On the Solutions of Homogeneous Matrix Difference Equations

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Abstract

In this paper, the solutions of a homogeneous matrix difference equation $A_m u_{j+m} + A_{m-1} u_{j+m-1} + \cdots + A_1 u_{j+1} + A_0 u_j = 0$ $(j = 0, 1, 2, \ldots)$ are investigated and it is obtained that they do not depend on the zeros of the equation at the infinity.

Keywords: matrix polynomial, Z-transformation, Smith-McMillan form, Laurent expansion.

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1 Introduction and preliminaries

Consider the *difference equation*

$$A_m u_{j+m} + A_{m-1} u_{j+m-1} + \cdots + A_1 u_{j+1} + A_0 u_j \;=\; f_j \;\; ; \quad j = 0, 1, 2, \ldots \quad (1)$$

where $A_0, A_1, \ldots, A_m$ are $n \times n$ complex matrices, $f_0, f_1, f_2, \ldots$ is a given sequence of vectors in $\mathbb{C}^n$, and $u_0, u_1, u_2, \ldots$ is a sequence to be found. If for some $j = 0, 1, 2, \ldots$, $f_j \neq 0$, then the equation (1) is called *inhomogeneous*, and if $f_j = 0$ for all $j = 0, 1, 2, \ldots,$ then the equation

$$A_m u_{j+m} + A_{m-1} u_{j+m-1} + \cdots + A_1 u_{j+1} + A_0 u_j \;=\; 0 \;\; ; \quad j = 0, 1, 2, \ldots \quad (2)$$

is said to be *homogeneous*.

A general solution $\mathbf{w} = \{w_0, w_1, w_2, \ldots\}$ of the inhomogeneous equation (1) is written in the form

$$\mathbf{w} \;=\; \mathbf{u} + \mathbf{v},$$

where $\mathbf{u} = \{u_0, u_1, u_2, \ldots\}$ is a general solution of the corresponding homogeneous equation (2) and $\mathbf{v} = \{v_0, v_1, v_2, \ldots\}$ is a fixed particular solution of

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(1) (see [1]). Using the notions of *standard* and *resolvent triples*, the general solution of (1) is given by Theorems 1.6 and 8.3 in [1].

In this paper, using a methodology in [5], we study the solutions of (2) and explain why they are independent of the zeros of the difference equation at the infinity (if any). In particular, the part of any solution $\mathbf{u} = \{u_0, u_1, u_2, \ldots\}$ of the homogeneous equation (2), which corresponds to the zeros at the infinity, coincides with the initial vector u_0. Our approach is based on the notions of the *Z-transformation* and the *Smith-McMillan form* of a matrix polynomial at the infinity. The suggested references are [2] and [3,5], respectively. Furthermore, the use of Laurent expansion of rational matrix functions is crucial.

For a vector sequence $y_0, y_1, y_2, \ldots \in \mathbb{C}^n$, its Z-transformation is defined by

$$Z\{y_j\} \,=\, \sum_{k=0}^{\infty} y_k \lambda^{-k},$$

where λ is a complex variable. The Z-transformation is *linear* and for any integer s,

$$Z\{y_{j+s}\} \,=\, \lambda^s Z\{y_j\} - \sum_{k=0}^{s-1} y_k \lambda^{s-k}. \tag{3}$$

It is also worth noting that for the Kronecker delta at a point $s \in \mathbb{Z}$, namely,

$$\delta_s(j) \,=\, \left\{ \begin{array}{ll} 1\,, & j = s \\ 0\,, & j \neq s \end{array} \right\}, \tag{4}$$

we have $Z\{\delta_s(j)\} = \lambda^{-s}$ $(\lambda \neq 0,\, \infty)$.

