# The Perron eigenspace of nonnegative almost skew-symmetric matrices and Levinger's transformation

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**Abstract**

Let $A$ be a nonnegative square matrix whose symmetric part has rank one. Tournament matrices are of this type up to a positive shift by $1/2\, I$. When the symmetric part of $A$ is irreducible, the Perron value and the left and right Perron vectors of $\mathcal{L}(A,\alpha) = (1-\alpha)A + \alpha A^t$ are studied and compared as functions of $\alpha \in [\,0,\, 1/2]$. In particular, upper bounds are obtained for both the Perron value and its derivative as functions of the parameter $\alpha$ via the notion of the $q$-numerical range.

*Keywords:* Almost skew-symmetric matrix, Perron value, Perron vector, Levinger's transformation, $q$-numerical range, tournament

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# 1 Introduction

An almost skew-symmetric matrix is a square matrix whose symmetric part has rank one. The initial interest in almost skew-symmetric matrices can be largely attributed to their association with tournament matrices; indeed, if $T$ is a $(0,1)$-tournament matrix, i.e., $T+T^t = J-I$, where $J$ is the all ones matrix, then $T+1/2\,I$ is a nonnegative almost skew-symmetric matrix. The discovery that the eigenvalues of almost skew-symmetric matrices satisfy interesting inequalities forced upon by their structure [4, 5, 11] is sustaining the interest in these matrices and their properties.

In this article, we consider an entrywise nonnegative almost skew-symmetric matrix $A$ and study the spectral radius of $\mathcal{L}(A,\alpha) = (1-\alpha)A + \alpha A^t$ as a function of $\alpha \in [0,\,1/2]$. The affine transformation $(1-\alpha)A+\alpha A^t$ and, in particular, its spectral radius when $A$ is nonnegative were initially considered by Levinger [7], who showed that the spectral radius is non-decreasing in $[0,\,1/2]$. We thus refer to $\mathcal{L}(A,\alpha)$ as Levinger's transformation. This result, re-proved and extended by Fiedler [2, 3], is regarded as one of the few results on the behavior of the spectral radius as a function of matrix elements and has proven useful in many instances.

More recently, use of Levinger's transformation is made in [11] in order to study the spectrum of general almost skew-symmetric matrices. In addition, $\mathcal{L}(A,\alpha)$ is used to study the shape of the numerical range of nonnegative matrices [10]. It appears that Levinger's transformation is a piece of many puzzles and so we are motivated to continue studying its role. Here, we do it in the context of nonnegative almost skew-symmetric matrices with irreducible symmetric parts. Specifically, we study the rate of change of the spectral radius of $\mathcal{L}(A,\alpha)$ as a function of $\alpha$, as well as the behavior of the corresponding left and right (Perron) eigenvectors.

# 2 Preliminaries

Let $x,\,y \in \mathbb{R}^n$. We call $x$ a *unit* vector if its Euclidean norm $\|x\|_2 = 1$. The angle between two nonzero vectors $x$ and $y$ is defined to be

$$(\widehat{x,y}) = \cos^{-1} \frac{x^t y}{\|x\|_2 \|y\|_2}$$

and is measured in $[\,0,\,\pi\,]$. Consider an $n \times n$ real matrix $A$ (denoted by $A \in M_n(\mathbb{R})$). The spectrum of $A$ is denoted by $\sigma(A)$ and its spectral radius by $\rho(A) = \max\{|\lambda| \,:\, \lambda \in \sigma(A)\}$. Recall that $A$ can be written as

$$A = S(A) + K(A),$$

where

$$S(A) = \frac{A + A^t}{2} \quad \text{and} \quad K(A) = \frac{A - A^t}{2}$$

are the (real) *symmetric part* and the (real) *skew-symmetric part* of $A$, respectively. We define *Levinger's transformation* of $A$ by

$$\mathcal{L}(A, \alpha) = (1 - \alpha)A + \alpha A^t\,; \quad \alpha \in [\,0,\ 1/2]$$

and *Levinger's function* by

$$\phi(A, \alpha) = \rho(\mathcal{L}(A, \alpha)) = \rho((1 - \alpha)A + \alpha A^t)\,; \quad \alpha \in [\,0,\ 1/2]. \tag{2.1}$$

Notice that the analysis of Levinger's function in $[\,0,\ 1/2]$ extends naturally to $[\,1/2,\ 1\,]$ as $\mathcal{L}(A, \alpha)^t = \mathcal{L}(A, 1 - \alpha)$. One can also see that

$$\mathcal{L}(A, 0) = A\,, \quad \mathcal{L}(A, 1/2) = S(A)$$

and that for every $\alpha \in (0,\ 1/2)$,

$$\mathcal{L}(A, \alpha) = S(A) + (1 - 2\alpha)K(A).$$

Suppose now that $A$ is entrywise nonnegative (denoted by $A \geq 0$). Then $S(A) \geq 0$. Moreover, suppose $S(A)$ is of rank one. This means that $\sigma(S(A))$ consists of the eigenvalue $0$ with multiplicity $n - 1$ and a simple eigenvalue $\delta(A)$. By the Perron-Frobenius Theorem (see e.g., [1]), $\delta(A) = \rho(S(A)) > 0$. It follows that there is a nonzero nonnegative vector $w \in \mathbb{R}^n$ such that

$$S(A) = w\,w^t \quad \text{and} \quad \delta(A) = w^t\,w.$$

We refer to a matrix $A$ having the above features as a *nonnegative almost skew-symmetric matrix* and assume the reader recalls the above associated notation.

