The Electronic Journal of Linear Algebra.
A publication of the International Linear Algebra Society.
Volume 7, pp. 182-190, December 2000.
ISSN 1081-3810. http://math.technion.ac.il/iic/ela

ELA

SEPARABLE CHARACTERISTIC POLYNOMIALS OF PENCILS AND PROPERTY L*

JOHN MAROULAS†, PANAYIOTIS J. PSARRAKOS‡, AND MICHAEL J. TSATSOMEROS‡

Abstract. The condition (SC): $\det(I-sA-tB) = \det(I-sA)\det(I-tB)$ for all scalars s, t, has naturally and long been connected to eigenvalue properties of the matrix pair A, B. In particular, Taussky used the notion of property L to generalize the Craig-Sakamoto Theorem by showing that when A and B are normal, (SC) is equivalent to $AB = 0$. The relation of (SC) to the eigenspaces of A, B and $sA + tB$ is examined in order to obtain necessary and/or sufficient conditions in terms of eigenspaces and space decompositions. A general criterion for (SC) based on the spectrum of the $n \times n$ matrix polynomial $\lambda^{2n+1}I - \lambda^{2n}A - B$ is also presented.

Key words. Characteristic polynomial, Property L, Craig-Sakamoto Theorem, Matrix polynomial.

AMS subject classifications. 15A15, 15A18, 15A22

1. Introduction.

Two quadratic forms x^tAx and x^tBx, where x is a vector of normally distributed independent random variables, are independent if and only if $AB = 0$. This is known as the Craig-Sakamoto Theorem and can be equivalently stated as follows. Given $n \times n$ real symmetric matrices A and B,

$$\det(I - sA - tB) = \det(I - sA)\ \det(I - tB) \quad \text{for all} \quad s, t \in \mathbb{R} \quad \textit{if and only if} \quad AB = 0.$$

There are several proofs of this result in the literature, most recently by Olkin [8] and by Li [1]. An extensive bibliography on this theorem and related topics is available from Dumais and Styan [2]. Taussky [9] provided a generalization of the Craig-Sakamoto Theorem with the assumption of symmetry replaced by normality of A and B and the field of scalars extended to $\mathbb{C}$. The key idea in [9] was to connect the *separable characteristic polynomial* condition (cf. Theorem 2.1), namely,

(SC) $$\det(I - sA - tB) = \det(I - sA)\ \det(I - tB) \quad \text{for all} \quad s, t \in \mathbb{C},$$

with pairs of matrices having property L. Recall that according to Motzkin and Taussky [7] (see also [5, p. 96]), if A has eigenvalues $\mu_1 \ldots, \mu_n$ and B has eigenvalues $\nu_1, \ldots, \nu_n$, the pair of matrices A, B has *property* L (L stands for linear) if for all scalars s, t and for some fixed *pairing* of the μ_i and ν_i, the matrix $sA + tB$ has eigenvalues $s\mu_i + t\nu_i$ $(i = 1, \ldots, n)$. In [9] (SC) is transformed via homogeneous coordinates into an equation about the *characteristic polynomial*

$$\det(\lambda I - sA - tB)$$

*Received by the editors on 15 May 2000. Accepted for publication on 17 December 2000. Handling Editor: Daniel Hershkowitz.

†Department of Mathematics, National Technical University of Athens, Zografou Campus, Athens 15780, Greece (maroulas@math.ntua.gr).

‡Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2 (panos@math.uregina.ca, tsat@math.uregina.ca). The work of the third author was supported by an NSERC research grant.

of the *pencil* $sA+tB$. It is then shown that (SC) implies that the pair A, B has property L with an additional requirement: either $\mu_i=0$ or $\nu_i=0$ in the said pairing of the eigenvalues of A and B.

In passing we mention that property L is a specialization of *property P*, namely that every polynomial function $f(A,B)$ has eigenvalues $f(\mu_i,\nu_i)$ for some fixed pairing of the eigenvalues μ_i of A and ν_i of B. It is known that property P is equivalent to simultaneous triangularizability of A and B [6]. Property P implies property L, but the converse is not true [7].

