On the Location of the Spectrum of Hypertournament Matrices

S. Kirkland*† P. J. Psarrakos*‡ M. J. Tsatsomeros*§

April 16, 2001

Abstract

Inclusion regions for the spectrum of a hypertournament matrix A are obtained, based on a complex curve that relates the real and imaginary parts of the eigenvalues. These results generalize and in certain cases improve work of S. Kirkland. The bounds obtained depend on the variance of the score vector; their tightness is investigated using the notion of numerical range.

Keywords : hypertournament matrix, spectrum, numerical range.
AMS subject classifications : 15A18, 15A48, 15A60, 05C20.
Abbreviated title : Hypertournament Matrices.

*Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada.
†Work supported by an NSERC grant. E-mail: kirkland@math.uregina.ca
‡E-mail: panos@math.uregina.ca
§Work supported by an NSERC grant. E-mail: tsat@math.uregina.ca

1 Introduction

Both a tournament matrix and its corresponding directed graph arise as a record of the outcomes of a round robin competition. The need and desire to come up with player ranking schemes has motivated an extensive study of the combinatorial and spectral properties of tournament matrices and their generalizations (see [2, 3, 10, 11, 12]). Hypertournament and generalized tournament matrices not only provide a means for inquiring into the properties of tournament matrices but also are the source of matrix analytic challenges of independent interest.

We proceed with some basic definitions and notation needed to describe our results. Let $\mathcal{M}_n(\mathbb{R})$ be the algebra of all $n \times n$ real matrices. Matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called an *h-hypertournament* if it has zero diagonal entries and $A + A^t = hh^t - I$ for some nonzero $h \in \mathbb{R}^n$. When $h = \mathbf{1}$, the all ones vector, an h-hypertournament matrix A satisfies $A + A^t = J - I$, where J denotes the all ones matrix. If all the entries of a $\mathbf{1}$-hypertournament matrix $A \in \mathcal{M}_n(\mathbb{R})$ are in $\{0, 1\}$, then A is called a *tournament matrix*, and if all the entries of A are nonnegative, then A is called a *generalized tournament matrix.*

In [10], Maybee and Pullman show that every h-hypertournament matrix is (diagonally) similar to a $\mathbf{1}$-hypertournament matrix. Thus, the discussion of the spectral properties of an h-hypertournament matrix can be reduced to the case of $\mathbf{1}$-hypertournament matrices. It is further shown in [10] that $-1/2 \le \mathrm{Re}\lambda \le (n-1)/2$ whenever λ is an eigenvalue of an h-hypertournament matrix. Moreover, the eigenvalues of a generalized tournament matrix satisfy $|\mathrm{Im}\lambda| \le (1/2)\cot(\pi/(2n))$ (see [4]).

For the purposes of our work, we introduce the quantity

$$\mathrm{v}(A) \;=\; \frac{1}{n}\left\| s - \left(\frac{n-1}{2}\right)\mathbf{1}\right\|^2,$$

associated with the *score vector* $s = A\mathbf{1}$ of an $n \times n$ $\mathbf{1}$-hypertournament matrix A. Notice the interpretation of $\mathrm{v}(A)$ as the variance of the score vector. Thus we refer to $\mathrm{v}(A)$ as the *score variance* of A. Moreover, it can be verified that

$$\mathrm{v}(A) \;=\; \frac{s^t s}{n} - \frac{(n-1)^2}{4}\,.$$

If A is an $n \times n$ generalized tournament matrix, then $0 \le \mathrm{v}(A) \le (n^2-1)/12$. The score variance $\mathrm{v}(A)$ is zero when A is a tournament matrix and in each row of A the number of off-diagonal zeros is equal to the number of ones. Also $\mathrm{v}(A) = (n^2-1)/12$ when A is triangular.

In this paper, we continue the work in [7] by providing inclusion regions for the spectra of $\mathbf{1}$-hypertournament matrices. These inclusion regions are described by a curve relating the real part and the imaginary part of an eigenvalue to each other and to the score variance. The bounds we obtain imply bounds in [7] on the real parts of the eigenvalues of a **1**-hypertournament matrix, and also give information on the imaginary parts as well. Our approach relies on Schur's Lemma and basic facts about the *numerical range* (also known as the *field of values*) of a matrix A,

$$F(A) \;=\; \{v^*Av \in \mathbb{C} : v \in \mathbb{C}^n \text{ with } v^*v = 1\}.$$

Recall that a matrix $A \in \mathcal{M}_n(\mathbb{R})$ with nonnegative entries is called *primitive* if there is a positive integer k such that all the entries of A^k are positive. Furthermore, by the Perron-Frobenius Theorem, a primitive entrywise nonnegative matrix A has a (simple) real positive eigenvalue ρ such that $\rho > |\lambda|$ for all eigenvalues λ of A, [1]. The eigenvalue ρ is known as the *Perron value* of A, and the corresponding eigenvector, called the *Perron vector*, can be taken to have all positive entries.

