q^2 - q + 1)\right].
\end{aligned}
$$

Proof. We prove relation (i) by induction on i. All other relations follow similarly. For $i = 1$ we have: $g_1^2 t = g_1 g_1 t g_1 g_1^{-1} = g_1 t_1 g_1^{-1} = (q-1) t_1 g_1^{-1} + q t$. Suppose that the relation holds for $i = n$. Then, for $i = n + 1$ we have:

$$
\begin{aligned}
& \left(g_1 \dots g_n g_{n+1}^2 g_n \dots g_1\right) \cdot t \ = \ (q-1)\left(g_1 \dots g_{n+1} g_n \dots g_1\right) \cdot t \ + \\
& + \ q\left(g_1 \dots g_{n-1} g_n^2 g_{n-1} \dots g_1\right) \cdot t \ = \\
& = \ (q-1) g_1 \dots g_n t_{n+1} g_{n+1}^{-1} \dots g_1^{-1} + q \textstyle\sum_{k=1}^{n} q^{n-k} (q-1) t_k \cdot \\
& \left(g_1 \dots g_{k-1} g_k^{-1} \dots g_1^{-1}\right) + q^{n+1} t \ = \\
& = \ (q-1) t_{n+1} \left(g_1 \dots g_n g_{n+1}^{-1} \dots g_1^{-1}\right) \ + \ \textstyle\sum_{k=1}^{n} q^{n+1-k} (q-1) t_k \cdot \\
& \left(g_1 \dots g_{k-1} g_k^{-1} \dots g_1^{-1}\right) + q^{n+1} t \ = \\
& = \ \textstyle\sum_{k=1}^{n+1} q^{n+1-k} (q-1) t_k \left(g_1 \dots g_{k-1} g_k^{-1} \dots g_1^{-1}\right) + q^{n+1} t. \qquad \square
\end{aligned}
$$

4.2. Converting elements in Λ' to elements in Σ_n

We are now in the position to prove a set of relations converting monomials of t_i''s to expressions containing the t_i's. In [13] we provide lemmas converting monomials of t_i's to monomials of t_i''s in the context of giving a simple proof that the sets Σ_n' form bases of $\mathrm{H}_{1,n}(q)$.

Lemma 9. *The following relations hold in* $\mathrm{H}_{1,n}(q)$ *for* $k \in \mathbb{N}$*:*

$$\begin{array}{lll} (i) & {t_1'}^{-k} & = q^k t_1^{-k} \ + \ \sum_{j=1}^{k} q^{k-j}(q-1)t^{-j}t_1^{j-k} \cdot g_1^{-1}, \\ (ii) & {t_1'}^{k} & = q^{-k} t_1^{k} \ + \ \sum_{j=1}^{k} q^{-(k-j)}(q^{-1}-1)t^{j-1}t_1^{k+1-j} \cdot g_1^{-1}. \end{array}$$

Proof. We prove relation (i) by induction on k. Relation (ii) follows similarly. For $k=1$ we have: ${t_1'}^{-1} = \underline{g_1}\ t^{-1}\ g_1^{-1} \ = \ q\ \underline{g_1^{-1}\ t^{-1}\ g_1^{-1}} \ + \ (q-1)\ t^{-1}\ g_1^{-1} \ = \ q\ t_1^{-1} \ + \ (q-1)\ t^{-1}\ g_1^{-1}$.

Suppose that the relation holds for $k-1$. Then, for k we have:

$$\begin{aligned} {t_1'}^{-k} &= {t_1'}^{-(k-1)}{t_1'}^{-1} \overset{\text{ind.}}{\underset{\text{step}}{=}} q^{k-1}t_1^{-(k-1)}{t_1'}^{-1} + \\ &+ \sum_{j=1}^{k-1} q^{k-1-j}(q-1)t^{-j}t_1^{j-(k-1)}g_1^{-1}{t_1'}^{-1} \ = \\ &= q^k t_1^{-k} + q^{k-1}t^{-1}t_1^{-(k-1)}g_1^{-1} + \sum_{j=1}^{k-1} q^{k-1-j}(q-1)t^{-j}t_1^{j-(k-1)}t^{-1}g_1^{-1} \\ &= q^k t_1^{-k} + q^{k-1}(q-1)t^{-1}t_1^{-(k-1)}g_1^{-1} + \\ &+ \sum_{j=1}^{k-1} q^{k-1-j}(q-1)t^{-j-1}t_1^{j-(k-1)}g_1^{-1} \ = \\ &= q^k t_1^{-k} + \sum_{j=1}^{k} q^{k-j}(q-1)t^{-j}t_1^{j-k}g_1^{-1}. \qquad \square \end{aligned}$$

Lemma 10. *The following relations hold in* $\mathrm{H}_{1,n}(q)$ *for* $k \in \mathbb{N}$*:*

$${t_k'}^{-1} \ = \ q^k\ t_k^{-1} \ + \ (q-1)\ \sum_{i=0}^{k-1} q^i\ t_i^{-1}\ (\ g_k\ g_{k-1}\ \ldots\ g_{i+2}\ g_{i+1}^{-1}\ \ldots\ g_{k-1}^{-1}\ g_k^{-1}\ ).$$

Proof. We prove the relations by induction on k. For $k=1$ we have:

$${t_1'}^{-1} = \underline{g_1}\ t^{-1}\ g_1^{-1} \ = \ q\ \underline{g_1^{-1}\ t^{-1}\ g_1^{-1}} \ + \ (q-1)\ t^{-1}\ g_1^{-1} \ = \ q\ t_1^{-1} \ + \ (q-1)\ t^{-1}\ g_1^{-1}.$$

Suppose that the relations hold for $k=n$. Then, for $k=n+1$ we have that:

$$\begin{aligned} {t_{n+1}'}^{-1} &= g_{n+1}\ \underline{{t_n'}^{-1}}\ g_{n+1}^{-1} \overset{\text{ind. step}}{=} \\ &= \ g_{n+1}\left[q^n t_n^{-1} \ + \ (q-1)\sum_{i=0}^{n-1} q^i\ t_i^{-1}(g_n \ldots g_{i+2}g_{i+1}^{-1}\cdots g_n^{-1})\right]g_{n+1}^{-1} = \\ &= \ q^n\ \underline{g_{n+1}\ t_n^{-1}}\ g_{n+1}^{-1} \ + \ (q-1)\sum_{i=0}^{n-1} q^i \underline{g_{n+1}t_i^{-1}}(g_n \ldots g_{i+2}g_{i+1}^{-1}\cdots g_n^{-1}g_{n+1}^{-1}) \ = \\ &= \ q^n\left[qt_{n+1}^{-1}g_{n+1} \ + \ (q-1)t_n^{-1}\right]g_{n+1}^{-1} \ + \ (q-1)\sum_{i=0}^{n-1} q^i t_i^{-1} \cdot \\ &\quad (g_{n+1}\ldots g_{i+2}g_{i+1}^{-1}\ldots g_{n+1}^{-1}) \ = \\ &= \ q^{n+1}t_{n+1}^{-1} \ + \ q^n(q-1)t_n^{-1}g_{n+1}^{-1} \ + \ (q-1)\sum_{i=0}^{n-1} q^i t_i^{-1} \cdot \\ &\quad (g_{n+1}\ldots g_{i+2}g_{i+1}^{-1}\ldots g_{n+1}^{-1}) \ = \\ &= \ q^{n+1}t_{n+1}^{-1} \ + \ (q-1)\sum_{i=0}^{n} q^i t_i^{-1}\ (g_{n+1}\ldots\ g_{i+2}g_{i+1}^{-1}\ldots g_{n+1}^{-1}). \qquad \square \end{aligned}$$

Lemma 11. *The following relations hold in* $\mathrm{H}_{1,n}(q)$ *for* $k \in \mathbb{Z}\backslash\{0\}$*:*

$${t_m'}^{k} \ = \ q^{-mk}t_m^k \ + \ \sum_i f_i(q)t_m^k w_i \ + \ \sum_i g_i(q)t^{\lambda_0}t_1^{\lambda_1}\ldots t_m^{\lambda_m}u_i,$$

where $w_i, u_i \in \mathrm{H}_{m+1}(q)$, $\forall i$, $\sum_{i=0}^m \lambda_i = k$ *and* $\lambda_i \geq 0$, $\forall i$, *if* $k > 0$ *and* $\lambda_i \leq 0$, $\forall i$, *if* $k < 0$.

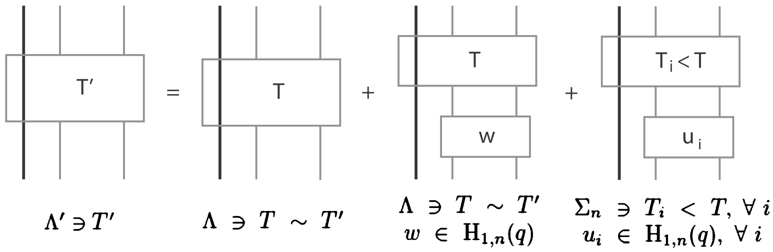

Fig. 13. Illustrating Theorem 7.