The *variance* of a nonnegative almost skew-symmetric matrix $A$ is

$$\mathrm{var}(A) = \frac{\|K(A)w\|_2^2}{\|w\|_2^2} = \frac{w^t(K(A)^tK(A))w}{w^tw}\,.$$

Note that the variance of $\mathcal{L}(A, \alpha)$ is given by

$$\mathrm{var}(\mathcal{L}(A, \alpha)) = (1 - 2\alpha)^2\mathrm{var}(A).$$

It is also clear that

$$\phi(A, 0) = \rho(A) \quad \text{and} \quad \phi(A, 1/2) = \rho(S(A)) = \delta(A)\,.$$

Some more terminology is in order for a nonnegative matrix $X$ (for details and general background see [1]). We refer to $\rho(X) \in \sigma(X)$ as the *Perron root* of $X$. Recall that $X$ is irreducible if its directed graph is strongly connected or, equivalently, if it cannot be symmetrically permuted to a block upper triangular matrix having non-vacuous, square diagonal blocks. When $X$ is irreducible, the Perron-Frobenius Theorem states that $\rho(X)$ is a *simple* eigenvalue. In this case, we denote the corresponding unit right and left eigenvectors of $X$ by $x_r(X)$ and $x_l(X)$, and refer to them as the unit *right Perron vector* and the unit *left Perron vector*, respectively.

Let us now assume that $A$ is nonnegative almost skew-symmetric and that $S(A) = ww^t$ is irreducible. As $S(A) \geq 0$, we have that $w$ must be a strictly positive vector and thus $S(A)$ is a strictly positive matrix. It follows that all $\mathcal{L}(A,\alpha)$ $(\alpha \in (0,\, 1))$ are also strictly positive (and thus irreducible) almost skew-symmetric matrices. As a consequence, $\phi(A,\alpha)$ is a simple eigenvalue and thus the right and left unit Perron vectors of $\mathcal{L}(A,\alpha)$ are well-defined; in particular notice that

$$x_r(\mathcal{L}(A,1/2)) \;=\; x_l(\mathcal{L}(A,1/2)) \;=\; x_r(S(A)) \;=\; x_l(S(A)) \;=\; w/\|w\|_2.$$

In addition, when $S(A)$ is irreducible, $\phi(A,\alpha)$ is a differentiable function of $\alpha \in (0,\, 1)$. These facts underlie most of our statements in Sections 3 and 4.

**Example 2.1** This example illustrates the above definitions and that $S(A)$ being irreducible is indeed an assumption weaker than $A$ being irreducible, even for nonnegative almost skew-symmetric matrices. The matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

is a reducible nonnegative almost skew-symmetric matrix with

$$K(A) = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad S(A) = ww^t, \quad \text{where} \quad w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

It satisfies $\delta(A) = 3$ and $\operatorname{var}(A) = 8/3$. As $S(A)$ is irreducible, the main results herein apply to this matrix $A$.

We proceed with a brief mention of some results from [11] that are relevant to our discussion. Notice that they apply to skew-symmetric matrices that are not necessarily nonnegative; in [11] an almost skew-symmetric matrix is by definition a matrix $A$ whose symmetric part has rank one and $\delta(A) > 0$.

**Theorem 2.2** *Let $A \in M_n(\mathbb{R})$ be an almost skew-symmetric matrix and $\lambda$ an eigenvalue of $A$. Then*

$$(\mathrm{Im}\lambda)^2\ \mathrm{Re}\lambda \leq (\delta(A) - \mathrm{Re}\lambda)\ \left[\operatorname{var}(A) + \mathrm{Re}\lambda\ (\mathrm{Re}\lambda - \delta(A))\right]. \qquad (2.2)$$

Prompted by the above theorem, the *shell* of an almost skew-symmetric matrix $A$ is defined in [11] as the curve in the complex plane given by

$$\Gamma(A) \;=\; \left\{ x+iy \in \mathbb{C} : x, y \in \mathbb{R} \;\text{ and }\; y^2 = (\delta(A) - x)\left(\frac{\operatorname{var}(A)}{x} + x - \delta(A)\right)\right\}.$$

By Theorem 2.2, $\Gamma(A)$ yields a localization of the spectrum of $A$, as specified by (2.2). The various possible configurations of the shell of an almost skew-symmetric matrix and how to achieve them via Levinger's transformation are illustrated in the next example.

**Example 2.3** We have taken a $5 \times 5$ almost skew-symmetric matrix $A$ with $\delta(A) = 2.6636$ and variance $\operatorname{var}(A) = 2.4219$, and found the almost skew-symmetric matrices $B = \mathcal{L}(A, 0.072)$ and $C = \mathcal{L}(A, 0.1)$ having variances $\operatorname{var}(B) = 1.7746$ and $\operatorname{var}(C) = 1.55$. Of course, $\delta(B) = \delta(C) = 2.6636$. Their shells are shown in Figure 1, where the eigenvalues of each matrix are marked with $*$'s. The shell $\Gamma(A)$ is connected and all the eigenvalues of $A$ are located in the region between $\Gamma(A)$ and the imaginary axis. The shell $\Gamma(C)$ consists of one bounded and one unbounded branch. The bounded branch surrounds a real eigenvalue of $C$ and the unbounded branch isolates the rest of the spectrum. The shell of $B$ is the transition in a continuous transformation from the connected shell to the shell with two branches.

