On the Pseudospectra of Matrix Polynomials

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0. Prologue

Consider the differential equation

$$A_m\, u^{(m)}(t) + A_{m-1}\, u^{(m-1)} + \cdots +$$

$$+ \cdots + A_1\, u'(t) + A_0\, u(t) = f(t) \quad (1)$$

and the difference equation

$$A_m\, u_{j+m} + A_{m-1}\, u_{j+m-1} + \cdots +$$

$$+ \cdots + A_1\, u_{j+1} + A_0\, u_j = f_j, \quad (2)$$

where $A_j \in \mathbb{C}^{n\times n}$, $u(t), u_j \in \mathbb{C}^n$ and $\underline{\det A_m \neq 0}$.

Applying the Laplace transformation to (1) or the Z-transformation to (2) yields the *matrix polynomial*

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0.$$

A $0 \neq x_0 \in \mathbb{C}^n$ is an *eigenvector* of $P(\lambda)$ corresponding to the *eigenvalue* $\lambda_0 \in \mathbb{C}$ if

$$P(\lambda_0)x_0 = 0 \quad (eigenproblem).$$

If, in addition, $x_1, x_2, \ldots, x_k \in \mathbb{C}^n$ satisfy

$$\sum_{j=1}^{\xi} \frac{1}{j!} P^{(j)}(\lambda_0)\, x_{\xi-j} = 0 \ ; \quad \xi = 1, 2, \ldots, k,$$

then $x_0, x_1, \ldots, x_k$ is a *Jordan chain* of $P(\lambda)$.

The solution of (1) is of the form

$$u(t) = X_P\, e^{t\, J_P}\, c + \int_{t_0}^{t} X_P\, e^{(t-s)\, J_P}\, Y_P\, f(s)\, ds$$

and the solution of (2) is of the form

$$u_j = X_P\, J_P^j\, c + \sum_{i=0}^{\nu-1} X_P\, J_P^{\nu-i-1}\, Y_P\, f_i\, .$$

1. Introduction

The *spectrum* of

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$$

is the set of all eigenvalues of $P(\lambda)$,

$$\sigma(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\},$$

where $\det P(\lambda)$ is a scalar polynomial of degree $n\,m$, with leading coefficient $\underline{\det A_m \neq 0}$.

We are interested in the spectra of perturbations of $P(\lambda)$ of the form

$$P_\Delta(\lambda) = (A_m + \Delta_m)\lambda^m + (A_{m-1} + \Delta_{m-1})\lambda^{m-1} + \cdots + (A_1 + \Delta_1)\lambda + A_0 + \Delta_0,$$

where $\Delta_0, \Delta_1, \ldots, \Delta_m \in \mathbb{C}^{n\times n}$ are arbitrary.

For a given $\varepsilon > 0$ and a given set of non-negative weights $\mathbf{w} = \{w_0, w_1, \ldots, w_m\}$, the *$\varepsilon$-pseudospectrum* of $P(\lambda)$ with respect to $\mathbf{w}$ (introduced by Tisseur and Higham, 2001) is

$$\sigma_{\varepsilon,\mathbf{w}}(P) = \{\lambda \in \mathbb{C} : \det P_\Delta(\lambda) = 0, \\ \|\Delta_j\| \le \varepsilon\, w_j,\ j = 0, 1, \ldots, m\}.$$

$w_0, w_1, \ldots, w_m \ge 0$ allow freedom in how perturbations are measured; for example, in an absolute sense when $w_0 = w_1 = \cdots = w_m = 1$, or in a relative sense when $w_j = \|A_j\|$. Different values for w_j admit different levels of confidence in A_j.

Note that for $\varepsilon = 0$, $\sigma_{0,\mathbf{w}}(P) = \sigma(P)$.

For $P(\lambda) = I\lambda - A$ ($A \in \mathbb{C}^{n\times n}$), $\sigma(P)$ coincides with the spectrum of A, $\sigma(A)$. If, in addition, $\mathbf{w} = \{w_0, w_1\} = \{1, 0\}$, then $\sigma_{\varepsilon,\mathbf{w}}(P)$ coincides with the *ε-pseudospectrum* of A,

$$\sigma_\varepsilon(A) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(A + \Delta_0),\ \|\Delta_0\| \leq \varepsilon\}.$$

For the spectral norm, defining the scalar

$$q_{\mathbf{w}}(\lambda) = w_m\lambda^m + w_{m-1}\lambda^{m-1} + \cdots + w_1\lambda + w_0,$$

one of the main tools is the formula (Tisseur-Higham, 2001) ($s_{\min}$: min. singular value)

$$\sigma_{\varepsilon,\mathbf{w}}(P) = \{\lambda \in \mathbb{C} : s_{\min}(P(\lambda)) \leq \varepsilon\, q_{\mathbf{w}}(|\lambda|)\}.$$

As the eigenvalues of $P_\Delta(\lambda)$ are continuous,

$$\partial\sigma_{\varepsilon,\mathbf{w}}(P) = \{\lambda \in \mathbb{C} : s_{\min}(P(\lambda)) = \varepsilon\, q_{\mathbf{w}}(|\lambda|)\}.$$