Proof. We prove relations by induction on m. The case $m = 1$ is Lemma 9. Suppose now that the relations hold for $m-1$. Then, for m we have:

$$\begin{array}{rl} {t'_m}^k & = g_m {t'_{m-1}}^k g_m^{-1} \overset{\text{ind.}}{\underset{\text{step}}{=}} q^{-(m-1)k} \underline{g_m t^k_{m-1}} g_m^{-1} + \sum_i f_i(q) \underline{g_m t^k_{m-1}} w_i g_m^{-1} + \\ & + \sum_i g_i(q) t^{\lambda_0} t_1^{\lambda_1} \dots t_{m-2}^{\lambda_{m-2}} \underline{g_m t_{m-1}^{\lambda_{m-1}}} u_i g_m^{-1} \overset{(\text{L. } 4)}{=} \\ & = q^{-(m-1)k} q^{-(k-1)} t^k_m \underline{g_m^{-2}} + \sum_{j=1}^{k-1} q^{-(k-1-j)} (q^{-1}-1) t^j_{m-1} t^{k-j}_m g_m^{-1} = \\ & = q^{-mk} t^k_m + \sum_i f_i(q) t^k_m w_i + \sum_i g_i(q) t^{\lambda_0} t_1^{\lambda_1} \dots t_m^{\lambda_{m_i}} u_i. \qquad \square \end{array}$$

Using now Lemma 11 we have that every element $u \in \Lambda'$ can be expressed to linear combinations of elements $v_i \in \Sigma_n$, where $\exists\, j : v_j \sim u$. More precisely:

Theorem 7. *The following relations hold in* $\mathrm{H}_{1,n}(q)$ *for* $k \in \mathbb{Z}$*:*

$$\begin{array}{rl} t^{k_0} {t'_1}^{k_1} \dots {t'_m}^{k_m} & = q^{-\sum_{n=1}^{m} n k_n} \cdot t^{k_0} t_1^{k_1} \dots t_m^{k_m} + \sum_i f_i(q) \cdot t^{k_0} t_1^{k_1} \dots t_m^{k_m} \cdot w_i + \\ & + \sum_j g_j(q) \tau_j \cdot u_j, \end{array}$$

where $w_i, u_j \in \mathrm{H}_{m+1}(q)$, $\forall i$, $\tau_j \in \Sigma_n$, *such that* $\tau_j < t^{k_0} t_1^{k_1} \dots t_m^{k_m}, \forall j$. *(See Fig. 13.)*

Proof. We prove relations by induction on m. Let $k_1 \in \mathbb{N}$, then for $m = 1$ we have:

$$\begin{array}{rl} t^{k_0} {t'_1}^{k_1} & \overset{(\text{L. } 9)}{=} q^{-k_1} t^{k_0} t_1^{k_1} + \sum_{j=1}^{k_1} q^{-(k_1-j)} (q^{-1}-1) t^{k_0+j-1} t_1^{k_1+1-j} g_1^{-1} = \\ & = q^{-k_1} t^{k_0} t_1^{k_1} + q^{-k_1} (q^{-1}-1) t^{k_0} t_1^{k_1} g_1^{-1} + \\ & + \sum_{j=2}^{k_1} q^{-(k_1-j)} (q^{-1}-1) t^{k_0+j-1} t_1^{k_1+1-j} g_1^{-1}. \end{array}$$

On the right hand side we obtain a term which is the homologous word of $t^{k_0} {t'_1}^{k_1}$ with scalar $q^{-k_1} \in \mathbb{C}$, the homologous word again followed by $g_1^{-2} \in \mathrm{H}_2(q)$ and with scalar $q^{-(k_1-1)}(q^{-1}-1) \in \mathbb{C}$ and the terms $t^{k_0+j-1} t_1^{k_1+1-j}$, which are of less order than the homologous word $t^{k_0} t_1^{k_1}$, since $k_1 > k_1 + 1 - j$, for all $j \in \{2, 3, \dots, k_1\}$. So the statement holds for $m = 1$ and $k_1 \in \mathbb{N}$. The case $m = 1$ and $k_1 \in \mathbb{Z} \backslash \mathbb{N}$ is similar.

Suppose now that the relations hold for $m-1$. Then, for m we have:

$$\begin{array}{rl} t^{k_0} {t'_1}^{k_1} \dots {t'_m}^{k_m} & \overset{\text{ind.}}{\underset{\text{step}}{=}} q^{-\sum_{n=1}^{m-1} n k_n} \cdot t^{k_0} \dots t_{m-1}^{k_{m-1}} \cdot {t'_m}^{k_m} + \\ & + \sum_i f_i(q) \cdot t^{k_0} t_1^{k_1} \dots t_{m-1}^{k_{m-1}} \cdot w_i \cdot {t'_m}^{k_m} \\ & + \sum_j g_j(q) \tau_j \cdot u_j \cdot {t'_m}^{k_m}. \end{array}$$

Now, since $w_i, u_i \in \mathrm{H}_m(q)$, $\forall i$ we have that $w_i {t'_m}^{k_m} = {t'_m}^{k_m} w_i$ and $u_i {t'_m}^{k_m} = {t'_m}^{k_m} u_i$, $\forall i$. Applying now Lemma 11 to ${t'_m}^k$ we obtain the requested relation. $\square$

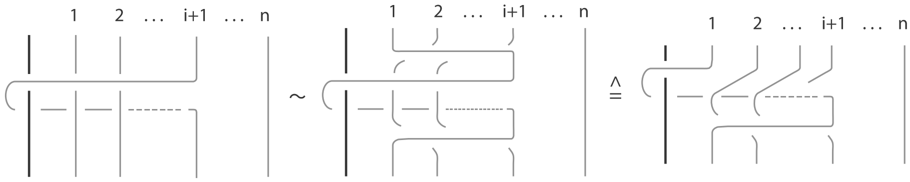

Fig. 14. Conjugating t_i by $g_1^{-1}\dots g_i^{-1}$.

Example 1. We convert the monomial $tt_1' {t_2'}^{-2} \in \Lambda'$ to linear combination of elements in Σ_n. We have that:

$$\begin{aligned} t_1' &= q^{-1}t_1 \ + \ (q^{-1}-1)t_1 g_1^{-1} \ \text{(Lemma 9)}, \\ {t_2'}^{-2} &= q^4 t_2^{-2} \ + \ q^3(q-1)t_1^{-1}t_2^{-1}g_2^{-1} \ + \ q^2(q-1)t^{-1}t_2^{-1}g_2g_1^{-1}g_2^{-1} \ + \\ &+ q^2(q-1)t_1^{-2}g_2^{-1} \ + \ q(q-1)^2t^{-1}t_1^{-1}g_1^{-1}g_2^{-1} \ + \ (q-1)t^{-2}g_2g_1^{-1}g_2^{-1} \ \text{(Lemma 10)}, \end{aligned}$$

and so:

$$\begin{aligned} tt_1'{t_2'}^{-2} &= q^3\cdot tt_1t_2^{-2} + q^4(q^{-1}-1)\cdot tt_1t_2^{-2}\cdot g_1^{-1} \ + 1\cdot u + \\ &+ tt_1^{-1}\cdot\left((q-1)(q^2-q+1)\cdot g_2^{-1} - (q-1)^2\cdot g_1g_2g_1^{-1}g_2^{-1}\right) + \\ &+ tt_2^{-1}\cdot\left(q^2(q-1)\cdot g_2^{-1} + q(q-1)^3\cdot g_2^{-1} - q(q-1)^2\cdot g_2g_1^{-1}g_2^{-1}\right) + \\ &+ t_1t_2^{-1}\cdot\left(q(q-1)\cdot g_2g_1^{-1}g_2^{-1} - q(q-1)^2\cdot g_1^{-1}g_2^{-1}\right) + \\ &+ t^{-1}t_1\cdot\left(-(q-1)\cdot g_2g_1^{-1}g_2^{-1} - q^{-1}(q-1)^2\cdot g_1^{-1}g_2^{-1}\right) \end{aligned}$$

where $u = (q-1)^2g_1^{-1}g_2^{-1} - (q-1)^3g_1^{-2}g_2^{-1} - q^{-1}(q-1)^3g_2g_1^{-1}g_2^{-1} + q^{-1}(q-1)^3g_2^{-1}$.

We obtain the homologous word $w = tt_1t_2^{-2}$, the homologous word again followed by the braiding generator g_1^{-1} and terms in Σ_n of less order than w, since either their index is less that $ind(w)$ (the terms tt_1^{-1}, 1 and $t^{-1}t_1$), either they contain gaps in the indices (the terms tt_2^{-1} and $t_1t_2^{-1}$).

5. From Σ_n to Λ

In order to prove Theorem 2 we need to show that the set Λ is a spanning set of $\mathcal{S}(\mathrm{ST})$ and also that it is linearly independent. In this section we show that every element in Λ' can be expressed in terms of elements in the set Λ. Linear independence of the set Λ is shown in the next section.

Before proceeding we need to discuss the following situation. According to Lemma 9, for a word $w' = t^k{t_1'}^{-\lambda} \in \Lambda'$, where $k, \lambda \in \mathbb{N}$ and $k < \lambda$ we have that:

$$\begin{aligned} w' \ = \ t^k{t_1'}^{-\lambda} &= t^{k-1}{t_1}^{-\lambda+1}\alpha_1 + t^{k-2}{t_1}^{-\lambda+2}\alpha_2 \ + \ \dots \ + \\ &+ \ t^0{t_1}^{-\lambda+k}\alpha_k \ + \ t^{-1}{t_1}^{-\lambda+k+1}\alpha_{k+1} \ + \ \dots \ + \ t^{-\lambda+k}\alpha_\lambda, \end{aligned}$$

where $\alpha_i \in \mathrm{H}_n(q)$, $\forall i$. We observe that in this particular case, in the right hand side there are terms which do not belong to the set Λ. These are the terms of the form $t^qt_1^p$, where $p > q$ and the term t_1^m. So these elements cannot be compared with the highest order term $w \sim w'$. The point now is that these terms are elements in the basis Σ_n on the Hecke algebra level, but, when we are working in $\mathcal{S}(\mathrm{ST})$, such elements must be considered up to conjugation by any braiding generator and up to stabilization moves. Topologically, conjugation corresponds to closing the braiding part of a mixed braid. Conjugating t_1 by g_1^{-1} we obtain tg_1^2 (view Fig. 14) and similarly conjugating t_1^m by g_1^{-1} we obtain $tg_1^2tg_1^2\dots tg_1^2$. Then, applying Lemma 3 we obtain the expression $\sum_{k=1}^{m-1} t^kt_1^{m-k}v_k$, where $v_k \in \mathrm{H}_n(q)$, for all k, that is, we obtain now elements with consecutive indices but not necessarily with ordered exponents.

We shall first deal with elements where the looping generators do not have consecutive indices, and then with elements where the exponents are not in decreasing order. For the expressions that we obtain after appropriate conjugations we shall use the notation $\widehat{=}$.

5.1. Managing the gaps

We will call *gaps* in monomials of the t_i's, gaps occurring in the indices and *size* of the gap $t_i^{k_i} t_j^{k_j}$ the number $s_{i,j} = j - i \in \mathbb{N}$.