Let $Q(\lambda)$ be an $n \times n$ rational matrix function. Then there exist two $n \times n$ rational matrix functions $U(\lambda)$ and $V(\lambda)$ with constant nonzero determinants such that

$$U(\lambda)\, Q(\lambda)\, V(\lambda) \,=\, \mathrm{diag}\left\{ \lambda^{p_1}, \lambda^{p_2}, \ldots, \lambda^{p_\mu}, \frac{1}{\lambda^{q_{\mu+1}}}, \frac{1}{\lambda^{q_{\mu+2}}}, \ldots, \frac{1}{\lambda^{q_\nu}} \right\},$$

where $p_1 \,\geq\, p_2 \,\geq\, \cdots \,\geq\, p_\mu \,\geq\, 0$ and $0 \,\leq\, q_{\mu+1} \,\leq\, q_{\mu+1} \,\leq\, \cdots \,\leq\, q_\nu$ are the orders of the poles and the zeros of $Q(\lambda)$ at the infinity, respectively [3,5]. The diagonal matrix function

$$S^{\infty}_{Q(\lambda)} \,=\, \mathrm{diag}\left\{ \lambda^{p_1}, \lambda^{p_2}, \ldots, \lambda^{p_\mu}, \frac{1}{\lambda^{q_{\mu+1}}}, \frac{1}{\lambda^{q_{\mu+2}}}, \ldots, \frac{1}{\lambda^{q_\nu}} \right\} \tag{5}$$

is known as the Smith-McMillan form of $Q(\lambda)$ at $\lambda = \infty$.

Finally, we recall that a rational matrix function $Q(\lambda) = [q_{jk}(\lambda)]$ is called *strictly proper (proper)* if for every element $q_{jk}(\lambda)$, the degree of the denominator is greater (resp. not less) than the degree of the corresponding numerator (see also [4]).

2 Zeros at the infinity

Consider the homogeneous difference equation in (2). Applying the Z-transformation (for every $j = 0, 1, 2, \dots$) yields

$$Z\{A_m u_{j+m} + \cdots + A_1 u_{j+1} + A_0 u_j\} \;=\; 0$$

or equivalently,

$$A_m Z\{u_{j+m}\} + \cdots + A_1 Z\{u_{j+1}\} + A_0\{u_j\} \;=\; 0.$$

Hence by (3),

$$A_m \left(\lambda^m Z\{u_j\} - \sum_{k=0}^{m-1} u_k \lambda^{m-k} \right) + \cdots + A_1 \left(\lambda Z\{u_j\} - u_0 \lambda \right) + A_0\{u_j\} \;=\; 0,$$

and thus,

$$(A_m \lambda^m + \cdots + A_1 \lambda + A_0) Z\{u_j\} \;=\; \alpha(\lambda),$$

where the vector

$$\alpha(\lambda) \;=\; \sum_{\rho=1}^{m} \left(A_\rho \sum_{k=0}^{\rho-1} u_k \lambda^{\rho-k} \right)$$

is the *initial condition vector* associated with $u_0, u_1, \dots, u_{m-1}$. Consequently, for the $n \times n$ matrix polynomial

$$L(\lambda) \;=\; A_m \lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1 \lambda + A_0,$$

it is clear that

$$Z\{u_j\} \;=\; L(\lambda)^{-1}\alpha(\lambda) \qquad (\det L(\lambda) \neq 0). \tag{6}$$

Let $S_{L(\lambda)}^{\infty} \;=\; \mathrm{diag}\left\{\lambda^{p_1}, \lambda^{p_2}, \dots, \lambda^{p_\mu}, 1/\lambda^{q_{\mu+1}}, 1/\lambda^{q_{\mu+2}}, \dots, 1/\lambda^{q_\nu}\right\}$ be the Smith-McMillan form of $L(\lambda)$ as in (5), and assume that $L(\lambda)$ has at least one zero at ∞, i.e., $q_\nu \geq 1$. Then by Proposition 2.1 in [5], the Laurent expansion at $\lambda = \infty$ of the rational matrix function $L(\lambda)^{-1}$ is of the form

$$L(\lambda)^{-1} \;=\; B_{q_\nu}\lambda^{q_\nu} + \cdots + B_1\lambda + B_0 + B_{-1}\lambda^{-1} + B_{-2}\lambda^{-2} + \cdots.$$

Hence, (6) implies

$$Z\{u_j\} \;=\; \left(B_{q_\nu}\lambda^{q_\nu} + \cdots + B_1\lambda + B_0 + B_{-1}\lambda^{-1} + B_{-2}\lambda^{-2} + \cdots\right) \alpha(\lambda)$$

or equivalently,

$$Z\{u_j\} \;=\; \begin{bmatrix} I\lambda^{q_\nu} & I\lambda^{q_\nu - 1} & \cdots & I\lambda & I & I\lambda^{-1} & \cdots \end{bmatrix} \begin{bmatrix} B_{q_\nu} \\ \vdots \\ B_1 \\ B_0 \\ B_{-1} \\ \vdots \end{bmatrix} \alpha(\lambda).$$