An $n \times n$ tournament matrix A corresponds to a round robin competition involving n players, with $a_{ij} = 1$ if player i defeats player j, and $a_{ij} = 0$ otherwise (tie games are not allowed). In the case that A is primitive, a ranking scheme of Kendall and Wei (see [11], for example) considers the sequence $(A^k\mathbf{1})/(\mathbf{1}^T A^k \mathbf{1})$ $(k \in \mathbb{N})$; it turns out that this sequence converges to the Perron vector of A, which is then used to rank the players. Further, the rate of convergence is governed by the quantities $|\lambda|/\rho$, where ρ is the Perron value of A and λ is a non-Perron eigenvalue of A. Section 3 applies some of the results of Section 2 to the problem of bounding the quantities $|\lambda|/\rho$.

2 1-Hypertournament matrices

Suppose that $A \in \mathcal{M}_n(\mathbb{R})$ is a **1**-hypertournament matrix and let $H = (A + A^t)/2$ and $K = (A - A^t)/2$ be the Hermitian and the skew-Hermitian part of A, respectively. Clearly, $A = H + K$ and $H = (J - I)/2$. The matrix H has exactly two eigenvalues, $\lambda_1 = (n-1)/2$ with multiplicity 1 and $\lambda_2 = -1/2$ with multiplicity $n-1$. The vector $(1/\sqrt{n})\mathbf{1}$ is a unit eigenvector of H corresponding to λ_1. By Schur's Lemma [5, Theorem 2.3.1], there exists a unitary $U \in \mathcal{M}_n(\mathbb{R})$ whose first column is $(1/\sqrt{n})\mathbf{1}$ such that

$$U^tHU \;=\; \mathrm{diag}\{(n-1)/2, -1/2, -1/2, ..., -1/2\}.$$

Moreover, it is easy to see that

$$U^tKU \;=\; \begin{bmatrix} 0 & -u^t \\ u & K_1 \end{bmatrix},$$

where $K_1 \in \mathcal{M}_{n-1}(\mathbb{R})$ is skew-Hermitian and $u \in \mathbb{R}^{n-1}$. Consequently,

$$U^tAU \;=\; \begin{bmatrix} (n-1)/2 & -u^t \\ u & K_1 - (1/2)I \end{bmatrix}. \tag{1}$$

The following theorem is the main result of this section.

Theorem 1 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a **1**-hypertournament matrix with score variance* $\mathrm{v}(A)$*. Let λ be an eigenvalue of A such that $\lambda \neq (n-1)/2$ and $\mathrm{Re}\lambda \neq -1/2$, and let $d = n - 1 - 2\mathrm{Re}\lambda$. Then*

$$(\mathrm{Im}\lambda)^2 \;\leq\; d\left(\frac{\mathrm{v}(A)}{n-d} - \frac{d}{4}\right). \tag{2}$$

Proof Since $A - \lambda I$ is singular, by equation (1), the matrix

$$U^t(A-\lambda I)U \;=\; \begin{bmatrix} (n-1)/2-\lambda & -u^t \\ u & K_1 - (1/2+\lambda)I \end{bmatrix}$$

is also singular. It follows that the Schur complement of the leading entry is singular [5, p. 21], that is, $0 \in \sigma(S)$, the spectrum of

$$S \;=\; K_1 - \left(\frac{1}{2}+\lambda\right)I + \frac{1}{(n-1)/2-\lambda}\,uu^t.$$

The Hermitian and skew-Hermitian parts of S are

$$M \;=\; \frac{2d}{d^2+4(\mathrm{Im}\lambda)^2}\,uu^t - \left(\frac{1}{2}+\mathrm{Re}\lambda\right)I$$

and

$$N \;=\; K_1 - i\,\mathrm{Im}\lambda\, I + \frac{4i\,\mathrm{Im}\lambda}{d^2+4(\mathrm{Im}\lambda)^2}uu^t, \tag{3}$$

respectively. Since $\sigma(S) \subseteq F(S)$ and $F(M) = \mathrm{Re}F(S)$ (see [6, Properties 1.2.5, 1.2.6]), it follows that $0 \in F(M)$, which, in turn, implies

$$\frac{1}{2}+\mathrm{Re}\lambda \;\in\; \frac{2d}{d^2+4(\mathrm{Im}\lambda)^2}F(uu^t).$$