Lemma 12. *For $k_0, k_1 \dots k_i \in \mathbb{Z}$, $\epsilon = 1$ or $\epsilon = -1$ and $s_{i,j} > 1$ the following relation holds in $\mathrm{H}_{1,n}(q)$:*

$$t^{k_0} t_1^{k_1} \dots t_{i-1}^{k_{i-1}} t_i^{k_i} \cdot t_j^{\epsilon} \mathrel{\widehat{=}} t^{k_0} t_1^{k_1} \dots t_{i-1}^{k_{i-1}} t_i^{k_i} \cdot t_{i+1}^{\epsilon} \left(g_{i+2}^{\epsilon} \dots g_{j-1}^{\epsilon} g_j^{2\epsilon} g_{j-1}^{\epsilon} \dots g_{i+2}^{\epsilon} \right).$$

Proof. We have that $t_j^{\epsilon} = \left(g_j^{\epsilon} \dots g_{i+2}^{\epsilon} \right)\ t_{i+1}^{\epsilon}\ \left(g_{i+2}^{\epsilon} \dots g_j^{\epsilon} \right)$ and so:

$$\begin{aligned} t^{k_0} t_1^{k_1} \dots t_{i-1}^{k_{i-1}} t_i^{k_i} t_j^{\epsilon} &= t^{k_0} t_1^{k_1} \dots t_{i-1}^{k_{i-1}} t_i^{k_i} (g_j^{\epsilon} \dots g_{i+2}^{\epsilon})\ t_{i+1}^{\epsilon}\ (g_{i+2}^{\epsilon} \dots g_j^{\epsilon}) &&= \\ &= (g_j^{\epsilon} \dots g_{i+2}^{\epsilon})\ t^{k_0} t_1^{k_1} \dots t_{i-1}^{k_{i-1}} t_i^{k_i} t_{i+1}^{\epsilon} (g_{i+2}^{\epsilon} \dots g_j^{\epsilon}) &&\widehat{=} \\ &\widehat{=} t^{k_0} \dots t_{i-1}^{k_{i-1}} t_i^{k_i} t_{i+1}^{\epsilon} (g_{i+2}^{\epsilon} \dots g_{j-1}^{\epsilon} g_j^{2\epsilon} g_{j-1}^{\epsilon} \dots g_{i+2}^{\epsilon}). && \square \end{aligned}$$

In order to pass to a general way for managing gaps in monomials of t_i's we first deal with gaps of size one. For this we have the following.

Lemma 13. *For $k \in \mathbb{N}$, $\epsilon = 1$ or $\epsilon = -1$ and $\alpha \in \mathrm{H}_{1,n}(q)$ the following relations hold:*

$$t_i^{\epsilon k} \cdot \alpha \mathrel{\widehat{=}} \sum_{u=1}^{k-1} q^{\epsilon(u-1)} (q^{\epsilon} - 1) t_{i-1}^{\epsilon u} t_i^{\epsilon(k-u)} (\alpha g_i^{\epsilon})\ +\ q^{\epsilon(k-1)} t_{i-1}^{\epsilon k} (g_i^{\epsilon} \alpha g_i^{\epsilon}).$$

Proof. We prove the relations by induction on k. For $k = 1$ we have $t_i^{\epsilon} \cdot \alpha \mathrel{\widehat{=}} g_i^{\epsilon} t_{i-1}^{\epsilon} g_i^{\epsilon} \cdot \alpha \mathrel{\widehat{=}} t_{i-1}^{\epsilon} g_i^{\epsilon} \cdot \alpha \cdot g_i^{\epsilon}$. Suppose that the assumption holds for $k - 1 > 1$. Then for k we have:

$$\begin{aligned} t_i^{\epsilon k} \cdot \alpha\ &\widehat{=}\ t_i^{\epsilon(k-1)} (t_i^{\epsilon} \cdot \alpha) \overset{(t_i^{\epsilon} \cdot \alpha\, =\, \beta)}{=} t_i^{\epsilon(k-1)} \cdot \beta \underset{\text{ind. step}}{\widehat{=}} \\ &= \textstyle\sum_{u=1}^{k-2} q^{\epsilon(u-1)} (q^{\epsilon} - 1) t_{i-1}^{\epsilon u} t_i^{\epsilon(k-1-u)} (\beta g_i^{\epsilon})\ +\ q^{\epsilon(k-2)} t_{i-1}^{\epsilon(k-1)} (g_i^{\epsilon} \beta g_i^{\epsilon}) \overset{(\beta\, =\, t_i^{\epsilon} \cdot \alpha)}{=} \\ &= \textstyle\sum_{u=1}^{k-2} q^{\epsilon(u-1)} (q^{\epsilon} - 1) t_{i-1}^{\epsilon u} t_i^{\epsilon(k-1-u)} t_i^{\epsilon} (\alpha g_i^{\epsilon})\ +\ q^{\epsilon(k-2)} t_{i-1}^{\epsilon(k-1)} (g_i^{\epsilon} t_i^{\epsilon} \alpha g_i^{\epsilon})\ = \\ &= \textstyle\sum_{u=1}^{k-2} q^{\epsilon(u-1)} (q^{\epsilon} - 1) t_{i-1}^{\epsilon u} t_i^{\epsilon(k-u)} (\alpha g_i^{\epsilon})\ +\ q^{\epsilon(k-2)} t_{i-1}^{\epsilon(k-1)} t_i^{\epsilon} \alpha g_i^{\epsilon}\ + \\ &+\ q^{\epsilon(k-1)} t_{i-1}^{\epsilon(k-1+1)} (g_i^{\epsilon} t_i^{\epsilon} \alpha g_i^{\epsilon})\ = \\ &= \textstyle\sum_{u=1}^{k-1} q^{\epsilon(u-1)} (q^{\epsilon} - 1) t_{i-1}^{\epsilon u} t_i^{\epsilon(k-u)} (\alpha g_i^{\epsilon})\ +\ q^{\epsilon(k-1)} t_{i-1}^{\epsilon k} (g_i^{\epsilon} \alpha g_i^{\epsilon}). \qquad \square \end{aligned}$$

We now introduce the following notation.

Notation 1. We set $\tau_{i,i+m}^{k_{i,i+m}} := t_i^{k_i} t_{i+1}^{k_{i+1}} \dots t_{i+m}^{k_{i+m}}$, where $m \in \mathbb{N}$ and $k_j \neq 0$ for all j and

$$\delta_{i,j} := \begin{cases} g_i g_{i+1} \dots g_{j-1} g_j & \text{if } i < j \\ g_i g_{i-1} \dots g_{j+1} g_j & \text{if } i > j, \end{cases} \qquad \delta_{i,\widehat{k},j} := \begin{cases} g_i g_{i+1} \dots g_{k-1} g_{k+1} \dots g_{j-1} g_j & \text{if } i < j \\ g_i g_{i-1} \dots g_{k+1} g_{k-1} \dots g_{j+1} g_j & \text{if } i > j \end{cases}$$

We also set $w_{i,j}$ an element in $\mathrm{H}_{j+1}(q)$ where the minimum index in w is i.

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Using now the notation introduced above, we apply Lemma 13 $s_{i,j}$-times to 1-gap monomials of the form $\tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j}$ and we obtain monomials with no gaps in the indices, followed by words in $\mathrm{H}_n(q)$.

Example 2. For $s_{i,j} > 1$ and $\alpha \in \mathrm{H}_n(q)$ we have:

$$
\begin{array}{lrcl}
(i) & \tau_{0,i}^{k_i} \cdot t_j \cdot \alpha & \widehat{=} & \tau_{0,i}^{k_i} \cdot t_{i+1} \cdot \delta_{i+2,j}\ \alpha\ \delta_{j,i+2} \\
(ii) & \tau_{0,i}^{k_i} \cdot t_j^2 \cdot \alpha & \widehat{=} & \tau_{0,i}^{k_i} \cdot t_{i+1}^2 \cdot \delta_{i+2,j}\ \alpha\ \delta_{j,i+2}\ +\ \tau_{0,i}^{k_i} \cdot t_{i+1}t_{i+2} \cdot \beta, \text{ where} \\
 & \beta & = & \left[(q-1)\sum_{s=i+2}^{j} q^{j-s}\delta_{i+3,s}\delta_{i+2,s-1}\delta_{s+1,j}\ \alpha\ \delta_{j,i+2}\delta_{s,i+3}\right] \\
(iii) & \tau_{0,i}^{k_i} \cdot t_j^3 \cdot \alpha & \widehat{=} & \left[q^{j-(i+2)+1}\right]^2 \tau_{0,i}^{k_i} \cdot t_{i+1}^3 \cdot \delta_{i+2,j}\ \alpha\ \delta_{j,i+2}\ + \\
 & & & +\ \tau_{0,i}^{k_i} \cdot t_{i+1}^2 t_{i+2} \cdot \beta\ +\ \tau_{0,i}^{k_i} \cdot t_{i+1}t_{i+2}^2 \cdot \gamma\ + \\
 & & & +\ \tau_{0,i}^{k_i} \cdot t_{i+1}t_{i+2}t_{i+3} \cdot \mu, \text{ where} \\
 & \gamma & = & q^{j-(i+3)+1}(q-1)\delta_{i+3,j}\delta_{i+2,s-1}\delta_{s+1,j}\ \alpha\ \delta_{j,i+2}\delta_{s,i+3}, \text{ and} \\
 & \mu & = & \sum_{s=i+2}^{j}\sum_{r=s+1}^{j} q^{2j-r-s}\ (q-1)^2\delta_{i+4,r}\delta_{i+2,s-1}\delta_{s+1,r-1}\delta_{r+1,j}\ \cdot \\
 & & & \alpha\ \delta_{j,i+2}\delta_{s,i+3}\delta_{r,i+4}\ +\ \sum_{s=i+2}^{j}\sum_{r=i+3}^{s} q^{2j-r-s}\ (q-1)^2 \cdot \\
 & & & \delta_{i+4,r}\delta_{i+3,r-1}\delta_{r+1,s}\delta_{i+2,s-1}\delta_{s+1,j}\ \alpha\ \delta_{j,i+2}\delta_{s,i+3}.
\end{array}
$$

Applying Lemma 13 to the one gap word $\tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j}$, where $k_j \in \mathbb{Z}\backslash\{0\}$ and $\alpha \in \mathrm{H}_n(q)$ we obtain:

$$
\tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j}\alpha \widehat{=} \begin{cases} \sum_\lambda \tau_{0,i}^{k_{0,i}} t_{i+1}^{\lambda_{i+1}} \dots t_{i+k_j}^{\lambda_{i+k_j}}\alpha' & \text{if } k_j < s_{i,j} \\ \sum_\lambda \tau_{0,i}^{k_{0,i}} t_{i+1}^{\lambda_{i+1}} \dots t_j^{\lambda_j}\beta' & \text{if } k_j \geq s_{i,j}, \end{cases}
$$

where $\alpha', \beta' \in \mathrm{H}_n(q)$, $\sum_{\mu=i+1}^{i+k_j} \lambda_\mu = k_j$, $\lambda_\mu \geq 0$, $\forall \mu$ and if $\lambda_u = 0$, then $\lambda_v = 0$, $\forall v \geq u$.

