A Primer of Perron-Frobenius Theory for Matrix Polynomials

Panayiotis J. Psarrakos[1] and Michael J. Tsatsomeros[2]

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Abstract

We present an extension of Perron-Frobenius theory to the spectra and numerical ranges of Perron polynomials, namely, matrix polynomials of the form

$$L(\lambda) = I\lambda^m - A_{m-1}\lambda^{m-1} - \cdots - A_1\lambda - A_0,$$

where the coefficient matrices are entrywise nonnegative. Our approach relies on the companion matrix linearization. First, we recount the generalization of the Perron-Frobenius Theorem to Perron polynomials and report some of its consequences. Subsequently, we examine the role of $L(\lambda)$ in multistep difference equations and provide a multistep version of the Fundamental Theorem of Demography. Finally, we extend Issos' results on the numerical range of nonnegative matrices to Perron polynomials.

Keywords: matrix polynomial, nonnegative matrix, Perron-Frobenius, Perron polynomial, spectral radius, multistep difference equation, numerical range.

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1 Introduction

Matrices with nonnegative entries arise in a wide variety of areas including dynamical systems theory, economics, statistics and optimization to name a few. The basic tenant of their theory, that the spectral radius of a nonnegative matrix is necessarily an eigenvalue, is historically attributed to Perron and Frobenius. Hence the term Perron-Frobenius theory is frequently being used to describe the development of a spectral theory for nonnegative matrices and, more

[1]Department of Mathematics, National Technical University, Zografou Campus 15780, Athens, Greece (`ppsarr@math.ntua.gr`).

[2]Mathematics Department, Washington State University, Pullman, Washington 99164-3113, U.S.A. (`tsat@math.wsu.edu`).

generally, for matrices that leave a proper cone invariant. The effort to extend the reach of nonnegative matrices has led to various generalizations and to a systematic study of related matrix classes. Here we explore the extension of Perron-Frobenius theory to (monic) matrix polynomials of the form

$$L(\lambda) \;=\; I\lambda^m - A_{m-1}\lambda^{m-1} - \cdots - A_1\lambda - A_0. \tag{1.1}$$

When the matrices A_j $(j = 0, 1, \ldots, m-1)$ in (1.1) are entrywise nonnegative $n \times n$ matrices, we shall refer to $L(\lambda)$ as an $n \times n$ *Perron polynomial* of degree m. Recall that the spectrum of a matrix polynomial $L(\lambda)$ consists of the scalars λ such that $\det L(\lambda) = 0$ and so it coincides with the spectrum of A_0 when $m = 1$. It is therefore natural to seek a generalization of Perron-Frobenius theory to Perron polynomials.

Entrywise nonnegativity associated with generalized eigenproblems has been studied before; see e.g., [2, 21] where matrix pencils are considered. An extension of Perron-Frobenius theory to positive operators on Banach lattices is pursued in [29]. The peripheral spectrum of the monic polynomial in (1.1) when the coefficients are positive (compact) operators in a Banach lattice are considered in [7, 18, 27]. In addition, the spectral properties of Perron polynomials are examined in [8, 9] via a partial linearization based on expansion graphs that were introduced in [10]. Finally, elements of a spectral theory for Perron polynomials appear in the study of the stability of linear difference equations with positive coefficients [24].

Our present goal is to take a comprehensive look at Perron-Frobenius theory for Perron polynomials, based on the companion matrix global linearization. The presentation is organized as follows: In Section 2 we recall definitions, notation and include some preliminaries regarding matrix polynomials. Section 3 contains a recount of Perron-Frobenius theory for Perron polynomials. Specifically, it contains the extension of the two classical parts of the Perron-Frobenius Theorem as well as a discussion of irreducibility, primitivity, stochastic Perron polynomials and bounds for the spectral radius. We also discuss $L(\lambda)^{-1}$ and consider its positivity, relating our discussion to the theory of M-matrices. In Section 4 we consider the association of $L(\lambda)$ in (1.1) with the multistep matrix difference equations

$$u_{j+m} \;=\; A_{m-1}u_{j+m-1} + \cdots + A_1 u_{j+1} + A_0 u_j \quad (j = 0, 1, \ldots),$$

which is of interest in the study of higher order linear differential equations, dynamical systems theory, economic models and queuing theory [4, 13, 24, 28]. We provide a generalization of the Fundamental Theorem of Demography (see [17]) pursuant to the asymptotic behavior of u_j. Finally, in Section 5, we extend the results on the numerical range of nonnegative matrices to Perron polynomials; Section 5 concludes with two illustrative examples and some directions for future study.

2 Definitions and preliminaries

Given a vector $z = [z_i] \in \mathbb{C}^n$, we use the norms $\|z\|_1 = \Sigma_{i=1}^n |z_i|$ and $\|z\|_2 = \sqrt{\Sigma_{i=1}^n |z_i|^2}$. By $\mathbf{1}$ we denote an all ones vector of appropriate size. The spectrum of $A \in \mathbb{C}^{n\times n}$ is denoted by $\sigma(A)$ and its *spectral radius* by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

The *nonnegative orthant* in $\mathbb{R}^n$, that is, the set of all nonnegative vectors in $\mathbb{R}^n$, is denoted by $\mathbb{R}^n_+$. Entrywise ordering of arrays of the same size is indicated by $\geq$. We write $A > B$ if A, B are real and every entry of $A - B$ is positive. When $A \geq 0$ (resp., $A > 0$), we refer to A as *nonnegative* (resp., *positive*). An *M-matrix* is a square matrix of the form $M = sI - A$, where $A \geq 0$ and $s > \rho(A)$. It is well-known [3, Chapter 6] that M-matrices are inverse nonnegative, that is, $M^{-1} \geq 0$.

We call a square matrix A *irreducible* if there does not exist a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square, non-vacuous matrices. We call A *k-cyclic* (or just *cyclic*) if for some permutation matrix P,

$$PAP^T = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k,1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.1}$$

where the zero blocks along the diagonal are square. Notice that a k-cyclic matrix is also m-cyclic for any divisor m of k. The largest positive integer k for which A is k-cyclic is referred to as the *cyclic index* of A.