2. Examples (using the spectral norm)

Example 1 *(A wing problem)*

The eigenproblem of the matrix polynomial

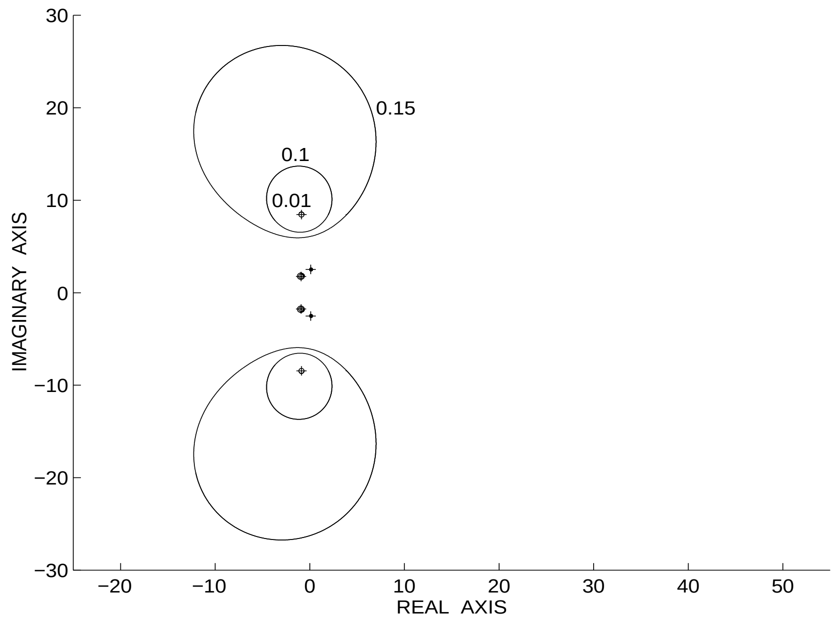
$$Q(\lambda) = \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix} \lambda^2 +$$

$$+ \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{bmatrix} \lambda + \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}.$$

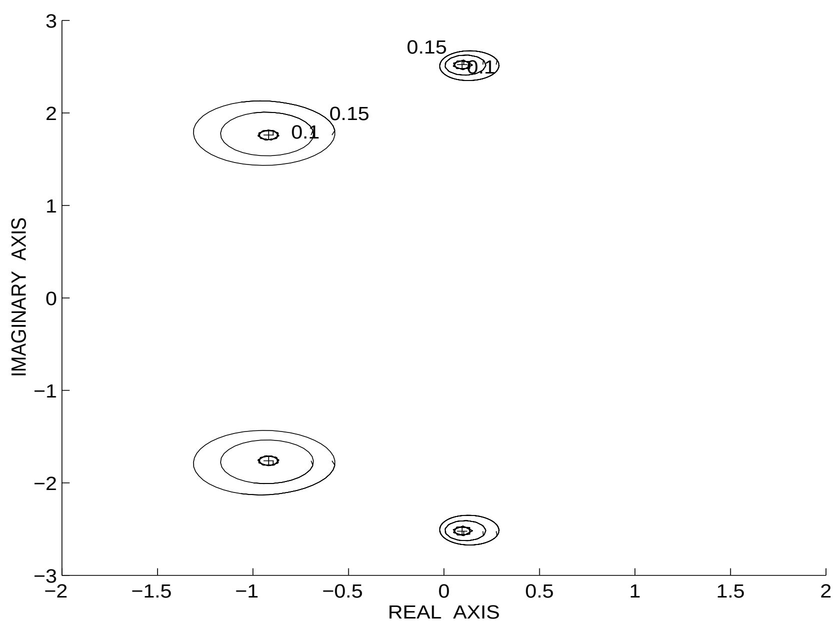
arose from a study of the oscillations of a wing in an airstream. The eigenvalues of $Q(\lambda)$ are

$$-0.88 \pm \mathrm{i}\,8.44,\ 0.09 \pm \mathrm{i}\,2.52,\ -0.92 \pm \mathrm{i}\,1.76.$$

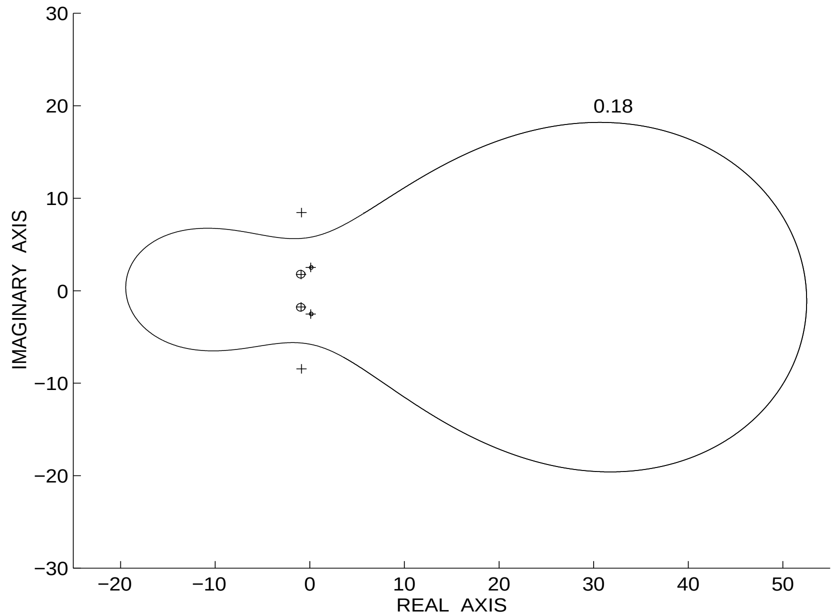
Perturbations are measured in the absolute sense, i.e., $w_0 = w_1 = w_2 = 1$.