More precisely:

Lemma 14. *For the 1-gap word* $A = \tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j} \cdot \alpha$, *where* $\alpha \in \mathrm{H}_n(q)$ *we have:*

$$
\begin{array}{llcl}
(i) & \text{If } |k_j| < s_{i,j}, \text{ then: } A & \widehat{=} & (q^{k_j-1})^{j-(i+1)}\tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j}\ \delta_{i+2,j}\ \alpha\ \delta_{j,i+2}\ + \\
 & & & +\ \sum_{k_j} f(q)\tau_{0,i}^{k_{0,i}}\tau_{i+1,i+k_j}^{k_{i+1,i+k_j}} \cdot \beta\alpha\beta'. \\
(ii) & \text{If } |k_j| \geq s_{i,j}, \text{ then: } A & \widehat{=} & (q^{k_j-1})^{j-(i+1)}\tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j}\ \delta_{i+2,j}\ \alpha\ \delta_{j,i+2}\ + \\
 & & & +\ \sum_{k_j} f(q)\tau_{0,i}^{k_{0,i}} \cdot \tau_{i+1,j}^{k_{i+1,j}} \cdot \beta\alpha\beta',
\end{array}
$$

where β *and* β' *are of the form* $w_{i+1,j} \in \mathrm{H}_{j+1}(q)$, $\sum_{k_j} f(q,z)\tau_{0,i}^{k_{0,i}}\tau_{i+1,i+k_j}^{k_{i+1,i+k_j}}$ *means a sum of elements in* Σ_n, *such that in each one of them, the sum of the exponents of the looping generators* $t_{i+1}, \dots, t_{i+k_j}$ *is equal to* k_j, *and such that* $|k_{i+1}| < |k_j|$. *Moreover, if* $k_\mu = 0$, *for some index* μ, *then* $k_s = 0$ *for all* $s > \mu$.

Proof. We prove the relations by induction on k_j. Let $0 < k_j < j - i$.

For $k_j = 1$ we have $A \widehat{=} \left[q^{(1-1)}\right]^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1}\delta_{i+2,j}\ \alpha\ \delta_{j,i+2}$ (Lemma 12). Suppose that the relation holds for $k_j - 1 > 1$. Then for k_j we have:

$$
\begin{aligned}
A = \tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j-1} \cdot (t_j\ \alpha) \underset{\text{ind. step}}{\widehat{=}} & \underbrace{\left[q^{k_j-2}\right]^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j-1}\underline{\delta_{i+2,j}\ t_j}\ \alpha\ \delta_{j,i+2}}_{B}\ + \\
+ & \underbrace{\sum_{k_{i_1,i+k_j-1}} f(q)\tau_{0,i}^{k_{0,i}} \cdot \tau_{i+1,i+k_j-1}^{k_{i_1,i+k_j-1}}\underline{\beta\ t_j}\ \beta'}_{C}.
\end{aligned}
$$

We now consider B and C separately and apply Lemma 4 to both expressions:

$$\begin{aligned}
B &\overset{(\text{L. }4)}{=} \\
&= \left[q^{k_j-2}\right]^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j-1} \cdot \\
&\quad \left[(q-1)\sum_{k+i+2}^{j} q^{j-k} t_k \delta_{i+2,k-1}\delta_{k+1,j} + q^{j-(i+2)+1} t_{i+1}\delta_{i+2,j}\right] \alpha\delta_{j,i+2} \\
&= \left[q^{k_j-2}\right]^{j-(i+1)} (q-1)\tau_{0,i}^{k_{0,i}} t_{i+1} \cdot \sum_{k+i+2}^{j} q^{j-k} t_k \delta_{i+2,k-1}\delta_{k+1,j}\alpha\delta_{j,i+2} + \\
&+ \left[q^{k_j-1}\right]^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j}\delta_{i+2,j}\alpha\delta_{j,i+2}.
\end{aligned}$$

We now do conjugation on the $(j-(i+3))$-one gap words that occur and since $t_k \cdot \beta \widehat{=} t_{i+2} \cdot \delta_{i+3,k}\ \beta\ \delta_{k,i+3}$ we obtain:

$$\begin{aligned}
B \widehat{=}& \left[q^{k_j-1}\right]^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j}\delta_{i+2,j}\ \alpha\ \delta_{j,i+2} + \\
&+ \tau_{0,i}^{k_{0,i}} t_{i+1}t_{i+2}\sum_{k=i+2}^{j} f(q,z)\delta_{i+3,k}\delta_{i+2,k-1}\delta_{k+1,j}\alpha\delta_{j,i+2}\delta_{k,i+3} = \\
&= \left[q^{k_j-1}\right]^{j-(i+1)} \tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j}\delta_{i+2,j}\ \alpha\ \delta_{j,i+2} + \tau_{0,i}^{k_i} t_{i+1}t_{i+2}\cdot\beta_1,
\end{aligned}$$

where $\beta_1 \in \mathrm{H}_{j+1}(q)$.

Moreover, $C = \sum_{k_r} f(q)\tau_{0,i}^{k_{0,i}} \cdot \tau_{i+1,i+k_j-1}^{k_{i+1,i+k_j-1}} \beta\ t_j\ \beta'$ and since $\beta = w_{i+k_j-1,j}$, we have that: $\beta \cdot t_j \overset{(L.\ 4)}{=} \sum_{s=i+k_j-1}^{j} t_s \cdot \gamma_s$, where $\gamma_s \in \mathrm{H}_{j+1}(q)$ and so: $C \widehat{=} \sum_{v_r} f(q)\tau_{0,i}^{k_{0,i}} \cdot \tau_{i+1,i+k_j}^{v_{i+1,i+k_j}} \cdot \beta_2$, where $\beta_2 \in \mathrm{H}_{j+1}(q)$.

This concludes the proof. □

We now pass to the general case of one-gap words.

Proposition 3. *For the 1-gap word $B = \tau_{0,i}^{k_{0,i}} \cdot \tau_{j,j+m}^{k_{j,j+m}} \cdot \alpha$, where $\alpha \in \mathrm{H}_n(q)$ we have:*

$$\begin{aligned}
B \widehat{=}& \prod_{s=0}^{m} (q^{k_{j+s}-1})^{j-(i+1)} \cdot \tau_{0,i}^{k_{0,i}} \tau_{i+1,i+m}^{k_{j,j+m}} \\
&\cdot \prod_{s=0}^{m}(\delta_{i+m+2-s,j+s}) \cdot \alpha \cdot \prod_{s=0}^{m}(\delta_{j+s,i+m+2-s}) + \\
&+ \sum_{u_r} f(q)\tau_{0,i}^{k_{0,i}} \cdot (\tau_{i+1,i+m}^{u_{1,m}}) \cdot \alpha'
\end{aligned}$$

where $\alpha' \in \mathrm{H}_n(q)$, $\sum u_{1,m} = k_j$ such that $u_1 < k_j$ and if $u_\mu = 0$, then $u_s = 0$, $\forall s > \mu$.

Proof. The proof follows from Lemma 14. The idea is to apply Lemma 14 on the expression $\underline{\tau_{0,i}^{k_{0,i}} \cdot t_j^{k_j}} \cdot \rho_1$, where $\rho_1 = \tau_{j+1,j+m}^{k_{j+1,j+m}}$ and obtain the terms $\tau_{0,i}^{k_{0,i}} \cdot t_{i+1}^{k_j} \cdot \rho_2$ and $\tau_{0,i}^{k_{0,i}} \cdot \tau_{i+1,i+q}^{k_{i+1,i+q}} \cdot \rho_2$ and follow the same procedure until there is no gap in the word. □

We are now ready to deal with the general case, that is, words with more than one gap in the indices of the generators.

Theorem 8. *For the ϕ-gap word:*

$$C = \tau_{0,i}^{k_{0,i}} \cdot \tau_{i+s_1,i+s_1+\mu_1}^{k_{i+s_1,i+s_1+\mu_1}} \cdot \tau_{i+s_2,i+s_2+\mu_2}^{k_{i+s_2,i+s_2+\mu_2}} \dots \tau_{i+s_\phi,i+s_\phi+\mu_\phi}^{k_{i+s_\phi,i+s_\phi+\mu_\phi}} \cdot \alpha,$$

where $k_i \in \mathbb{Z}\backslash\{0\}$ for all i, $\alpha \in \mathrm{H}_n(q)$, $s_j, \mu_j \in \mathbb{N}$, such that $s_1 > 1$ and $s_j > s_{j-1} + \mu_{j-1}$ for all j we have:

$$\begin{aligned}
C \widehat{=}& \prod_{j=1}^{\phi} \left(q^{k_{i+s_j}-1}\right)^{s_j-j-\sum_{p=1}^{j-1}\mu_p} \cdot \tau_{0,i+\phi+\sum_{p=1}^{\phi}\mu_p}^{u_{0,i+\phi+\sum_{p=1}^{\phi}\mu_p}} \cdot \left(\prod_{p=0}^{\phi-1}\alpha_{\phi-p}\right) \cdot \alpha \cdot \\
&\left(\prod_{p=1}^{\phi}\alpha'_p\right) + \sum_v f_v(q)\tau_{0,v}^{k_{0,v}} \cdot w_v,\ \textit{where}
\end{aligned}$$

(i) $\alpha_j = \prod_{\lambda_j=0}^{\mu_j} \delta_{i+j+1+\sum_{k=1}^{j}\mu_k-\lambda_j,\ i+s_j+\mu_j-\lambda_j}$, $j = \{1,2,\ldots,\phi\}$,
(ii) $\alpha'_j = \prod_{\lambda_j=0}^{\mu_j} \delta_{i+j+1+\sum_{k=1}^{j-1}\mu_k+\lambda_j,\ i+s_j+\lambda_j}$, $j = \{1,2,\ldots,\phi\}$,
(iii) $\tau_{0,i+\phi+\sum_{p=1}^{\phi}\mu_p}^{u_{0,i+\phi+\sum_{p=1}^{\phi}\mu_p}} = \tau_{0,i}^{k_{0,i}} \cdot \prod_{j=1}^{\phi} \tau_{i+j+\sum_{p=1}^{j-1}\mu_p,i+j+\sum_{p=1}^{j}\mu_p}^{k_{i+s_j,i+s_j+\mu_j}}$,
(iv) $\tau_{0,v}^{u_{0,v}} < \tau_{0,i+\phi+\sum_{p=1}^{\phi}\mu_p}^{u_{0,i+\phi+\sum_{p=1}^{\phi}\mu_p}}$, *for all* v,
(v) w_v *of the form* $w_{i+2,i+s_\phi+\mu_\phi} \in \mathrm{H}_{i+s_\phi+\mu_\phi+1}(q)$, *for all* v,
(vi) the scalars $f_v(q)$ *are expressions of* $q \in \mathbb{C}$ *for all* v.

