Perron-Frobenius Type Results on the Numerical Range

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October 31, 2001

Abstract

We present results connecting the shape of the numerical range to intrinsic properties of a matrix A. When A is a nonnegative matrix, these results are to a large extent analogous to the Perron-Frobenius theory, especially as it pertains to irreducibility and cyclicity in the combinatorial sense. Special attention is given to polygonal, circular and elliptic numerical ranges. The main vehicles for obtaining these results are the Hermitian and skew-Hermitian parts of A, as well as Levinger's transformation $aA + (1-a)A^*$.

Key words. Numerical range, numerical radius, nonnegative matrix, Perron-Frobenius, k-cyclic matrix.

AMS subject classification. 15A48, 15A60, 47A12

1 Introduction

The numerical range of a matrix may be used to draw surprisingly strong conclusions about the spectrum and the combinatorial structure of a matrix, in particular a nonnegative matrix. In this paper, we will recall and develop results on the numerical range of a real or a nonnegative matrix, which directly relate its shape to intrinsic properties of the matrix, and thus provide means of detecting such properties. The motivation is largely provided

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by the unpublished doctoral thesis of J. N. Issos [7] in 1966; also by [14, 15], and a recent paper [16] in which the authors thoroughly examine matrices whose numerical ranges have rotational symmetries.

The structure and goals of this paper are as follows.

In Section 3 we look at the basic conclusions that can be drawn about/from the numerical range of a nonnegative matrix that are analogous to the Perron-Frobenius Theorem. These are mainly the results of Issos [7] generalized by relaxing or disposing of the assumption of irreducibility. We have attempted to provide as simple proofs as possible, combining the approach of Issos with results found in [6] and [16]. We also report some progress on a question raised in [16] (Proposition 3.8), and a result based on Fan's Theorem that relates the numerical ranges of a matrix and its entrywise absolute value (Theorem 3.10).

In Section 4, motivated by the work of Levinger [10] and subsequent work of Fiedler [4], we consider the numerical range of $(1-a)A + aA^*$ as a function of $a \in \mathbb{R}$. When the numerical range of $A \in \mathcal{M}_n(\mathbb{C})$ is an elliptic or circular disk, this approach allows us to obtain results on the irreducibility of a nonnegative matrix A (Theorem 4.5) or to describe the spectral radius as a function of a (Theorem 4.8). We also generalize a result by J. Anderson from circles to ellipses (Theorem 4.12).

The reader is alerted to the forthcoming article [11] in which many of the topics herein are treated in a different but related manner.

2 Notation and preliminaries

Let $\mathcal{M}_n(\mathbb{C})$ ($\mathcal{M}_n(\mathbb{R})$) be the algebra of all $n \times n$ complex (real) matrices. For $A \in \mathcal{M}_n(\mathbb{C})$, the *numerical range* (also known as the *field of values*) of A is defined and denoted by

$$W(A) = \{x^*Ax \ : \ x \in \mathbb{C}^n, \ x^*x = 1\},$$

and is known to be a compact and convex subset of $\mathbb{C}$. We refer to a vector x as *unit* if $x^*x = 1$. The *numerical radius* of A is $r(A) = \max\{|a| : a \in W(A)\}$. We refer to $a \in W(A)$ as *maximal* if $|a| = r(A)$. The *spectrum* of A is denoted by $\sigma(A)$ and its *spectral radius* by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

Let also $H(A) = (A + A^*)/2$ denote the Hermitian part and $S(A) = (A - A^*)/(2i)$, i.e., $iS(A)$ is the skew-Hermitian part of A. Clearly, $W(H(A)) = \{\mathrm{Re} z : z \in W(A)\}$ and if $A \in \mathcal{M}_n(\mathbb{R})$, then $W(A)$ is symmetric with respect to the real axis.

It is assumed that the reader is familiar with the basic elements of the Perron-Frobenius theory for nonnegative matrices. The suggested references are [2, 5, 6]. We will only selectively recall some notions and notation.

Let ≥ 0 be the symbol for real arrays with nonnegative entries, and $\geq$ the symbol for the induced entrywise order. The symbol $|\cdot|$ is reserved for the entrywise absolute value of an array. Matrix A is called *irreducible* if there does not exist a permutation matrix P such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square, non-vacuous matrices. Matrix A is called k*-cyclic* if for some permutation matrix P,

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \dots & \dots & 0 \\ 0 & 0 & A_{23} & 0 & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & A_{k-1,k} \\ A_{k,1} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{1}$$

where the zero blocks along the diagonal are square. Notice that a k-cyclic matrix is also m-cyclic for any divisor m of k. The largest positive integer k for which A is k-cyclic is referred to as the *cyclic index* of A. When $A_{k,1} = 0$ and $k \geq 2$ in (1), A is said to be permutationally similar to a *block-shift matrix.*

Note that, by the (second part of the) Perron-Frobenius Theorem (see [2, Theorem 2.20, p. 32]), the cyclic index of a nonnegative irreducible matrix A coincides with the *index of imprimitivity* of A, namely, the number of eigenvalues of A of maximal modulus.