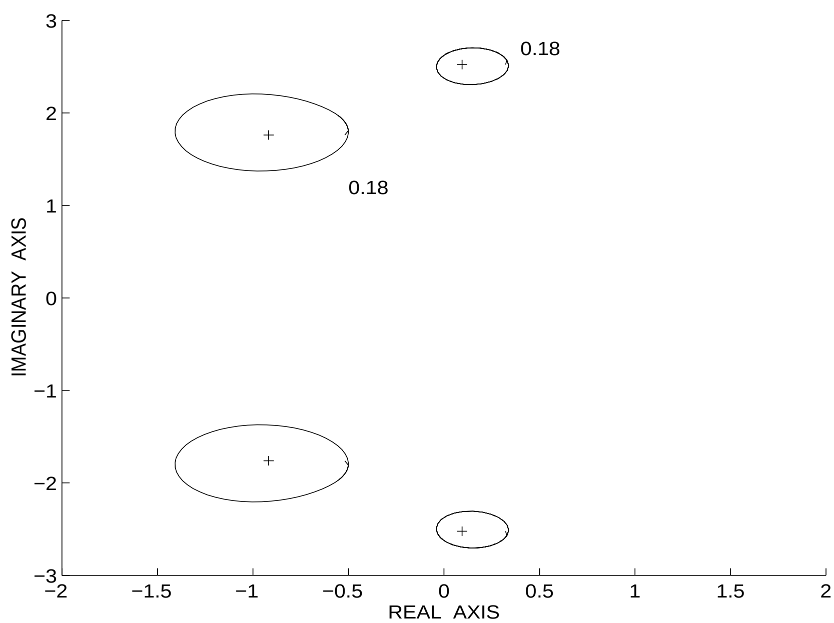
$\partial \sigma_{\varepsilon,\mathrm{w}}(Q)$ for $\varepsilon = 0.01,\ 0.1,\ 0.15 < s_{\min}(A_2)$.



The less sensitive eigenvalues.



$\partial\sigma_{\varepsilon,\mathbf{w}}(Q)$ for $\varepsilon = 0.18 > s_{\min}(A_2)$.



The less sensitive eigenvalues.