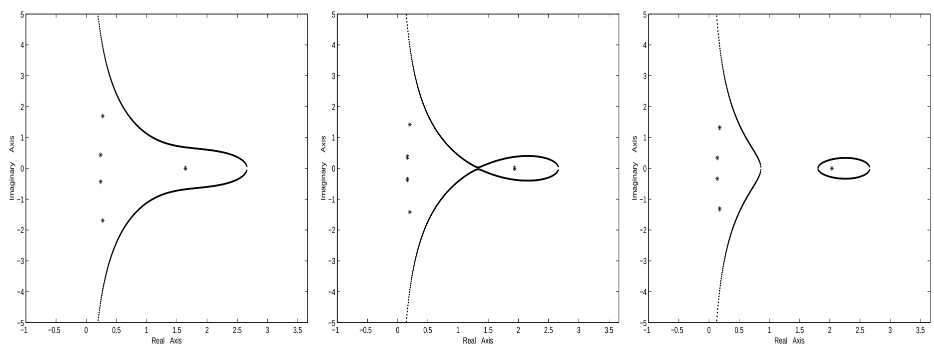


Figure 1: The shells $\Gamma(A)$, $\Gamma(B)$ and $\Gamma(C)$.

When $A$ is almost skew-symmetric, applying Theorem 2.2 to $\mathcal{L}(A, \alpha)$, which is also almost skew-symmetric, the following can be shown [11].

**Proposition 2.4** *Let $A \in M_n(\mathbb{R})$ be an almost skew-symmetric matrix. Then for every $\alpha \in [0,\, 1/2]$ such that $\delta^2(A) > 4(1-2\alpha)^2 \operatorname{var}(A)$, the matrix $\mathcal{L}(A, \alpha)$ has a real eigenvalue*

$$\lambda(A, \alpha) \geq \frac{\delta(A) + \sqrt{\delta^2(A) - 4(1-2\alpha)^2 \operatorname{var}(A)}}{2}$$

*and* $n-1$ *complex eigenvalues whose real parts are not greater than*

$$\frac{\delta(A)-\sqrt{\delta^2(A)-4(1-2\alpha)^2\,\mathrm{var}(A)}}{2}\,.$$

We conclude with an outline of our results to follow. In Section 3 we obtain bounds for the angles between the vectors $w$, $x_r(\mathcal{L}(A,\alpha))$ and $x_l(\mathcal{L}(A,\alpha))$ for appropriate values of $\alpha \in [\,0,\,1/2]$. In Section 4, using the notion of *q-numerical range* and a result in [3], we construct an upper bound for the derivative of Levinger's function in (2.1), which, in turn, yields an upper bound for $\phi(A,\alpha)$. Finally, an illustrative example is presented in Section 5.

# 3 Right and left Perron vectors of $\mathcal{L}(A,\alpha)$

Let $A \in M_n(\mathbb{R})$ be an irreducible nonnegative almost skew-symmetric matrix with symmetric part $S(A) = w\,w^t$. Then there exists an *orthonormal basis* of $\mathbb{C}^n$, $\{v_1,\, v_2,\, \ldots,\, v_{n-1},\, w/\|w\|_2\}$, where $v_1, v_2, \ldots, v_{n-1} \in \mathrm{Ker}S(A)$ and $w/\|w\|_2$ is the unit eigenvector of $S(A)$ corresponding to the eigenvalue $\delta(A) = w^t w$.

For any $\alpha \in [\,0,\,1/2]$, consider the unit right and left Perron vectors of $\mathcal{L}(A,\alpha)$ and decompose them as

$$x_r(\alpha) \;=\; y_r(\alpha) + z_r(\alpha) \tag{3.1}$$

and

$$x_l(\alpha) \;=\; y_l(\alpha) + z_l(\alpha), \tag{3.2}$$

where $y_r(\alpha), y_l(\alpha) \in \mathrm{span}\,\{v_1, v_2, \ldots, v_{n-1}\}$ and $z_r(\alpha), z_l(\alpha) \in \mathrm{span}\,\{w\}$.

**Proposition 3.1** *Let* $A \in M_n(\mathbb{R})$ *be an irreducible nonnegative almost skew-symmetric matrix. Then for every* $\alpha \in [\,0,\,1/2]$, *Levinger's function satisfies*

$$\phi(A,\alpha) \;=\; \delta(A)\,\|z_r(\alpha)\|_2^2 \;=\; \delta(A)\,\|z_l(\alpha)\|_2^2,$$

*where* $z_r(\alpha)$ *and* $z_l(\alpha)$ *are defined in* (3.1) *and* (3.2), *respectively.*

**Proof.** It is enough to obtain that $\phi(A,\alpha) = \delta(A)\,\|z_r(\alpha)\|_2^2$. The proof of the equality $\phi(A,\alpha) = \delta(A)\,\|z_l(\alpha)\|_2^2$ is similar. Since the eigenvector $x_r(\alpha)$ of $\mathcal{L}(A,\alpha)$ in (3.1) is unit, it follows that

$$\phi(A,\alpha) \;=\; x_r(\alpha)^t \mathcal{L}(A,\alpha) x_r(\alpha).$$

Hence,

$$\begin{aligned}\phi(A,\alpha) &= x_r(\alpha)^t(S(A)+(1-2\alpha)K(A))x_r(\alpha)\\ &= (y_r(\alpha)^t+z_r(\alpha)^t)S(A)(y_r(\alpha)+z_r(\alpha))\\ &\quad +(1-2\alpha)x_r(\alpha)^tK(A)x_r(\alpha)\\ &= z_r(\alpha)^tS(A)z_r(\alpha)+(1-2\alpha)x_r(\alpha)^tK(A)x_r(\alpha)\\ &= \delta(A)\,\|z_r(\alpha)\|_2^2+(1-2\alpha)x_r(\alpha)^tK(A)x_r(\alpha).\end{aligned}$$