Our goal is to find necessary, sufficient or equivalent conditions to (SC), under assumptions on A and B other than normality or no assumptions at all. This is done in Section 2 by considering the implications of (SC) with regard to the eigenspaces of A and B. In particular, when 0 is a semisimple eigenvalue of A and B, we show that (SC) implies (and in many cases is equivalent to) a pairing of the nullspaces of A and B so as to form a decomposition of $\mathbb{C}^n$. Also, as $AB=0$ is clearly sufficient for (SC) to hold, we investigate under what conditions, other than normality, $AB=0$ is necessary for (SC). In Section 3, we relate (SC) to an appropriately defined matrix polynomial and to a factorization of its characteristic polynomial, resulting in a new practical criterion for (SC).

We comment that as in (SC) the roles of A and B are interchangeable, it should be clear that certain results in the sequel hold with the roles of A and B swapped, even if this is not explicitly mentioned. Also notice that (SC) holds if and only if it holds for A^t and B^t, resulting in some straightforward interpretations of the results to follow.

Some notational conventions are now in order. For $X\in\mathcal{M}_n(\mathbb{C})$, denote its spectrum by $\sigma(X)$, viewed as a multiset with the eigenvalues of X repeated according to their multiplicities. The spectral radius of X is denoted by $\rho(X)$, its image by $\mathrm{Im}(X)$, its null space by $\mathrm{Nul}(X)$, and its generalized eigenspace corresponding to an eigenvalue λ by

$$E_X(\lambda)=\mathrm{Nul}\left((X-\lambda I)^m\right),$$

where $m=\mathrm{ind}_\lambda(X)$ is the size of the largest Jordan block associated with λ in the Jordan canonical form of X. (By convention, $\mathrm{ind}_\lambda(X)=0$ means $\lambda\notin\sigma(X)$.) Recall that λ is a semisimple eigenvalue of X if $\mathrm{ind}_\lambda(X)=1$, in which case the geometric multiplicity $\mathrm{dimNul}(A)$ equals to the algebraic multiplicity $\mathrm{dim}E_X(\lambda)$. Also, $\mathbb{C}^n$ always admits the direct sum decomposition

$$\mathbb{C}^n=\bigoplus_{\lambda\in\sigma(X)}E_X(\lambda).$$

2. Condition (SC) and the eigenspaces of A and B.

The following fundamental characterization of matrices that satisfy (SC) is essentially found within the proof of the main result in [9].

THEOREM 2.1. *Let $A,B\in\mathcal{M}_n(\mathbb{C})$. The following are equivalent.*

(i) *Condition* (SC) *holds.*

(ii) *For every $s,t\in\mathbb{C}$, $\sigma(sA\oplus tB)=\sigma\left((sA+tB)\oplus O_n\right)$, where O_n denotes the zero*

matrix in $\mathcal{M}_n(\mathbb{C})$.
(iii) *The pair* A, B *has property L and for the associated pairing of the eigenvalues* μ_i *and* ν_i *of* A *and* B, *respectively, either* $\mu_i = 0$ *or* $\nu_i = 0$.

Proof. Substituting s/λ for s and t/λ for t in (SC), we obtain the following expression that is equivalent to (SC) :

$$\lambda^n \det(\lambda I - sA - tB) = \det(\lambda I - sA)\det(\lambda I - tB). \tag{1}$$

Consequently, (SC) is equivalent to each eigenvalue of $sA + tB$ being either of the form $s\mu$ with $\mu \in \sigma(A)$, or of the form $t\nu$ with $\nu \in \sigma(B)$, for every s and t in $\mathbb{C}$. Also the polynomial in λ in the left-hand side of (1) has n additional zero roots, proving the implication (i) $\Rightarrow$ (ii). The converse is straightforward as (ii) clearly implies (1). The equivalence of (ii) and (iii) follows readily from the definition of property L. □

COROLLARY 2.2. *Let* $A, B \in \mathcal{M}_n(\mathbb{C})$ *satisfy* (SC). *Let* k_1 *and* k_2 *be the algebraic multiplicities of the eigenvalue* 0 *of* A *and* B, *respectively. Then the following hold.*
(i) $k_1 + k_2 \geq n$.
(ii) *If* A *is nonsingular, then* B *must be nilpotent.*
(iii) *If* $\text{ind}_0(A) \leq 1$ *and* $\text{ind}_0(B) \leq 1$, *then* $\text{rank}(A) + \text{rank}(B) \leq n$.