Example 2 *(A vibrating system)*

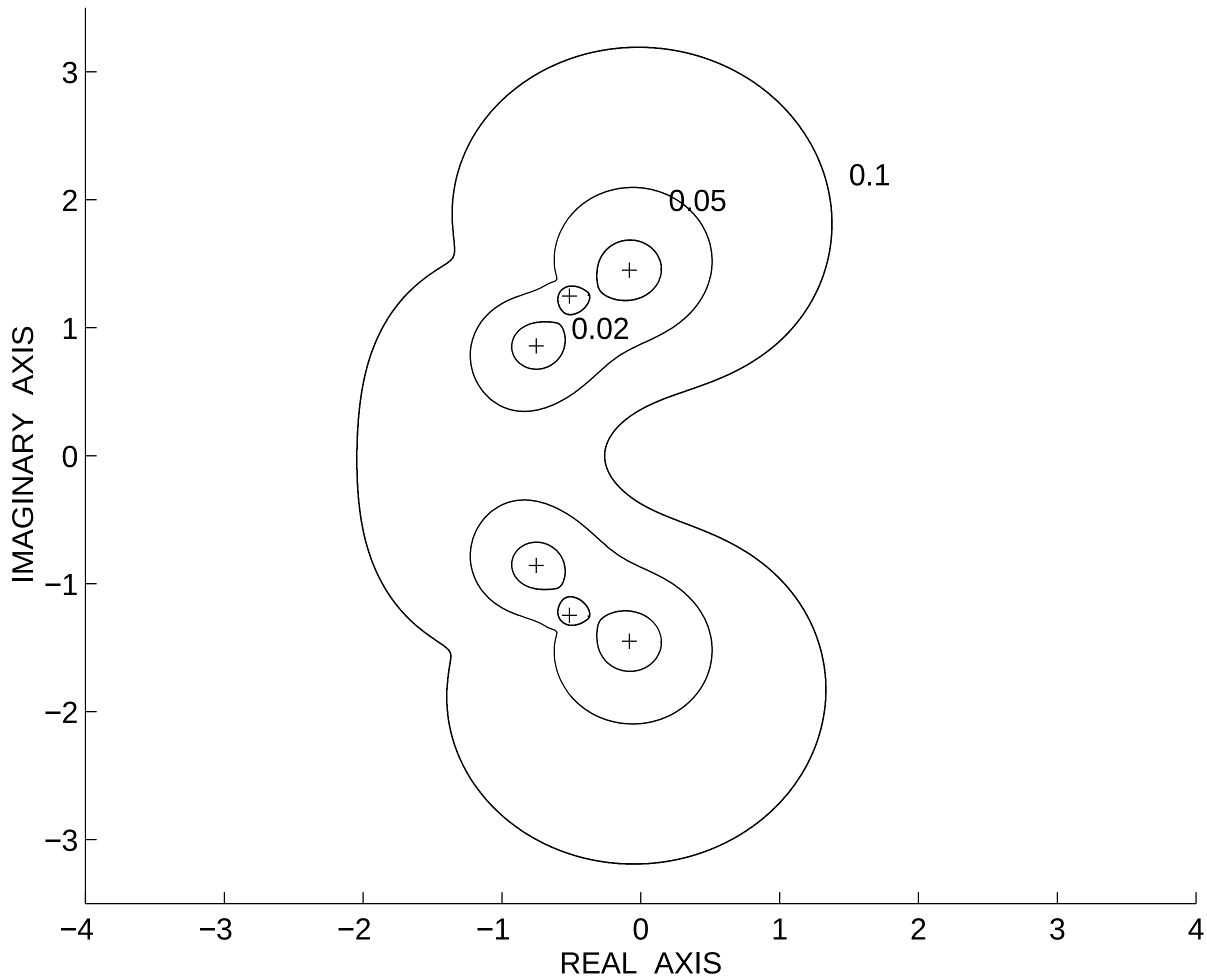
The 3×3 selfadjoint matrix polynomial

$$P(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \lambda^2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 6 \end{bmatrix} \lambda + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

corresponds to a mass-spring model described by Falk (1960). The eigenvalues of $P(\lambda)$ are

$$-0.08 \pm \mathrm{i}\,1.45,\ -0.75 \pm \mathrm{i}\,0.86,\ -0.51 \pm \mathrm{i}\,1.25.$$

Perturbations are measured in a relative sense, i.e., $w_0 = \|A_0\| = 10$, $w_1 = \|A_1\| = 6.3$ and $w_2 = \|A_2\| = 5$.



$\partial\sigma_{\varepsilon,\mathbf{w}}(P)$ for $\varepsilon = 0.02, 0.05, 0.1.$

Example 3 *(A gyroscopic system)*

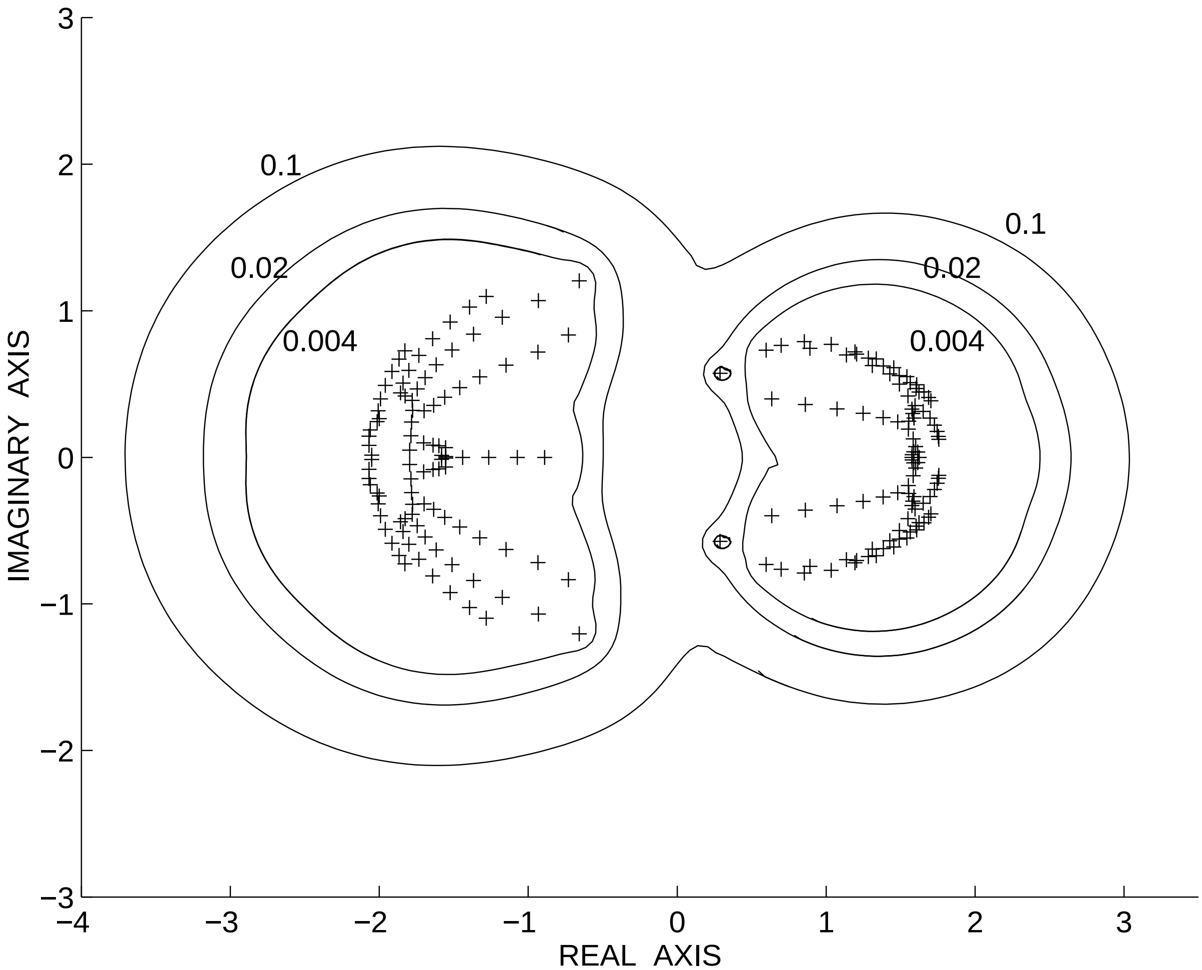
Let B be the 10×10 nilpotent matrix with ones on the subdiagonal and zeros elsewhere. Define $\widehat{M} = (4I_{10} + B + B^T)/6$, $\widehat{G} = B - B^T$ $\widehat{K} = B + B^T - 2I_{10}$, and set