Note that $\phi(A,\alpha)\in\mathbb{R}$ and the matrix $K(A)$ is skew-symmetric. Thus, $x_r(\alpha)^tK(A)x_r(\alpha)=0$ and consequently, $\phi(A,\alpha)=\delta(A)\,\|z_r(\alpha)\|_2^2$. □

**Corollary 3.2** *Let $A\in M_n(\mathbb{R})$ be an irreducible nonnegative almost skew-symmetric matrix. Then for every $\alpha\in[0,\,1/2]$, the right and left Perron vectors of $\mathcal{L}(A,\alpha)$ have the same orthogonal projection onto $\omega$, i.e., $z_r(\alpha)=z_l(\alpha)$. Moreover, Levinger's function satisfies*

$$\phi(A,\alpha)\;=\;\delta(A)\left(\frac{w^tx_r(\alpha)}{\|w\|_2}\right)^2\;=\;\delta(A)\left(\frac{w^tx_l(\alpha)}{\|w\|_2}\right)^2.$$

**Proof.** From Proposition 3.1, it follows that for every $\alpha\in[0,\,1/2]$, the vectors $z_r(\alpha)$ and $z_l(\alpha)$ have the same modulus. Also, since the Perron vectors $x_r(\alpha)$ and $x_l(\alpha)$ are nonnegative, and since $z_r(\alpha)$ (resp., $z_l(\alpha)$) is the orthogonal projection of $x_r(\alpha)$ (resp., $x_l(\alpha)$) onto the vector $w$, the claimed expression for $\phi(A,\alpha)$ follows readily, as well as that

$$\cos(\widehat{w,x_r(\alpha)})\;=\;\cos(\widehat{w,x_l(\alpha)})$$

or, equivalently,

$$(\widehat{w,x_r(\alpha)})\;=\;(\widehat{w,x_l(\alpha)}).$$

The latter equality and as $\|z_r(\alpha)\|_2=\|z_l(\alpha)\|_2$, implies that $z_r(\alpha)=z_l(\alpha)$. □

**Remark 3.3** Note that Proposition 3.1 and Corollary 3.2 hold for $\alpha\in(0,\,1/2]$ when the assumption of irreducibility of $A$ is relaxed to irreducibility of the symmetric part of $A$.

We will now apply the results in [11] in order to investigate the behavior of the Perron vectors of $\mathcal{L}(A,\alpha)$. First, define the interval

$$\mathcal{X}_A\;=\;\left(\max\left\{0,\,\frac{1}{2}-\sqrt{\frac{\delta^2(A)}{16\,\mathrm{var}(A)}}\right\},\,\frac{1}{2}\right] \tag{3.3}$$

and observe that $\alpha\in(0,\,1/2]$ lies in $\mathcal{X}_A$ if and only if $\delta^2(A)>4(1-2\alpha)^2\mathrm{var}(A)$. Notice also that since $0\notin\mathcal{X}_A$, if the symmetric part of $A$ is irreducible, then for every $\alpha\in\mathcal{X}_A$, $\mathcal{L}(A,\alpha)$ is irreducible.

**Theorem 3.4** *Let* $A \in M_n(\mathbb{R})$ *be a nonnegative almost skew-symmetric matrix with irreducible symmetric part. Then for every* $\alpha \in \mathcal{X}_A$*, the cosine of the angle* $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$ *is greater than or equal to the quantity*

$$R_A = \sqrt{\frac{1}{2} + \sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{4\,\delta^2(A)}}}\,.$$

**Proof.** We prove that $\cos(\widehat{w, x_r(\alpha)}) \geq R_A$. For any $\alpha \in \mathcal{X}_A$, by Proposition 2.4,

$$\phi(A, \alpha) \geq \frac{\delta(A) + \sqrt{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}}{2}\,.$$

Hence, by Corollary 3.2 applied to $\mathcal{L}(A, \alpha)$, it follows that

$$\frac{\delta(A)(w^t x_r(\alpha))^2}{\|w\|_2^2} \geq \frac{\delta(A)}{2} + \delta(A)\sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{4\,\delta^2(A)}}$$

or, equivalently,

$$\frac{w^t x_r(\alpha)}{\|w\|_2} \geq \sqrt{\frac{1}{2} + \sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{4\,\delta^2(A)}}}\,.$$

Since the vector $x_r(\alpha)$ is unit, it is clear that

$$\cos(\widehat{w, x_r(\alpha)}) \geq \sqrt{\frac{1}{2} + \sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{4\,\delta^2(A)}}}\,. \quad \square$$

**Remark 3.5** These remarks refer to the above theorem.

(i) The following interpretation of $\mathrm{var}(A)$ is possible. For any $\alpha \in \mathcal{X}_A$, define the angle $\vartheta_A(\alpha) \in [0, \pi/4]$ such that

$$\cos \vartheta_A(\alpha) = R_A = \sqrt{\frac{1}{2} + \sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{4\,\delta^2(A)}}}\,. \tag{3.4}$$

Consider now $\alpha$ varying in $\mathcal{X}_A$, starting from $1/2$ and taking values toward the left endpoint of $\mathcal{X}_A$. Then the Perron vectors $x_r(\alpha)$ and $x_l(\alpha)$ of $\mathcal{L}(A, \alpha)$ lie in the cone

$$\mathcal{K}_A(\alpha) = \{u \in \mathbb{R}^n : 0 \leq (\widehat{w, u}) \leq \vartheta_A(\alpha)\},$$

which contains $w$. Notice that the smaller $\mathrm{var}(A)$ is, the slower $\mathcal{K}_A(\alpha)$ dilates, and the slower $x_r(\alpha)$ and $x_l(\alpha)$ dilate away from $w$.

