The polar decomposition of block companion matrices

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Abstract

Let $L(\lambda) = I_n\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be an $n \times n$ monic matrix polynomial, and let C_L be the corresponding block companion matrix. In this note, we extend a known result on scalar polynomials to obtain a formula for the polar decomposition of C_L when the matrices A_0 and $\sum_{j=1}^{m-1} A_j A_j^*$ are nonsingular.

Keywords: block companion matrix, matrix polynomial, eigenvalue, singular value, polar decomposition.

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1 Introduction and notation

Consider the monic *matrix polynomial*

$$L(\lambda) \;=\; I_n\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \tag{1}$$

where $A_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, m-1,\ m \geq 2)$, λ is a complex variable and I_n denotes the $n \times n$ identity matrix. The study of matrix polynomials, especially with regard to their spectral analysis, has a long history and plays an important role in systems theory [1, 2, 3, 4]. A scalar $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $L(\lambda)$ if the system $L(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $L(\lambda)$ corresponding to λ_0. The set of all eigenvalues of $L(\lambda)$ is the *spectrum* of $L(\lambda)$, namely, $\mathrm{sp}(L) = \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\}$, and contains no more than nm distinct (finite) elements.

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Define $\Delta = [\ A_1 \ \ A_2 \ \cdots \ A_{m-1}\ ] \in \mathbb{C}^{n\times n(m-1)}$. The $nm \times nm$ matrix

$$C_L = \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & I_{n(m-1)} \\ -A_0 & -\Delta \end{bmatrix} \qquad (2)$$

(where the zero matrices are of appropriate size) is known as the *block companion matrix* of $L(\lambda)$, and its spectrum, $\mathrm{sp}(C_L)$, coincides with $\mathrm{sp}(L)$. Moreover, C_L and $L(\lambda)$ are strongly connected since they have similar Jordan structures and define equivalent dynamical systems; for example, see [2, 3] for these comments and general background on block companion matrices of matrix polynomials.

Let $C_L = PU$ be the (left) polar decomposition of C_L, where the $nm\times nm$ matrix $P = (C_L C_L^*)^{1/2}$ is positive semidefinite and $U \in \mathbb{C}^{nm\times nm}$ is unitary. Then the eigenvalues of P are the *singular values* of C_L and (recalling that $\mathrm{sp}(C_L) = \mathrm{sp}(L)$) yield bounds for the eigenvalues and for products of eigenvalues of $L(\lambda)$ [5, 6].

In [7], van den Driessche and Wimmer obtained an explicit formula for the polar decomposition of the companion matrix corresponding to a monic scalar polynomial $p(\lambda)$ (i.e., for $n = 1$) in terms of the coefficients of $p(\lambda)$. In this article, extending their methodology, we prove that their results are also valid for the matrix polynomial $L(\lambda)$ in (1) and its block companion matrix C_L in (2) when the matrices A_0 and $\Delta\Delta^* = \sum_{j=1}^{m-1} A_j A_j^*$ are nonsingular. An important feature of our generalization is that the construction of the polar decomposition of the $nm \times nm$ matrix C_L is reduced to the computation of the (positive definite) square root of a $2n\times 2n$ positive definite matrix. If in addition, $A_0A_0^*$ and $\Delta\Delta^*$ commute, then the polar decomposition of C_L is further reduced to the computation of the $n \times n$ positive definite square roots

$$P_0 = (A_0A_0^*)^{1/2}, \quad P_1 = (\Delta\Delta^*)^{1/2} = \left(\sum_{j=1}^{m-1} A_j A_j^* \right)^{1/2} \qquad (3)$$

and

$$\Psi = (\Delta\Delta^* + A_0A_0^* + I_n + 2P_0)^{1/2} = \left(P_1^2 + (P_0 + I_n)^2\right)^{1/2}. \qquad (4)$$

2 Singular values and polar decomposition

Suppose $L(\lambda) = I_n\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ is an $n \times n$ matrix polynomial with $n \geq 2$ and $\det A_0 \neq 0$ (or equivalently, $0 \notin \mathrm{sp}(L)$), and let C_L be the corresponding (nonsingular) block companion matrix in (2).