$$\begin{aligned} M &= I_{10} \otimes \widehat{M} + 1.30\widehat{M} \otimes I_{10}, \\ G &= 1.35 I_{10} \otimes \widehat{G} + 1.10\widehat{G} \otimes I_{10}, \\ K &= I_{10} \otimes \widehat{K} + 1.20\widehat{K} \otimes I_{10}. \end{aligned}$$

The 100×100 matrix polynomial $M\lambda^2 + G\lambda + K$ corresponds to a gyroscopic system (Mehrmann-Watkins, 2001). Adding the damping matrix $D = \text{tridiag}\{-0.1, 0.3, -0.1\}$ to G yields

$$R(\lambda) = M\lambda^2 + (G + D)\lambda + K.$$

Perturbations are measured in the absolute sense.



$\partial\sigma_{\varepsilon,\mathbf{w}}(R)$ for $\varepsilon = 0.004, 0.02, 0.1.$

3. General Properties

Consider an $n \times n$ matrix polynomial

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0.$$

Proposition 1 If the coefficients of $P(\lambda)$ are all real or all hermitian, then for any $\varepsilon > 0$ and $\mathbf{w} = \{w_0, w_1, \ldots, w_m\}$, $\sigma_{\varepsilon,\mathbf{w}}(P)$ is symmetric with respect to the real axis.

Proof Based on the observation

$$\|\Delta_j\| = \|\overline{\Delta}_j\| = \|\Delta_j^*\|. \quad \square$$

Theorem 2 The pseudospectrum $\sigma_{\varepsilon,\mathbf{w}}(P)$ is bounded if and only if $0 \notin \sigma_{\varepsilon w_m}(A_m)$, i.e., if and only if

$$s_{\min}(A_m) \geq \varepsilon\, w_m.$$

Proof Suppose $0 \notin \sigma_{\varepsilon w_m}(A_m)$ and define

$$\zeta_\varepsilon = \min\{|\det(A_m + \Delta_m)| : \|\Delta_m\| \leq \varepsilon\, w_m\} > 0.$$

Then there is an $M_\varepsilon > 0$ such that for any associated perturbation

$$P_\Delta(\lambda) = (A_m + \Delta_m)\lambda^m + \cdots + (A_1 + \Delta_1)\lambda + A_0 + \Delta_0$$

and for any $\lambda \in \mathbb{C}$ with $|\lambda| > M_\varepsilon$,

$$|\det P_\Delta(\lambda) - \det(A_m + \Delta_m)\lambda^{nm}| < \zeta_\varepsilon |\lambda^{mn}|$$

$$\leq |\det(A_m + \Delta_m)\lambda^{mn}|,$$

i.e., $\det P_\Delta(\lambda) \neq 0$. Hence, $\sigma_{\varepsilon,\mathbf{w}}(P)$ is bounded.

To prove the converse, assume that $\sigma_{\varepsilon,\mathbf{w}}(P)$ is bounded but there is an associated

$$P_{\widehat{\Delta}}(\lambda) = (A_m + \widehat{\Delta}_m)\lambda^m + \cdots + (A_1 + \widehat{\Delta}_1)\lambda + A_0 + \widehat{\Delta}_0,$$

with $\det(A_m + \widehat{\Delta}_m) = 0$. One of the coefficients of $\det P_{\widehat{\Delta}}(\lambda)$, let of λ^τ, is $\beta_\tau \neq 0$. Construct a sequence $\{\widehat{\Delta}_{m,k}\}_{k\in\mathbb{N}} \subset \mathbb{C}^{n\times n}$ such that $\lim_{k\to\infty} \widehat{\Delta}_{m,k} = \widehat{\Delta}_m$, and for every $k \in \mathbb{N}$,

$$\det(A_m + \widehat{\Delta}_{m,k}) \neq 0 \quad \text{and} \quad \|\widehat{\Delta}_{m,k}\| \leq \varepsilon\, w_m.$$

Since $\sigma_{\varepsilon,\mathbf{w}}(P)$ is bounded, the $(nm - \tau)$th elementary symmetric function of the zeros of

$$\det[(A_m + \widehat{\Delta}_{m,k})\lambda^m + \cdots + (A_1 + \widehat{\Delta}_1)\lambda + A_0 + \widehat{\Delta}_0],$$

which is equal to $\pm\beta_\tau / \det(A_m + \widehat{\Delta}_{m,k})$, is bounded for all k; this is a contradiction. $\square$

Theorem 3 If $\sigma_{\varepsilon,\mathbf{w}}(P)$ is bounded, then it has no more than $n\,m$ con. components, and any associated $P_\Delta(\lambda)$ has the same number (≥ 1) of eigenvalues with $P(\lambda)$ in each one of these components, counting multiplicities.