Proof. (i) According to part (ii) of Theorem 2.1, $\sigma(sA \oplus tB)$, viewed as a multiset, contains at least n elements equal to 0, and equals $\sigma(sA) \cup \sigma(tB)$.
(ii) From (i) we have that if $k_1 = 0$, then $k_2 = n$.
(iii) By the index assumptions, $k_1 + \text{rank}(A) = n$ and $k_2 + \text{rank}(B) = n$. Thus from (i) it follows that $\text{rank}(A) + \text{rank}(B) \leq n$. □

Assuming that 0 is a semisimple eigenvalue of A and B, we can now show that (SC) implies a pairing of the nullspaces of A and B analogous to the pairing of the eigenvalues of matrices that satisfy (SC); cf. Theorem 2.1 (iii).

THEOREM 2.3. *Let* $A, B \in \mathcal{M}_n(\mathbb{C})$ *with* $\text{ind}_0(A) \leq 1$ *and* $\text{ind}_0(B) \leq 1$ *and suppose that* (SC) *holds. Then* $\mathbb{C}^n = \text{Nul}(A) + \text{Nul}(B)$.

Proof. First assume that $\text{Nul}(A) \cap \text{Nul}(B) = \{0\}$. Since $\text{ind}_0(A) \leq 1$, $\dim\text{Nul}(A)$ equals the algebraic multiplicity of 0. A similar assertion is true for B. So by Corollary 2.2, $\dim\text{Nul}(A) + \dim\text{Nul}(B) \geq n$. However, by assumption, $\text{Nul}(A) \cap \text{Nul}(B) = \{0\}$, and thus $\mathbb{C}^n = \text{Nul}(A) \oplus \text{Nul}(B)$.
Now if $\text{Nul}(A) \cap \text{Nul}(B) \neq \{0\}$, let $\{u_1, \ldots, u_p\}$ be a basis for $\text{Nul}(A) \cap \text{Nul}(B)$ and complete it to a basis $\{u_1, \ldots, u_n\}$ of $\mathbb{C}^n$. Define $S = (u_1 \mid \ldots \mid u_n) \in \mathcal{M}_n(\mathbb{C})$ and observe that since $\text{ind}_0(A) \leq 1$ and $\text{ind}_0(B) \leq 1$, we have

$$S^{-1}AS = O_p \oplus A_1 \quad \text{and} \quad S^{-1}BS = O_p \oplus B_1. \tag{2}$$

As (SC) is invariant under common similarity transformations of A and B, we may without loss of generality assume that A and B are equal to the right-hand sides in (2), respectively. Then (SC) holds if and only if

$$\det(I - sA_1 - tB_1) = \det(I - sA_1)\det(I - tB_1) \quad \text{for all} \quad s, t \in \mathbb{C}.$$

By construction, $\text{Nul}(A_1) \cap \text{Nul}(B_1) = \{0\}$. By the first part of the proof applied to A_1 and B_1,

$$\mathbb{C}^{n-p} = \text{Nul}(A_1) \oplus \text{Nul}(B_1)$$

and by (2), the proof is complete. □

The immediate question is whether the necessary condition for (SC) in the above theorem is sufficient. As the following example shows, this is not the case in general.

EXAMPLE 2.4. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} .5 & 0 & -.5 \\ -1 & 2 & -1 \\ -.5 & 0 & .5 \end{bmatrix}.$$

These matrices are diagonalizable with spectra $\{0,0,1\}$ and $\{0,1,2\}$, respectively, and $\mathbb{C}^n = \mathrm{Nul}(A) \oplus \mathrm{Nul}(B)$. However, $A+B$ has spectrum $\{2, 1.7071, 0.2929\}$ and thus the pair A, B does not have property L; that is, by Theorem 2.1, they do not satisfy (SC).

Nevertheless, the condition $\mathbb{C}^n = \mathrm{Nul}(A) + \mathrm{Nul}(B)$ is necessary and sufficient for (SC) under special assumptions.

PROPOSITION 2.5. *Let* $A, B \in \mathcal{M}_n(\mathbb{C})$ *and suppose that* $\mathrm{ind}_0(A) \leq 1$, $\mathrm{ind}_0(B) \leq 1$ *and that* $AB = BA$. *Then* (SC) *holds if and only if* $\mathbb{C}^n = \mathrm{Nul}(A) + \mathrm{Nul}(B)$.