3 Perron-Frobenius and the results of Issos

It is perhaps well known, but not readily accessible in the standard literature, that when $A \geq 0$, $r(A)$ is a maximal element of $W(A)$ attained by the quadratic form of $H(A)$ at a unit eigenvector of $H(A)$ corresponding to $\rho(H(A))$. That is, when $A \geq 0$, $r(A) = \rho(H(A))$. It is of interest, however, that primarily in the case of irreducible nonnegative matrices, there are further analogies between the Perron-Frobenius Theorem and the shape of the numerical range. These analogies were first presented in [7] and are summarized and paraphrased next.

Theorem 3.1 (Issos [7]) *Let* $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$*, be an irreducible matrix.*

(i) *There exists a unit positive vector x such that $x^T Ax = r(A)$. Moreover, $y^* Ay = r(A)$ with $y^* y = 1$ if and only if $y = e^{i\theta} x$ for some $\theta \in [0, 2\pi)$.*

(ii) *If A is k-cyclic, then*

$$W(A) = W(e^{i\frac{2t\pi}{k}} A) \quad (t = 0, 1, \ldots, k-1).$$

Moreover, if $\lambda \in \sigma(A)$, then $e^{i\frac{2t\pi}{k}} \lambda \in \sigma(A)$ with the same multiplicity as λ.

(iii) *If $W(A)$ has exactly k $(2 \le k \le n)$ maximal elements, then A has cyclic index k.*

(iv) *There exists a positive integer $k \le n$ such that the maximal elements of $W(A)$ are of the form*

$$r(A) e^{i\frac{2t\pi}{k}} \quad (t = 0, 1, \ldots, k-1).$$

Moreover, the maximal elements of $\sigma(A)$ are of the form

$$\rho(A) e^{i\frac{2t\pi}{k}} \quad (t = 0, 1, \ldots, k-1).$$

Remarks The following remarks refer to Theorem 3.1.

(1) In fact, Issos showed that (ii) holds for all complex matrices.

(2) The integer k in clause (iv) equals the cyclic index and the index of imprimitivity of A.

(3) Notice the striking similarities of Theorem 3.1 with the various clauses of the Perron-Frobenius Theorem for irreducible and k-cyclic matrices. As a matter of fact, in the proof attributed to Wielandt of the second part of the Perron-Frobenius Theorem ([2, Theorem 2.20, p.32]), it is observed that a k-cyclic matrix A is invariant under the transformation $A \longrightarrow e^{i\theta} DAD^*$ for a suitable angle θ and a diagonal unitary matrix D. Such a transformation simply rotates $W(A)$ by θ and so Wielandt's approach can be used as an alternate method of proof for some of the clauses of the above theorem and their extensions herein.

Our first goal is to generalize Theorem 3.1 (i) by relaxing the assumption of irreducibility. The proof of the first part of the next theorem is essentially contained in [7] and also in [1].

Theorem 3.2 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \neq 0, A \geq 0$. Then there exists a unit vector $x_a \geq 0$ such that $x_a^T Ax_a = r(A)$. Moreover, if $\rho(H(A))$ is a simple eigenvalue of $H(A)$, then y is a unit vector with $y^* Ay = r(A)$ if and only if $y = e^{i\theta} x_a$ for some $\theta \in [0, 2\pi)$.*

Proof. We readily have that for every $x \in \mathbb{C}^n$, $|x^* Ax| \leq |x|^T A|x|$. Consequently,

$$r(A) = \max\{x^T Ax : x^T x = 1, x \geq 0\}$$

and there exists $x_a \geq 0$ such that $x_a^T A x_a = r(A)$. Suppose now that $r(A) = \rho(H(A))$ is a simple eigenvalue of $H(A)$. By [6, Lemma 1.5.7], any unit vector $y \in \mathbb{C}^n$ such that $y^*Ay = r(A)$ is an eigenvector of $H(A)$ corresponding to $\rho(H(A))$. Thus, as $\rho(H(A))$ is simple, $y = e^{i\theta} x_a$ for some $\theta \in [0, 2\pi)$. ∎

Note that if $H(A)$ is irreducible, x_a in the above theorem is indeed the unit Perron vector of $H(A)$.

Next we aim to generalize Theorem 3.1 (iv). We first need two lemmas of independent interest, which are also generalizations of results in [7].