Since $F(uu^t)$ coincides with the interval $[0, u^tu]$,

$$\frac{1}{2}+\mathrm{Re}\lambda \;\leq\; \frac{2d(u^tu)}{d^2+4(\mathrm{Im}\lambda)^2}$$

or equivalently,

$$(\mathrm{Im}\lambda)^2 \;\leq\; d\left(\frac{u^tu}{n-d} - \frac{d}{4}\right).$$

Observe that

$$u^tu \;=\; \left\|\frac{1}{\sqrt{n}}K\mathbf{1}\right\|^2 \;=\; \frac{1}{n}\left\|s - \left(\frac{n-1}{2}\right)\mathbf{1}\right\|^2 \;=\; \mathrm{v}(A)$$

and the proof is complete. ■

For the real parts of the eigenvalues and for any purely imaginary eigenvalues of A, we have the following results.

Corollary 2 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a* ***1****-hypertournament matrix with score variance* $\mathrm{v}(A) < n^2/16$. *Then for every eigenvalue λ of A,*

$$\mathrm{Re}\lambda \;\notin\; \left(\frac{n-2-\sqrt{n^2-16\mathrm{v}(A)}}{4}\,,\;\; \frac{n-2+\sqrt{n^2-16\mathrm{v}(A)}}{4} \right).$$

Proof Suppose that λ is an eigenvalue of A. Since

$$-1/2 \;<\; \frac{n-2-\sqrt{n^2-16\mathrm{v}(A)}}{4} \;<\; \frac{n-1}{2},$$

consider $\mathrm{Re}\lambda \neq -1/2$ and $\lambda \neq (n-1)/2$. Then by (2),

$$d\left(\frac{\mathrm{v}(A)}{n-d} - \frac{d}{4}\right) \;\geq\; 0,$$

where $d = n-1-2\mathrm{Re}\lambda > 0$. Hence,

$$\frac{\mathrm{v}(A)}{1+2\mathrm{Re}\lambda} - \frac{d}{4} \;\geq\; 0,$$

that is,

$$(\mathrm{Re}\lambda)^2 - \left(\frac{n-2}{2}\right)\mathrm{Re}\lambda + \mathrm{v}(A) - \frac{n-1}{4} \;\geq\; 0.$$

Since $\mathrm{v}(A) < n^2/16$, the proof is complete. ■

Remark 1 Corollary 2 is a special case of [7, Theorem 1]. There, under the same assumptions as in Corollary 2, it is shown that A has one real eigenvalue

$$\rho(A) \;\in\; \left[\frac{n-2+\sqrt{n^2-16\mathrm{v}(A)}}{4},\;\; \frac{n-1}{2} \right]$$

and $n-1$ complex eigenvalues with real parts in the interval

$$\left[-\frac{1}{2}\,,\;\; \frac{n-2-\sqrt{n^2-16\mathrm{v}(A)}}{4} \right].$$

Corollary 3 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a* ***1****-hypertournament matrix with score variance* $\mathrm{v}(A) > (n-1)/4$. *Then for any purely imaginary eigenvalue $\lambda = ir$ $(r \in \mathbb{R})$ of A,* $|r| \leq \sqrt{(n-1)[\mathrm{v}(A)-(n-1)/4]}$.

Proof Follows directly from (2) for $\mathrm{Re}\lambda = 0$ and $d = n-1$. ■

Prompted by (2), we define the *shell* of a **1**-hypertournament matrix A to be the curve

$$\Gamma(A) \;=\; \left\{ x+iy \in \mathbb{C} : x, y \in \mathbb{R} \;\text{ and }\; y^2 = d\left(\frac{\mathrm{v}(A)}{n-d} - \frac{d}{4}\right)\right\},$$

where $d = n-1-2x$. This curve is symmetric with respect to the real axis and is asymptotic to the line $\mathrm{Re}z = -1/2$. It is clear that $\Gamma(A)$ depends only on the order n and the score variance $\mathrm{v}(A)$ of the matrix A. Moreover, $\Gamma(A)$ always intersects the real axis at the point $(n-1)/2$. If in addition, $\mathrm{v}(A) < n^2/16$, then $\Gamma(A)$ also intersects the real axis at the points $(n-2 \pm \sqrt{n^2 - 16\mathrm{v}(A)})/4$.

If $\mathrm{v}(A) < n^2/16$, then $\Gamma(A)$ has two branches (one bounded and one unbounded), and if $\mathrm{v}(A) \geq n^2/16$, then $\Gamma(A)$ consists of one unbounded branch. By Theorem 1, the shell $\Gamma(A)$ yields a localization of the spectrum of A specified by (2) (see Example 1 below).