Consider the $n\times n$ positive definite matrix $S = \sum_{j=0}^{m-1} A_j A_j^* = \Delta\Delta^* + A_0A_0^*$ and the $nm \times nm$ positive definite matrix

$$C_L C_L^* = \begin{bmatrix} I_{n(m-1)} & -\Delta^* \\ -\Delta & S \end{bmatrix}.$$

The square roots of the eigenvalues of $C_L C_L^*$ are the singular values of C_L. Keeping in mind the Schur complement of the leading $n(m-1) \times n(m-1)$ block of the linear pencil $I_{nm}\lambda - C_L C_L^*$, one can see that

$$\begin{aligned} \det(I_{nm}\lambda - C_L C_L^*) &= \det[I_{n(m-1)}(\lambda - 1)] \det\left(I_n\lambda - S - \frac{1}{\lambda - 1}\,\Delta\Delta^*\right) \\ &= (\lambda - 1)^{n(m-2)} \det\left[I_n\lambda^2 - (I_n + S)\lambda + A_0A_0^*\right]. \end{aligned}$$

Hence, the singular values of C_L are $\sigma = 1$ (of multiplicity at least $n(m-2)$) and the square roots of the eigenvalues of the quadratic selfadjoint matrix polynomial

$$R_L(\lambda) \;=\; I_n\lambda^2 - (I_n + S)\lambda + A_0A_0^*. \tag{5}$$

The hermitian matrices $I_n + S$ and $A_0A_0^*$ are positive definite, and the quantity $[x^*(I_n + S)x]^2 - 4x^*A_0A_0^*x$ is nonnegative for every unit vector $x \in \mathbb{C}^n$. Thus, by [4, Section IV.31] (see also [8]), the eigenvalues of the matrix polynomial $R_L(\lambda)$ are real positive and their minimum and maximum values are given by

$$\lambda_{\min}(R_L) \;=\; \min\left\{\frac{1 + x^*Sx - \sqrt{(1 + x^*Sx)^2 - 4x^*A_0A_0^*x}}{2} \;:\; x \in \mathbb{C}^n,\, x^*x = 1\right\}$$

and

$$\lambda_{\max}(R_L) \;=\; \max\left\{\frac{1 + x^*Sx + \sqrt{(1 + x^*Sx)^2 - 4x^*A_0A_0^*x}}{2} \;:\; x \in \mathbb{C}^n,\, x^*x = 1\right\},$$

respectively. Moreover, we have the following known result [5, Lemma 2.7] (see also [6, p. 336] and [7, Lemma 2.2]).

Proposition 1 *The singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{nm}$ of the block companion matrix C_L fall into three groups:*

(i) $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 1$,

(ii) $\sigma_{n+1} = \sigma_{n+2} = \cdots = \sigma_{n(m-1)} = 1$ *(if $m \geq 3$), and*

(iii) $1 \geq \sigma_{n(m-1)+1} \geq \sigma_{n(m-1)+2} \geq \cdots \geq \sigma_{nm} > 0$.

The $2n$ singular values of C_L in (i) *and* (iii) *are the square roots of the eigenvalues of $R_L(\lambda)$ in (5).*

Corollary 2 *For any eigenvalue μ of $L(\lambda)$,*

$$\lambda_{\min}(R_L)^{1/2} \;=\; \sigma_{nm} \;\leq\; |\mu| \;\leq\; \sigma_1 \;=\; \lambda_{\max}(R_L)^{1/2}.$$

Next we characterize the case when 1 is an eigenvalue of $R_L(\lambda)$, i.e., when C_L has more than $n(m-2)$ singular values equal to 1 (see also [5, Lemma 2.8]).