Lemma 3.3 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ and assume $\rho(H(A))$ is a simple eigenvalue of $H(A)$. If $y \in \mathbb{C}^n$ is a unit vector such that $|y^*Ay| = r(A)$ and $x_a \geq 0$ is the vector of Theorem 3.2, then $|y| = x_a$.*

Proof. As in the proof of Theorem 3.2, $r(A) = |y^*Ay| \leq |y|^T A|y|$ and thus $r(A) = |y|^T A|y|$. By Theorem 3.2 it now follows that $|y| = x_a$. ∎

Lemma 3.4 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ and assume $\rho(H(A))$ is a simple eigenvalue of $H(A)$. If there are angles $\phi, \theta \in [0, 2\pi)$, $\phi \neq \theta$, such that $r(A)e^{i\phi}, r(A)e^{i\theta} \in W(A)$, then $r(A)e^{i(\phi+\theta)} \in W(A)$.*

Proof. Let $u = (u_j), v = (v_j) \in \mathbb{C}^n$ be two unit vectors such that $u^*Au = r(A)e^{i\phi}$ and $v^*Av = r(A)e^{i\theta}$. If $x_a = (x_j)$ is the nonnegative vector of Theorem 3.2, then by Lemma 3.3, $|u| = |v| = x_a$. Moreover, precisely as in the proofs of [7, Theorems 2 and 3], by the ensuing relations

$$r(A) = |u^*Au| = |\sum_{i,j=1}^{n} a_{ij}\overline{u}_i u_j| \leq \sum_{i,j=1}^{n} a_{ij}\, |\overline{u}_i|\, |u_j| = \sum_{i,j=1}^{n} a_{ij} x_i x_j = r(A),$$

it follows that every non-zero term of $\sum a_{ij}\overline{u}_i u_j$ has the same argument ϕ. Thus, corresponding to every positive entry a_{ij} of A, $\arg u_j - \arg u_i = \phi$. Similarly, $\arg v_j - \arg v_i = \theta$. Define now the unit vector $w = (w_j)$ by

$$w_j = x_j\ e^{i(\arg u_j + \arg v_j)} \quad (j = 1, 2, \ldots, n).$$

It is straightforward to verify that for every $a_{ij} > 0,\ \arg(a_{ij}\overline{w}_i w_j) = \phi + \theta$, and so $w^*Aw = r(A)e^{i(\phi+\theta)}$. ∎

We can now generalize Theorem 3.1 (iv).

Theorem 3.5 *Let $A \geq 0$ and assume $\rho(H(A))$ is a simple eigenvalue of $H(A)$. Assume also that $W(A)$ is not a circular disk centered at the origin. Then there exists a positive integer $k \leq n$ such that the maximal elements of $W(A)$ are of the form $r(A)e^{i\frac{2t\pi}{k}}$ $(t = 0, 1, \ldots, k-1)$.*

Proof. By Theorem 3.2, $W(A)$ has at least one maximal element, $r(A)$. Moreover, by a result attributed to J. Anderson (see [16, Lemma 6]), $W(A)$ cannot have more than n maximal elements. Thus its maximal elements are $a_1, a_2, \ldots, a_k$ with arguments $\phi_1 = 0, \phi_2, \ldots, \phi_k$ for some positive integer $k \leq n$. By Lemma 3.4, the set

$$\{\phi_1(\mathrm{mod} 2\pi), \phi_2(\mathrm{mod} 2\pi), \ldots, \phi_k(\mathrm{mod} 2\pi)\}$$

is a finite additive abelian group and thus a cyclic group. ■

Corollary 3.6 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ such that $H(A)$ is irreducible. If $W(A)$ has a finite number $k \geq 2$ of maximal elements, then the following (equivalent) conditions hold.*

(i) *A is diagonally similar to $e^{i\frac{2\pi}{k}}A$.*

(ii) *A is k-cyclic.*

(iii) $e^{i\frac{2t\pi}{k}}W(A) = W(A)$ $(t = 0, 1, \ldots, k-1)$.

(iv) $e^{i\frac{2t\pi}{k}}r(A) \in W(A)$ $(t = 0, 1, \ldots, k-1)$.

In particular, the cyclic index of A equals k.

Proof. Conditions (i)-(iv) are shown to be equivalent in [16, Theorem 2], when $A \geq 0$, $H(A)$ is irreducible and A is not permutationally similar to a block-shift matrix. The latter assumption holds here as $W(A)$ is assumed to have a finite number of maximal elements; cf. Theorem 4.1. Moreover, as $H(A)$ is assumed to be irreducible, $\rho(H(A))$ is a simple eigenvalue of $H(A)$ and thus by Theorem 3.5, (iv) holds. ■

Example 3.7 Suppose it is known that the matrices with numerical ranges as given by Figure 1 are nonnegative and have irreducible Hermitian parts. Their numerical ranges have 3 and 4 maximal elements and thus they must have cyclic index 3 and 4, respectively. Their eigenvalues are also marked to illustrate the corresponding symmetries of the spectrum and the numerical range.