Consider now a **1**-hypertournament matrix $A \in \mathcal{M}_n(\mathbb{R})$ with score variance $\mathrm{v}(A) > 0$, and the function

$$f(t) \;=\; \frac{n\mathrm{v}(A)}{t} - \frac{(n-t)^2}{4} - \mathrm{v}(A)\;;\;\; t \in (0, n].$$

Observe that for $t = 2x+1$,

$$f(2x+1) = (n-1-2x)\left(\frac{\mathrm{v}(A)}{2x+1} - \frac{n-1-2x}{4}\right)\;;\;\; x \in (-1/2,\; (n-1)/2].$$

Moreover, $f(t)$ is decreasing on $(0, n]$ if and only if $t^3 - nt^2 + 2n\mathrm{v}(A) > 0$ on $(0, n]$. The latter inequality holds if and only if it holds at the minimum $t_0 = 2n/3$. Hence, $f(t)$ is decreasing on the interval $(0, n]$ if and only if $\mathrm{v}(A) > 2n^2/27$. In this case the curve $\Gamma(A) \cap \{z \in \mathbb{C} : \mathrm{Im}z \geq 0\}$ is decreasing (see $\Gamma(B)$ in Figure 1).

If $\mathrm{v}(A) < n^2/16$, we are interested only in values of t such that

$$t^2 - nt + 4\mathrm{v}(A) \;\geq\; 0$$

(these values correspond to $-1/2 \leq x \leq (n-2-\sqrt{n^2-16\mathrm{v}(A)})/4$). But then we have $t^3 - nt^2 \geq -4n\mathrm{v}(A)$, and since $t < n/2$ for $x \leq (n-2-\sqrt{n^2-16\mathrm{v}(A)})/4$,

$$t^3 - nt^2 + 2n\mathrm{v}(A) \;\geq\; 2\mathrm{v}(A)(n-2t) > 0.$$

Consequently, if $\mathrm{v}(A) \in (0, n^2/16)$, then the curve

$$\Gamma(A) \;\cap\; \{z \in \mathbb{C} : \mathrm{Im}z \geq 0,\; \mathrm{Re}z \leq (n-2-\sqrt{n^2-16\mathrm{v}(A)})/4\}$$

is decreasing (see the unbounded branch of $\Gamma(A)$ in Figure 1).

If $\mathrm{v}(A) \in [n^2/16, 2n^2/27]$, then one can see that the curve $\Gamma(A) \cap \{z \in \mathbb{C} : \mathrm{Im}z \geq 0\}$ is decreasing, then increasing, then decreasing again, (Figure 2).

Example 1 A and B are two 8×8 **1**-hypertournament matrices with score variances $\mathrm{v}(A) = 2.25$ and $\mathrm{v}(B) = 6.75$. The shell $\Gamma(A)$ in Figure 1 consists of one bounded and one unbounded branch. The bounded branch surrounds a real eigenvalue of A and the unbounded branch isolates the rest of the spectrum of A. The shell $\Gamma(B)$ is connected and all the eigenvalues of B are located in the region between $\Gamma(B)$ and the line $\mathrm{Re}z = -1/2$.

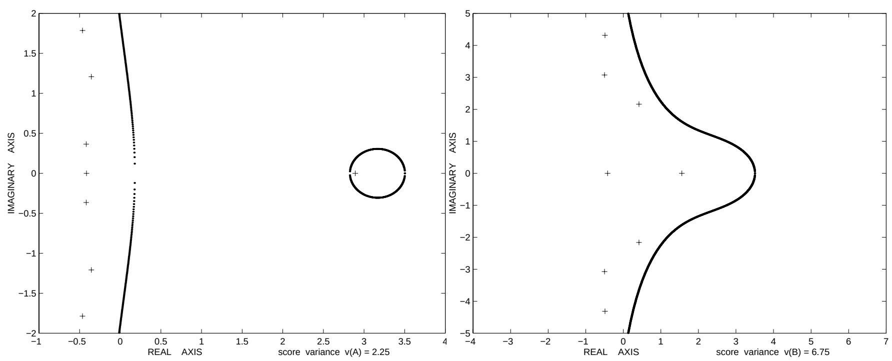

Figure 1: The shells $\Gamma(A)$ and $\Gamma(B)$ for different score variances.

Example 2 A and B are two 8×8 **1**-hypertournament matrices with score variances $\mathrm{v}(A) = 8^2/16 = 4$ and $\mathrm{v}(B) = 4.32$. Their spectra and shells are sketched in Figure 2.