Proposition 3 *The following statements are equivalent:*

(i) *The matrix polynomial $R_L(\lambda)$ has an eigenvalue $\lambda = 1$.*

(ii) *The matrix $\Delta\Delta^* = \sum_{j=1}^{m-1} A_j A_j^*$ is singular.*

(iii) *The matrices $A_1, A_2, \ldots, A_{m-1}$ are singular and have a common left eigenvector corresponding to zero.*

Proof Since $\det R_L(1) = \det(\Delta\Delta^*)$, the equivalence (i)$\Leftrightarrow$(ii) follows readily. Moreover, if $y \in \mathbb{C}^n$ is a nonzero vector such that $A_j^* y = 0$ $(j = 1, 2, \ldots, m-1)$, then $\Delta\Delta^* y = \sum_{j=1}^{m-1} A_j A_j^* y = 0$. Thus, it is enough to prove the part (ii)$\Rightarrow$(iii).

Suppose $\Delta\Delta^*$ is singular, and let $x_0 \in \mathbb{C}^n$ be a unit eigenvector of $\Delta\Delta^*$ corresponding to 0. Then

$$x_0^* \Delta\Delta^* x_0 \;=\; \sum_{j=1}^{m-1} x_0^* A_j A_j^* x_0 \;=\; 0,$$

where $A_j A_j^*$ is positive semidefinite and satisfies $x_0^* A_j A_j^* x_0 \geq 0$ for every $j = 1, 2, \ldots, m-1$. Hence, $x_0^* A_j A_j^* x_0 = 0$ for every $j = 1, 2, \ldots, m-1$, and thus,

$$A_j^* x_0 \;=\; 0 \;\; ; \quad j = 1, 2, \ldots, m-1. \qquad \square$$

If the minimum or the maximum singular value of C_L is equal to 1, then the polar decomposition of C_L is equivalent to the polar decomposition of the matrix A_0.

Proposition 4 *Suppose the minimum or the maximum singular value of C_L is $\sigma = 1$. Let P_0 be the positive definite matrix in (3) (recall that* $\det A_0 \neq 0$*). Then $A_1 = A_2 = \cdots = A_{m-1} = 0$, and the polar decomposition of C_L is given by $C_L = PU$, where*

$$P \;=\; \begin{bmatrix} I_{n(m-1)} & 0 \\ 0 & P_0 \end{bmatrix} \quad \textit{and} \quad U \;=\; \begin{bmatrix} 0 & I_{n(m-1)} \\ -P_0^{-1} A_0 & 0 \end{bmatrix}.$$

Proof Clearly, $\lambda = 1$ is the minimum or the maximum eigenvalue of the matrix polynomial $R_L(\lambda)$ in (5). Suppose $\lambda = 1$ is the minimum eigenvalue of $R_L(\lambda)$. Then for every unit vector $x \in \mathbb{C}^n$, the equation

$$x^* R_L(\lambda) x \;=\; \lambda^2 - (1 + x^* S x)\lambda + x^* A_0 A_0^* x \;=\; 0 \tag{6}$$

has a real root [4, 8]

$$\rho_1 \;=\; \frac{1 + x^* S x - \sqrt{(1 + x^* S x)^2 - 4 x^* A_0 A_0^* x}}{2} \;\geq\; 1.$$

Hence,

$$x^* \left(\sum_{j=0}^{m-1} A_j A_j^* \right) x \;\le\; x^* A_0 A_0^* x,$$

where the equality holds for x a unit eigenvector of $R_L(\lambda)$ corresponding to 1. Consequently, the matrix $\Delta\Delta^* = \sum_{j=1}^{m-1} A_j A_j^*$ is singular negative semidefinite. Since $\Delta\Delta^*$ is always positive semidefinite, this means that $\Delta = 0$, and thus, $C_L C_L^* = I_{n(m-1)} \oplus A_0 A_0^*$ and $P = (C_L C_L^*)^{1/2} = I_{n(m-1)} \oplus P_0$.