Proof. We prove the relations by induction on the number of gaps. For the 1-gap word $\tau_{0,i}^{k_{0,i}} \cdot \tau_{i+s,i+s+\mu}^{k_{i+s,i+s+\mu}} \cdot \alpha$, where $\alpha \in \mathrm{H}_n(q)$, we have:

$$A \,\widehat{=}\, \left[\prod_{\lambda=0}^{\mu} \left(q^{k_{i+s+\lambda}-1}\right)^{s-1}\right] \cdot \tau_{0,i}^{k_{0,i}} \cdot \tau_{i+1,i+1+\mu}^{k_{i+s,i+s+\mu}} \cdot \prod_{\lambda=0}^{\mu} \delta_{i+2+\mu-\lambda,i+s+\mu-\lambda} \cdot \alpha \cdot \prod_{\lambda=0}^{\mu} \delta_{i+2+\mu+\lambda,i+s+\lambda} + \sum_v f_v(q) \cdot \tau_{0,v}^{u_{0,v}} \cdot w_v,$$

which holds from Proposition 3.

Suppose that the relation holds for $(\phi-1)$-gap words. Then for a ϕ-gap word we have:

$$\begin{aligned}
&\underline{\left(\tau_{0,i}^{k_{0,i}} \cdot \tau_{i+s_1,i+s_1+\mu_1}^{k_{i+s_1,i+s_1+\mu_1}} \cdot \tau_{i+s_2,i+s_2+\mu_2}^{k_{i+s_2,i+s_2+\mu_2}} \ldots \tau_{i+s_{\phi-1},i+s_{\phi-1}+\mu_{\phi-1}}^{k_{i+s_{\phi-1},i+s_{\phi-1}+\mu_{\phi-1}}}\right)} \cdot \tau_{i+s_\phi,i+s_\phi+\mu_\phi}^{k_{i+s_\phi,i+s_\phi+\mu_\phi}} \cdot \alpha \underset{\text{ind. step}}{\widehat{=}}\\
&\prod_{j=1}^{\phi-1}\left(q^{k_{i+s_j}-1}\right)^{s_j-j-\sum_{k=1}^{j-1}\mu_k} \cdot \tau_{0,i+\phi-1+\sum_{k=1}^{\phi-1}\mu_k}^{u_{0,i+\phi-1+\sum_{k=1}^{\phi-1}\mu_k}} \cdot \underline{\prod_{k=0}^{\phi-2}\alpha_{\phi-1-k} \cdot \tau_{i+s_\phi,i+s_\phi+\mu_\phi}^{k_{i+s_\phi,i+s_\phi+\mu_\phi}}} \cdot \alpha \cdot \prod_{k=1}^{\phi-1}\alpha'_k +\\
&\sum_v f_v(q) \cdot \tau_{0,v}^{u_{0,v}} \cdot \underline{w \cdot \tau_{i+s_\phi,i+s_\phi+\mu_\phi}^{k_{i+s_\phi,i+s_\phi+\mu_\phi}}} \overset{s_\phi > s_{\phi-1}+\mu_{\phi-1}}{=}\\
&\prod_{j=1}^{\phi-1}\left(q^{k_{i+s_j}-1}\right)^{s_j-j-\sum_{k=1}^{j-1}\mu_k} \cdot \underline{\tau_{0,i+\phi-1+\sum_{k=1}^{\phi-1}\mu_k}^{u_{0,i+\phi-1+\sum_{k=1}^{\phi-1}\mu_k}} \cdot \tau_{i+s_\phi,i+s_\phi+\mu_\phi}^{k_{i+s_\phi,i+s_\phi+\mu_\phi}}} \cdot \prod_{k=0}^{\phi-2}\alpha_{\phi-1-k} \cdot \alpha \cdot \prod_{k=1}^{\phi-1}\alpha'_k +\\
&\sum_v f_v(q) \cdot \tau_{0,v}^{u_{0,v}} \cdot \tau_{i+s_\phi,i+s_\phi+\mu_\phi}^{k_{i+s_\phi,i+s_\phi+\mu_\phi}} \cdot w \overset{(\text{Prop. } 3)}{=}\\
&\prod_{j=1}^{\phi-1}\left(q^{k_{i+s_j}-1}\right)^{s_j-j-\sum_{k=1}^{j-1}\mu_k} \cdot \prod_{p=0}^{\mu_\phi}\left(q^{k_{i+s_\phi+p}-1}\right)^{s_\phi-\phi-\sum_{k=1}^{\phi-1}\mu_k} \tau_{0,i+\phi-1+\sum_{k=1}^{\phi-1}\mu_k}^{u_{0,i+\phi-1+\sum_{k=1}^{\phi-1}\mu_k}} \cdot\\
&\tau_{i+\phi+\sum_{k=1}^{\phi-1}\mu_k,i+\phi+\sum_{k=1}^{\phi-1}\mu_k+\mu_\phi}^{k_{i+s_\phi,i+s_\phi+\mu_\phi}} \cdot \prod_{k=0}^{\phi-1}\alpha_{\phi-1-k} \cdot \alpha \cdot \prod_{k=1}^{\phi-1}\alpha'_k + \sum_v f_v(q) \cdot \underline{\tau_{0,v}^{u_{0,v}} \cdot \tau_{i+s_\phi,i+s_\phi+\mu_\phi}^{k_{i+s_\phi,i+s_\phi+\mu_\phi}}} \cdot w \overset{(\text{Prop. } 3)}{=}\\
&\left[\prod_{\lambda=0}^{\mu}\left(q^{k_{i+s+\lambda}-1}\right)^{s-1}\right] \cdot \tau_{0,i}^{k_{0,i}} \cdot \tau_{i+1,i+1+\mu}^{k_{i+s,i+s+\mu}} \cdot \prod_{\lambda=0}^{\mu}\delta_{i+2+\mu-\lambda,i+s+\mu-\lambda} \cdot \alpha \cdot \prod_{\lambda=0}^{\mu}\delta_{i+2+\mu+\lambda,i+s+\lambda} +\\
&\sum_v f_v(q) \cdot \tau_{0,v}^{u_{0,v}} \cdot w_v. \qquad \square
\end{aligned}$$

All results are best demonstrated in the following example on a word with two gaps.

Example 3. For the 2-gap word $t^{k_0}t_1^{k_1}t_3t_5^2t_6^{-1} \in \Sigma_n$ we have:

$$\begin{aligned}
&t^{k_0}\underline{t_1^{k_1}t_3}t_5^2t_6^{-1} = t^{k_0}t_1^{k_1}\underline{g_3t_2g_3}t_5^2t_6^{-1} = g_3t^{k_0}t_1^{k_1}t_2t_5^2t_6^{-1}g_3 \,\widehat{=}\, t^{k_0}t_1^{k_1}\underline{t_2t_5^2}t_6^{-1}g_3^2 =\\
&= t^{k_0}t_1^{k_1}t_2\underline{t_5}t_5t_6^{-1}g_3^2 = t^{k_0}t_1^{k_1}t_2\underline{g_5g_4}t_3g_4g_5t_5t_6^{-1}g_3^2 =\\
&= \underline{g_5g_4}t^{k_0}t_1^{k_1}t_2t_3g_4g_5t_5t_6^{-1}g_3^2 \,\widehat{=}\, t^{k_0}t_1^{k_1}t_2t_3\underline{g_4g_5t_5}t_6^{-1}g_3^2g_5g_4 =\\
&= t^{k_0}t_1^{k_1}t_2t_3\left[q^2t_3g_4g_5 + q(q-1)t_4g_5 + (q-1)t_5g_4\right]t_6^{-1}g_3^2g_5g_4 =\\
&= q^2t^{k_0}t_1^{k_1}t_2t_3^2\underline{g_4g_5t_6^{-1}}g_3^2g_5g_4 + q(q-1)t^{k_0}t_1^{k_1}t_2t_3t_4\underline{g_5t_6^{-1}}g_3^2g_5g_4 +\\
&+ (q-1)t^{k_0}t_1^{k_1}t_2t_3t_5\underline{g_4t_6^{-1}}g_3^2g_5g_4 = q^2t^{k_0}t_1^{k_1}t_2t_3^2\underline{t_6^{-1}}g_4g_5g_3^2g_5g_4 +\\
&+ (q-1)t^{k_0}t_1^{k_1}t_2t_3\underline{t_5}t_6^{-1}g_4g_3^2g_5g_4 + q(q-1)t^{k_0}t_1^{k_1}t_2t_3t_4\underline{t_6^{-1}}g_5g_3^2g_5g_4 \,\widehat{=}\\
&\widehat{=}\, q^2t^{k_0}t_1^{k_1}t_2t_3^2\underline{g_6^{-1}g_5^{-1}}t_4^{-1}g_5^{-1}g_6^{-1}g_4g_5g_3^2g_5g_4 +
\end{aligned}$$