The *directed graph*, $\mathcal{D}(A)$, of $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ consists of the set of vertices $\{1, 2, \ldots, n\}$ and a set of directed edges (i, j) connecting vertex i to vertex j if and only if $a_{ij} \neq 0$. We say vertex i has *access* to vertex j if there is a path $(i, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k), (i_k, j)$ connecting the two vertices. We say $\mathcal{D}(A)$ is *strongly connected* if any two distinct vertices i, j have access to each other. It is well-known that A is irreducible if and only if $\mathcal{D}(A)$ is strongly connected. A *cycle of length k* in $\mathcal{D}(A)$ consists of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$, where the vertices $i_1, i_2, \ldots, i_k$ are distinct. The nonzero diagonal entries of A correspond to cycles of length 1 in $\mathcal{D}(A)$.

The main clause of the Perron-Frobenius Theorem ([1, 3, 14, 31]) states that if $A \geq 0$, then $\rho(A) \in \sigma(A)$. An irreducible matrix $A \geq 0$ is called *primitive* if $\rho(A)$ is the only eigenvalue of A of maximum modulus. Note that the following are equivalent statements for a nonnegative irreducible matrix A: (i) A is primitive; (ii) the cyclic index of A equals 1; (iii) the greatest common divisor of the cycle lengths in $\mathcal{D}(A)$ equals 1; (iv) $A^m > 0$ for some positive integer m.

We continue with some preliminary facts on matrix polynomials. A scalar λ_0 is said to be an *eigenvalue* of $L(\lambda)$ in (1.1) if the system $L(\lambda_0)y = 0$ has a nonzero solution $y_0 \in \mathbb{C}^n$; y_0 is referred to as a *right eigenvector* of $L(\lambda)$ corresponding to λ_0. A nonzero vector w_0 is said to be a *left eigenvector* of $L(\lambda)$ corresponding to λ_0 if $L(\lambda_0)^T w_0 = 0$. The *spectrum* of $L(\lambda)$ is the set of all eigenvalues of $L(\lambda)$, $\sigma(L) = \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\}$; The *companion matrix* of $L(\lambda)$,

$$C_L = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ A_0 & A_1 & A_2 & \cdots & A_{m-1} \end{bmatrix} \in \mathbb{C}^{nm \times nm}, \tag{2.2}$$

provides a *linearization* of $L(\lambda)$ (see [11, 16]); this means that

$$E(\lambda)(I\lambda - C_L)G(\lambda) = \begin{bmatrix} L(\lambda) & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}, \tag{2.3}$$

where $E(\lambda)$ and $G(\lambda)$ are $nm \times nm$ matrix polynomials with constant nonzero determinants. More specifically,

$$E(\lambda) = \begin{bmatrix} E_1(\lambda) & E_2(\lambda) & \cdots & E_{m-1}(\lambda) & I \\ -I & 0 & \cdots & 0 & 0 \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \end{bmatrix}$$

with

$$\begin{array}{rcl} E_{m-1}(\lambda) & = & I\lambda - A_{m-1} \\ E_{m-2}(\lambda) & = & I\lambda^2 - A_{m-1}\lambda - A_{m-2} \\ \vdots & \vdots & \vdots \\ E_2(\lambda) & = & I\lambda^{m-2} - A_{m-1}\lambda^{m-3} - \cdots - A_3\lambda - A_2 \\ E_1(\lambda) & = & I\lambda^{m-1} - A_{m-1}\lambda^{m-2} - \cdots - A_2\lambda - A_1, \end{array} \tag{2.4}$$

and

$$G(\lambda) = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ I\lambda & I & \cdots & 0 & 0 \\ I\lambda^2 & I\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I\lambda^{m-1} & I\lambda^{m-2} & \cdots & I\lambda & I \end{bmatrix}.$$

As a consequence of (2.3) and the nature of $E(\lambda)$ and $G(\lambda)$, the spectrum of $L(\lambda)$ and its linearization coincide, that is, $\sigma(L) = \sigma(C_L)$. The *spectral radius* of $L(\lambda)$ is $\rho(L) = \max\{|\lambda| : \lambda \in \sigma(L)\}$, and thus it also coincides with the spectral radius of C_L, $\rho(C_L)$. An eigenvalue $\lambda \in \sigma(L) = \sigma(C_L)$ is referred to as *maximal* if $|\lambda| = \rho(L)$.

3 Perron-Frobenius for polynomials

The relation between the eigenvectors of $L(\lambda)$ in (1.1) and the eigenvectors of the companion matrix C_L in (2.2) is obtained directly via (2.3) as follows.

Lemma 3.1 *Let μ be an eigenvalue of the matrix polynomial $L(\lambda)$ in* (1.1) *and C_L be its companion matrix. Then*

(i) *a nonzero vector $y \in \mathbb{C}^n$ is a right eigenvector of $L(\lambda)$ corresponding to $\mu \in \sigma(L)$ if and only if $\begin{bmatrix} y \\ \mu y \\ \vdots \\ \mu^{m-1} y \end{bmatrix} \in \mathbb{C}^{nm}$ is a right eigenvector of C_L corresponding to μ;*

(ii) *a nonzero vector $w \in \mathbb{C}^n$ is a left eigenvector of $L(\lambda)$ corresponding to $\mu \in \sigma(L)$ if and only if $\begin{bmatrix} E_1(\mu)^T w \\ E_2(\mu)^T w \\ \vdots \\ E_{m-1}(\mu)^T w \\ w \end{bmatrix} \in \mathbb{C}^{nm}$ is a left eigenvector of C_L corresponding to μ; the matrix polynomials $E_1(\lambda), E_2(\lambda), \ldots, E_{m-1}(\lambda)$ are those defined in* (2.4).

The Perron-Frobenius Theorem for nonnegative matrices can now be stated for matrix polynomials.