If $\lambda = 1$ is the maximum eigenvalue of $R_L(\lambda)$, then the proof is similar, using the real root

$$\rho_2 \;=\; \frac{1 + x^* S x + \sqrt{(1 + x^* S x)^2 - 4 x^* A_0 A_0^* x}}{2} \;\le\; 1$$

of the quadratic equation (6). In both cases, the matrix

$$U \;=\; \begin{bmatrix} 0 & I_{n(m-1)} \\ -P_0^{-1} A_0 & 0 \end{bmatrix}$$

satisfies

$$U U^* \;=\; \begin{bmatrix} 0 & I_{n(m-1)} \\ -P_0^{-1} A_0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -A_0^* P_0^{-1} \\ I_{n(m-1)} & 0 \end{bmatrix} \;=\; I_{nm}$$

and

$$P U \;=\; \begin{bmatrix} I_{n(m-1)} & 0 \\ 0 & P_0 \end{bmatrix} \begin{bmatrix} 0 & I_{n(m-1)} \\ -P_0^{-1} A_0 & 0 \end{bmatrix} \;=\; C_L.$$

The proof is complete. □

Note that if all the coefficient matrices $A_1, A_2, \ldots, A_{m-1}$ of $L(\lambda)$ are zero (this is the case in the above proposition), then $C_L C_L^* = I_{n(m-1)} \oplus A_0 A_0^*$ and the spectrum of $R_L(\lambda) = (\lambda - 1)(I_n \lambda - A_0 A_0^*)$ coincides with the union $\mathrm{sp}(A_0 A_0^*) \cup \{1\}$.

Consider the $2n \times 2n$ hermitian matrix

$$H_L \;=\; \begin{bmatrix} I_n & \left(\sum_{j=1}^{m-1} A_j A_j^* \right)^{1/2} \\ \left(\sum_{j=1}^{m-1} A_j A_j^* \right)^{1/2} & \sum_{j=0}^{m-1} A_j A_j^* \end{bmatrix} \;=\; \begin{bmatrix} I_n & P_1 \\ P_1 & S \end{bmatrix}.$$

Then by straightforward computations, we see that $\det(I_{2n}\lambda - H_L) = \det R_L(\lambda)$, i.e., $\mathrm{sp}(R_L) = \mathrm{sp}(H_L)$. As a consequence, H_L is positive definite.

Assuming that $\Delta\Delta^* = \sum_{j=1}^{m-1} A_j A_j^*$ is nonsingular, we also define the $nm \times 2n$ matrix

$$M_L \;=\; \begin{bmatrix} -\Delta^* P_1^{-1} & 0 \\ 0 & I_n \end{bmatrix}$$

and observe that

$$M_L^* M_L = \begin{bmatrix} -P_1^{-1}\Delta & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} -\Delta^* P_1^{-1} & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} P_1^{-1} P_1^2 P_1^{-1} & 0 \\ 0 & I_n \end{bmatrix} = I_{2n}.$$

Now we can prove the main result of the paper, generalizing [7, Theorem 2.1].

Theorem 5 *Let $L(\lambda) = I_n\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be an $n\times n$ matrix polynomial with $\det A_0 \neq 0$, and suppose $\Delta\Delta^* = \sum_{j=1}^{m-1} A_j A_j^*$ is nonsingular. Define M_L as above, and let $H_L^{1/2}$ be the positive definite square root of H_L. Then the polar decomposition of the block companion matrix C_L is given by $C_L = PU$, where*

$$P = I_{nm} + M_L(H_L^{1/2} - I_{2n})M_L^*$$

and

$$U = \left(I_{nm} + M_L(H_L^{1/2} - I_{2n})M_L^*\right) \begin{bmatrix} -\Delta^*(A_0^{-1})^* & I_{n(m-1)} \\ -(A_0^{-1})^* & 0 \end{bmatrix}.$$