Proof. By Theorem 2.3, if (SC) holds, then $\mathbb{C}^n = \mathrm{Nul}(A) + \mathrm{Nul}(B)$. For the converse, suppose $\mathbb{C}^n = \mathrm{Nul}(A) + \mathrm{Nul}(B)$ and let $x \neq 0$, $s \neq 0$ and $t \neq 0$ such that

$$(sA + tB)x = \lambda x. \tag{3}$$

Write $x = x_1 + x_2$, where $x_1 \in \mathrm{Nul}(A)$, $x_2 \in \mathrm{Nul}(B)$. Then, as $AB = BA$, multiplying (3) by A we obtain

$$sA^2 x_2 = \lambda A x_2.$$

If $Ax_2 \neq 0$, we have that $\lambda/s \in \sigma(A)$. Otherwise, $tBx_1 = \lambda(x_1 + x_2)$ or $tB^2 x_1 = \lambda B x_1$. In the latter case, if $Bx_1 \neq 0$, then $\lambda/t \in \sigma(B)$; if $Bx_1 = 0$, then $\lambda = 0$. That is, in all cases every eigenvalue of $sA + tB$ is of the form prescribed in Theorem 2.1 (iii) and thus (SC) holds. □

PROPOSITION 2.6. *Let* $A, B \in \mathcal{M}_n(\mathbb{C})$ *and suppose that* $\mathrm{ind}_0(A) \leq 1$ *and* $\mathrm{ind}_0(B) \leq 1$ *and that* $B\,\mathrm{Nul}(A) \subseteq \mathrm{Nul}(A)$. *Then the following are equivalent.*

(i) *Condition* (SC) *holds.*

(ii) $\mathbb{C}^n = \mathrm{Nul}(A) + \mathrm{Nul}(B)$.

(iii) $AB = 0$.

Proof. (i) ⇒ (ii) This proof proceeds as the second part of the proof of Proposition 2.5, except for using the assumption that $B\,\mathrm{Nul}(A) \subseteq \mathrm{Nul}(A)$ instead of commutativity at the appropriate step.

(ii) ⇒ (iii) For every $x = x_1 + x_2 \in \mathbb{C}^n$, where $x_1 \in \mathrm{Nul}(A), x_2 \in \mathrm{Nul}(B)$, we have

$$ABx = AB(x_1 + x_2) = ABx_1 = 0$$

since $Bx_1 \in \mathrm{Nul}(A)$. Thus $AB = 0$.

(iii) ⇒ (i) This implication holds in general and is a consequence of determinantal properties. □

The above result provides an instance other than normality of A and B in which (SC) is equivalent to $AB = 0$. Moreover, if (SC) holds we can state the following condition equivalent to $AB = 0$. (Recall that it is generally true that $AB = 0$ if and only if $\mathrm{Im}(B) \subseteq \mathrm{Nul}(A)$.)

COROLLARY 2.7. *Let* $A, B \in \mathcal{M}_n(\mathbb{C})$ *satisfy* (SC) *and assume that* $\mathrm{ind}_0(A) \le 1$ *and* $\mathrm{ind}_0(B) \le 1$. *Then* $AB = 0$ *if and only if* $B\,\mathrm{Nul}(A) \subseteq \mathrm{Nul}(A)$.

Proof. If $AB = 0$, then clearly $B\,\mathrm{Nul}(A) \subseteq \mathrm{Nul}(A)$. The converse follows from the equivalence of (i) and (iii) of Proposition 2.6. □

EXAMPLE 2.8. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

These matrices satisfy $\mathrm{ind}_0(A) = \mathrm{ind}_0(B) = 1$, $\mathbb{C}^n = \mathrm{Nul}(A) + \mathrm{Nul}(B)$ and (SC). However, $A\,\mathrm{Nul}(B) \not\subseteq \mathrm{Nul}(B)$, $B\,\mathrm{Nul}(A) \not\subseteq \mathrm{Nul}(A)$ and $AB \neq BA$. That is, the assumptions implicated in Propositions 2.5 and 2.6 are sufficient but not necessary for (SC).

The pairing of the eigenvalues and nullspaces of A and B imposed by (SC) suggests a relation among the eigenspaces of A, B and $sA + tB$. Indeed, as we see next, any eigenvectors of B belonging to $\bigoplus_{\mu \in \sigma(A)\setminus\{0\}} E_{sA+tB}(s\mu)$ are necessarily null vectors of B.