Theorem 3.2 **[Perron-Frobenius, Part I]** *Let $L(\lambda)$ be a Perron polynomial as in* (1.1)*. Then the following hold:*

(a) *$\rho(L) \in \sigma(L)$;*

(b) *$L(\lambda)$ has an entrywise nonnegative right eigenvector and an entrywise nonnegative left eigenvector corresponding to $\rho(L)$;*

(c) *$\rho(L)$ is a nondecreasing function of the entries of each A_j $(j = 0, 1, \dots, m-1)$.*

If, in addition, the companion matrix C_L is irreducible, then

(a′) *$\rho(L)$ is an algebraically simple eigenvalue of $L(\lambda)$;*

(b′) *$L(\lambda)$ has an entrywise positive right eigenvector and an entrywise positive left eigenvector corresponding to $\rho(L)$;*

(c′) *$\rho(L)$ is an increasing function of the entries of each A_j $(j = 0, 1, \dots, m-1)$.*

Proof. Recall that $\sigma(L) = \sigma(C_L)$. Statements (a) and (b) follow directly from Lemma 3.1 and [14, Theorem 8.3.1], and statement (c) from [14, Theorem 8.1.18]. Lemma 3.1 and [14, Theorem 8.4.4] yield immediately (a′) and (b′). Statement (c′) follows from the first part of [14, Theorem 8.4.5]. $\square$

Theorem 3.3 [Perron-Frobenius, Part II] *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in (1.1) and suppose its companion matrix C_L is irreducible. Then there exists an integer $1 \leq k \leq nm$ such that the maximal eigenvalues of $L(\lambda)$ are the roots of $\lambda^k - \rho(L)^k = 0$. Moreover,*

$$\sigma(L) \;=\; e^{\mathrm{i}\frac{2t\pi}{k}}\sigma(L) \quad (t = 0, 1, \dots, k-1)$$

and C_L is cyclic of index k. When $k = 1$, C_L is a primitive matrix.

Proof. The result follows directly from [3, Chapter 2, Theorem 2.20] applied to C_L. $\square$

Next we consider the irreducibility of the companion matrix, raised as a condition in Theorems 3.2 and 3.3.

Theorem 3.4 *The companion matrix C_L in (2.2) of an mth degree $n \times n$ matrix polynomial $L(\lambda)$ as in (1.1) is irreducible if and only if each one of the vertices $\{n(m-1)+1, \dots, nm\}$ in $\mathcal{D}(C_L)$ has access to each one of the vertices $\{1, 2, \dots, n\}$. In particular, if A_0 is irreducible, so is C_L.*

Proof. Group the vertices in $\mathcal{D}(C_L)$ in m sets

$$J_k \;=\; \{kn+1, kn+2, \dots, kn+n\} \quad (k = 0, 1, \dots, m-1).$$

If C_L is irreducible, clearly every vertex in J_{m-1} has access to every vertex in J_0. To prove the converse, assume that every vertex in J_{m-1} has access to every vertex in J_0. Since every vertex in J_k for $k = 1, 2, \ldots, m-1$ is accessed by some vertex in J_0, it follows that every vertex in J_{m-1} has access to every vertex in $\mathcal{D}(C_L)$. In addition, each of the vertices in J_k for $k = 0, 1, \ldots, m-2$ has access to every vertex in J_{m-1}. Hence, $\mathcal{D}(C_L)$ is strongly connected and C_L is irreducible. The condition for irreducibility of C_L clearly holds when A_0 is irreducible. $\square$

Remark 3.5 The specific block structure of an irreducible companion matrix C_L does not seem to be of any tractable consequence to its primitivity. C_L is primitive if and only if the greatest common divisor of the cycle lengths in $\mathcal{D}(C_L)$ is 1. E.g., if $\operatorname{trace}(A_{m-1}) > 0$ and C_L is irreducible, then C_L is primitive.

We proceed with a generalization of the notion of a stochastic matrix. Recall that a (row) stochastic matrix is a nonnegative square matrix A that satisfies $A\mathbf{1} = \mathbf{1}$. Let then $L(\lambda)$ be the $n \times n$ matrix polynomial in (1.1). We call $L(\lambda)$ *stochastic* if $A_j \geq 0$ $(j = 0, 1, \ldots, m-1)$ and $L(1)\mathbf{1} = 0$; that is, $L(\lambda)$ is a Perron polynomial having $1 \in \sigma(L)$ with corresponding eigenvector $\mathbf{1}$. Note that $L(\lambda)$ is stochastic if and only if C_L is a stochastic matrix. As a consequence, we have the following extension of a well-known fact about nonnegative matrices.

Theorem 3.6 *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *Suppose that $\rho := \rho(L) > 0$ and that $L(\rho)z = 0$ for some vector $z > 0$. Then*

$$\hat{L}(\lambda) = I\lambda^m - \frac{A_{m-1}}{\rho}\lambda^{m-1} - \cdots - \frac{A_1}{\rho^{m-1}}\lambda - \frac{A_0}{\rho^m}$$

is similar to a stochastic matrix polynomial.

Proof. Let $L(\lambda)$, ρ, $z = [z_i]$ be as prescribed and let $D = \operatorname{diag}\{z_1, z_2, \ldots, z_n\}$. Let also

$$D_L = \begin{bmatrix} D & 0 & \cdots & 0 \\ 0 & \rho D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho^{m-1} D \end{bmatrix} \in \mathbb{R}^{nm \times nm}.$$

Notice that by Lemma 3.1, $D_L^{-1} C_L D_L \mathbf{1} = \rho \mathbf{1}$; i.e., $D_L^{-1} C_L D_L / \rho$ is a stochastic matrix. It follows that the matrix polynomial $D^{-1}\hat{L}(\lambda)D$ is stochastic. $\square$

Next we review some results and bounds for the spectral radius of a Perron polynomial relative to the rational matrix function

$$S(\lambda) = A_{m-1} + \frac{1}{\lambda}A_{m-2} + \cdots + \frac{1}{\lambda^{m-2}}A_1 + \frac{1}{\lambda^{m-1}}A_0. \tag{3.1}$$

In the next proposition the operators $\min(\cdot)$ and $\max(\cdot)$ applied to a vector return the minimum and maximum entry of that vector, respectively.

Proposition 3.7 *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *Then*

$$\min\{\min(S(1)\,\mathbf{1}\,),\ 1\} \ \leq\ \rho(L)\ \leq\ \max\{\max(S(1)\,\mathbf{1}\,)\},\ 1\}\,. \tag{3.2}$$

Moreover, if the companion matrix of $L(\lambda)$ is irreducible, equality in any one of the inequalities in (3.2) *holds if and only if $L(\lambda)$ is a stochastic matrix polynomial.*

Proof. The two inequalities and the equality cases follow from [3, Chapter 2, Theorem 2.35] applied to C_L. □

The following proposition obtains directly from [27, Proposition 2.1, Corollary 2.2] and [7, Proposition 1.1]. The main idea used in the proofs of the cited results is indeed the monotonicity of $\rho(S(\mu))$ as a function of $\mu \in (0,\infty)$ (a nonincreasing function).