In [7] the following question is asked: What are necessary and sufficient conditions for the numerical range of an irreducible nonnegative matrix to be a regular polygon? In [16] a

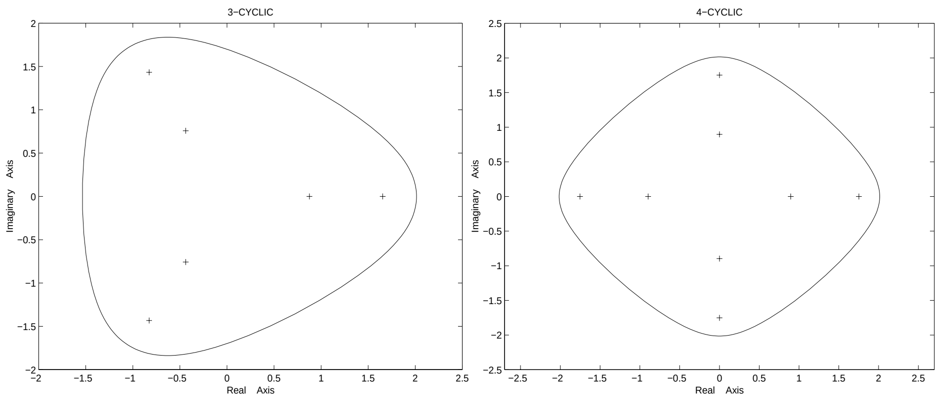

Figure 1: Numerical ranges of cyclic matrices.

similar problem is raised, namely, to characterize nonnegative matrices whose numerical ranges are regular (convex) polygons centered at the origin. The following shed some light in this regard.

Let $A \geq 0$ be a square matrix. By considering the graph of $H(A)$ and its connected components, it is easy to see that A is permutationally similar to a direct sum $A_1 \oplus A_2 \oplus \ldots \oplus A_m$ so that $H(A_j)$ is irreducible for every $j = 1, 2, \ldots, m$, and $H(A)$ is permutationally similar to $H(A_1) \oplus H(A_2) \oplus \ldots \oplus H(A_m)$. Notice now that, on one hand, $W(A)$ being a regular polygon centered at the origin with k vertices means

$$W(A) \;=\; \text{conv.hull}\,\{r(A)e^{2\pi ti/k} \;:\; t = 0, 1, \ldots, k-1\}.$$

On the other hand, by [6, Property 1.2.10],

$$W(A) \;=\; \text{conv.hull}\,\{W(A_1) \cup W(A_2) \cup \ldots \cup W(A_m)\}.$$

By Theorem 3.2, $r(A) \in W(A)$ and so there exists s such that $r(A) = r(A_s) \in W(A_s)$. The matrix A_s is not necessarily unique in having numerical radius equal $r(A)$. As $r(A)e^{2\pi i/k}$ is also an extreme point of $W(A)$, we may without loss of generality assume that $r(A)e^{2\pi i/k} \in W(A_s)$. Then, by Lemma 3.4 applied to A_s, $W(A_s)$ contains all points $r(A)e^{2\pi ti/k}$, $t = 0, 1, \ldots, k-1$. Thus, by the convexity of the numerical range, $W(A_s) = W(A)$. As a consequence, in considering nonnegative matrices whose numerical ranges are regular polygons centered at the origin, one can restrict attention to matrices with irreducible Hermitian parts, as is done in the next result.

Proposition 3.8 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ such that $H(A)$ is irreducible. Then $W(A)$ is a regular polygon $\mathcal{P}_k$ with k vertices centered at the origin if and only if*

(i) *A has cyclic index k, and*

(ii) *there exists a unitary $U \in \mathcal{M}_n(\mathbb{C})$ such that U^*AU is the direct sum of a $k \times k$ diagonal matrix D with a matrix B such that $W(B) \subseteq W(D)$.*

Proof. By Corollary 3.6, if $W(A)$ has exactly k maximal elements then A has cyclic index k. If we assume that $W(A)$ is a regular polygon $\mathcal{P}_k$ with k vertices centered at the origin, then the vertices of the polygon are necessarily (normal) eigenvalues of A (see [6, Theorem 1.6.3]) and so (ii) is due to [6, Theorem 1.6.8]. Conversely, if (i) and (ii) hold, then $W(A) = W(D)$ is the claimed polygon by [6, Property 1.2.10] and because A has cyclic index k. ■

Note that condition (ii) of Proposition 3.8 is essential as illustrated by the following example.