THEOREM 2.9. *Let* $A, B \in \mathcal{M}_n(\mathbb{C})$ *satisfy* (SC). *If for some* $\nu \in \mathbb{C}$

$$\bigoplus_{\mu \in \sigma(A)\setminus\{0\}} E_{sA+tB}(s\mu) \ \cap \ \mathrm{Nul}(B - \nu I) \ \neq \ \{0\},$$

then $\nu = 0$.

Proof. By Theorem 2.1, for every $\mu \in \sigma(A) \setminus \{0\}$ and every scalar s, $s\mu \in \sigma(sA + tB)$. Consider then $sA + tB$ with $s \neq 0$ fixed and view

$$W(t) := \bigoplus_{\mu \in \sigma(A)\setminus\{0\}} E_{sA+tB}(s\mu)$$

as a subspace of $\mathbb{C}^n$ that depends on t. Clearly, $W(t)$ is a non-trivial $(sA + tB)$-invariant subspace and thus we may consider the restriction of $sA + tB$ to $W(t)$. Combining Lemma 2.1.4 and Theorem 1.8.3 in [4], we have that the restriction of $sA + tB$ to $W(t)$ has spectrum

$$\sigma((sA + tB)\,|_{W(t)}) = \{s\mu : \mu \in \sigma(A) \setminus \{0\}\}.$$

Suppose that for some $\hat{t} \in \mathbb{C}$, there exists nonzero $x \in W(\hat{t})$ such that $Bx = \nu x$. Then $(sA + \hat{t}B)x \in W(\hat{t})$ and thus $sAx \in W(\hat{t})$. It follows that $sAx + t\nu x = (sA + tB)x \in W(\hat{t})$ for all t. Let now $t \to \infty$ in the expression

$$\left(\frac{s}{t}A + B\right) x \in W(\hat{t})\ .$$

The eigenvalues of $(\frac{s}{t}A + B)\,|_{W(\hat{t})}$ are of the form $\frac{s}{t}\mu$, where $\mu \in \sigma(A) \setminus \{0\}$ and thus they are converging to zero. On the other hand, $(\frac{s}{t}A + B)x$ converges to $Bx = \nu x$, i.e., $(\frac{s}{t}A+B)\,|_{W(\hat{t})}$ has an eigenvalue arbitrarily close to ν for t large enough. This implies that $\nu = 0$, proving the claim that if such an eigenvector x of B exists, it must be a null vector of B. □

We continue with some illustrative examples.

EXAMPLE 2.10. Let

$$A = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

so that

$$sA + tB = \begin{bmatrix} t & -(s+t) \\ 0 & -s \end{bmatrix}.$$

Thus the pair A, B has property L with its eigenvalues being $s\,(-1) + t\,0$ and $s\,0 + t\,1$. It follows that (SC) holds. As the conditions of Theorem 2.3 are satisfied, we in fact have $\mathbb{C}^n = \mathrm{Nul}(A) \oplus \mathrm{Nul}(B)$. Also $AB = 0$ and $B\mathrm{Nul}(A) \subseteq \mathrm{Nul}(A)$; cf. Proposition 2.6 and Corollary 2.7.

EXAMPLE 2.11. This example is due to Wielandt [10]. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then $sA + tB$ is nilpotent, as are A and B. It follows that the pair A, B has property L and satisfies (SC) (with the pairing of the eigenvalues being immaterial); however, $AB \neq 0$. This does not contradict our results as $\mathrm{ind}_0(A) = \mathrm{ind}_0(B) = 3$.

REMARK 2.12. Let $A, B \in \mathcal{M}_n(\mathbb{C})$, $s \in (0, 1/\rho(A))$ and consider $H_s = (I - sA)^{-1}B$. Notice that $AB = 0$ if and only if $H_s = B$. Thus, under the assumption that (SC) holds, it is not in general true that $H_s = B$ (see Example 2.8, where $AB \neq 0$). Nevertheless, if (SC) holds, then (i) $\sigma(B) = \sigma(H_s)$ and (ii) $\mathrm{Im}(H)$ is the minimal A-invariant subspace over $\mathrm{Im}(B)$; see [4, Section 2.8].