(ii) For $\alpha = 1/2$, we have

$$1 \,=\, \cos 0 \,=\, \cos(\widehat{w, w}) \,\geq\, \sqrt{\frac{1}{2} + \sqrt{\frac{\delta^2(A)}{4\,\delta^2(A)}}} \,=\, 1.$$

Notice also that for every $\alpha \in \mathcal{X}_A$, the angle $(\widehat{w, x_r(\alpha)}) = (\widehat{w, x_l(\alpha)})$ is not greater than $\pi/4$.

Theorem 3.4 also allows us to estimate *a priori* the angle between $x_r(\alpha)$ and $x_l(\alpha)$ and subsequently the Euclidean distance between these two vectors for any $\alpha \in \mathcal{X}_A$.

**Theorem 3.6** *Let $A \in M_n(\mathbb{R})$ be a nonnegative almost skew-symmetric matrix with irreducible symmetric part. Then for every $\alpha \in \mathcal{X}_A$,*

$$\cos(\widehat{x_r(\alpha), x_l(\alpha)}) \,\geq\, \sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{\delta^2(A)}}\,.$$

**Proof.** With $\vartheta_A(\alpha)$ as defined in (3.4) and for the angles $(\widehat{x_r(\alpha), x_l(\alpha)})$, $(\widehat{w, x_r(\alpha)})$ and $(\widehat{w, x_l(\alpha)})$, we have

$$(\widehat{x_r(\alpha), x_l(\alpha)}) \,\leq\, (\widehat{w, x_r(\alpha)}) + (\widehat{w, x_l(\alpha)}) \,\leq\, 2\,\vartheta_A(\alpha) \,\leq\, \frac{\pi}{2}\,.$$

Thus,

$$\cos(\widehat{x_r(\alpha), x_l(\alpha)}) \,\geq\, \cos(2\,\vartheta_A(\alpha)) \,\geq\, 0,$$

where

$$\begin{aligned} \cos(2\,\vartheta_A(\alpha)) &= 2\,\cos^2\vartheta_A(\alpha) - 1 \\ &= \sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{\delta^2(A)}}\,. \quad \square \end{aligned}$$

**Corollary 3.7** *Let $A \in M_n(\mathbb{R})$ be a nonnegative almost skew-symmetric matrix with irreducible symmetric part. Then for every $\alpha \in \mathcal{X}_A$, the Perron vectors $x_r(\alpha)$ and $x_l(\alpha)$ satisfy*

$$\|x_r(\alpha) - x_l(\alpha)\|_2^2 \,\leq\, 2\left(1 - \sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{\delta^2(A)}}\right).$$

**Proof.** Since the vectors $x_r(\alpha)$ and $x_l(\alpha)$ are unit, one can see that

$$\begin{aligned} \|x_r(\alpha) - x_l(\alpha)\|_2^2 &= (x_r(\alpha)^t - x_l(\alpha)^t)(x_r(\alpha) - x_l(\alpha)) \\ &= 2 - 2\,\cos(\widehat{x_r(\alpha), x_l(\alpha)}). \end{aligned}$$

Hence, by the above theorem,

$$\|x_r(\alpha) - x_l(\alpha)\|_2^2 \;\le\; 2 - 2\sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{\delta^2(A)}}\,. \quad \square$$

# 4 Bounds for Levinger's function and its derivative

Let $A \in M_n(\mathbb{R})$ be a nonnegative almost skew-symmetric matrix with irreducible symmetric part, and recall Levinger's function

$$\phi(A,\alpha) \;=\; \rho(\mathcal{L}(A,\alpha)) \;=\; \rho((1-\alpha)A + \alpha A^t)\,; \quad \alpha \in [0,\, 1/2]$$

and the interval $\mathcal{X}_A \subseteq (0,\, 1/2]$ defined in (3.3). Proposition 2.4 applied to $A$, and since then $\lambda(A,\alpha) = \phi(A,\alpha)$, yields a lower bound on Levinger's function. In this section, we focus on upper bounds for $\phi(A,\alpha)$ and for the 'rate of change' of $\phi(A,\alpha)$ as a function of $\alpha$. The rate of change of Levinger's function has been studied systematically by Fiedler [3]. By the proof of [3, Theorem 1.2, p. 176], we have that for every $\alpha \in (0,\, 1/2)$,

$$\begin{aligned} 0 \;\le\; \phi^{'}(A,\alpha) \;&=\; \frac{1}{1-2\alpha}\,\frac{x_l(\alpha)^t(\mathcal{L}(A,\alpha)^t - \mathcal{L}(A,\alpha))x_r(\alpha)}{x_l(\alpha)^t x_r(\alpha)} \\ &=\; -2\,\frac{x_l(\alpha)^t K(A) x_r(\alpha)}{\cos(\widehat{x_r(\alpha), x_l(\alpha)})}\,. \end{aligned} \tag{4.1}$$

This expression for $\phi^{'}(A,\alpha)$ will lead us to an upper bound for $\phi^{'}(A,\alpha)$ that depends on $\mathrm{var}(A)$ and $\delta(A)$; it is contained in Theorem 4.1. Subsequently, this upper bound will be used to obtain an upper bound for $\phi(A,\alpha)$.