Example 3.9 Consider the 3-cyclic 9×9 matrix

$$A = \begin{pmatrix} 0 & C & 0 \\ 0 & 0 & C \\ C & 0 & 0 \end{pmatrix}, \quad \text{where} \quad C = \begin{pmatrix} 1.5 & 2 & 0 \\ 0 & 1.6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and see $W(A)$ in Figure 2. The spectrum of A is

$$\sigma(A) = \{1.5e^{\frac{2t\pi}{3}},\ 1.6e^{\frac{2t\pi}{3}},\ 3e^{\frac{2t\pi}{3}} : t = 0, 1, 2\}.$$

Moreover, A is unitarily similar to $\text{diag}\{3,\ 3e^{\frac{2\pi}{3}},\ 3e^{\frac{4\pi}{3}}\} \oplus B$, where $W(B)$ is not a subset of $\mathcal{P}_3 = \text{conv.hull}\{3,\ 3e^{\frac{2\pi}{3}},\ 3e^{\frac{4\pi}{3}}\}$. Therefore, $W(A) = \text{conv.hull}\{W(B) \cup \mathcal{P}_3\} \neq \mathcal{P}_3$.

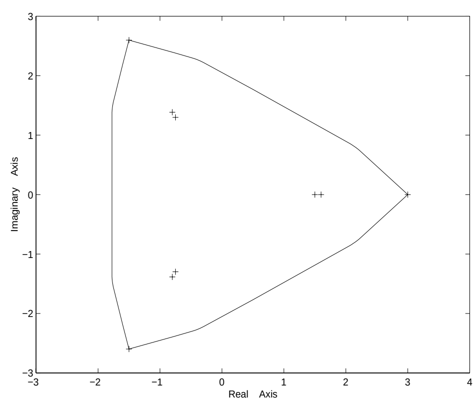

Figure 2: The vertices are the maximal elements.

One of the applications of the Perron-Frobenius Theorem is Fan's Theorem [5, Theorem 8.2.12], which can be used to obtain the following result about the numerical range.

Theorem 3.10 *Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{C})$ and $B = (b_{ij}) \in \mathcal{M}_n(\mathbb{R}), B \geq 0$ such that $B \geq |A|$. Denote $\xi_j(B) = r(B) - b_{jj},\ j = 1, 2, \dots, n$ and*

$$m_\theta = \min_j \left(\mathrm{Re}(e^{i\theta} a_{jj}) - \xi_j(B)\right), \quad M_\theta = \max_j \left(\mathrm{Re}(e^{i\theta} a_{jj}) + \xi_j(B)\right).$$

Then

$$W(A) \subseteq \bigcap_{\theta \in [0, 2\pi]} \{z \in \mathbb{C} : m_\theta \leq \cos\theta \,\mathrm{Re} z - \sin\theta \,\mathrm{Im} z \leq M_\theta\}.$$

Proof. Since $|A| \leq B$, it is easy to see that for any $\theta \in [0, 2\pi]$, $|H(e^{i\theta}A)| \leq H(|A|) \leq H(B)$. By Fan's Theorem,

$$\sigma(H(e^{i\theta}A)) \subseteq \bigcup_{j=1}^{n} \{z \in \mathbb{C} : |z - \mathrm{Re}(e^{i\theta} a_{jj})| \leq \rho(H(B)) - b_{jj}\}$$

and consequently,

$$\mathrm{Re}\, W(e^{i\theta}A) \subseteq \left[\min_j \left(\mathrm{Re}(e^{i\theta} a_{jj}) - r(B) + b_{jj}\right),\ \max_j \left(\mathrm{Re}(e^{i\theta} a_{jj}) + r(B) - b_{jj}\right)\right].$$

Hence, $W(A)$ lies in the zone $\{z \in \mathbb{C} : m_\theta \leq \mathrm{Re}(e^{i\theta} z) \leq M_\theta\}$. The proof is completed by taking the intersection of these zones for all $\theta \in [0, 2\pi]$. ∎

4 Circular and elliptic disks

For an $n \times n$ matrix A, $W(A)$ has either at most n maximal elements or, according to [16, Lemma 6], it is a circular disk *centered at the origin* (and thus has infinitely many maximal elements.) In fact, the following holds for nonnegative matrices.

Theorem 4.1 [16, Theorem 1 (a), (r)] *Let $A \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ such that $H(A)$ is irreducible. Then $W(A)$ is a circular disk centered at the origin if and only if A is permutationally similar to a block-shift matrix.*

This result provides our motivation to look at numerical ranges that are circular or elliptic and at the consequences of such an assumption.

It was shown in [9, 12] that the boundary of the numerical range of a complex matrix A is the real part of an algebraic curve whose equation in line coordinates is

$$\det(uI + vH(A) + wS(A)) = 0.$$

This algebraic curve is of class n and has n foci that coincide with the eigenvalues of A (see [12]). Taking into consideration that the dual curve of a conic is a conic, we can now state the following theorem (cf. [13, Theorem 2]).