Proof. Observe that since $s \in (0, \frac{1}{\rho(A)})$, $(I - sA)^{-1}$ exists and has the series expansion

$$(I - sA)^{-1} = \sum_{j=0}^{\infty} s^j A^j. \tag{4}$$

We then have

$$I - sA - tB = (I - sA)(I - t(I - sA)^{-1}B)$$

and thus by (SC),

$$\det(I - t(I - sA)^{-1}B) = \det(I - tB).$$

Letting $\lambda = 1/t$, we obtain

$$\det(\lambda I - (I - sA)^{-1}B) = \det(\lambda I - B)$$

or equivalently, $\sigma((I-sA)^{-1}B)=\sigma(B)$. Also

$$\mathrm{Im}((I-sA)^{-1}B)=\mathrm{Im}(\sum_{j=0}^{\infty} s^j A^j B),$$

which coincides with the minimal A-invariant subspace over $\mathrm{Im}(B)$. □

REMARK 2.13. We consider a variant of (SC) for the case where 0 is an eigenvalue of A or B but is not necessarily semisimple. When $p \geq \mathrm{ind}_0(A)$ and $q \geq \mathrm{ind}_0(B)$, 0 is an eigenvalue of A^p and B^q of index at most 1. Thus we may consider

(SC') $\det(I-sA^p-tB^q)=\det(I-sA^p)\,\det(I-tB^q)$ for all $s,t\in\mathbb{C}$.

From our results (cross-referenced in parentheses) we readily deduce the following three results.

(i) (SC') $\Longrightarrow$ $\mathbb{C}^n=E_A(0)+E_B(0)$ (cf. Theorem 2.3).

(ii) If $B\,E_A(0)\subseteq E_A(0)$, then (cf. Proposition 2.6)

$$\text{(SC')} \iff \mathbb{C}^n=E_A(0)+E_B(0) \iff A^pB^q=0.$$

(iii) If (SC') holds, then $A^pB^q=0 \iff B\,E_A(0)\subseteq E_A(0)$ (cf. Corollary 2.7).

3. A criterion for (SC)**.** The only necessary and sufficient conditions for (SC) so far are those of Theorem 2.1, which involve quantifiers for the parameters s and t. In this section we strive for a parameter-free method to determine whether two arbitrary matrices A and B satisfy (SC) or not. Since (SC) induces a polynomial $p(s,t)$ of degree at most $2n$, there should exist a finite number of (at most $2(n+1)$) zeros (s,t), whose existence theoretically suffices to imply that $p(s,t)$ is identically zero. In general, however, such a polynomial has an infinite number of zeros and so one might have to consider a randomized test to check whether (SC) holds or not. Alternatively, we can devise a parameter-free test based solely on the matrices A and B as follows.

First, we relate (SC) to the eigenvalue problem for the matrix polynomial

$$P(\lambda)=\lambda^{2n+1}I-\lambda^{2n}A-B,\quad A,B\in\mathcal{M}_n(\mathbb{C}).$$

Recall that the spectrum of $P(\lambda)$ is defined as the set

$$\sigma(P(\lambda))=\{\lambda\in\mathbb{C}:\det(P(\lambda))=0\}.$$

We need the following auxiliary result.

LEMMA 3.1. *Consider a polynomial in two variables $p(s,t)\in\mathbb{C}[s,t]$ of total degree k, and let $m\geq k$ be an integer. Then $p(s,t)$ is the zero polynomial in $\mathbb{C}[s,t]$ if and only if $p(s,s^{m+1})=0$ for every $s\in\mathbb{C}$.*

Proof. Let

$$p(s,t)=\sum_{\alpha=0}^{k}\sum_{\beta=0}^{k-\alpha} c_{\alpha,\beta}\, s^{\alpha}\, t^{\beta}$$

so that

$$p(s, s^{m+1}) = \sum_{\alpha=0}^{k} \sum_{\beta=0}^{k-\alpha} c_{\alpha,\beta}\ s^{\alpha+(m+1)\beta}.$$

Note that in the expression for $p(s, s^{m+1})$ above there are no repeated exponents of s and thus if $p(s, s^{m+1})$ is identically zero, all coefficients $c_{\alpha,\beta} = 0$; that is, $p(s,t)$ is also identically zero. The converse is obvious. □