Proof Since $M_L^* M_L = I_{2n}$, the matrix $P = I_{nm} + M_L(H_L^{1/2} - I_{2n})M_L^*$ is positive definite and satisfies

$$\begin{aligned}
P^2 &= \left(I_{nm} + M_L(H_L^{1/2} - I_{2n})M_L^*\right)^2 \\
&= I_{nm} + 2M_L(H_L^{1/2} - I_{2n})M_L^* + M_L(H_L^{1/2} - I_{2n})^2 M_L^* \\
&= I_{nm} + M_L(2H_L^{1/2} - 2I_{2n} + H_L - 2H_L^{1/2} + I_{2n})M_L^* \\
&= I_{nm} + M_L(H_L - I_{2n})M_L^* \\
&= I_{nm} + \begin{bmatrix} -\Delta^* P_1^{-1} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & P_1 \\ P_1 & S - I_n \end{bmatrix} \begin{bmatrix} -P_1^{-1}\Delta & 0 \\ 0 & I_n \end{bmatrix} \\
&= I_{nm} + \begin{bmatrix} 0 & -\Delta^* \\ -\Delta & S - I_n \end{bmatrix} \\
&= C_L C_L^*.
\end{aligned}$$

Furthermore, by the relation $C_L = PU$, we have that $U^* = C_L^{-1}P$, or equivalently, $U = P(C_L^{-1})^*$, where

$$(C_L^{-1})^* = \begin{bmatrix} -A_1^*(A_0^{-1})^* & I_n & 0 & \cdots & 0 \\ -A_2^*(A_0^{-1})^* & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{m-1}^*(A_0^{-1})^* & 0 & 0 & \cdots & I_n \\ -(A_0^{-1})^* & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} -\Delta^*(A_0^{-1})^* & I_{n(m-1)} \\ -(A_0^{-1})^* & 0 \end{bmatrix}.$$

The proof is complete. □

Notice that by the assumption $\det A_0 \neq 0$, it follows that C_L is nonsingular and the matrices P and U are unique [9].

Next we obtain that the matrices $C_L C_L^*$ and $I_{n(m-2)} \oplus H_L$ are unitarily similar. One can easily see that this result leads directly to a second proof of Theorem 5.

Proposition 6 *Suppose $\Delta\Delta^*$ is nonsingular and let $\omega_1, \omega_2, \ldots, \omega_{n(m-2)}$ be an orthonormal system of $n(m-2)$ eigenvectors of $C_L C_L^*$ corresponding to the eigenvalue $\lambda = 1$. Then the $nm \times nm$ matrix $V = \begin{bmatrix} \omega_1 & \omega_2 & \ldots & \omega_{n(m-2)} & M_L \end{bmatrix}$ is unitary and satisfies $V^*(C_L C_L^*)V = I_{n(m-2)} \oplus H_L$.*

Proof Consider the eigenvectors

$$\omega_1 = \begin{bmatrix} \omega_{1,1} \\ \omega_{1,2} \\ \vdots \\ \omega_{1,m} \end{bmatrix}, \quad \omega_2 = \begin{bmatrix} \omega_{2,1} \\ \omega_{2,2} \\ \vdots \\ \omega_{2,m} \end{bmatrix}, \quad \ldots, \quad \omega_{n(m-2)} = \begin{bmatrix} \omega_{n(m-2),1} \\ \omega_{n(m-2),2} \\ \vdots \\ \omega_{n(m-2),m} \end{bmatrix} \in \mathbb{C}^{nm},$$

where $\omega_{i,j} \in \mathbb{C}^n$ $(i = 1, 2, \ldots, n(m-2),\ j = 1, 2, \ldots, m)$, and define the $nm \times n(m-2)$ matrix $W = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_{n(m-2)} \end{bmatrix}$. Since $\Delta\Delta^*$ is nonsingular, by Proposition 3, $A_1, A_2, \ldots, A_{m-1}$ cannot have a common left eigenvector corresponding to zero. As a consequence, the equations