$$
\begin{aligned}
&+ q(q-1)t^{k_0}t_1^{k_1}t_2t_3t_4\underline{g_6^{-1}}t_5^{-1}g_6^{-1}g_5g_3^2g_5g_4 + (q-1)t^{k_0}t_1^{k_1}t_2t_3\underline{g_5}t_4\underline{g_5}t_6^{-1}\cdot \\
&\cdot (g_4g_3^2g_5g_4) = \ q^2g_6^{-1}g_5^{-1}t^{k_0}t_1^{k_1}t_2t_3^2t_4^{-1}g_5^{-1}g_6^{-1}g_4g_5g_3^2g_5g_4 \ + \\
&+ q(q-1)g_6^{-1}t^{k_0}t_1^{k_1}t_2t_3t_4t_5^{-1}g_6^{-1}g_5g_3^2g_5g_4 \ + \\
&+ (q-1)g_5t^{k_0}t_1^{k_1}t_2t_3t_4t_6^{-1}g_5g_4g_3^2g_5g_4 \ \widehat{=} \\
&\widehat{=}\ q^2t^{k_0}t_1^{k_1}t_2t_3^2t_4^{-1}g_5^{-1}g_6^{-1}g_4g_5g_3^2g_5g_4g_6^{-1}g_5^{-1} \ + \\
&+ q(q-1)t^{k_0}t_1^{k_1}t_2t_3t_4t_5^{-1}g_6^{-1}g_5g_3^2g_5g_4g_6^{-1} \ + \ (q-1)t^{k_0}t_1^{k_1}t_2t_3t_4\underline{t_6^{-1}}g_5\cdot \\
&\cdot (g_4g_3^2g_5g_4g_5) \ = \ q^2t^{k_0}t_1^{k_1}t_2t_3^2t_4^{-1}g_5^{-1}g_6^{-1}g_4g_5g_3^2g_5g_4g_6^{-1}g_5^{-1} \ + \\
&+ q(q-1)t^{k_0}t_1^{k_1}t_2t_3t_4t_5^{-1}g_6^{-1}g_5g_3^2g_5g_4g_6^{-1} + \\
&+ (q-1)t^{k_0}t_1^{k_1}t_2t_3t_4\underline{g_6^{-1}}t_5^{-1}g_6^{-1}g_5g_4g_3^2g_5g_4g_5 \ \widehat{=} \\
&\widehat{=}\ q^2t^{k_0}t_1^{k_1}t_2t_3^2t_4^{-1}g_5^{-1}g_6^{-1}g_4g_5g_3^2g_5g_4g_6^{-1}g_5^{-1} \ + \\
&+ q(q-1)t^{k_0}t_1^{k_1}t_2t_3t_4t_5^{-1}g_6^{-1}g_5g_3^2g_5g_4g_6^{-1} \ + \\
&+ (q-1)t^{k_0}t_1^{k_1}t_2t_3t_4t_5^{-1}g_6^{-1}g_5g_4g_3^2g_5g_4g_5g_6^{-1}.
\end{aligned}
$$

5.2. *Ordering the exponents*

We now deal with elements in Σ_n, where the looping generators have consecutive indices but their exponents are not in decreasing order. More precisely, we will show that these elements can be expressed as sums of elements in the $\bigcup_n \mathrm{H}_n(q)$-module Λ, namely, as sums of elements in Λ followed by a braiding tail.

We will need the following lemma.

Lemma 15. *The following relations hold in $\mathrm{H}_{1,n}(q)$ for $\lambda \in \mathbb{N}$:*

$$t_i^k \cdot t_{i+1}^{k+\lambda} \ \widehat{=}\ \sum_j t_i^{u_j} t_{i+1}^{v_j} \cdot w_j,$$

where $u_j + v_j = 2k + \lambda,\ u_j \geq v_j$ and $w_j \in \mathrm{H}_n(q),\ \forall j$.

Proof. We have that

$$
\begin{aligned}
t_i^k \cdot t_{i+1}^{k+\lambda} \ &= \ t_i^k \cdot t_{i+1}^k \underline{t_{i+1}^{\lambda}} \ \overset{\mathrm{L.}\,13}{=} \\
&= \ t_i^k \cdot t_{i+1}^k \cdot \left(q^{\lambda-1} g_{i+1} t_i^{\lambda} g_{i+1} \ + \ \textstyle\sum_{j=0}^{\lambda-2} q^j (q-1) t_i^{j+1} t_{i+1}^{\lambda-1-j} \right) \ = \\
&= \ q^{\lambda-1} t_i^k \cdot t_{i+1}^k \cdot g_{i+1} t_i^{\lambda} g_{i+1} \ + \ \textstyle\sum_{j=0}^{\lambda-2} q^j (q-1) t_i^{k+j+1} t_{i+1}^{k+\lambda-1-j}.
\end{aligned}
$$

We obtained the term $t_i^k \cdot t_{i+1}^k \cdot g_{i+1} t_i^{\lambda} g_{i+1}$, terms where the exponent of t_i is greater than the exponent of t_{i+1} and terms of the form $t_i^{p_1} t_{i+1}^{p_2}$, where $k < p_1 > p_2 < k + \lambda$. We apply Lemma 13 on the terms of the last form and repeat the same procedure until there are only elements of the form $t_i^{u_1} t_{i+1}^{u_2},\ u_1 > u_2$ left in each sum. Note that each time Lemma 13 is performed, a term of the form $t_i^{m_1} \cdot t_{i+1}^{m_1} \cdot g_{i+1} t_i^{m_2} g_{i+1}$ appears. For these elements we have:

$$
\begin{aligned}
& t_i^{m_1} \cdot \underline{t_{i+1}^{m_1} \cdot g_{i+1}} t_i^{m_2} g_{i+1} \ \overset{\mathrm{L.}\,3}{=} \ t_i^{m_1} \cdot \left((q-1) \textstyle\sum_{j=0}^{m_1-1} q^j t_i^j t_{i+1}^{m_1-j} \ + \ q^{m_1} g_{i+1} t_i^{m_1} \right) \cdot t_i^{m_2} g_{i+1} \\
& = \ (q-1) \textstyle\sum_{j=0}^{m_1-1} q^j t_i^{m_1+m_2+j} t_{i+1}^{m_1-j} g_{i+1} \ + q^{m_1} t_i^{m_1} \cdot g_{i+1} t_i^{m_1+m_2} \cdot g_{i+1}.
\end{aligned}
$$

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We have obtained now elements where the exponent of t_i is greater than the exponent of t_{i+1} and the term

$$\begin{array}{l} \underline{t_i^{m_1} \cdot g_{i+1} t_i^{m_1+m_2}} \cdot g_{i+1} \;\widehat{=}\; t_i^{m_1+m_2} \cdot \underline{g_{i+1} t_i^{m_1}} g_{i+1} \overset{\text{L. 4}}{=} \\ = \; t_i^{m_1+m_2} \cdot \left(q^{-m_1+1} t_{i+1}^{m_1} g_{i+1}^{-1} \; + \; \sum_{j=1}^{m_1-1} q^{-m_1+1-j} (q^{-1}-1) t_i^j t_{i+1}^{m_1-j} \right) \end{array}$$

and this concludes the proof. $\square$

Remark 5. Let $\tau_{0,m}^{k_{0,m}} \in \Sigma_n$ such that $k_i < k_{i+1}$. Applying Lemma 15 on $\tau_{0,m}^{k_{0,m}}$ we obtain a sum of elements $\tau_j \in \Sigma_n$, such that $\tau_j < \tau$, $\forall j$, since the exponent of the generator t_{i+1} in τ_j is less than k_{i+1} for all j (see Definition 2).

Example 4. Consider the element $tt_1^2 t_2^3 \in \Sigma_n$ and apply Lemma 15 on the first "bad" exponent occurring in the word, starting from right to left.

$$t\underline{t_1^2 t_2^3} \;\widehat{=}\; f_1(q) \cdot tt_1^3 t_2^2 \cdot w_1 \; + \; f_2(q) \cdot tt_1^4 t_2 \cdot w_2.$$

The terms obtained are still in Σ_n but they have one "bad" exponent less. We apply Lemma 15 again and obtain:

$$\begin{array}{l} \underline{tt_1^3} t_2^2 \;\widehat{=}\; f_3(q) \cdot t^3 t_1 t_2^2 \cdot w_3 \; + \; f_4(q) \cdot t^2 t_1^2 t_2^2 \cdot w_4 \\ \underline{tt_1^4} t_2 \;\widehat{=}\; f_5(q) \cdot t^4 t_1 t_2 \cdot w_5 \; + \; f_6(q) \cdot t^3 t_1^2 t_2 \cdot w_6 \end{array}$$

All terms obtained now are in the $\bigcup_n \mathrm{H}_n(q)$-module Λ except from the element $t^3 t_1 t_2^2$. We apply Lemma 15 again and obtain:

$$t^3 \underline{t_1 t_2^2} \;\widehat{=}\; f_7(q) \cdot t^3 t_1^2 t_2 \cdot w_7.$$

So:

$$tt_1^2 t_2^3 \;\widehat{=}\; g_1(q) \cdot t^3 t_1^2 t_2 \cdot u_1 \; + \; g_2(q) \cdot t^2 t_1^2 t_2^2 \cdot u_2 \; + \; g_3(q) \cdot t^4 t_1 t_2 \cdot u_3$$

where $u_1, \ldots, u_5 \in \mathrm{H}_n(q)$ and $g_1(q), \ldots, g_5(q) \in \mathbb{C}$.

Theorem 9. *Applying conjugation on an element in Σ_n we have that:*

$$\tau_{0,m}^{k_{0,m}} \cdot w \;\widehat{=}\; \sum_j \tau_{0,j}^{\lambda_{0,j}} \cdot w_j,$$

where $\tau_{0,j}^{\lambda_{0,j}} \in \Lambda$ and $w, w_j \in \mathrm{H}_n(q)$, $\forall j$.

Proof. We prove the statement by induction on the order of $\tau_{0,m}^{k_{0,m}} \cdot w \in \Sigma_n$, where order of an element in Σ_n denotes the position of this element in Σ_n with respect to total-ordering.

The base of the induction is Lemma 15 for $i = 0$. Suppose that the relation holds for all $\tau_j \cdot u_j \in \Sigma_n$ of less order than $\tau_{0,m}^{k_{0,m}} \cdot w$. Then, for $\tau_{0,m}^{k_{0,m}} \cdot w$ we have:

Let $k_0 > k_1 > \ldots > k_i < k_{i+1}$. Applying Lemma 15 on $\tau_{0,m}^{k_{0,m}} \cdot w$ we obtain:

$$\tau_{0,m}^{k_{0,m}} \cdot w \; := \; t_0^{k_0} t_1^{k_1} \ldots \underline{t_i^{k_i} t_{i+1}^{k_{i+1}}} \ldots t_m^{k_m} \cdot w \; = \; \sum_j t_0^{k_0} t_1^{k_1} \ldots t_i^{u_j} t_{i+1}^{v_j} \ldots t_m^{k_m} \cdot w_j,$$

where $u_j > v_j < k_{i+1}$, $\forall j$, that is, a sum of lower order terms than $\tau_{0,m}^{k_{0,m}} \cdot w$ (see Remark 5). So, by the induction hypothesis, the relation holds. $\square$

5.3. Eliminating the tails

So far we have seen how to convert elements in the basis Λ' to sums of elements in Σ_n and then, using conjugation, how these elements are expressed as sums of elements in the $\bigcup_n \mathrm{H}_n(q)$-module Λ. We will show now that using conjugation and stabilization moves all these elements of the $\bigcup_n \mathrm{H}_n(q)$-module Λ are expressed to sums of elements in the set Λ with scalars in the field $\mathbb{C}$. We will use the symbol $\simeq$ when a stabilization move is performed and $\widehat{\cong}$ when both stabilization moves and conjugation are performed.