Proposition 3.8 *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *Let $S(\lambda)$ be as in* (3.1) *and denote $\rho := \rho(L)$. Then the following hold:*

(i) *The function $\phi : (0,\infty) \longrightarrow \mathbb{R}$ defined by $\phi(\mu) = \rho(S(\mu))$ is continuous and nonincreasing. Moreover, $\rho = 0$ if and only if $\phi \equiv 0$; if $\rho > 0$, then $\rho = \phi(\rho)$.*

(ii) $\rho(S(1)) \ \leq\ \rho\ \leq\ \rho(S(1))^{1/m}$ *if* $\rho(S(1)) \leq 1$.

(iii) $\rho(S(1))^{1/m} \ \leq\ \rho\ \leq\ \rho(S(1))$ *if* $\rho(S(1)) \geq 1$.

(iv) $\rho < 1$ *if and only if* $\rho(S(1)) < 1$.

(v) $\rho = 1$ *if and only if* $\rho(S(1)) = 1$.

(vi) $\rho > 0$ *if and only if there exists a $\tau > 0$ such that $\tau = \rho(S(\tau))$; in this case $\tau = \rho$.*

Next we extend a basic component of Perron-Frobenius theory. A Perron polynomial of degree $m = 1$ is of the form $L(\lambda) = I\lambda - A$, where $A \geq 0$. Thus, for $\lambda > \rho(A) = \rho(L)$, $L(\lambda)$ is an M-matrix and $L(\lambda)^{-1} = (I\lambda - A)^{-1} \geq 0$. An analogous result holds for Perron polynomials of arbitrary degree as shown below (see also [7, Proposition 2.1]).

Theorem 3.9 *Let $L(\lambda)$ be a Perron polynomial as in* (1.1). *For every $\mu > \rho(L)$, $L(\mu)$ is a nonsingular M-matrix and*

$$L(\mu)^{-1} \;=\; \sum_{k=0}^{\infty} \frac{P\,C_L^k\,R}{\mu^{k+1}} \;\geq\; 0,$$

where

$$P \;=\; [I\ 0\ \cdots\ 0] \in \mathbb{C}^{n\times nm} \quad \textit{and} \quad R \;=\; \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \in \mathbb{C}^{nm\times n}.$$

Proof. First, from [12, Proposition 5.1.2], for every $\mu \notin \sigma(L)$, the inverse of $L(\mu)$ and the resolvent of its companion matrix C_L satisfy $L(\mu)^{-1} = P(I\mu - C_L)^{-1}R$. Now observe that for $\mu > \rho(L) = \rho(C_L)$, $I\mu - C_L$ is an M-matrix and so $(I\mu - C_L)^{-1} \geq 0$. It follows that for $\mu > \rho(L)$, $L(\mu)$ has non-positive off-diagonal entries and is inverse positive. That is, $L(\mu)$ is a nonsingular M-matrix; see [3, Chapter 6, Theorem (2.3)]. In particular, using the Neumann expansion for $(I\mu - C_L)^{-1}$ when $\mu > \rho(L) = \rho(C_L)$, we obtain

$$L(\mu)^{-1} = P(I\mu - C_L)^{-1}R = \mu^{-1} P \left(I - \frac{C_L}{\mu}\right)^{-1} R = \sum_{k=0}^{\infty} \frac{P\, C_L^k\, R}{\mu^{k+1}}. \quad \square$$

4 Multistep matrix difference equations

Motivated by iterative methods and the ubiquitous nature of the difference equations

$$u_{j+1} = A\, u_j \quad (j = 0, 1, \ldots),$$

where A is a nonnegative matrix (e.g., in population dynamics), the convergence of the powers of a nonnegative matrix have been studied extensively. A classical result is the following.

Theorem 4.1 *Let $A \geq 0$ be a primitive matrix with spectral radius $\rho := \rho(A)$. Let $y, w > 0$ be right and left eigenvectors of A corresponding to ρ, respectively, such that $w^T y = 1$. Then*

(i)

$$\lim_{j\to\infty} \frac{A^j}{\rho^j} = y\, w^T.$$

(ii) *Referring to the process $u_{j+1} = A\, u_j$ $(j = 0, 1, \ldots)$, $u_0 \neq 0$, we have that*

$$\lim_{j\to\infty} \|u_j\|_1 = \begin{cases} 0 & \text{if } \rho < 1 \\ \|(w^T u_0) y\|_1 & \text{if } \rho = 1 \\ \infty & \text{if } \rho > 1 \end{cases}.$$

A proof of Theorem 4.1 (i) can be found in [3, 14, 17]. Part (ii) is referred to as the Fundamental Theorem of Demography; see [5, Theorem 1.1.2] and [17].

As mentioned in the introduction, the generalization from matrices to matrix polynomials $L(\lambda)$ as in (1.1) is relevant to a matrix model of the form

$$u_{j+m} = A_{m-1}u_{j+m-1} + \cdots + A_1 u_{j+1} + A_0 u_j \quad (j = 0, 1, \ldots), \tag{4.1}$$

where $u_0, u_1, \ldots, u_{m-1} \in \mathbb{R}^n$ are initial states that determine the solution $\{u_j\}_{j=0}^{\infty}$. In this section we will establish a generalization of Theorem 4.1 to the multistep difference equations in (4.1). Indeed, if C_L is the companion matrix of $L(\lambda)$, then (4.1) is equivalent to

$$x_{j+1} = C_L x_j \quad (j = 0, 1, \ldots),$$

where

$$x_j = \begin{bmatrix} u_j \\ u_{j+1} \\ \vdots \\ u_{j+m-1} \end{bmatrix} \quad (j = 0, 1, \ldots).$$

By [11, Theorem 1.6], for a given initial vector x_0, (4.1) has the unique solution

$$u_j = \underbrace{[I \ 0 \ \cdots \ 0]}_{\in \mathbb{C}^{n \times nm}} C_L^j x_0 \quad (j = 0, 1, \ldots). \tag{4.2}$$

Recall that $A \in \mathbb{R}^{n \times n}$ is nonnegative if and only if $A\,\mathbb{R}_+^n \subseteq \mathbb{R}_+^n$. Similarly, one can easily establish that for the scheme in (4.1) and the proper cone

$$W_+^{n,m} = \mathbb{R}_+^n \times \mathbb{R}_+^n \times \cdots \times \mathbb{R}_+^n,$$

the $n \times n$ matrix polynomial $L(\lambda)$ in (1.1) is Perron if and only if $x_j \geq 0$ $(j = 0, 1, \ldots)$ whenever $x_0 \geq 0$. Our analysis yields the following interesting generalization of the Fundamental Theorem of Demography.