The initial condition vector $\alpha(\lambda)$ is written

$$\alpha(\lambda) = \begin{bmatrix} I\lambda^m & I\lambda^{m-1} & \cdots & I\lambda \end{bmatrix} \begin{bmatrix} A_m & 0 & \cdots & 0 \\ A_{m-1} & A_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_m \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix}$$

and by straightforward computations, one can see that

$$\begin{aligned} Z\{u_j\} = & \begin{bmatrix} I\lambda^{q_\nu+m} & I\lambda^{q_\nu+m-1} & \cdots & I\lambda & I & I\lambda^{-1} & \cdots \end{bmatrix} \\ & \times \begin{bmatrix} B_{q_\nu} & 0 & \cdots & 0 & 0 \\ B_{q_\nu-1} & B_{q_\nu} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ B_{q_\nu-m+1} & B_{q_\nu-m+2} & \cdots & B_{q_\nu-1} & B_{q_\nu} \\ B_{q_\nu-m} & B_{q_\nu-m+1} & \cdots & B_{q_\nu-2} & B_{q_\nu-1} \\ \vdots & \vdots & & \vdots & \vdots \\ B_{-m} & B_{-m+1} & \cdots & B_{-2} & B_{-1} \\ B_{-m-1} & B_{-m} & \cdots & B_{-3} & B_{-2} \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix} \\ & \times \begin{bmatrix} A_m & 0 & \cdots & 0 \\ A_{m-1} & A_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_m \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix}. \end{aligned}$$

As a consequence, $Z\{u_j\}$ is of the form

$$Z\{u_j\} = \beta_{pol}(\lambda) + \beta_{sp}(\lambda), \tag{7}$$

where

$$\begin{aligned} \beta_{pol}(\lambda) = & \begin{bmatrix} I\lambda^{q_\nu+m} & I\lambda^{q_\nu+m-1} & \cdots & I\lambda & I \end{bmatrix} \\ & \times \begin{bmatrix} B_{q_\nu} & 0 & \cdots & 0 & 0 \\ B_{q_\nu-1} & B_{q_\nu} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ B_{q_\nu-m+1} & B_{q_\nu-m+2} & \cdots & B_{q_\nu-1} & B_{q_\nu} \\ B_{q_\nu-m} & B_{q_\nu-m+1} & \cdots & B_{q_\nu-2} & B_{q_\nu-1} \\ \vdots & \vdots & & \vdots & \vdots \\ B_{-m} & B_{-m+1} & \cdots & B_{-2} & B_{-1} \end{bmatrix} \\ & \times \begin{bmatrix} A_m & 0 & \cdots & 0 \\ A_{m-1} & A_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_m \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \end{aligned}$$

is the *polynomial part* of $Z\{u_j\}$, and

$$\beta_{sp}(\lambda) \;=\; \begin{bmatrix} I\lambda^{-1} & I\lambda^{-2} & \cdots \end{bmatrix} \begin{bmatrix} B_{-m-1} & B_{-m} & \cdots & B_{-3} & B_{-2} \\ B_{-m-2} & B_{-m-1} & \cdots & B_{-4} & B_{-3} \\ B_{-m-3} & B_{-m-2} & \cdots & B_{-5} & B_{-4} \\ \vdots & \vdots & & \vdots & \vdots \end{bmatrix}$$

$$\times \begin{bmatrix} A_m & 0 & \cdots & 0 \\ A_{m-1} & A_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_m \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \tag{8}$$

is the *strictly proper part* of $Z\{u_j\}$. Notice that $\beta_{sp}(\lambda)$ is of the form

$$\beta_{sp}(\lambda) \;=\; \gamma_1\lambda^{-1} + \gamma_2\lambda^{-2} + \gamma_3\lambda^{-3} + \cdots \tag{9}$$