$$(C_L C_L^*)\,\omega_k = \begin{bmatrix} I_n & 0 & \cdots & 0 & -A_1^* \\ 0 & I_n & \cdots & 0 & -A_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & -A_{m-1}^* \\ -A_1 & -A_2 & \cdots & -A_{m-1} & S \end{bmatrix} \begin{bmatrix} \omega_{k,1} \\ \omega_{k,2} \\ \vdots \\ \omega_{k,m} \end{bmatrix} = \begin{bmatrix} \omega_{k,1} \\ \omega_{k,2} \\ \vdots \\ \omega_{k,m} \end{bmatrix}$$

$(k = 1, 2, \ldots, n(m-2))$ yield

$$\omega_{k,m} = 0 \;; \quad k = 1, 2, \ldots, n(m-2)$$

and

$$M_L^*\,\omega_k = \begin{bmatrix} -P_1^{-1}\Delta & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \omega_{k,1} \\ \omega_{k,2} \\ \vdots \\ \omega_{k,m} \end{bmatrix} = 0 \;; \quad k = 1, 2, \ldots, n(m-2).$$

Furthermore, $M_L^* M_L = I_{2n}$, and thus, the $nm \times nm$ matrix $V = [\, W \ \ M_L \,]$ is unitary. If W_1 is the $n(m-1) \times n(m-2)$ submatrix of W obtained by striking out the last n zero rows of W, then straightforward computations imply that

$$\begin{aligned} V^*(C_L C_L^*)V &= \begin{bmatrix} W^* \\ M_L^* \end{bmatrix} \begin{bmatrix} I_{n(m-1)} & -\Delta^* \\ -\Delta & S \end{bmatrix} [\, W \ \ M_L \,] \\ &= \begin{bmatrix} W_1^* & 0 \\ -P_1^{-1}\Delta & P_1 \\ -\Delta & S \end{bmatrix} \begin{bmatrix} W_1 & -\Delta^* P_1^{-1} & 0 \\ 0 & 0 & I_n \end{bmatrix} \\ &= \begin{bmatrix} I_{n(m-2)} & 0 & 0 \\ 0 & I_n & P_1 \\ 0 & P_1 & S \end{bmatrix} = I_{n(m-2)} \oplus H_L. \quad \Box \end{aligned}$$

When the matrices $A_0A_0^*$ and $\Delta\Delta^*$ commute, the problem of computing $H_L^{1/2}$ arisen in Theorem 5 can be reduced to the computation of the positive definite matrices P_0, P_1 and Ψ in (3) and (4). The following lemma is necessary for our discussion.

Lemma 7 *Suppose $A_0A_0^*$ and $\Delta\Delta^*$ commute. Then the matrices $A_0A_0^*$, $\Delta\Delta^*$, P_0, P_1, Ψ and their inverses are mutually commuting.*

Proof By [9, Theorem 4.1.6], there exists a unitary $V_0 \in \mathbb{C}^{n\times n}$ such that $V_0^*(A_0A_0^*)V_0$ and $V_0^*(\Delta\Delta^*)V_0$ are diagonal. The result follows readily. □

Proposition 8 *Suppose $\Delta\Delta^*$ is nonsingular and commutes with $A_0A_0^*$. Then the positive definite square root of H_L is given by*

$$H_L^{1/2} = \begin{bmatrix} \Psi^{-1} & 0 \\ 0 & \Psi^{-1} \end{bmatrix} \begin{bmatrix} I_n + P_0 & P_1 \\ P_1 & P_0 + S \end{bmatrix}.$$