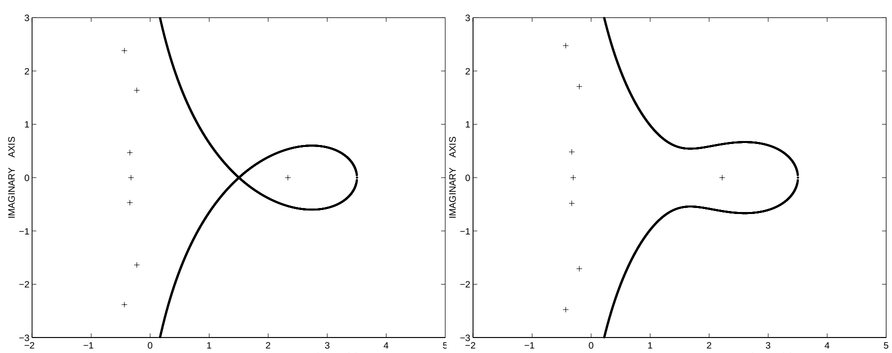

Figure 2: The shells $\Gamma(A)$ and $\Gamma(B)$ for score variances on $[n^2/16, 2n^2/27]$.

Furthermore, the following results hold.

Corollary 4 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a **1**-hypertournament matrix with score variance $\mathrm{v}(A) < n^2/16$. If A has k eigenvalues with nonnegative parts, then $k \le (n + 2 - \sqrt{n^2 - 16\mathrm{v}(A)})/2$.*

Proof The matrix A has a real eigenvalue $\rho(A) \ge (n - 2 + \sqrt{n^2 - 16\mathrm{v}(A)})/4$ (see Remark 1) and exactly $n - k$ eigenvalues with real parts in the interval $[-1/2, 0)$. Since $\mathrm{trace}(A) = 0$, we have that

$$\frac{n - 2 + \sqrt{n^2 - 16\mathrm{v}(A)}}{4} \le \frac{n-k}{2},$$

which implies that

$$k \le \frac{n + 2 - \sqrt{n^2 - 16\mathrm{v}(A)}}{2}. \qquad \blacksquare$$

Note that Corollary 4 implies a result of Katzenberger and Shader [9] for a **1**-hypertournament matrix A, namely that if $\mathrm{v}(A) < (n-1)/4$, then A is nonsingular.

Corollary 5 *Suppose that $A \in \mathcal{M}_n(\mathbb{R})$ is a **1**-hypertournament matrix with $\mathrm{v}(A) < (n-2)/2$. Then A has at least $n - 2$ eigenvalues with negative real parts and 1 or 2 real positive eigenvalues.*

Proof Since $(n-2)/2 < n^2/16$, by the above corollary,

$$k < \frac{n + 2 - \sqrt{n^2 - 8n + 16}}{2} = 3. \qquad \blacksquare$$

Next we characterize the case of equality in (2).

Theorem 6 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a **1**-hypertournament matrix with score variance $\mathrm{v}(A) > 0$, and let λ be an eigenvalue of A. Then equality holds in (2) if and only if $\mathrm{v}(A) \le n^2/16$ and λ is real and equal to either*

$$\frac{n - 2 + \sqrt{n^2 - 16\mathrm{v}(A)}}{4} \quad or \quad \frac{n - 2 - \sqrt{n^2 - 16\mathrm{v}(A)}}{4}.$$

Further, equality holds in (2) for some eigenvalue if and only if A has $n - 2$ eigenvalues having real parts equal to $-1/2$.

Proof Let λ be an eigenvalue of A and let S, M and N be as defined in the proof of Theorem 1. Equality in (2), namely,

$$(\mathrm{Im}\lambda)^2 = d\left(\frac{\mathrm{v}(A)}{n-d} - \frac{d}{4}\right)$$

holds if and only if

$$\frac{1}{2} + \mathrm{Re}\lambda \;=\; \frac{2d(u^t u)}{d^2 + 4(\mathrm{Im}\lambda)^2}$$

or equivalently, if and only if the matrix M is singular negative semidefinite. In this case, the eigenvalue $0 \in \sigma(M)$ is simple and corresponds to the eigenvector u in (1). Moreover, the matrix S is singular and since $0 \in \partial F(S)$ (the boundary of the numerical range), 0 must be a normal eigenvalue of S (see [6, Theorem 1.6.6]); every corresponding eigenvector belongs to

$$\mathrm{null}(M) \cap \mathrm{null}(N) \;=\; \mathrm{span}\{u\} \; .$$

Hence, u is an eigenvector of N in (3) corresponding to the eigenvalue 0. Furthermore, the vector u is an eigenvector of the rank one matrix

$$\frac{4\mathrm{Im}\lambda}{d^2 + 4(\mathrm{Im}\lambda)^2}\, u u^t$$

corresponding to the simple eigenvalue

$$\frac{4\mathrm{Im}\lambda (u^t u)}{d^2 + 4(\mathrm{Im}\lambda)^2} \; .$$

As a consequence, the quantity

$$i\;\mathrm{Im}\lambda - \frac{4i\;\mathrm{Im}\lambda (u^t u)}{d^2 + 4(\mathrm{Im}\lambda)^2} \;=\; \frac{i\;\mathrm{Im}\lambda (n - 2 - 4\mathrm{Re}\lambda)}{d}$$

is an eigenvalue of the matrix K_1 in (1) with corresponding eigenvector u. Thus,

$$\frac{u^t K_1 u}{u^t u} \;=\; \frac{i\;\mathrm{Im}\lambda (n - 2 - 4\mathrm{Re}\lambda)}{d} \; .$$