Proof By Theorem 2, for any associated

$$P_\Delta(\lambda) = (A_m + \Delta_m)\lambda^m + \cdots + (A_1 + \Delta_1)\lambda + A_0 + \Delta_0,$$

$\det(A_m + \Delta_m) \neq 0$. Thus, $P_\Delta(\lambda)$ has exactly $n\,m$ eigenvalues, counting multiplicities, as does every

$$P_{\Delta,t}(\lambda) = (1-t)\,P(\lambda) + tP_\Delta(\lambda)\;;\;\; t \in [0,1].$$

As t varies from 0 to 1, the eigenvalues of $P_{\Delta,t}(\lambda)$ trace continuous paths from the eigenvalues of $P(\lambda)$ to the eigenvalues of $P_\Delta(\lambda)$. $\square$

4. A Curve-Tracing Algorithm

Recall that for the spectral norm,

$$\partial\sigma_{\varepsilon,\mathbf{w}}(P) = \{\lambda \in \mathbb{C} : s_{\min}(P(\lambda)) = \varepsilon\, q_{\mathbf{w}}(|\lambda|)\}\,.$$

For convenience, define

$$g_P(x,y) = s_{\min}(P(x+\mathrm{i}y)) \;\; ; \quad x,y \in \mathbb{R}$$

and

$$g_P(\lambda) = s_{\min}(P(\lambda)) \;\; ; \quad \lambda \in \mathbb{C}.$$

Theorem 4 (Sun, 1988) Let $\lambda_0 = x_0 + \mathrm{i}y_0 \in \mathbb{C} \setminus \sigma(P)$. If $s_{\min}(P(\lambda_0))$ is a simple singular value of $P(\lambda_0)$, and u_0, v_0 are associated left and right singular vectors, respectively, then

$$\nabla g_P(\lambda_0) = \left(\mathrm{Re}\left(u_0^*\,\frac{\partial P(\lambda_0)}{\partial x}\,v_0\right), \mathrm{Re}\left(u_0^*\,\frac{\partial P(\lambda_0)}{\partial y}\,v_0\right)\right).$$

Our continuation method for drawing

$$\partial\sigma_{\varepsilon,\mathbf{w}}(P) = \{\lambda \in \mathbb{C} : g_P(\lambda) - \varepsilon\, q_{\mathbf{w}}(|\lambda|) = 0\}$$

is an extension of (Brühl, 1996), and consists of an initial step to find a starting point on $\partial\sigma_{\varepsilon,\mathbf{w}}(P)$ followed by a sequence of "predictor" steps tangential to $\partial\sigma_{\varepsilon,\mathbf{w}}(P)$ and "corrector" steps to go back to $\partial\sigma_{\varepsilon,\mathbf{w}}(P)$.

Initial Step: For calculation of a first point on $\partial\sigma_{\varepsilon,\mathbf{w}}(P)$, let $\lambda_0 \in \sigma_{\varepsilon,\mathbf{w}}(P) \setminus \sigma(P)$ and $d_0 \in \mathbb{C}$ be nonzero. Then use Newton's method to solve

$$g_P(\lambda_0 + t\, d_0) - \varepsilon\, q_{\mathbf{w}}(|\lambda_0 + t\, d_0|) = 0$$

along the straight line $\{\lambda_0 + t\, d_0 : t \in \mathbb{R}\}$. Set $t_0 = 0$, and assume that g_P is differentiable at λ_0 and $\nabla g_P(\lambda_0)$ is given by Theorem 4.

The first Newton iterate gives

$$t_1 = -\frac{g_P(\lambda_0) - \varepsilon\, q_{\mathbf{w}}(|\lambda_0|)}{(g_P(\lambda_0 + t\, d_0) - \varepsilon\, q_{\mathbf{w}}(|\lambda_0 + t\, d_0|))'}$$

and the point

$$z_1 = \lambda_0 - \frac{g_P(\lambda_0) - \varepsilon\, q_{\mathbf{w}}(\lambda_0)}{(\mathrm{Re}\, d_0, \mathrm{Im}\, d_0) \cdot \nabla[g_P(\lambda_0) - \varepsilon\, q_{\mathbf{w}}(|\lambda_0|)]}\, d_0. \tag{3}$$

Since $\lambda_0 \in \sigma_{\varepsilon,\mathbf{w}}(P)$, for suitable direction d_0, we estimate a point of $\partial\sigma_{\varepsilon,\mathbf{w}}(P)$ by repeating (3) until $|s_{\min}(P(z)) - \varepsilon\, q_{\mathbf{w}}(|z|)|$ is small enough. In our examples, only a few iterations are required.