Proof Since $A_0A_0^*$ and $\Delta\Delta^*$ commute, using Lemma 7, it is straightforward to see that

$$\begin{aligned} & \left(\begin{bmatrix} \Psi^{-1} & 0 \\ 0 & \Psi^{-1} \end{bmatrix} \begin{bmatrix} I_n + P_0 & P_1 \\ P_1 & P_0 + S \end{bmatrix} \right)^2 \\ = & \begin{bmatrix} \Psi^{-1}(I_n + P_0) & \Psi^{-1}P_1 \\ \Psi^{-1}P_1 & \Psi^{-1}(P_0 + S) \end{bmatrix}^2 \\ = & \begin{bmatrix} \Psi^{-2}[(I_n + P_0)^2 + \Delta\Delta^*] & \Psi^{-2}[I_n + 2P_0 + S]P_1 \\ \Psi^{-2}[I_n + 2P_0 + S]P_1 & \Psi^{-2}[I_n + 2P_0 + S]S \end{bmatrix} \\ = & \begin{bmatrix} I_n & P_1 \\ P_1 & S \end{bmatrix} = H_L. \end{aligned}$$

Moreover, by Lemma 7 and [9, Theorem 7.7.6], we obtain that the matrices

$$\begin{bmatrix} \Psi^{-1} & 0 \\ 0 & \Psi^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_n + P_0 & P_1 \\ P_1 & P_0 + S \end{bmatrix}$$

are commuting positive definite. Hence, their product is also a positive definite matrix, completing the proof. □

It is worth mentioning that if C_L is nonsingular with polar decomposition $C_L = PU$ and the $nm \times nm$ matrix P is written in the form $P = [\, Q_1 \; Q_2 \; \cdots \; Q_m \,]$, where $Q_k \in \mathbb{C}^{nm\times n}$ $(k = 1, 2, \ldots, m)$, then

$$\begin{aligned} U &= [\, Q_1 \; Q_2 \; \cdots \; Q_m \,] \begin{bmatrix} -\Delta^*(A_0^{-1})^* & I_{n(m-1)} \\ -(A_0^{-1})^* & 0 \end{bmatrix} \\ &= \left[-\left(Q_m(A_0^{-1})^* + \sum_{j=1}^{m-1} Q_j A_j^* (A_0^{-1})^* \right) \; Q_1 \; Q_2 \; \cdots \; Q_{m-1} \right]. \end{aligned}$$

Our results are illustrated in the following example.

Example Consider the 2×2 matrix polynomial

$$L(\lambda) = I_2\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = I_2\lambda^3 + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\lambda^2 + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\lambda + \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

and its block companion matrix C_L. The spectrum $\mathrm{sp}(L) = \{1, -1, -0.5 \pm 0.866\,\mathrm{i}\}$ and the singular values of C_L, namely, $2.4171, 1.8354, 1, 1, 0.8477, 0.2659$, clearly confirm Proposition 1 and Corollary 2. The matrices A_0 and $\Delta\Delta^* = A_1A_1^* + A_2A_2^* = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ are nonsingular and do not commute. It is easy to compute

$$P_0 = \begin{bmatrix} 1.3416 & -0.4472 \\ -0.4472 & 0.8944 \end{bmatrix},\ P_1 = \begin{bmatrix} 0.9391 & 0.3437 \\ 0.3437 & 1.9702 \end{bmatrix},\ \Psi = \begin{bmatrix} 2.3094 & -0.0906 \\ -0.0906 & 2.6242 \end{bmatrix}$$

and

$$M_L = \begin{bmatrix} -\Delta^* P_1^{-1} & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 0.1984 & -0.5422 & 0 & 0 \\ 0.1984 & -0.5422 & 0 & 0 \\ -0.9391 & -0.3437 & 0 & 0 \\ 0.1984 & -0.5422 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The positive definite square root of H_L is

$$H_L^{1/2} = \begin{bmatrix} 0.91 & -0.1089 & 0.3726 & 0.1459 \\ -0.1089 & 0.6629 & 0.1787 & 0.7189 \\ 0.3726 & 0.1787 & 1.6813 & -0.0482 \\ 0.1459 & 0.7189 & -0.0482 & 2.1118 \end{bmatrix},$$