Theorem 4.2 *Let $A \in \mathcal{M}_n(\mathbb{C})$. If $W(A)$ is an elliptic disk, then its foci are eigenvalues of A. In particular, if $W(A)$ is a circular disk, then its center is an eigenvalue of A.*

We proceed by considering the numerical range of the matrix

$$T(a, A) := (1-a)A + aA^*,$$

whose spectral radius as a function of $a \in [0, 1]$ has been studied by Levinger [10] and Fiedler [4]. For any $a \in \mathbb{R}$, it is easy to verify that $H(T(a, A)) = H(A)$ and $S(T(a, A)) = (1-2a)S(A)$. Consequently,

$$W(T(a, A)) = \{u + i(1-2a)v : u, v \in \mathbb{R}, u + iv \in W(A)\}. \tag{2}$$

Theorem 4.3 *Let $A \in \mathcal{M}_n(\mathbb{C})$ such that $W(A)$ is symmetric with respect to the real axis. Then the following are equivalent.*

(i) *$W(A)$ is a circular disk centered at $\mu \in \mathbb{R}$.*

(ii) *$W(T(a, A))$ is an elliptic disk with center $\mu \in \mathbb{R}$ and foci lying on the real axis if and only if $a \in (0, 1)$.*

(iii) *$W(T(a, A))$ is an elliptic disk with center $\mu \in \mathbb{R}$ and foci lying on the line $\{\mu + i\lambda : \lambda \in \mathbb{R}\}$ if and only if $a \in (-\infty, 0) \cup (1, \infty)$.*

Proof. By continuity of the boundary of $W(T(a, A))$ in a (relative to the Hausdorff metric) and its symmetry with respect to the real axis, it is enough to prove that (i) implies (ii) and (iii). Assume that the boundary of $W(A)$ is a circle with equation $(u-\mu)^2 + v^2 = r^2$ in the (u, v)-plane. Then the boundary of $W(T(a, A))$ is obtained by (2). For $a \in \mathbb{R} \setminus \{0, 1/2, 1\}$, via the change of variables

$$X = u \text{ and } Y = (1-2a)v,$$

the boundary of $W(T(a, A))$ is seen to be the ellipse

$$\frac{(X-\mu)^2}{r^2} + \frac{Y^2}{r^2(1-2a)^2} = 1,$$

whose foci are $\mu \pm 2r\sqrt{a(1-a)}$ if $a \in (0, 1/2) \cup (1/2, 1)$, or $\mu \pm i2r\sqrt{a(a-1)}$ if $a \in (-\infty, 0) \cup (1, \infty)$. For $a = 1/2$, the line segment $W(T(1/2, A)) = W(H(A))$ is indeed a degenerate ellipse. ∎

From Theorems 4.2 and 4.3, and the continuity of the eigenvalues of $T(a, A)$ in a, we immediately obtain the following.

Corollary 4.4 *Let $A \in \mathcal{M}_n(\mathbb{C})$ such that $W(A)$ is a circular disk centered at $\mu \in \mathbb{C}$. Then $\mu \in \sigma(A)$ with multiplicity at least 2.*

Suppose now that the boundary of $W(A)$ in the (u, v)-plane has equation

$$\frac{u^2}{k^2} + \frac{v^2}{\lambda^2} = 1 \quad (k, \lambda > 0).$$

If $k > \lambda$ (resp., $k < \lambda$), then $W(A)$ is an elliptic disk with foci lying on the real axis (resp., off the real axis). For any $a \in (-\infty, 1/2)$, the boundary of $W(T(a, A))$ has equation

$$\frac{u^2}{k^2} + \frac{v^2}{\lambda^2(1-2a)^2} = 1,$$

which becomes a circle when $a = (\lambda - k)/(2\lambda)$. It is also worth noting that for $a, b, \in [0, 1]$,

$$T(b, T(a, A)) = [1 - (a + b - 2ab)A] + (a + b - 2ab)A^T = T(a + b - 2ab, A),$$

where $a + b - 2ab \in [0, 1]$. In conclusion, if the boundary of $W(A)$ is a conic, then the boundary of $W(T(a, A))$ is transformed as follows as a varies in $\mathbb{R}$:

$$[\text{Ellipse with foci} \notin \mathbb{R}] \longleftrightarrow [\text{Circle}] \longleftrightarrow [\text{Ellipse with foci} \in \mathbb{R}] \overset{a=1/2}{\longleftrightarrow} [\text{Line segment}]\,.$$

For nonnegative matrices, we have the following.