Theorem 4.2 *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1)*, and let C_L be primitive. Suppose $y > 0$ and $w > 0$ are right and left eigenvectors of $L(\lambda)$ corresponding to $\rho := \rho(L)$, respectively, normalized so that*

$$\begin{bmatrix} w^T E_1(\rho) & \cdots & w^T E_{m-1}(\rho) & w^T \end{bmatrix} \begin{bmatrix} y \\ \rho y \\ \vdots \\ \rho^{m-1} y \end{bmatrix} = 1,$$

where $E_1(\lambda), \ldots, E_{m-1}(\lambda)$ are defined in (2.4)*. Consider a nonzero initial vector $x_0 = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \in \mathbb{C}^{nm}$ and let $\{u_0, u_1, \ldots\}$ be the solution of* (4.1) *given in* (4.2)*. Taking $E_m(\lambda) = I$, we have*

$$\lim_{j \to \infty} \frac{1}{\rho^j} u_j = \left[w^T \left(\sum_{k=1}^{m} E_k(\rho) u_{k-1} \right) \right] y. \tag{4.3}$$

Furthermore,

$$\lim_{j\to\infty} \|u_j\|_1 = \begin{cases} 0 & \textit{if} \quad \rho < 1 \\ \left\| \left[w^T \left(\sum_{k=1}^m E_k(\rho) u_{k-1} \right) \right] y \right\|_1 & \textit{if} \quad \rho = 1 \\ \infty & \textit{if} \quad \rho > 1 \end{cases} .$$

Proof. Since the nonnegative companion matrix C_L is primitive, by Lemma 3.1 and by [14, Theorem 8.5.1],

$$\lim_{j\to\infty} \left[\frac{1}{\rho(C_L)} C_L \right]^j = \begin{bmatrix} y \\ \rho y \\ \vdots \\ \rho^{m-1} y \end{bmatrix} \begin{bmatrix} w^T E_1(\rho) & \cdots & w^T E_{m-1}(\rho) & w^T \end{bmatrix}$$

$$= \begin{bmatrix} y w^T E_1(\rho) & \cdots & y w^T E_{m-1}(\rho) & y w^T \\ \rho y w^T E_1(\rho) & \cdots & \rho y w^T E_{m-1}(\rho) & \rho y w^T \\ \vdots & & \vdots & \vdots \\ \rho^{m-1} y w^T E_1(\rho) & \cdots & \rho^{m-1} y w^T E_{m-1}(\rho) & \rho^{m-1} y w^T \end{bmatrix}.$$

Thus, it follows that

$$\begin{aligned} \lim_{j\to\infty} \frac{1}{\rho^j} u_j &= \underbrace{[I \ 0 \ \cdots \ 0]}_{\in \mathbb{C}^{n\times nm}} \lim_{j\to\infty} \left[\frac{1}{\rho^j} C_L^j \right] x_0 \\ &= [I \ 0 \ \cdots \ 0] \end{aligned}$$

$$\times \begin{bmatrix} y w^T E_1(\rho) & \cdots & y w^T E_{m-1}(\rho) & y w^T \\ \rho y w^T E_1(\rho) & \cdots & \rho y w^T E_{m-1}(\rho) & \rho y w^T \\ \vdots & & \vdots & \vdots \\ \rho^{m-1} y w^T E_1(\rho) & \cdots & \rho^{m-1} y w^T E_{m-1}(\rho) & \rho^{m-1} y w^T \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} y w^T E_1(\rho) & \cdots & y w^T E_{m-1}(\rho) & y w^T \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \\ &= y \left(w^T E_1(\rho) u_0 + \cdots + w^T E_{m-1}(\rho) u_{m-2} + w^T u_{m-1} \right), \end{aligned}$$

completing the proof. $\square$

5 The numerical range of a Perron polynomial

In two recent papers [19, 23], the results in James Nestor Issos' thesis [20] on the numerical range of an entrywise nonnegative matrix were reviewed, extended and utilized. In this section we generalize these results to the numerical range of a Perron polynomial. For that purpose, consider an $n \times n$ polynomial $L(\lambda)$ of degree m as in (1.1) and the corresponding $nm \times nm$ companion matrix C_L in (2.2). The *numerical range* of $L(\lambda)$ is defined by

$$W(L) \,=\, \{\lambda \in \mathbb{C} : x^* L(\lambda) x = 0 \;\text{ for some nonzero }\; x \in \mathbb{C}^n\},$$

which is a compact subset of $\mathbb{C}$ that contains the spectrum $\sigma(L)$. For a linear pencil $I\lambda - A$, where $A \in \mathbb{C}^{n\times n}$, $W(I\lambda - A)$ coincides with the classical *numerical range* (or *field of values*) of the matrix A, namely, $F(A) = \{x^* A\, x : x \in \mathbb{C}^n,\, x^*x = 1\}$, and it is always convex; see e.g., [15]. The *numerical radius* of $L(\lambda)$ is defined by $r(L) = \max\{|\lambda| : \lambda \in W(L)\}$. In the remainder of this paper, we assume that $r(L) > 0$, or equivalently, that at least one of the coefficients A_j $(j = 0, 1, \ldots, m-1)$ of $L(\lambda)$ is nonzero.