The same arguments applied to $\overline{\lambda} \in \sigma(A)$ yield

$$\frac{u^t K_1 u}{u^t u} \;=\; \frac{i\;\mathrm{Im}\overline{\lambda} (n - 2 - 4\mathrm{Re}\lambda)}{d}$$

and hence $\mathrm{Im}\lambda = 0$. Thus the existence of eigenvalues on $\Gamma(A)$ hinges on the equation in d (here $d = n - 1 - 2\lambda$)

$$d\left(\frac{\mathrm{v}(A)}{n - d} - \frac{d}{4}\right) \;=\; 0$$

having real solutions. If $\mathrm{v}(A) > n^2/16$, this equation has no real solutions. Otherwise, the real solutions lead to eigenvalues as stated in the theorem.

If equality holds in (2) for some eigenvalue λ, then from our argument above, λ is one of $(n-2+\sqrt{n^2-16\mathrm{v}(A)})/4$ and $(n-2-\sqrt{n^2-16\mathrm{v}(A)})/4$, and in [8] it is shown that this implies that A has $n-2$ eigenvalues having real parts equal to $-1/2$. Conversely, if A has $n-2$ eigenvalues having real parts equal to $-1/2$, then by [7, Theorem 2], A has two eigenvalues with real parts equal to $(n-2+\sqrt{n^2-16\mathrm{v}(A)})/4$ and $(n-2-\sqrt{n^2-16\mathrm{v}(A)})/4$, respectively. From (2) it follows that those eigenvalues are necessarily real, so that equality holds in (2) for both eigenvalues. ∎

3 Generalized tournament matrices

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a generalized tournament matrix, i.e., all the entries of A are nonnegative and $A + A^t = J - I$. By Pick's inequality (see [4]), every eigenvalue λ of A satisfies $|\mathrm{Im}\lambda| \leq (1/2)\cot(\pi/(2n))$. Hence, it is natural to ask where the shell $\Gamma(A)$ intersects the horizontal line $\mathrm{Im}z = (1/2)\cot(\pi/(2n))$. In essence, we are asking for $x \in \mathbb{R}$ such that

$$\frac{1}{4}\cot^2\left(\frac{\pi}{2n}\right) \;=\; (n-1-2x)\frac{\mathrm{v}(A)}{2x+1} - \frac{(n-1-2x)^2}{4}\,.$$

Letting $y = 2x+1$, we have

$$y^3 - 2ny^2 + \left(n^2 + \cot^2\left(\frac{\pi}{2n}\right) + 4\mathrm{v}(A)\right) y - 4n\mathrm{v}(A) \;=\; 0.$$

Observe that the left part of this equation is increasing as a function of $y \in [0, n]$; it follows from the Implicit Function Theorem that

$$\frac{\partial y}{\partial \mathrm{v}(A)} \;>\; 0.$$

Thus, the largest possible root will occur when $\mathrm{v}(A)$ is as large as possible, i.e., when A is triangular, in which case $\mathrm{v}(A) = (n^2-1)/12$. Considering this maximum possible value of $\mathrm{v}(A)$, we investigate the asymptotics of y as $n \to \infty$. It follows that

$$y^3 - 2ny^2 + \left(n^2 + \cot^2\left(\frac{\pi}{2n}\right) + \frac{n^2-1}{3}\right) y - \frac{n^3-n}{3} \;=\; 0,$$

which, in turn, implies

$$\left(\frac{y}{n}\right)^3 - 2\left(\frac{y}{n}\right)^2 + \left(1 + \frac{\cot^2(\pi/(2n))}{n^2} + \frac{n^2-1}{3n^2}\right)\frac{y}{n} - \frac{n^2-1}{3n^2} \;=\; 0.$$

Asymptotically we have $y/n \;\to\; \xi$, where

$$\xi^3 - 2\xi^2 + \left(1 + \frac{4}{\pi^2} + \frac{1}{3}\right)\xi - \frac{1}{3} \;=\; 0.$$

Solving this cubic gives $\xi \cong 0.2588$, so that for large n, y is asymptotic to $0.2588n$. As a consequence, for all sufficiently large n, the shell $\Gamma(A)$ of an $n \times n$ generalized tournament matrix A intersects the line $\mathrm{Im} z = (1/2)\cot(\pi/(2n))$ somewhere in the zone

$$\{z \in \mathbb{C} : -1/2 \leq \mathrm{Re} z \leq 0.1295n\}.$$

The Kendall-Wei ranking method for tournament matrices (see [11]) relies on the power method as a justification. The following results show that when the score variance is not too big, convergence of the power method is quite fast.