To begin, by (4.1) and our Theorem 3.6, it follows that for every $\alpha \in \mathcal{X}_A \setminus \{1/2\}$,

$$0 \;\le\; \phi^{'}(A,\alpha) \;\le\; \frac{2}{q(\alpha)}\,\left(-x_l(\alpha)^t K(A) x_r(\alpha)\right), \tag{4.2}$$

where

$$q(\alpha) \;=\; \sqrt{\frac{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)}{\delta^2(A)}}\,. \tag{4.3}$$

At this point, it is necessary to introduce the notion of *q-numerical range* of an $n \times n$ complex matrix $N$ for a real $q \in [0,\, 1]$, that is,

$$\begin{aligned} F_q(N) \;&=\; \{x^* N y \in \mathbb{C} : x, y \in \mathbb{C}^n,\; x^*x = y^*y = 1 \text{ and } x^*y = q\} \\ &=\; \left\{\frac{x^*}{\|x\|_2}\,N\,\frac{y}{\|y\|_2} \in \mathbb{C} : x, y \in \mathbb{C}^n \setminus \{0\} \text{ and } \cos(\widehat{x,y}) = q\right\}. \end{aligned}$$

For $q = 1$, $F_q(N)$ coincides with the classical numerical range

$$F(N) \equiv F_1(N) = \{x^* N x \in \mathbb{C} : x \in \mathbb{C}^n \text{ and } x^* x = 1\}.$$

The $q$-numerical range $F_q(N)$ $(q \in [0, 1])$ is always a compact and convex subset of the complex plane. For $0 \le q < 1$, $F_q(N)$ has a nonempty interior, and for $q = 1$, the range $F(N)$ is a line segment if and only if $N$ is a normal matrix with colinear eigenvalues. In particular, a complex matrix $N$ is Hermitian (resp., skew-Hermitian) if and only if $F(N) \subset \mathbb{R}$ (resp., $F(N) \subset \mathrm{i}\,\mathbb{R}$).

For the matrix $A$, it is obvious now that the quantity $-x_l(\alpha)^t K(A) x_r(\alpha)$ in (4.2) is positive and lies in the interval

$$[0, +\infty) \cap F_{\cos(\widehat{x_r(\alpha), x_l(\alpha)})}(K(A)).$$

As a consequence, it will be necessary to estimate the length of the intersection of the positive half-axis $[0, +\infty)$ with the $\cos(\widehat{x_r(\alpha), x_l(\alpha)})$-numerical range of the real skew-symmetric matrix $K(A)$.

First observe that in general, for any pair $0 < \alpha_1 < \alpha_2 < 1/2$, it follows that $q(\alpha_1) < q(\alpha_2)$, where $q(\alpha)$ is defined in (4.3). Thus, by [8, Theorem 2.5],

$$F_{q(\alpha_2)}(K(A)) \subseteq \frac{q(\alpha_2)}{q(\alpha_1)} F_{q(\alpha_1)}(K(A))$$

and consequently,

$$[0, +\infty) \cap F_{q(\alpha_2)}(K(A)) \subseteq \frac{q(\alpha_2)}{q(\alpha_1)} \left\{ [0, +\infty) \cap F_{q(\alpha_1)}(K(A)) \right\}.$$

Similarly we obtain that for every $\alpha \in \mathcal{X}_A \setminus \{1/2\}$,

$$\begin{aligned} [0, +\infty) \cap F_{\cos(\widehat{x_r(\alpha), x_l(\alpha)})}(K(A)) \quad &\subseteq \quad \frac{\cos(\widehat{x_r(\alpha), x_l(\alpha)})}{q(\alpha)} \\ &\qquad \times \left\{ [0, +\infty) \cap F_{q(\alpha)}(K(A)) \right\}. \end{aligned}$$

Hence, by (4.2) and Theorem 3.6,

$$\begin{aligned} 0 \le \phi'(A, \alpha) \quad &\le \quad \frac{2 \cos(\widehat{x_r(\alpha), x_l(\alpha)})}{q^2(\alpha)} \max \left\{ [0, +\infty) \cap F_{q(\alpha)}(K(A)) \right\} \\ &\le \quad \frac{2}{q^2(\alpha)} \max \left\{ [0, +\infty) \cap F_{q(\alpha)}(K(A)) \right\}. \end{aligned}$$

**Theorem 4.1** *Let $A \in M_n(\mathbb{R})$ be a nonnegative almost skew-symmetric matrix with irreducible symmetric part and let $s_1$ be the maximum singular value of the skew-symmetric matrix $K(A)$. Then for every $\alpha \in \mathcal{X}_A \setminus \{1/2\}$,*

$$0 \le \phi'(A, \alpha) \le \frac{4\, s_1 (1 - 2\alpha) \delta(A) \sqrt{\mathrm{var}(A)}}{\delta^2(A) - 4(1 - 2\alpha)^2 \mathrm{var}(A)}.$$

**Proof.** The numerical range of $K(A)$ is $F(K(A)) = [-\mathrm{i}\beta, \mathrm{i}\beta]$, where the real number $\beta$ is nonnegative [6]. Denote by $D(\lambda, r)$ the closed disk centered at $\lambda$ with radius $r$. By the results in [9], it is known that for any $q \in [0, 1]$,

$$F_q(K(A)) = \bigcup_{\substack{(\mathrm{i}\gamma,\, h) \\ \gamma, h \in \mathbb{R}}} D(\mathrm{i}\, q\gamma, \sqrt{(1-q^2)(h-\gamma^2)}),$$

where the union is taken over all the pairs

$$(\mathrm{i}\gamma, h) = (x^* K(A) x, x^*(K(A)^* K(A)) x) \; ; \quad x^* x = 1.$$

Since $K(A)$ is skew-symmetric, there is a unitary matrix $U$ such that

$$K(A) = \mathrm{i}\, U^* \mathrm{diag}\{\pm s_1, \pm s_2, \dots, \pm s_m\}\, U$$

and

$$K(A)^* K(A) = U^* \mathrm{diag}\{s_1^2, s_1^2, s_2^2, s_2^2, \dots, s_m^2\}\, U,$$

where $s_1 \geq s_2 \geq \dots \geq s_m \geq 0$ are the singular values of $K(A)$. (Note that every nonzero singular value appears an even number of times.)