Denote the *symmetric part* of a real square matrix A by $H(A) = (A + A^T)/2$. Also, for any $\mu \in \mathbb{C}$ and $x \in \mathbb{C}^n$, denote

$$y(\mu, x) \,=\, \begin{bmatrix} x \\ \mu x \\ \vdots \\ \mu^{m-1} x \end{bmatrix} \,\in\, \mathbb{C}^{nm}$$

and observe that $\|y(\mu, x)\|_2 = \|x\|_2 \sqrt{1 + |\mu| + \cdots + |\mu|^{m-1}}$. Vectors of the form $y(\mu, x)$ have been used to prove that the numerical range of the companion matrix C_L always contains $W(L)$ and the origin [22, Proposition 2.4]. Based on the same approach, we obtain in the next two lemmas the relation between the numerical range and the companion matrix of an arbitrary monic polynomial $L(\lambda)$, which is essential for the further investigation of $W(L)$.

Lemma 5.1 *The numerical range of $L(\lambda)$ in* (1.1) *satisfies*

$$W(L)\backslash\{0\} \,=\, \left\{ \mu \neq 0 : \mu = y(\mu, x)^* C_L\, y(\mu, x),\; x \in \mathbb{C}^n,\; \|x\|_2 = \frac{1}{\sqrt{1 + |\mu| + \cdots + |\mu|^{m-1}}} \right\}.$$

Proof. For any scalar $\mu \in \mathbb{C}$ and for any nonzero vector $x \in \mathbb{C}^n$, by the results in [22], we have

$$y(\mu, x)^* \,(I\mu - C_L)\, y(\mu, x) \,=\, \begin{bmatrix} x^* & \overline{\mu} x^* & \cdots & \overline{\mu}^{m-1} x^* \end{bmatrix}$$

$$\times \begin{bmatrix} I\mu & -I & 0 & \cdots & 0 \\ 0 & I\mu & -I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -I \\ -A_0 & -A_1 & -A_2 & \cdots & I\mu - A_{m-1} \end{bmatrix} \begin{bmatrix} x \\ \mu x \\ \vdots \\ \mu^{m-1} x \end{bmatrix}$$

$$= \overline{\mu}^{m-1}\, x^* \, L(\mu)\, x.$$

Thus, a nonzero $\mu \in W(L)$ if and only if $\mu = y(\mu,x)^* C_L\, y(\mu,x)$ for some nonzero $x \in \mathbb{C}^n$ with norm $\|x\|_2 = (1 + |\mu| + \cdots + |\mu|^{m-1})^{-1/2}$. $\square$

Lemma 5.2 *If the matrix coefficients of $L(\lambda)$ in* (1.1) *are real and $\mu_0 = \max\{\mu : \mu \in W(L) \cap \mathbb{R}\}$, then $F(L(\mu_0))$ lies in the closed right half-plane of $\mathbb{C}$ and the symmetric part $H(L(\mu_0))$ is singular positive semidefinite.*

Proof. By the definitions of the numerical range of a matrix and of a matrix polynomial, it is clear that $0 \in F(L(\mu_0))$, and that for every $\mu > \mu_0$, $0 \notin F(L(\mu))$. Moreover, the numerical range of any real matrix $L(\mu)$ ($\mu \geq \mu_0$) is convex and symmetric with respect to the real axis, and depends continuously on μ with respect to the Hausdorff metric. Consequently, $F(L(\mu_0))$ lies in the right closed half-plane of $\mathbb{C}$ and has the origin as a boundary point. Hence, the numerical range of the symmetric part of $L(\mu_0)$, namely,

$$F(H(L(\mu_0))) \;=\; \{\mathrm{Re}\,\lambda : \lambda \in F(L(\mu_0))\}$$

is a real interval with the origin as its left endpoint. Thus, $H(L(\mu_0))$ is singular positive semidefinite and the proof is complete. $\square$

Next we generalize Theorem 3.2 in [23].

Theorem 5.3 *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1). *Then $r(L) \in W(L)$ and there exists a nonnegative vector $x_r \in \mathbb{R}^n$ such that*

$$\|x_r\|_2 \;=\; \frac{1}{\sqrt{1 + r(L) + \cdots + r(L)^{m-1}}} \quad \textit{and} \quad x_r^T L(r(L)) x_r \;=\; 0.$$

If, in addition, 0 is a simple eigenvalue of $H(L(r(L)))$, then a vector $\hat{x} \in \mathbb{C}^n$ satisfies

$$\|\hat{x}\|_2 \;=\; \frac{1}{\sqrt{1 + r(L) + \cdots + r(L)^{m-1}}} \quad \textit{and} \quad \hat{x}^* L(r(L)) \hat{x} \;=\; 0$$

if and only if $\hat{x} = e^{\mathrm{i}\theta} x_r$ for some $\theta \in [0, 2\pi)$.

Proof. Let $\mu \in W(L)$. Then for every $x \in \mathbb{C}^n$ such that $\|x\|_2 = (1 + |\mu| + \cdots + |\mu|^{m-1})^{-1/2}$ and $\mu = y(\mu, x)^* C_L\, y(\mu, x)$, we readily have that

$$\begin{aligned} |\mu| &= |y(\mu,x)^* C_L\, y(\mu,x)| \\ &\leq |y(\mu,x)|^T C_L\, |y(\mu,x)| \\ &= \begin{bmatrix} |x|^T & |\mu|\,|x|^T & \cdots & |\mu|^{m-1}|x|^T \end{bmatrix} C_L \begin{bmatrix} |x| \\ |\mu|\,|x| \\ \vdots \\ |\mu|^{m-1}|x| \end{bmatrix}. \end{aligned}$$

Since the vectors x and $|x|$ have the same norm, by Lemma 5.1, it follows that the nonnegative number

$$\begin{bmatrix} |x|^T & |\mu|\,|x|^T & \cdots & |\mu|^{m-1}|x|^T \end{bmatrix} C_L \begin{bmatrix} |x| \\ |\mu|\,|x| \\ \vdots \\ |\mu|^{m-1}|x| \end{bmatrix} \;\geq\; |\mu|$$

also lies in $W(L)$. Hence, the numerical radius $r(L)$ belongs to $W(L)$ and there is a nonnegative vector $x_r \in \mathbb{R}^n$ such that $\|x_r\|_2 = (1 + r(L) + \cdots + r(L)^{m-1})^{-1/2}$ and $x_r^T L(r(L)) x_r = 0$.