Theorem 7 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a generalized tournament matrix with $n \geq 12$ and score variance*

$$\mathrm{v}(A) \;\leq\; \frac{1}{2(n+2)}\left(n^2 - 4n + 3 + \frac{4n^2}{\pi^2}\right).$$

If λ is an eigenvalue of A such that $\mathrm{Re}\lambda \leq (n - 2 - \sqrt{n^2 - 16\mathrm{v}(A)})/4$, then

$$|\lambda| \;\leq\; \left|-\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\pi}{2n}\right)\right| .$$

Proof Since $-1/2 \leq \mathrm{Re}\lambda$ the inequality is straightforward if $\mathrm{Re}\lambda \leq 1/2$, so suppose that $1/2 < \mathrm{Re}\lambda \leq (n - 2 - \sqrt{n^2 - 16\mathrm{v}(A)})/4$. By Theorem 1, it follows that

$$|\lambda|^2 \;=\; (\mathrm{Re}\lambda)^2 + (\mathrm{Im}\lambda)^2 \;\leq\; \frac{n\mathrm{v}(A)}{2\mathrm{Re}\lambda + 1} + (n-1)\mathrm{Re}\lambda - \mathrm{v}(A) - \frac{(n-1)^2}{4} .$$

Hence, it is enough to prove that

$$\frac{n\mathrm{v}(A)}{2\mathrm{Re}\lambda + 1} + (n-1)\mathrm{Re}\lambda - \mathrm{v}(A) - \frac{(n-1)^2}{4} \;\leq\; \left|-\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\pi}{2n}\right)\right|^2 ; \qquad (4)$$

observe that

$$\left|-\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\pi}{2n}\right)\right|^2 \;=\; \frac{1}{4\sin^2(\pi/(2n))} .$$

Considering the left part of (4) as a function of $\mathrm{Re}\lambda$, it is easy to see that it is concave up with at most one critical point. Thus, (4) will hold for

$$\mathrm{Re}\lambda \;\in\; \left[\frac{1}{2}\,,\;\; \frac{n - 2 - \sqrt{n^2 - 16\mathrm{v}(A)}}{4}\right]$$

provided that it holds for $\mathrm{Re}\lambda = 1/2$ and $\mathrm{Re}\lambda = (n - 2 - \sqrt{n^2 - 16\mathrm{v}(A)})/4$.

Notice that for $\mathrm{Re}\lambda = 1/2$, inequality (4) can be written as

$$\frac{n\mathrm{v}(A)}{2} + \frac{n-1}{2} - \mathrm{v}(A) - \frac{(n-1)^2}{4} \;\leq\; \frac{1}{4\sin^2(\pi/(2n))}$$

or equivalently,

$$\mathrm{v}(A) \;\le\; \left(\frac{1}{n-2}\right)\left(\frac{n^2-4n+3}{2} + \frac{1}{2\sin^2(\pi/(2n))}\right).$$

This is implied by our hypothesis.

For $\mathrm{Re}\lambda = (n-2-\sqrt{n^2-16\mathrm{v}(A)})/4$, (4) is written as

$$\frac{2n\mathrm{v}(A)}{n-\sqrt{n^2-16\mathrm{v}(A)}} + \frac{(n-1)(n-2-\sqrt{n^2-16\mathrm{v}(A)})}{4} - \mathrm{v}(A) - \frac{(n-1)^2}{4}$$

$$\le\; \frac{1}{4\sin^2(\pi/(2n))}\;,$$

which is equivalent to

$$\frac{n^2-2n+2}{8} - \frac{(n-2)\sqrt{n^2-16\mathrm{v}(A)}}{8} - \mathrm{v}(A) \;\le\; \frac{1}{4\sin^2(\pi/(2n))}\;.$$

Since $1/2 < (n-2-\sqrt{n^2-16\mathrm{v}(A)})/4$, $\mathrm{v}(A) > (n-2)/2$. Moreover, our hypothesis implies that $\mathrm{v}(A) \le (3n)/4$ as well. After straightforward computations, we find that

$$\frac{n^2-2n+2}{8} - \frac{(n-2)\sqrt{n^2-16\mathrm{v}(A)}}{8} - \mathrm{v}(A) \le \frac{n^2-8n+2-(n-2)\sqrt{n^2-12n}}{8}\;.$$