where the coefficients $\gamma_1, \gamma_2, \gamma_3, \ldots \in \mathbb{C}^n$ are obtained by (8). In particular,

$$\begin{aligned} \gamma_1 &= \left(\sum_{k=1}^{m} B_{-k-1}A_k\right) u_0 + \left(\sum_{k=2}^{m} B_{-k}A_k\right) u_1 + \cdots + B_{-2}A_m u_{m-1} \\ \gamma_2 &= \left(\sum_{k=1}^{m} B_{-k-2}A_k\right) u_0 + \left(\sum_{k=2}^{m} B_{-k-1}A_k\right) u_1 + \cdots + B_{-3}A_m u_{m-1} \\ &\qquad \vdots \qquad\qquad\qquad \vdots \\ \gamma_j &= \left(\sum_{k=1}^{m} B_{-k-j}A_k\right) u_0 + \left(\sum_{k=2}^{m} B_{-k-j+1}A_k\right) u_1 + \cdots + B_{-j-1}A_m u_{m-1} \\ &\qquad \vdots \qquad\qquad\qquad \vdots \end{aligned} \tag{10}$$

The equation $L(\lambda)^{-1}L(\lambda) = I$ $(\det L(\lambda) \neq 0)$ is written

$$\left(B_{q_\nu}\lambda^{q_\nu} + \cdots + B_1\lambda + B_0 + B_{-1}\lambda^{-1} + \cdots\right)\left(A_m\lambda^m + \cdots + A_1\lambda + A_0\right) \;=\; I$$

and yields the equations

$$\begin{aligned} 0 &= B_{q_\nu}A_m \\ 0 &= B_{q_\nu}A_{m-1} + B_{q_\nu-1}A_m \\ 0 &= B_{q_\nu}A_{m-2} + B_{q_\nu-1}A_{m-1} + B_{q_\nu-2}A_m \\ \vdots & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ 0 &= B_{q_\nu}A_1 + B_{q_\nu-1}A_2 + \cdots + B_{q_\nu-m+1}A_m \\ 0 &= B_{q_\nu}A_0 + B_{q_\nu-1}A_1 + \cdots + B_{q_\nu-m}A_m \\ \vdots & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ I &= B_0A_0 + B_{-1}A_1 + \cdots + B_{-m}A_m \\ \vdots & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{aligned} \tag{11}$$

Then we can verify that

$$\begin{bmatrix} B_{q_\nu} & 0 & \cdots & 0 & 0 \\ B_{q_\nu-1} & B_{q_\nu} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ B_{q_\nu-m+1} & B_{q_\nu-m+2} & \cdots & B_{q_\nu-1} & B_{q_\nu} \\ B_{q_\nu-m} & B_{q_\nu-m+1} & \cdots & B_{q_\nu-2} & B_{q_\nu-1} \\ \vdots & \vdots & & \vdots & \vdots \\ B_{-m} & B_{-m+1} & \cdots & B_{-2} & B_{-1} \end{bmatrix} \begin{bmatrix} A_m & 0 & \cdots & 0 \\ A_{m-1} & A_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_m \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ B_{q_\nu} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ B_{q_\nu-1} & B_{q_\nu} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ B_0 & B_1 & \cdots & B_{q_\nu} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ 0 & A_0 & \cdots & A_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix}.$$

Consequently,

$$\begin{aligned} \beta_{pol}(\lambda) &= \left([\, I \;\, 0 \;\, \cdots \;\, 0\,] - [\, I\lambda^{q_\nu} \;\; I\lambda^{q_{\nu-1}} \;\; \cdots \;\; I\lambda \;\; I\,]\, \mathbf{B}_L \right) \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \\ &= u_0 - [\, I\lambda^{q_\nu} \;\; I\lambda^{q_{\nu-1}} \;\; \cdots \;\; I\lambda \;\; I\,]\, \mathbf{B}_L \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix}, \end{aligned} \tag{12}$$

where $\mathbf{B}_L$ is the $n(q_\nu+1) \times nm$ matrix

$$\mathbf{B}_L = \begin{bmatrix} B_{q_\nu} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ B_{q_\nu-1} & B_{q_\nu} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ B_0 & B_1 & \cdots & B_{q_\nu} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ 0 & A_0 & \cdots & A_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix}$$