For $d_0 = \nabla[g_P(\lambda_0) - \varepsilon\, q_{\mathbf{w}}(|\lambda_0|)]$, (3) implies

$$z_1 = \lambda_0 - (g_P(\lambda_0) - \varepsilon\, q_{\mathbf{w}}(|\lambda_0|))\, \overline{d_0}^{-1}. \tag{4}$$

Prediction: Assuming that $z_{k-1} \in \partial\sigma_{\varepsilon,\mathbf{w}}(P)$ has been computed and τ_k is the corresponding step-length, the (tangential) prediction for the kth boundary point of $\sigma_{\varepsilon,\mathbf{w}}(P)$, z_k, is

$$\widehat{z}_k = z_{k-1} + \tau_k \left(\mathrm{i}\, \frac{\nabla \left[g_P(z_{k-1}) - \varepsilon\, q_{\mathbf{w}}(|z_{k-1}|) \right]}{\left| \nabla \left[g_P(z_{k-1}) - \varepsilon\, q_{\mathbf{w}}(|z_{k-1}|) \right] \right|} \right).$$

Correction: For small τ_k, the correction step is a single Newton iterate for the equation $g_P(\widehat{z}_k + t\, d_k) - \varepsilon\, q_{\mathbf{w}}(|\widehat{z}_k + t\, d_k|) = 0$, with an appropriate direction d_k and initial $t_0 = 0$. In our examples, has been found that one Newton step gives satisfactory performance, although the effect of taking more steps could be a subject for further investigation.

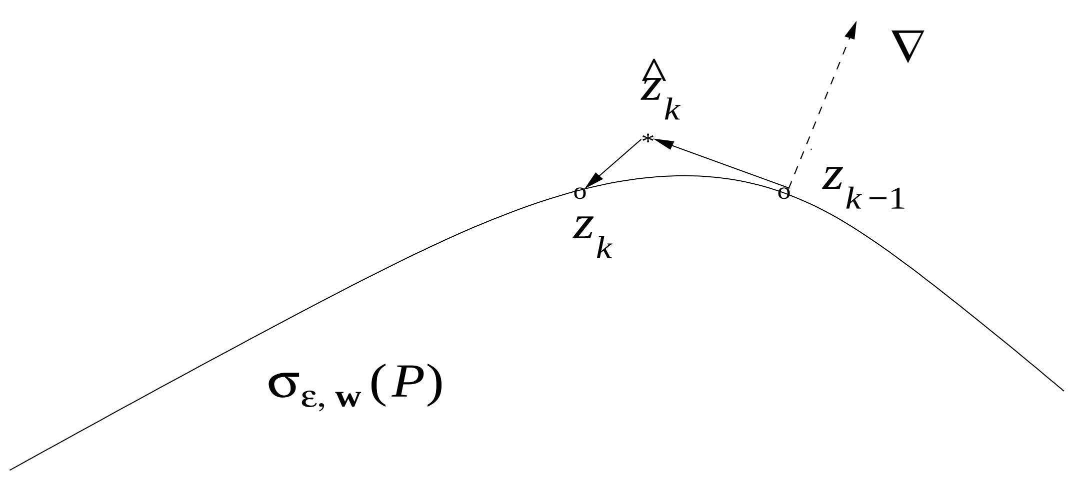
A natural choice for $d_k \in \mathbb{C}$ is

$$\widehat{d}_k = \nabla \left[g_P(\widehat{z}_k) - \varepsilon\, q_{\mathbf{w}}(|\widehat{z}_k|) \right].$$

In this case, the step (4) is written

$$z_k = \widehat{z}_k - \left(g_P(z_{k-1}) - \varepsilon\, q_{\mathbf{w}}(|z_{k-1}|) \right) \overline{\widehat{d}_k}^{\,-1}$$

and the estimation of z_k requires the computation of $s_{\min}(P(z_{k-1}))$, $s_{\min}(P(\widehat{z}_k))$ and their associated left and right singular vectors.



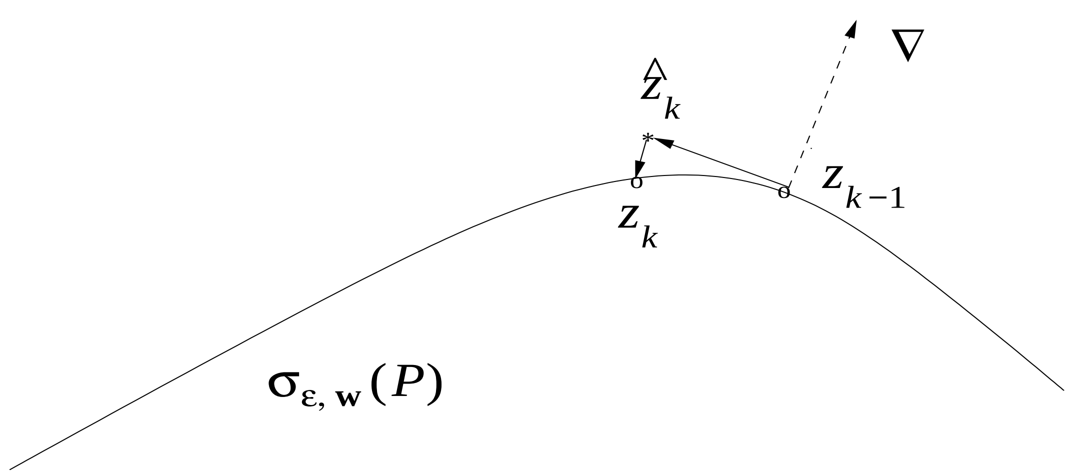
Choosing the direction $\widehat{d}_k$ in the correction step.