This last quantity is seen to be at most $(n/\pi)^2$ ($\le$ $(1/4)\ \sin^{-2}(\pi/(2n))$). ■

Corollary 8 *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a generalized tournament matrix with score variance*

$$\mathrm{v}(A) \;\le\; \frac{1}{2(n-2)}\left(n^2-4n+3+\frac{4n^2}{\pi^2}\right)$$

and $n \ge 12$. If in addition, A is primitive with Perron value ρ, then for every eigenvalue $\lambda \neq \rho$,

$$\frac{|\lambda|}{\rho} \;\le\; \frac{2}{\sin(\pi/(2n))(n-2+\sqrt{n^2-16\mathrm{v}(A)})}\;.$$

(Observe that for large n, this bound is asymptotic to $2/\pi$.)

Proof By Remark 1,

$$\rho \;\ge\; \frac{n-2+\sqrt{n^2-16\mathrm{v}(A)}}{4}$$

and by Theorem 7,

$$|\lambda| \;\leq\; \left| -\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\pi}{2n}\right) \right| \;=\; \frac{1}{2\sin(\pi/(2n))}\ .$$

The result follows immediately. ∎

Example 3 Let T be the circulant matrix of order $2k+1$ given by

$$T \;=\; \mathrm{Circ}([\ 0\ \overbrace{1\ 1\ \ldots\ 1}^{k}\ \overbrace{0\ 0\ \ldots\ 0}^{k}\ ])$$

(see [5] for the definition of a circulant), and observe that T is a tournament matrix. Now let A be the $(4k+2)\times(4k+2)$ tournament matrix

$$A \;=\; \left[\begin{array}{cc} T & T^t + I \\ T^t & T \end{array}\right].$$

Letting $n = 4k+2$, one can verify that A has score variance $\mathrm{v}(A) = 1/4$, Perron value $(n^2 - 2 + \sqrt{n^2-4})/4$ and $-1/2 \pm (i/2)\cot(\pi/(2n))$ as eigenvalues. So this matrix actually achieves equality on the above corollary.

We conclude by posing the open problem of determining those primitive generalized tournament matrices A with score variance

$$\mathrm{v}(A) \;\leq\; \frac{1}{2(n-2)}\left(n^2 - 4n + 3 + \frac{4n^2}{\pi^2}\right)$$

such that both

$$\lambda \;=\; -\frac{1}{2} + \frac{1}{2}\cot\left(\frac{\pi}{2n}\right) \quad \text{and} \quad \rho \;=\; \frac{n^2 - 2 + \sqrt{n^2 - 16\mathrm{v}(A)}}{4}$$

are eigenvalues of A. (Note that each such matrix provides an example for which equality holds in Corollary 8.) We note that [4] provides a constructive characterization of generalized tournament matrices having $-1/2 + (1/2)\cot(\pi/(2n))$ as an eigenvalue, while [8] provides a characterization when $(n^2 - 2 + \sqrt{n^2 - 16\mathrm{v}(A)})/4$ is an eigenvalue of A. Thus our problem can be reduced to looking at the intersection of those two classes of matrices.

Acknowledgment The third author would like to acknowledge early discussions with Prof. Ahmed Sourour, which led to the use of Schur's Lemma in the present work.

References

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia (1994).

[2] A. Brauer and I. C. Gentry, On the characteristic roots of tournament matrices, *Bulletin of the American Mathematical Society*, **74** (1968), pp. 1133-1135.

[3] A. Brauer and I. C. Gentry, Some remarks on tournament matrices, *Linear Algebra Appl.*, **5** (1972), pp. 311-318.

[4] D. A. Gregory, S. J. Kirkland and B. L. Shader , Pick's inequality and tournaments, *Linear Algebra Appl.*, **186** (1993), pp. 15-36.

[5] R. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press (1990).

[6] R. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press (1991).

[7] S. Kirkland, Hypertournament matrices, score vectors and eigenvalues, *Linear and Multilinear Algebra*, **30** (1991), pp. 261-274.

[8] S. Kirkland and B. Shader, Tournament matrices with extremal spectral properties, *Linear Algebra Appl.*, **196** (1994), pp. 1-17.

[9] G. S. Katzenberger and B. L. Shader, Singular tournament matrices. Proceedings of the Twentieth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1989). *Congr. Numer.*, **72** (1990), pp. 71-80.

[10] J. S. Maybee and N. J. Pullman, Tournament matrices and their generalizations, *Linear and Multilinear Algebra*, **28** (1990), pp. 57-70.

[11] J. W. Moon, *Topics on Tournaments* Holt, Rinehart and Winston (1968).

[12] J. W. Moon and N. J. Pullman, On generalized tournaments matrices, *SIAM Review*, **12** (1970), pp. 384-399.