Theorem 4.5 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ such that $W(A)$ is a circular disk or an elliptic disk with foci off the real line. Then A is reducible.*

Proof. Suppose that A is as prescribed and, by way of contradiction, that A is irreducible. By (2) and Theorem 4.3, for some $a_0 \in [0, 1/2)$, $W(T(a_0, A))$ is a circular disk concentric to $W(A)$. Let $\mu \in \mathbb{R}$ be the center. Then, by Corollary 4.4, $\mu \in \sigma(T(a_0, A))$ has multiplicity at least 2. As $T(a_0, A) \geq 0$ must also be irreducible, its spectral radius is a simple eigenvalue and thus

$$\mu < \rho(T(a_0, A)). \tag{3}$$

Now for $a \in [a_0, 1/2]$, consider the right focus of $W(T(a, A))$ as a function $\psi(a, A)$. When $a = a_0$, $\psi(a, A)$ coincides with the center μ of $W(T(a_0, A))$. By Theorem 4.2, we also have $\psi(a, A) \in \sigma(T(a, A))$. We can now deduce that for every $a \in [a_0, 1/2)$,

$$\psi(a, A) < \rho(T(a, A)) \leq \rho(H(A)). \tag{4}$$

The first (strict) inequality in (4) follows by (3) and the simplicity of the eigenvalue $\rho(T(a, A))$ as $T(a, A)$ is irreducible for $0 \leq a \leq 1$. The second inequality in (4) was shown in [10] but also follows readily from the fact that $\operatorname{Re} W(T(a, A)) = \operatorname{Re} W(A) = W(H(A))$. Lastly, at $a = 1/2$, we have

$$\psi(1/2, A) = \rho(T(1/2, A)) = \rho(H(A)).$$

The latter equalities and (4) imply that $\rho(H(A))$ is not a simple eigenvalue of $H(A)$, a contradiction to the assumption that A is irreducible that completes the proof. ∎

Corollary 4.6 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ such that $W(A)$ is an elliptic disk with foci off the real line. Then $H(A)$ is reducible.*

Proof. If $H(A)$ is irreducible, then there exists an $a_0 \in (0, 1/2)$ such that $T(a_0, A) \geq 0$ is irreducible with $W(T(a_0, A))$ a circular disk. This is a contradiction by Theorem 4.5. ∎

Example 4.7 Consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \geq 0.$$

It is known that $W(A)$ is a circular or elliptic disk. The discriminant of the characteristic polynomial of A is

$$\Delta = (a - d)^2 + 4bc \geq 0.$$

If $\Delta = 0$, then $W(A)$ is a circular disk centered at $(a + d)/2$; if $\Delta > 0$, then $W(A)$ is an elliptic disk with foci $(a + d \pm \sqrt{\Delta})/2$. Moreover, A is irreducible if and only if $bc \neq 0$, in which case $\Delta > 0$ (see also Theorem 4.5).

When A is an $n \times n$ nonnegative matrix and $W(A)$ is an elliptic disk with foci on the real axis, the behavior of $\rho(T(a, A))$ can be described in detail, complementing (in this case) the work of Fiedler [3].

Theorem 4.8 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ such that $H(A)$ is irreducible. If $W(A)$ is an elliptic disk centered at $\mu \in \mathbb{R}$ with foci $\mu \pm k \in \mathbb{R}$ and minor axis length λ, then*

$$\rho(T(a, A)) = \mu + \sqrt{k^2 + 4\lambda^2 a(1 - a)}, \;\; a \in [0, 1/2].$$

Proof. Assume that the boundary of $W(A)$ is the ellipse (in the (u, v)-plane)

$$\frac{(u - \mu)^2}{k^2 + \lambda^2} + \frac{v^2}{\lambda^2} = 1,$$

which is centered at $\mu \in \mathbb{R}$ and has foci $\mu \pm k$. For every $a \in [0, 1/2)$, the boundary of $W(T(a, A))$ is also an ellipse given by

$$\frac{(u-\mu)^2}{k^2+\lambda^2} + \frac{v^2}{\lambda^2(1-2a)^2} = 1$$

having center μ and foci $\mu \pm \sqrt{k^2 + 4\lambda^2 a(1-a)}$ (see the proof of Theorem 4.3). As in the proof of Theorem 4.5, one can see that since $T(a, A) \geq 0$ is irreducible for every $a \in (0, 1/2]$, $\rho(T(a, A))$ must be the right focus $\psi(a, A)$ of $W(T(a, A))$. ■

Example 4.9 Suppose it is known that the matrices whose numerical ranges are the elliptic disks given by Figure 3 are nonnegative and have irreducible Hermitian parts. The first matrix must have cyclic index 2 for it has exactly two maximal elements (see Corollary 3.6); the second must be primitive (that is, some positive power of it is a strictly positive matrix). In either case, the right focus is an eigenvalue (see Theorem 4.2) and thus equals the spectral radius.