By Lemma 5.2, 0 is the smallest eigenvalue of $H(L(r(L)))$, and thus, by [15, Lemma 1.5.7], a nonzero vector $\hat{x} \in \mathbb{C}^n$ satisfies $\hat{x}^* L(r(L))\hat{x} = 0$ if and only if it is an eigenvector of $H(L(r(L)))$ corresponding to 0. Hence, if 0 is a simple eigenvalue of $H(L(r(L)))$, then $\hat{x}$ is a scalar multiple of x_r. $\square$

The following two results are direct generalizations of Lemmas 3.3 and 3.4 in [23], respectively.

Corollary 5.4 *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1). *Suppose 0 is a simple eigenvalue of $H(L(r(L)))$ and $x_r \geq 0$ is the vector in Theorem 5.3, and let $e^{i\theta} r(L) \in W(L)$ for some $\theta \in (0, 2\pi)$. Then for every nonzero vector $x_\theta \in \mathbb{C}^n$ such that*

$$\|x_\theta\|_2 \;=\; \frac{1}{\sqrt{1 + r(L) + \cdots + r(L)^{m-1}}} \quad \text{and} \quad x_\theta^* L(e^{i\theta} r(L)) x_\theta \;=\; 0,$$

we have that $|x_\theta| = x_r$.

Proof. Let $x_\theta \in \mathbb{C}^n$ be any nonzero vector such that $x_\theta^* L(e^{i\theta} r(L)) x_\theta = 0$. Then as in the proof of Theorem 5.3,

$$\begin{aligned} r(L) &= |e^{i\theta} r(L)| \;=\; \left| y(e^{i\theta} r(L), x_\theta)^* C_L\, y(e^{i\theta} r(L), x_\theta) \right| \\ &\leq \begin{bmatrix} |x_\theta|^T & r(L)\,|x_\theta|^T & \cdots & r(L)^{m-1}|x_\theta|^T \end{bmatrix} C_L \begin{bmatrix} |x_\theta| \\ r(L)\,|x_\theta| \\ \vdots \\ r(L)^{m-1}|x_\theta| \end{bmatrix} \end{aligned}$$

$$\leq \quad r(L).$$

Consequently, the vector $|x_\theta|$ satisfies

$$\|\, |x_\theta|\, \|_2 \;=\; \frac{1}{\sqrt{1 + r(L) + \cdots + r(L)^{m-1}}} \quad \text{and} \quad |x_\theta|^T L(r(L))|x_\theta| \;=\; 0,$$

and by Theorem 5.3, $|x_\theta| = x_r$. $\square$

Theorem 5.5 *Let* $L(\lambda)$ *be an* $n \times n$ *Perron polynomial as in* (1.1). *Suppose that* 0 *is a simple eigenvalue of* $H(L(r(L)))$ *and* $x_r \geq 0$ *is the vector in Theorem 5.3. If there are angles* $\theta, \phi \in [0, 2\pi)$ *such that* $e^{\mathrm{i}\theta} r(L), e^{\mathrm{i}\phi} r(L) \in W(L)$, *then* $e^{\mathrm{i}(\theta+\phi)} r(L)$ *also lies in* $W(L)$.

Proof. By Corollary 5.4, there exist two nonzero vectors $x_\theta, x_\phi \in \mathbb{C}^n$ such that

$$|x_\theta| \;=\; |x_\phi| \;=\; x_r \quad \text{and} \quad x_\theta^* L(e^{\mathrm{i}\theta} r(L)) x_\theta \;=\; x_\phi^* L(e^{\mathrm{i}\phi} r(L)) x_\phi \;=\; 0.$$

By the relations

$$\begin{aligned} r(L) \;=\; |e^{\mathrm{i}\theta} r(L)| \;&=\; |y(e^{\mathrm{i}\theta} r(L), x_\theta)^* C_L\, y(e^{\mathrm{i}\theta} r(L), x_\theta)| \\ &=\; y(r(L), |x_\theta|)^T C_L\, y(r(L), |x_\theta|) \end{aligned}$$

and

$$\begin{aligned} r(L) \;=\; |e^{\mathrm{i}\phi} r(L)| \;&=\; |y(e^{\mathrm{i}\phi} r(L), x_\phi)^* C_L\, y(e^{\mathrm{i}\phi} r(L), x_\phi)| \\ &=\; y(r(L), |x_\phi|)^T C_L\, y(r(L), |x_\phi|), \end{aligned}$$

and following exactly the steps in the proof of [23, Lemma 3.4], one can construct a vector $w \in \mathbb{C}^n$ such that $|w| = x_r$ and

$$e^{\mathrm{i}(\theta+\phi)} r(L) \;=\; y(e^{\mathrm{i}(\theta+\phi)} r(L), w)^* C_L\, y(e^{\mathrm{i}(\theta+\phi)} r(L), w),$$

or equivalently,

$$w^* L(e^{\mathrm{i}(\theta+\phi)} r(L)) w \;=\; 0. \qquad \square$$

Suppose the maximal elements of $W(L)$ are of the form

$$r(L)\, e^{\mathrm{i}\theta_j} \qquad (j = 1, 2, \ldots, k)$$

with $\theta_1 = 0$ and $\theta_2, \ldots, \theta_k \in (0, 2\pi)$ for some positive integer k. Then, by Theorem 5.5, the set

$$\{\theta_1 \,(\mathrm{mod}\, 2\pi),\, \theta_2 \,(\mathrm{mod}\, 2\pi),\, \ldots,\, \theta_k \,(\mathrm{mod}\, 2\pi)\}$$

is a (finite) additive abelian group and hence a cyclic group. As a consequence, we have the following generalization of [23, Theorem 3.5].