THEOREM 3.2. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$. Then the following are equivalent.*
(i) *Condition* (SC) *holds.*
(ii) $\det(I - sA - s^{2n+1}B) = \det(I - sA)\ \det(I - s^{2n+1}B)$ *for all* $s \in \mathbb{C}$.
(iii) $\sigma(K) \setminus \{0\} = \left[\sigma(A) \cup \{\mu : \mu^{2n+1} \in \sigma(B)\}\right] \setminus \{0\}$, *where*

$$K = \begin{bmatrix} 0_n & I_n & 0_n & \dots & \dots & 0_n \\ 0_n & 0_n & I_n & 0_n & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0_n \\ 0_n & 0_n & 0_n & \dots & 0_n & I_n \\ B & 0_n & 0_n & \dots & 0_n & A \end{bmatrix} \in \mathcal{M}_{n(2n+1)}(\mathbb{C}).$$

Proof. (i) $\Longleftrightarrow$ (ii) This equivalence follows from Lemma 3.1 applied to

$$p(s,t) = \det(I - sA - tB) - \det(I - sA)\det(I - tB),$$

which is a polynomial in s, t of total degree $k \le 2n$.
(ii) $\Longleftrightarrow$ (iii) With the transformation $s = 1/\lambda$ we obtain that the condition in (ii) is equivalent to

$$\det(I - \lambda^{-1}A - \lambda^{-(2n+1)}B) = \det(I - \lambda^{-1}A)\ \det(I - \lambda^{-(2n+1)}B),$$

which, in turn, is equivalent to

$$\lambda^n\ \det(I\lambda^{2n+1} - \lambda^{2n}A - B) = \det(\lambda I - A)\ \det(\lambda^{2n+1}I - B).$$

Since eigenvalues are counted according to their multiplicities, the above equality is equivalent to

$$\sigma(\lambda^{2n+1}I - \lambda^{2n}A - B) \setminus \{0\} = \left[\sigma(A) \cup \{\mu : \mu^{2n+1} \in \sigma(B)\}\right] \setminus \{0\}.$$

Notice that the number of the nonzero elements of $\sigma(A) \cup \{\mu : \mu^{2n+1} \in \sigma(B)\}$ ranges between n (when A is nonsingular) and $n(2n+1)$ (when B is nonsingular).
On the other hand, the eigenvalues of the matrix polynomial $\lambda^{2n+1}I - \lambda^{2n}A - B$ coincide with the spectrum of the block companion matrix K as given above (see [3, p. 4]), completing the proof of the theorem. □

The above theorem gives us a straightforward way to check whether (SC) holds or not: Compute the eigenvalues of A, B and of the block companion matrix K, and check whether Theorem (3.2) (iii) holds or not. Referring to the matrices A and B of Example 2.8, we compute the nonzero eigenvalues of the corresponding matrix K to be 2 and the ninth roots of 1. It follows that the pair A, B satisfies (SC) as expected.

REFERENCES

[1] C.-K. Li. A simple proof of the Craig-Sakamoto Theorem. *Linear Algebra and Its Applications*, 321:281–283, 2000.

[2] M. Dumais and G. P. H. Styan. A bibliography on the distribution of quadratic forms in normal variables, with special emphasis on the Craig-Sakamoto theorem and on Cochran's theorem. In George Styan, ed., *Three Bibliographies and a Guide*, Seventh International Workshop on Matrices and Statistics, Fort Lauderdale, 1998, pp. 1–9.

[3] I. Gohberg, P. Lancaster, L. Rodman. Matrix Polynomials, Academic Press, New York, 1982.

[4] I. Gohberg, P. Lancaster, L. Rodman. Invariant subspaces of matrices with applications, Wiley, New York, 1986.

[5] T. Kato. A Short Introduction to Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1982.

[6] N. H. McCoy. On the characteristic roots of matric polynomials. *Bulletin of the American Mathematical Society*, 42:592–600, 1936.

[7] T. S. Motzkin and O. Taussky. Pairs of matrices with property L. *Transactions of the American Mathematical Society*, 73:108–114, 1952.

[8] I. Olkin. A determinantal proof of the Craig-Sakamoto Theorem. *Linear Algebra and Its Applications*, 264:217–223, 1997.

[9] O. Taussky. On a matrix theorem of A.T. Craig and H. Hotelling. *Indigationes Mathematicae*, 20:139–141, 1958.

[10] H. Wielandt. Lineare scharen von matrizen mit reellen eigenwerten. *Mathematische Zeitschrift*, 53:219–225, 1950.