The computation of $s_{\min}(P(\hat{z}_k))$ and the corresponding singular vectors can be avoided (and the computational cost of the algorithm reduced by about a half) if the correction step is taken in the direction of

$$d_k = \nabla \left[g_P(z_{k-1}) - \varepsilon\, q_{\mathbf{w}}(|z_{k-1}|) \right]$$

and (4) is written in the form

$$z_k = \hat{z}_k - \left(g_P(z_{k-1}) - \varepsilon\, q_{\mathbf{w}}(|z_{k-1}|) \right) \overline{d_k}^{-1}.$$



Choosing the direction d_k in the correction step.

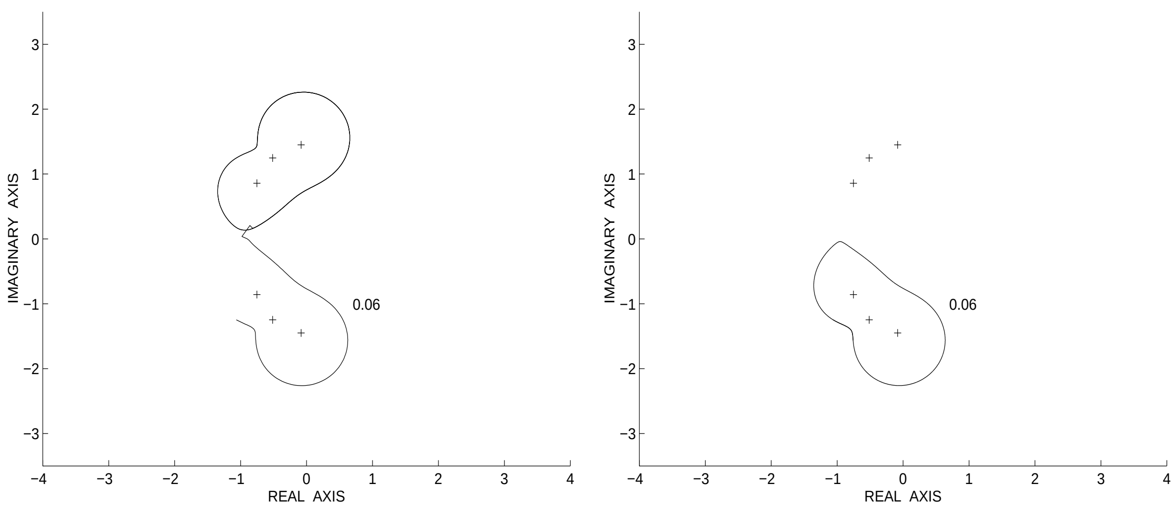
5. Comments on the Algorithm

(a) It tracks the boundary of that con. component of $\sigma_{\varepsilon,\mathbf{w}}(P)$ containing λ_0. For a complete picture, it may be necessary to repeat the procedure for several values of λ_0.

(b) It does not require *a priori* knowledge of the size of $\sigma_{\varepsilon,\mathbf{w}}(P)$, since it sketches the con. components of $\sigma_{\varepsilon,\mathbf{w}}(P)$ one after the other by using starting points close to eigenvalues.

(c) The size of the step-lengths, τ_k, in the prediction step affects the accuracy and the computational cost of the algorithm, and it is important to obtain criteria for their selection (Bekas-Gallopoulos, 2001).

(d) The algorithm may lose its path near boundary points where $\nabla\left[g_P(\lambda) - \varepsilon\, q_{\mathbf{w}}(|\lambda|)\right]$ does not exist or it is zero, and near points where the distance between con. components of $\sigma_{\varepsilon,\mathbf{w}}(P)$ becomes small. Some of these difficulties can be solved by choosing a smaller step-length (increasing the cost). See below $\sigma_{\varepsilon,\mathbf{w}}(P)$ of Example 2, for $\varepsilon = 0.06$.



Constant step-lengths $\tau = 0.03$ and $\tau = 0.003$.

6. Some Open Questions

(a) What else can we say about the topological and geometrical properties of $\sigma_{\varepsilon,\mathbf{w}}(P)$ and its con. components ?

(b) When $\sigma_{\varepsilon,\mathbf{w}}(P)$ is unbounded, how many con. components it may have ?

(c) Is it true that a bounded con. component $\mathcal{G}$ of $\sigma_{\varepsilon,\mathbf{w}}(P)$ contains 2 eigenvalues of $P(\lambda)$ if and only if an associated perturbation $P_{\Delta}(\lambda)$ has a multiple eigenvalue in $\mathcal{G}$?

(d) How can $\sigma_{\varepsilon,\mathbf{w}}(P)$ be used in studying the stability of the spectral factorization of $P(\lambda)$?

(e) Can the path-tracing algorithm be a part of a parallel algorithm ?

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