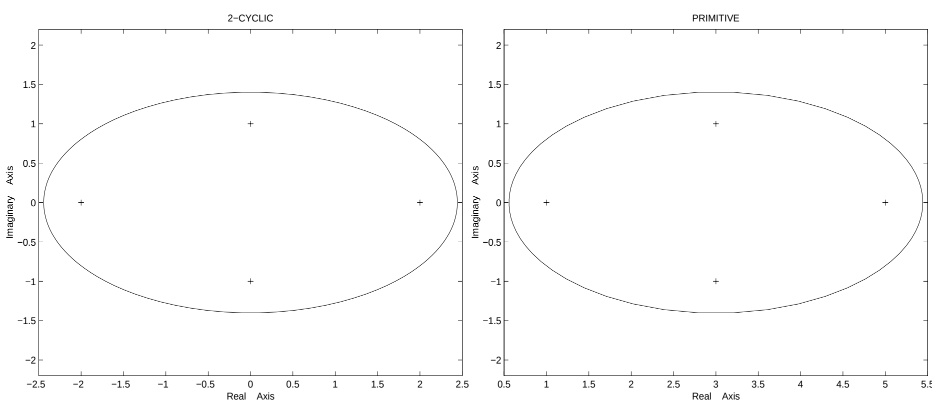

Figure 3: Elliptical disks.

The following result follows from Theorem 4.8 for $k = 0$ and $\lambda = r$.

Corollary 4.10 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ such that A is reducible and $H(A)$ is irreducible. If $W(A)$ is a circular disk centered at $\mu \in \mathbb{R}$ with radius r, then $\rho(A) = \mu$ and $\rho(T(a, A)) = \mu + 2r\sqrt{a(1-a)}$ for every $a \in [0, 1]$.*

Note that if $W(A)$ is a circular disk as in the above corollary, then in agreement with [3, Theorem 4.6],

$$\frac{\mathrm{d}^2 \rho(T(a, A))}{\mathrm{d}a^2} = -4r.$$

The next result follows from Theorem 4.8 for $\mu = 0$.

Corollary 4.11 *Let $A \in \mathcal{M}_n(\mathbb{R}), A \geq 0$ such that $H(A)$ is irreducible and $W(A)$ is an elliptic disk with foci $\pm k \in \mathbb{R}$ and minor axis length λ. Then A is 2-cyclic and for every $a \in [0,1]$, $T(a, A)$ has two maximal eigenvalues:*

$$\pm\sqrt{k^2 + 4\lambda^2 a(1-a)}.$$

From our results so far arises the need of identifying numerical ranges that are elliptic or circular. To contribute in this direction, we conclude by obtaining a generalization of a result attributed to J. Anderson (see [16, Lemma 6]).

Theorem 4.12 *Let $A \in \mathcal{M}_n(\mathbb{C})$ and suppose $W(A)$ is included in an elliptic disk. If $W(A)$ meets the boundary of the disk at $n+1$ or more points, then $W(A)$ coincides with the elliptic disk.*

Proof. Suppose that $W(A) \subseteq \mathcal{D}$, where $\mathcal{D}$ is an elliptic disk, and that $W(A)$ meets the boundary $\partial\mathcal{D}$ at $n+1$ or more points. The shape of $W(A)$ does not change if A is shifted by a multiple of the identity matrix or rotated via scalar multiplication. Thus, without loss of generality, we can assume that $\partial\mathcal{D}$ has equation (in the (u, v)-plane)

$$\frac{u^2}{k^2} + \frac{v^2}{\lambda^2} = 1 \quad (k > \lambda > 0).$$

As in the proof of Theorem 4.3, for $a = (\lambda - k)/(2\lambda)$, $W(T(a, A))$ is contained in the circular disk $\mathcal{D}_a$ whose boundary $\partial\mathcal{D}_a$ has equation

$$u^2 + v^2 = k^2.$$

Moreover, $W(T(a, A))$ meets $\partial\mathcal{D}_a$ at $n + 1$ or more points. Hence, by [16, Lemma 6], $W(T(a, A))$ coincides with the disk $\mathcal{D}_a$; applying the inverse affine transformation to $T(a, A)$ and $\mathcal{D}_a$, we obtain that $W(A)$ coincides with the elliptic disk $\mathcal{D}$ as claimed. ∎

Remark In our experience, many times a computer generated numerical range appears to be circular or elliptic, but a confirmation is lacking. In the case of a circular disk, the results of [14] are applicable. In the general case, and in view of Theorem 4.12, one can use the following tactic to obtain such a confirmation:

(i) Compute $n + 1$ boundary points of $W(A)$; this can be achieved by applying the basic step of the algorithm of [8] for $n + 1$ angles in $[0, \pi]$.

(ii) Using an interpolation method, confirm that these $n+1$ points belong to an ellipse/circle.

Acknowledgment The reduction argument to matrices with irreducible Hermitian parts preceding Proposition 3.8 was pointed out by a referee.

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