One can see that for the pair $(\mathrm{i}\gamma, h) = (0, s_1^2)$, the radius of the disk

$$D(\, \mathrm{i}\, q\gamma, \sqrt{(1-q^2)(h-\gamma^2)}\,)$$

attains its maximum value, that is,

$$r_{max}(A, q) = s_1 \sqrt{1-q^2}.$$

Indeed, let $y_1$ and $\hat{y}_1$ be two orthonormal eigenvectors of $K(A)$ corresponding to the eigenvalues $\mathrm{i}\, s_1$ and $-\mathrm{i}\, s_1$, respectively. Then the vector $y_0 = (y_1 + \hat{y}_1)/\sqrt{2}$ is a unit eigenvector of matrix $K(A)^* K(A)$ corresponding to $s_1^2$, and

$$\begin{aligned}
&(y_0^* K(A) y_0, y_0^*(K(A)^* K(A)) y_0) \\
&\qquad = \left( \frac{y_1^* K(A) y_1}{\sqrt{2}} + \frac{\hat{y}_1^* K(A) \hat{y}_1}{\sqrt{2}}, y_0^*(K(A)^* K(A)) y_0 \right) \\
&\qquad = (0, s_1^2).
\end{aligned}$$

Thus, for any $\alpha \in \mathcal{X}_A \setminus \{1/2\}$, the set $[0, +\infty] \cap F_{q(\alpha)}(K(A))$ coincides with the interval

$$\left[0,\, s_1 \sqrt{1 - q^2(\alpha)}\right] = \left[0,\, \frac{2\, s_1\, (1-2\alpha)}{\delta(A)} \sqrt{\mathrm{var}(A)}\right].$$

Consequently, for any $\alpha \in \mathcal{X}_A \setminus \{1/2\}$, we have

$$0 \leq \phi^{'}(A, \alpha) \leq \frac{2}{q^2(\alpha)} \frac{2\, s_1\, (1-2\alpha)}{\delta(A)} \sqrt{\mathrm{var}(A)},$$

where $q(\alpha)$ is given in (4.3), or equivalently,

$$0 \le \phi^{'}(A,\alpha) \le \frac{4\, s_1\, (1-2\alpha)\delta(A)\sqrt{\mathrm{var}(A)}}{\delta^2(A) - 4(1-2\alpha)^2 \mathrm{var}(A)} . \quad \square$$

Next we turn our attention to obtaining an upper bound for $\phi(A,\alpha)$. The only upper bound we know so far is Levinger's result, namely, $\phi(A,\alpha) \le \delta(A)$. However, our bound for $\phi^{'}(A,\alpha)$ can lead us to a better estimate for $\phi(A,\alpha)$.

**Theorem 4.2** *Let $A \in M_n(\mathbb{R})$ be a nonnegative almost skew-symmetric matrix with irreducible symmetric part and let $s_1$ be the maximum singular value of the skew-symmetric matrix $K(A)$. Then for every $\alpha_1, \alpha_2 \in \mathcal{X}_A \setminus \{1/2\}$ such that $\alpha_1 < \alpha_2$,*

$$0 \le \phi(A,\alpha_2) - \phi(A,\alpha_1) \le \frac{s_1\, \delta(A)}{4\sqrt{\mathrm{var}(A)}} \ln\left(\frac{\delta^2(A) - 4(1-2\alpha_2)^2\mathrm{var}(A)}{\delta^2(A) - 4(1-2\alpha_1)^2\mathrm{var}(A)}\right) .$$

**Proof.** From Theorem 4.1, for every $\alpha \in \mathcal{X}_A \setminus \{1/2\}$, we get

$$0 \le \phi^{'}(A,\alpha) \le \frac{4s_1(1-2\alpha)\delta(A)\sqrt{\mathrm{var}(A)}}{\delta^2(A) - 4(1-2\alpha)^2\mathrm{var}(A)} .$$

Integrating through the above inequality with respect to $\alpha$ in the interval $(\alpha_1, \alpha_2) \subseteq \mathcal{X}_A \setminus \{1/2\}$, and as $\phi(A,\alpha)$ is a non-decreasing function in $[0,\, 1/2]$ (see [3, 7]), we obtain the claimed inequality. $\square$

**Corollary 4.3** *Let $A \in M_n(\mathbb{R})$ be a nonnegative almost skew-symmetric matrix with irreducible symmetric part and let $s_1$ be the maximum singular value of the skew-symmetric matrix $K(A)$. If $\delta^2(A) > 4\mathrm{var}(A)$, then for every $\alpha \in [0,\, 1/2)$, we have*

$$\phi(A,\alpha) \le \rho(A) + \frac{s_1\delta(A)}{4\sqrt{\mathrm{var}(A)}} \ln\left(\frac{\delta^2(A) - 4(1-2\alpha)^2\mathrm{var}(A)}{\delta^2(A) - 4\mathrm{var}(A)}\right) .$$

*Moreover, for $\alpha = 1/2$, the following inequality obtains:*

$$\delta(A) - \rho(A) \le \frac{s_1\delta(A)}{4\sqrt{\mathrm{var}(A)}} \ln\left(\frac{\delta^2(A)}{\delta^2(A) - 4\mathrm{var}(A)}\right) .$$

Note that for values of $\alpha$ sufficiently close to 0, the upper bound for $\phi(A,\alpha)$ in the above corollary is less than $\delta(A)$, and for $\alpha = 0$, it coincides with $\rho(A)$.