Let us consider a generic word in $\mathrm{H}_{1,n+1}(q)$. This is of the form $\tau_{0,n}^{k_{0,n}} \cdot w_{n+1}$, where $w_{n+1} \in \mathrm{H}_{n+1}(q)$. Without loss of generality we consider the exponent of the braiding generator with the highest index to be (-1) when the exponent of the corresponding loop generator is in $\mathbb{N}$ and $(+1)$ when the exponent of the corresponding loop generator is in $\mathbb{Z}\backslash\mathbb{N}$. We then apply Lemmas 3 and 4 in order to interact $t_n^{\pm k_n}$ with $g_n^{\mp 1}$ and obtain words of the following form:

$$\begin{array}{llll} (1) & \tau_{0,p}^{\lambda_{0,p}} \cdot v, & \text{where} & \tau_{0,p}^{\lambda_{0,p}} < \tau_{0,n}^{k_{0,n}} \text{ and } v \in \mathrm{H}_{n+1}(q) \text{ of any length, or} \\ (2) & \tau_{0,q}^{k_{0,q}} \cdot u, & \text{where} & \tau_{0,q}^{\lambda_{0,q}} < \tau_{0,n}^{k_{0,n}} \text{ and } u \in \mathrm{H}_{n}(q) \text{ such that } l(u) < l(w). \end{array}$$

In the first case we obtain monomials of t_i's of less order than the initial monomial, followed by a word in $\mathrm{H}_{n+1}(q)$ of any length. After at most $(k_n + 1)$-interactions of t_n with g_n, the exponent of t_n will become zero and so by applying a stabilization move we obtain monomials of t_i's of less index, and thus of less order (Definition 2), followed by a word in $\mathrm{H}_n(q)$.

In the second case, we have monomials of t_i's of less order than the initial monomial followed by words $u \in \mathrm{H}_n(q)$ such that $l(u) < l(w)$. We interact the generator with the maximum index of u, g_m with the corresponding loop generator until the exponent of t_m becomes zero. A gap in the indices of the monomials of the t_i's occurs and we apply Theorem 8. This leads to monomials of t_i's of less order followed by words of the braiding generators of any length. We then apply stabilization moves and repeat the same procedure until the braiding 'tails' are eliminated.

Theorem 10. *Applying conjugation and stabilization moves on a word in the $\bigcup_\infty \mathrm{H}_n(q)$-module, Λ we have that:*

$$\tau_{0,m}^{k_{0,m}} \cdot w_n \ \widehat{\cong} \ \sum_j f_j(q,z) \cdot \tau_{0,u_j}^{v_{0,u_j}},$$

such that $\sum v_{0,u_j} = \sum k_{0,m}$ and $\tau_{0,u_j}^{v_{0,u_j}} < \tau_{0,m}^{k_{0,m}}$, for all j.

The logic for the induction hypothesis is explained above. We shall now proceed with the proof of the theorem.

Proof of Theorem 10. We prove the statement by double induction on the length of $w_n \in \mathrm{H}_n(q)$ and on the order of $\tau_{0,m}^{k_{0,m}} \in \Lambda$, where order of $\tau_{0,m}^{k_{0,m}}$ denotes the position of $\tau_{0,m}^{k_{0,m}}$ in Λ with respect to total-ordering.

For $l(w) = 0$, that is for $w = e$ we have that $\tau_{0,m}^{k_{0,m}} \ \widehat{\cong} \ \tau_{0,m}^{k_{0,m}}$ and there's nothing to show. Moreover, the minimal element in the set Λ is t^k and for any word $w \in \mathrm{H}_n(q)$ we have that $t^k \cdot w \ \simeq \ f(q,z) \cdot t^k$, by the quadratic relation and stabilization moves.

Suppose that the relation holds for all $\tau_{0,p}^{u_{0,p}} \cdot w'$, where $\tau_{0,p}^{u_{0,p}} \leq \tau_{0,m}^{k_{0,m}}$ and $l(w') = l$, and for all $\tau_{0,q}^{v_{0,q}} \cdot w$, where $\tau_{0,q}^{v_{0,q}} < \tau_{0,m}^{k_{0,m}}$ and $l(w) = l + 1$. We will show that it holds for $\tau_{0,m}^{k_{0,m}} \cdot w$. Let the exponent of t_r, $k_r \in \mathbb{N}$ and let $w \in \mathrm{H}_{r+1}(q)$. Then, w can be written as $w' \cdot g_r^{-1} \cdot \delta_{r-1,d}$, where $w' \in \mathrm{H}_r(q)$ and $d < r$. We have that:

$$
\begin{aligned}
\tau_{0,m}^{k_{0,m}} \cdot w &= \tau_{0,r-1}^{k_{0,r-1}} t_r^{k_r-1} \tau_{r+1,m}^{k_{r+1,m}} \cdot w' \cdot t_r g_r^{-1} \delta_{r-1,d} \ = \\
&= \tau_{0,r-1}^{k_{0,r-1}} t_r^{k_r-1} \tau_{r+1,m}^{k_{r+1,m}} \cdot w' \cdot g_r \underline{t_{r-1}\delta_{r-1,d}} \overset{\text{L. 6}}{=} \\
&= \tau_{0,r-1}^{k_{0,r-1}} t_r^{k_r-1} \tau_{r+1,m}^{k_{r+1,m}} \cdot w' \cdot g_r \\
&\cdot \left(\sum_{j=0}^{r-1-d} q^j (q-1) \delta_{r-1,\widehat{r-1-j},d} t_{r-1-j} \ + \ q^{l(\delta_{r-1,d})} \delta_{r-1,d} t_{d-1} \right) \widehat{=} \\
&\widehat{=} \sum_{j=0}^{r-1-d} q^j (q-1) \tau_{0,r-1}^{k_{0,r-1}} t_r^{k_r-1} \tau_{r+1,m}^{k_{r+1,m}} \cdot t_{r-1-j} \cdot w' \cdot g_r \delta_{r-1,\widehat{r-1-j},d} \ + \\
&+ \ q^{l(\delta_{r-1,d})} \tau_{0,r-1}^{k_{0,r-1}} t_r^{k_r-1} \tau_{r+1,m}^{k_{r+1,m}} \cdot t_{d-1} \cdot w.
\end{aligned}
$$

We have that $\left(\tau_{0,r-1}^{k_{0,r-1}} t_r^{k_r-1} \tau_{r+1,m}^{k_{r+1,m}} \cdot t_{r-1-j}\right) < \left(t_{0,m}^{k_{0,m}}\right)$, for all $j \in \{1,2,\dots,r-1-d\}$ and $l\left(w' \cdot g_r \delta_{r-1,\widehat{r-1-j},d}\right) = l$ and $\left(\tau_{0,r-1}^{k_{0,r-1}} t_r^{k_r-1} \tau_{r+1,m}^{k_{r+1,m}} \cdot t_{d-1}\right) < \left(t_{0,m}^{k_{0,m}}\right)$. So, by the induction hypothesis, the relation holds. □

Example 5. In this example we demonstrate how to eliminate the braiding 'tail' in a word in Σ_n.

$$
\begin{aligned}
t^{-1}\underline{t_1^2} t_2^{-1} g_1^{-1} &= t^{-1} t_1 t_2^{-1} \underline{t_1 g_1^{-1}} \ = \ t^{-1} t_1 t_2^{-1} g_1 \underline{t} \ \widehat{=} \ \underline{t_1} t_2^{-1} g_1 \ = \ t_2^{-1} \underline{t_1 g_1} \ = \\
&= (q-1)\underline{t_1} t_2^{-1} \ + \ q t_2^{-1} g_1 \underline{t} \ \widehat{=} \ (q-1) t \underline{t_2^{-1}} g_1^2 \ + \ q t \underline{t_2^{-1}} g_1 \ = \\
&= (q-1) t t_1^{-1} g_2^{-1} g_1^2 g_2^{-1} \ + \ q t t_1^{-1} g_2^{-1} g_1 g_2^{-1}.
\end{aligned}
$$

We have that:

$$
\begin{aligned}
g_2^{-1} g_1 g_2^{-1} &= q^{-2} g_1 g_2 g_1 \ + \ q^{-1}(q^{-1}-1) g_2 g_1 \ + \ q^{-1}(q^{-1}-1) g_1 g_2 \ + \\
&+ \ (q^{-1}-1)^2 g_1, \\
g_2^{-1} g_1^2 g_2^{-1} &= q^{-2}(q-1) g_1 g_2 g_1 \ - \ (q^{-1}-1)^2 g_2 g_1 \ - \ (q^{-1}-1)^2 g_1 g_2 \ + \\
&+ \ (q-1)(q^{-1}-1)^2 g_1 \ + \ q(q^{-1}-1) g_2^{-1} \ + \ 1,
\end{aligned}
$$

and so

$$
\begin{aligned}
(q-1) t t_1^{-1} g_2^{-1} g_1^2 g_2^{-1} &\widehat{\simeq} \left((q-1) + q^{-1}(q-1)^3\right) \cdot t t_1^{-1} - q^{-3}(q^{-1}-1)^3 z^2 \cdot 1 + \\
&+ 3q^{-3}(q-1)^4 z \cdot 1 - q^{-1}(q-1)^2 z \cdot 1 - q^{-3}(q-1)^5 \cdot 1, \\
q t t_1^{-1} g_2^{-1} g_1 g_2^{-1} &\widehat{\simeq} z \cdot t t_1^{-1} + q^{-1}(q^{-1}-1) z^2 \cdot 1 + 2(q^{-1}-1)^2 z \cdot 1 + \\
&+ q(q^{-1}-1)^3 \cdot 1.
\end{aligned}
$$

6. The basis Λ of $\mathcal{S}(\mathrm{ST})$

In this section we show that the set Λ is linearly independent. This is done in two steps:

- We first relate the two sets Λ and Λ' via an infinite lower triangular matrix with invertible elements in the diagonal.
- Then, using the matrix mentioned above, we prove that the set Λ is linearly independent.