and by Theorem 5, the polar decomposition of C_L is given by $C_L = PU$, where

$$P = \begin{bmatrix} 0.9208 & -0.0792 & -0.0941 & -0.0792 & -0.0229 & -0.3609 \\ -0.0792 & 0.9208 & -0.0941 & -0.0792 & -0.0229 & -0.3609 \\ -0.0941 & -0.0941 & 0.8105 & -0.0941 & -0.4113 & -0.3838 \\ -0.0792 & -0.0792 & -0.0941 & 0.9208 & -0.0229 & -0.3609 \\ -0.0229 & -0.0229 & -0.4113 & -0.0229 & 1.6813 & -0.0482 \\ -0.3609 & -0.3609 & -0.3838 & -0.3609 & -0.0482 & 2.1118 \end{bmatrix}$$

and

$$U = P(C_L^{-1})^* = \begin{bmatrix} -0.3074 & -0.1904 & 0.9208 & -0.0792 & -0.0941 & -0.0792 \\ -0.3074 & -0.1904 & -0.0792 & 0.9208 & -0.0941 & -0.0792 \\ -0.1445 & -0.5437 & -0.0941 & -0.0941 & 0.8105 & -0.0941 \\ -0.3074 & -0.1904 & -0.0792 & -0.0792 & -0.0941 & 0.9208 \\ 0.5283 & -0.7417 & -0.0229 & -0.0229 & -0.4113 & -0.0229 \\ -0.6453 & -0.2134 & -0.3609 & -0.3609 & -0.3838 & -0.3609 \end{bmatrix}.$$

Denoting the Frobenius norm by $\|\cdot\|_F$, we confirm our numerical results by calculating $\|C_L C_L^* - P^2\|_F < 10^{-14}$. Notice also that the last four columns of U are exactly the same with the first four columns of P, verifying our discussion. $\square$

Finally, we remark that since our results yield a strong reduction of the order of the problem of polar decomposition, they lead to better estimations of the factors P and U than the classical methods applied directly to C_L. For example, consider the 50×50 diagonal matrix polynomial

$$L(\lambda) = I_{50}\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + I_{50},$$

where $A_j = \mathrm{diag}\{1, 2^j, 3^j, \ldots, 50^j\}$ $(j = 1, 2, 3, 4)$. Two approximations of the positive definite square root of the 250×250 matrix $C_L C_L^*$, P and $\hat{P}$, were constructed by our methodology (using Theorem 5 and Proposition 8) and by a standard singular value decomposition of C_L, respectively. All the computations were performed in MATLAB. To compare the accuracy of the two techniques, we compute $\|C_L C_L^* - P^2\|_F \cong 0.0135$ and $\|C_L C_L^* - \hat{P}^2\|_F \cong 0.1562$. Hence, we conclude that our results add one more possibility for testing numerical algorithms relative to polar decomposition and singular value decomposition.

References

[1] F.R. Gantmacher, *The Theory of Matrices*, Chelsea, New York (1959).

[2] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, New York (1982).

[3] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd edition, Academic Press, Orlando (1985).

[4] A.S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Amer. Math. Society, Providence, Translations of Math. Monographs, Vol. 71 (1988).

[5] N.J. Higham and F. Tisseur, Bounds of eigenvalues of matrix polynomials. *Linear Algebra Appl.* **358** 5-22 (2003).

[6] F. Kittaneh, Singular values of companion matrices and bounds on zeros of polynomials. *SIAM J. Matrix Anal. Appl.* **16** 333-340 (1995).

[7] P. van den Driessche and H.K. Wimmer, Explicit polar decomposition of companion matrices. *Electr. J. Linear Algebra* **1** 64-69 (1996).

[8] P. Lancaster and P. Psarrakos, The numerical range of selfadjoint quadratic matrix polynomials. *SIAM J. Matrix Anal. Appl.* **23** 615-631 (2001).

[9] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge (1985).