Corollary 5.6 *Let $L(\lambda)$ be an $n \times n$ Perron polynomial as in* (1.1)*. Suppose that* 0 *is a simple eigenvalue of $H(L(r(L)))$ and that $W(L)$ has k maximal elements. Then these maximal elements are of the form*

$$r(L)\, e^{\mathrm{i}\frac{2t\pi}{k}} \quad (t = 0, 1, \ldots, k-1).$$

We proceed with an illustration of the numerical range of a Perron polynomial. Let A be an $n \times n$ nonnegative matrix with irreducible symmetric part $H(A)$, and consider the Perron polynomial of degree m,

$$L_A(\lambda) \,=\, I\lambda^m - A.$$

Suppose A is k-cyclic. Then by the Perron-Frobenius Theorem and the results in [23],

$$e^{\mathrm{i}\frac{2t\pi}{k}}\sigma(A) \,=\, \sigma(A) \quad \text{with} \quad e^{\mathrm{i}\frac{2t\pi}{k}}\rho(A) \,\in\, \sigma(A) \quad (t = 0, 1, \ldots, k-1)$$

and

$$e^{\mathrm{i}\frac{2t\pi}{k}}F(A) \,=\, F(A) \quad \text{with} \quad e^{\mathrm{i}\frac{2t\pi}{k}}r(A) \,\in\, F(A) \quad (t = 0, 1, \ldots, k-1).$$

The spectrum and the numerical range of $L_A(\lambda)$ are given, respectively, by

$$\sigma(L_A) \,=\, \{\mu \in \mathbb{C} : \mu^m \in \sigma(A)\} \quad \text{and} \quad W(L_A) \,=\, \{\mu \in \mathbb{C} : \mu^m \in F(A)\}\,.$$

Since the mth roots of a nonzero complex number $\alpha e^{\mathrm{i}\theta}$ ($\alpha > 0,\ \theta \in [0, 2\pi)$) are $\alpha^{\frac{1}{m}} e^{\mathrm{i}\frac{2\xi\pi+\theta}{m}}$ ($\xi = 0, 1, \ldots, m-1$) and the mth roots of $e^{\mathrm{i}\frac{2t\pi}{k}}$ are given by $e^{\mathrm{i}\frac{2\pi(t+\xi m)}{km}}$ ($\xi = 0, 1, \ldots, m-1$), we have that

$$e^{\mathrm{i}\frac{2\zeta\pi}{km}}\sigma(L_A) \,=\, \sigma(L_A) \quad \text{with} \quad e^{\mathrm{i}\frac{2\zeta\pi}{km}}\rho(L_A) \,\in\, \sigma(L_A) \quad (\zeta = 0, 1, \ldots, km-1)$$

and

$$e^{\mathrm{i}\frac{2\zeta\pi}{km}}W(L_A) \,=\, W(L_A) \quad \text{with} \quad e^{\mathrm{i}\frac{2\zeta\pi}{km}}r(L_A) \,\in\, W(L_A) \quad (\zeta = 0, 1, \ldots, km-1).$$

Example 1 The nonnegative matrix $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is 2-cyclic and has an irreducible symmetric part. The boundary of the (convex) numerical range of A is drawn in the left part of Figure 1. The numerical range of the Perron polynomial $L_A(\lambda) = I\lambda^3 - A$ is illustrated in the right part of the figure (the unshaded star-shaped region). It is obtained using the inclusion-exclusion algorithm described in [25] and confirms the above discussion. The eigenvalues of A and $L_A(\lambda)$ are marked with +'s.

Example 2 For the matrix A in the previous example, consider the Perron polynomial $L(\lambda) = I\lambda^3 - I\lambda^2 - I\lambda - A$. The numerical range $W(L)$ is illustrated in Figure 2 and has exactly one (real positive) maximal element. The eigenvalues of $L(\lambda)$ are marked with +'s. Note that the

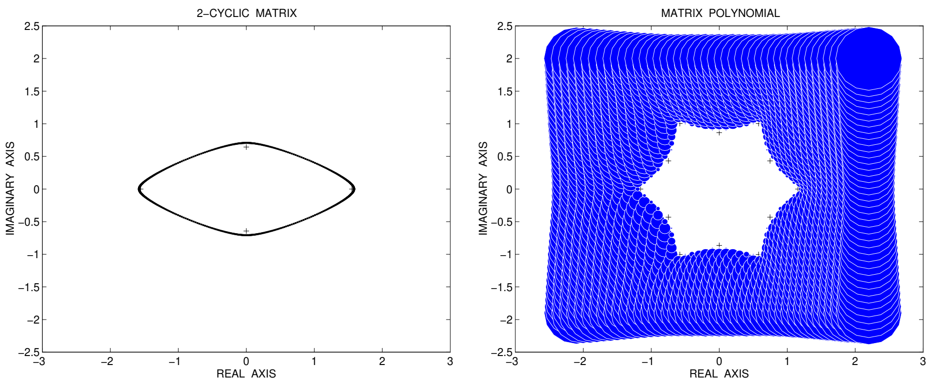

Figure 1: Spectra and numerical ranges of A and $I\lambda^3 - A$.

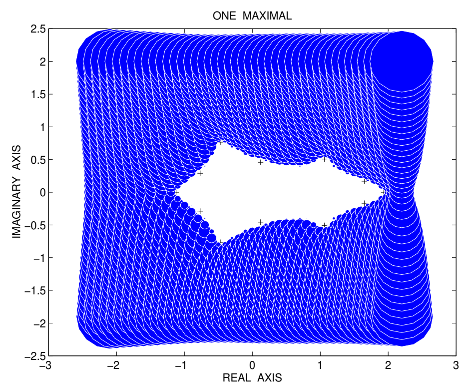

Figure 2: Spectrum and numerical range of $L(\lambda)$.

(nonnegative) companion matrix of $L(\lambda)$, $C_L = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ A & I & I \end{bmatrix}$, is primitive and the spectrum $\sigma(L) = \sigma(C_L)$ has also exactly one maximal element, which is approximately 1.9331.

In conclusion, we have pursued a discussion on the spectrum and the numerical range of a Perron polynomial, as well as an asymptotic analysis of the associated multistep finite differences scheme. Our discussion is based on the companion matrix linearization. We have not treated a number of subjects in Perron-Frobenius theory suitable for further research. They include a study of the Perron generalized eigenspace in the spirit of the results of Rothblum [26] (see also Schneider [30]) on the existence of a nonnegative basis with combinatorial structure. For such a direction it may be advantageous to consider the role of expansion graphs as presented in [8] and in [9], where the level and height characteristics are examined. It is also worthwhile pursuing a partition of Perron polynomials analogous to the partition of Z-matrices and Z-pencils in classes based on the spectral radii of submatrices; see [6] and [21], respectively.

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