6.1. The infinite matrix

With the orderings given in Definition 2 we shall show that the infinite matrix converting elements of the basis Λ' to elements of the set Λ is a block diagonal matrix, where each block is an infinite lower triangular matrix with invertible elements in the diagonal. Note that applying conjugation and stabilization moves on an element of some Λ_k followed by a braiding part won't alter the sum of the exponents of the loop generators and thus, the resulted terms will belong to the set of the same level Λ_k. Fixing the level k of a subset of Λ', the proof of Theorem 2 is equivalent to proving the following claims:

$$\Lambda' \ni \tau' \overset{Thm.7}{=} \underbrace{q^A \cdot \tau}_{\begin{array}{ccc}\Lambda \ni \tau & \sim & \tau' \\ q^A & \in & \mathbb{C}\end{array}} + \sum_i \underbrace{f_i(q,z)\cdot \tau \cdot w_i}_{\begin{array}{ccc}\Lambda \ni \tau & \sim & \tau' \\ f_i(q,z) & \in & \mathbb{C}, \forall i \\ w_i & \in & \mathrm{H}_n(q), \forall i\end{array}} + \sum_i \underbrace{g_i(q,z)\cdot \tau_i \cdot u_i}_{\begin{array}{ccc}\Sigma_n \ni \tau_i & < & \tau, \forall i \\ g_i(q,z) & \in & \mathbb{C}, \forall i \\ u_i & \in & \mathrm{H}_n(q), \forall i\end{array}}$$

$$\Big\downarrow Thm.10 \qquad\qquad\qquad \Big\downarrow Thm.8$$

$$\sum_j \underbrace{f_j(q,z)\cdot T_j}_{\begin{array}{ccc}\Sigma_n \ni T_j & < & \tau, \forall j \\ f_j(q,z) & \in & \mathbb{C}, \forall j\end{array}} \quad \xleftarrow{Thm.10} \quad \sum_j \underbrace{g_j(q,z)\cdot T_j \cdot u_j}_{\begin{array}{ccc}\Sigma_n \ni T_j & < & \tau, \forall j \\ g_j(q,z) & \in & \mathbb{C}, \forall j \\ u_j & \in & \mathrm{H}_n(q), \forall j\end{array}}$$

$$\Big\downarrow Thm.9$$

$$\sum_j \underbrace{h_j(q,z)\cdot \mathcal{T}_j \cdot v_j}_{\begin{array}{ccc}\Lambda \ni \mathcal{T}_j & < & \tau, \forall j \\ v_j & \in & \mathrm{H}_n(q), \forall j \\ h_j(q,z) & \in & \mathbb{C}, \forall j\end{array}}$$

$$\Big\downarrow Thm.10 \vdots Thm.9$$

$$\sum_i \underbrace{\mathcal{F}_i(q,z)\cdot \lambda_i}_{\begin{array}{ccc}\Lambda \ni \lambda_i & < & \tau, \forall i \\ \mathcal{F}_i(q,z) & \in & \mathbb{C}, \forall i\end{array}}$$

Fig. 15. From Λ' to Λ.

(1) A monomial $w' \in \Lambda'_k \subseteq \Lambda'$ can be expressed as linear combinations of elements in $\Lambda_k \subseteq \Lambda$, v_i, followed by monomials in $\mathrm{H}_n(q)$, with scalars in $\mathbb{C}$ such that $\exists\ j : v_j = w \sim w'$.
(2) Applying conjugation and stabilization moves on all v_i's results in obtaining elements in Λ_k, u_i's, such that $u_i < v_i$ for all i.
(3) The coefficient of w is an invertible element in $\mathbb{C}$.
(4) $\Lambda_k \ni w < u \in \Lambda_{k+1}$.

Indeed we have the following: Let $w' \in \Lambda'_k \subset \Lambda'$. Then, by Theorem 7 the monomial w' is expressed as a sum of elements in Σ_n, where the only term that isn't followed by a braiding part is the homologous monomial $w \in \Lambda_k \subset \Lambda$. Other terms in the sum involve lower order terms than w (with possible gaps in the indices and possible non-ordered exponents) followed by a braiding part and words of the form $w \cdot \beta$, where $\beta \in \mathrm{H}_n(q)$. Then, by Theorem 8 elements in Σ_n are expressed to linear combinations of elements in Σ_n with no gaps in the indices of the looping generators (regularizing elements with gaps) and obtaining words which are of less order than the initial word w. Then, by Theorem 9 we express these elements to linear combinations of elements in the $\mathrm{H}_n(q)$-module Λ, again of less order than w. In Theorem 10 all elements that are followed by a braiding part are expressed as sums of monomials in t_i's with coefficients in $\mathbb{C}$. It is essential to mention that when applying Theorem 10 to a word of the form $w \cdot \beta$ one obtains monomials in t_i's that are less ordered than w. Some of these monomials in t_i's are in Λ and some have their exponents in non-decreasing order, but all monomials are of less order than w. We apply again Theorem 9 on these monomials τ that don't belong in the set Λ and obtain words of less order than τ, followed by a braiding part. We only consider now the monomials not in Λ and perform Theorem 9. We obtain elements in the $\mathrm{H}_n(q)$-module Λ of less order than the initial monomials, followed by a braiding part. Eventually this procedure stops at the lower order term of Λ_k, t^k. So we have obtained elements in Λ of lower order terms than the initial element w, and thus, we obtain a lower triangular matrix with entries in the diagonal of the form q^{-A} (see Theorem 7), which are invertible elements in $\mathbb{C}$. The fourth claim follows directly from Definition 2. (See Fig. 15.)

If we denote as $[\Lambda_k]$ the block matrix converting elements in Λ'_k to elements in Λ_k for some k, then the change of basis matrix will be of the form:

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$$S = \begin{bmatrix} \ddots & 0 & 0 & 0 & 0 & 0 & \\ & [\Lambda_{k-2}] & 0 & 0 & 0 & 0 & \\ & 0 & [\Lambda_{k-1}] & 0 & 0 & 0 & \\ & 0 & 0 & [\Lambda_k] & 0 & 0 & \\ & 0 & 0 & 0 & [\Lambda_{k+1}] & 0 & \\ & 0 & 0 & 0 & 0 & [\Lambda_{k+2}] & \\ & 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix}$$

The infinite block diagonal matrix

6.2. Linear independence of Λ

Theorem 11. *The set* Λ *is linearly independent.*

Proof. Consider an arbitrary subset of Λ with finite many elements $\tau_1, \tau_2, \ldots, \tau_k$. Without loss of generality we consider $\tau_1 < \tau_2 < \ldots < \tau_k$ according to Definition 2. We convert now each element $\tau_i \in \Lambda$ to linear combination of elements in Λ' according to the infinite matrix. We have that

$$\tau_i \widehat{\cong} A_i \tau_i' \ + \sum_j A_j \tau_j' \ ,$$

where $\tau_i' \sim \tau_i$, $A_i \in \mathbb{C} \setminus \{0\}$, $\tau_j' < t_i'$ and $A_j \in \mathbb{C}$, $\forall j$.

So, we have that:

$$\begin{array}{rcl} \tau_1 & \widehat{\cong} & A_1\tau_1' + \sum_j A_{1j}\tau_{1j}' \\ \tau_2 & \widehat{\cong} & A_2\tau_2' + \sum_j A_{2j}\tau_{2j}' \\ \vdots & & \vdots \\ \tau_{k-1} & \widehat{\cong} & A_{k-1}\tau_{k-1}' + \sum_j A_{(k-1)j}\tau_{(k-1)j}' \\ \tau_k & \widehat{\cong} & A_k\tau_k' + \sum_j A_{kj}\tau_{kj}' \end{array}$$

Note that each τ_i' can occur as an element in the sum $\sum_j A_{pj}\tau_{pj}'$ for $p > i$. We consider now the equation $\sum_{i=1}^{k} \lambda_i \cdot \tau_i \ = \ 0$, $\lambda_i \in \mathbb{C}$, $\forall i$ and we show that this holds only when $\lambda_i = 0$, $\forall i$. Indeed, we have:

$$\sum_{i=1}^{k} \lambda_i \cdot \tau_i \ = \ 0 \ \Leftrightarrow \ \lambda_k A_k \tau_k' \ + \ \sum_{i=1}^{k} \sum_j \lambda_i A_{ij} \tau_{ij}' \ = \ 0,$$

where $\tau_k' > \tau_{ij}'$, $\forall i, j$. So we conclude that $\lambda_k \ = \ 0$. Using the same argument we have that:

$$\sum_{i=1}^{k} \lambda_i \cdot \tau_i \ = \ 0 \ \Leftrightarrow \ \sum_{i=1}^{k-1} \lambda_i \cdot \tau_i \ = \ 0 \ \Leftrightarrow \ \lambda_{k-1} A_{k-1} \tau_{k-1}' \ + \ \sum_{i=1}^{k-1} \sum_j \lambda_i A_{ij} \tau_{ij}' \ = \ 0,$$

where $\tau_{k-1}' > \tau_{ij}'$, $\forall i, j$. So, $\lambda_{k-1} \ = \ 0$. Retrospectively we get:

$$\sum_{i=1}^{k} \lambda_i \cdot \tau_i \ = \ 0 \ \Leftrightarrow \ \lambda_i \ = \ 0, \ \forall i,$$

and so an arbitrary finite subset of Λ is linearly independent. Thus, the set Λ is linearly independent. $\square$

6.3. The proof of the main result

By Theorems 7, 8, 9 and 10 the set Λ is a spanning set of $\mathcal{S}(\mathrm{ST})$. By Theorem 11 the set Λ is also linearly independent. Thus, it forms a basis for $\mathcal{S}(\mathrm{ST})$ and the proof of Theorem 2 is now concluded.

7. Conclusions

In this paper we gave a new basis Λ for $\mathcal{S}(\mathrm{ST})$, different from the Turaev–Hoste–Kidwell basis and the Morton–Aiston basis. The new basis is appropriate for describing the handle sliding moves, whilst the old basis Λ' is consistent with the trace rules [4]. In a sequel paper we use the bases Λ' and Λ of $\mathcal{S}(\mathrm{ST})$ and the change of basis matrix in order to compute the Homflypt skein module of the lens spaces $L(p, 1)$.

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