## 5 Illustrative example

Consider the (irreducible) nonnegative matrix

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.4 & 0.7 & 0.7 & 1 \\ 0.7 & 0.5 & 1 & 0.8 & 0.9 & 0.3 \\ 0.6 & 0 & 0.5 & 0.5 & 0.6 & 0.2 \\ 0.3 & 0.2 & 0.5 & 0.5 & 0.3 & 0.5 \\ 0.3 & 0.1 & 0.4 & 0.7 & 0.5 & 0.3 \\ 0 & 0.7 & 0.8 & 0.5 & 0.7 & 0.5 \end{bmatrix}.$$

This matrix has irreducible symmetric part

$$S(A) = w\,w^t, \quad \text{where} \quad w = (\sqrt{2}/2)\,[1,\ 1,\ 1,\ 1,\ 1,\ 1]^t\,.$$

Thus $A$ is almost skew-symmetric. The spectral radius of $A$ is $\rho(A) \cong 2.8128$, its variance is $\operatorname{var}(A) = 0.53$ and $\delta(A) = 3$. The maximum singular value of the skew-symmetric part of $A$,

$$K(A) = \begin{bmatrix} 0 & -0.2 & -0.1 & 0.2 & 0.2 & 0.5 \\ 0.2 & 0 & 0.5 & 0.3 & 0.4 & -0.2 \\ 0.1 & -0.5 & 0 & 0 & 0.1 & -0.3 \\ -0.2 & -0.3 & 0 & 0 & -0.2 & 0 \\ -0.2 & -0.4 & -0.1 & 0.2 & 0 & -0.2 \\ -0.5 & 0.2 & 0.3 & 0 & 0.2 & 0 \end{bmatrix},$$

is $s_1 \cong 0.8247$. We also have that $\mathcal{X}_A = (0,\,1/2]$ and that for every $\alpha \in [0,\,1/2]$,

$$9 = \delta^2(A) > 2.12 \geq 4(1-2\alpha)^2 \operatorname{var}(A).$$

If we choose $\alpha = 1/3$, then the Perron root of $\mathcal{L}(A, 1/3)$ is $\phi(A, 1/3) \cong 2.980254$ with corresponding unit Perron vectors

$$x_r(1/3) = [0.435139,\ 0.460558,\ 0.378913,\ 0.376042,\ 0.374113,\ 0.416649]^t.$$

and

$$x_l(1/3) = [0.380385,\ 0.351427,\ 0.433478,\ 0.439692,\ 0.437761,\ 0.398672]^t$$

By (4.1), we have

$$\phi^{'}(A, 1/3) = -2\,\frac{x_l(1/3)^t K(A) x_r(1/3)}{x_l(1/3)^t x_r(1/3)} \cong 0.238368.$$

Notice that

$$q(1/3) \cong 0.986827 < 0.986845 \cong \cos(\widehat{x_r(1/3), x_l}(1/3)),$$

confirming Theorem 3.6. The upper bound of the derivative of Levinger's function in Theorem 4.1 is

$$\frac{4\,s_1\,(1-2/3)\delta(A)\sqrt{\mathrm{var}(A)}}{\delta^2(A)-4(1-2/3)^2\mathrm{var}(A)} \cong 0.274012 > 0.238368 \cong \phi^{'}(A,1/3)\,.$$

In Figure 2, we illustrate the upper bound for $\phi(A,\alpha)$ in Corollary 4.3 and the lower bound in Proposition 2.4. The Perron roots $\phi(A,\alpha)$ for $\alpha = 0$, 0.1, 0.2, 0.3, 0.4, 0.5 are marked with o's. Notice that since the variance $\mathrm{var}(A) = 0.53$ is relatively small, the lower bound in Proposition 2.4 is evidently quite close to Levinger's function $\phi(A,\alpha)$.

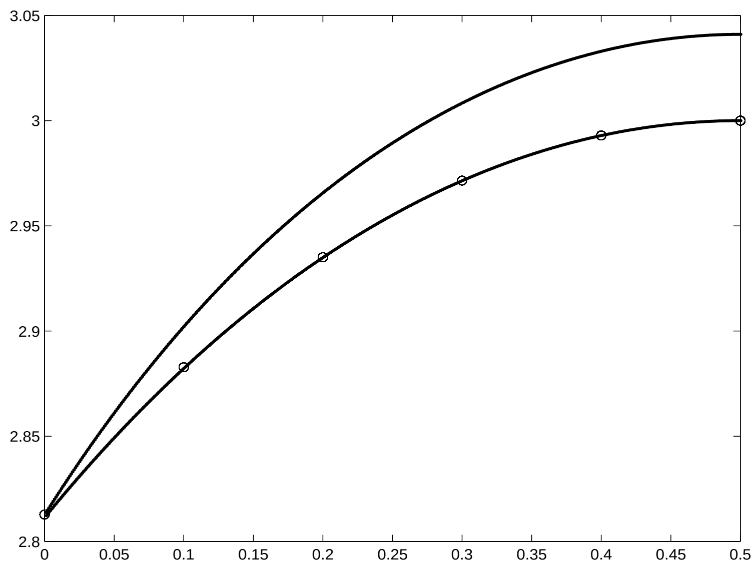


Figure 2: The lower and upper bound for $\phi(A,\alpha)$.

Finally, we remark that Levinger's function and our bounds for it have the same qualitative behavior in $\mathcal{X}_A$, that is, they are all increasing and concave functions, and their derivatives have a zero at $\alpha = 1/2$.

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