$$= \begin{bmatrix} B_{q_\nu}A_0 & B_{q_\nu}A_1 & \cdots & B_{q_\nu}A_{m-1} \\ B_{q_\nu-1}A_0 & B_{q_\nu-1}A_1 + B_{q_\nu}A_0 & \cdots & B_{q_\nu-1}A_{m-1} + B_{q_\nu}A_{m-2} \\ B_{q_\nu-2}A_0 & B_{q_\nu-2}A_1 + B_{q_\nu-1}A_0 & \cdots & B_{q_\nu-2}A_{m-1} + B_{q_\nu-1}A_{m-2} + B_{q_\nu}A_{m-3} \\ \vdots & \vdots & & \vdots \\ B_1A_0 & B_1A_1 + B_2A_0 & \cdots & \sum_{j=1}^{q_\nu} B_jA_{m-1-j} \\ B_0A_0 & B_0A_1 + B_1A_0 & \cdots & \sum_{j=0}^{q_\nu} B_jA_{m-1-j} \end{bmatrix}.$$

By straightforward computations, it follows

$$
[\,I\lambda^{q_\nu}\;\; I\lambda^{q_{\nu-1}}\;\; \cdots\;\; I\lambda\;\; I\,]\,\mathbf{B}_L \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \;=\; \left(\sum_{j=0}^{q_\nu} B_j\lambda^j\right) A_0u_0
$$

$$
+\left[\left(\sum_{j=0}^{q_\nu} B_j\lambda^j\right) A_1 + \left(\sum_{j=1}^{q_\nu} B_j\lambda^{j-1}\right) A_0\right] u_1 + \cdots
$$

$$
+\left[\left(\sum_{j=0}^{q_\nu} B_j\lambda^j\right) A_{m-1} + \left(\sum_{j=1}^{q_\nu} B_j\lambda^{j-1}\right) A_{m-2} + \cdots + B_{q_\nu}A_{m-q_\nu-1}\right] u_{m-1}
$$

$$
\begin{aligned}
= \quad & \left(\sum_{j=0}^{q_\nu} B_j\lambda^j\right)\left(\sum_{k=0}^{m-1} A_ku_k\right) + \left(\sum_{j=1}^{q_\nu} B_j\lambda^{j-1}\right)\left(\sum_{k=0}^{m-2} A_ku_{k+1}\right) + \cdots \\
& + (B_{q_\nu}\lambda + B_{q_\nu-1})\left(\sum_{k=0}^{m-q_\nu-2} A_ku_{k+q_\nu-1}\right) + B_{q_\nu}\left(\sum_{k=0}^{m-q_\nu-1} A_ku_{k+q_\nu}\right).
\end{aligned}
$$

Assume now that $u_0, u_1, \ldots, u_{m+q_\nu}$ satisfy (2). Then the equations (2) and (11) imply

$$
\begin{aligned}
& [\,I\lambda^{q_\nu}\;\; I\lambda^{q_{\nu-1}}\;\; \cdots\;\; I\lambda\;\; I\,]\,\mathbf{B}_L \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \;=\; -\,(B_{q_\nu}A_m)\lambda^{q_\nu}u_m \\
& \quad -\,(B_{q_\nu-1}A_m + B_{q_\nu}A_{m-1})\lambda^{q_\nu-1}u_m - (B_{q_\nu}A_m)\lambda^{q_\nu-1}u_{m+1} \\
& \quad -\,(B_{q_\nu}A_{m-2} + B_{q_\nu-1}A_{m-1} + B_{q_\nu-2}A_m)\lambda^{q_\nu-1}u_m - \cdots - (B_{q_\nu}A_m)\,u_{m+q_\nu} \\
= \quad & -\,(B_{q_\nu}A_m)\,(\lambda^{q_\nu}u_m + \lambda^{q_\nu-1}u_{m+1} + \cdots + u_{m+q_\nu}) \\
& -\,(B_{q_\nu}A_{m-1} + B_{q_\nu-1}A_m)\,(\lambda^{q_\nu-1}u_m + \lambda^{q_\nu-2}u_{m+1} + \cdots + u_{m+q_\nu-1}) \\
& -\,(B_{q_\nu}A_{m-2} + B_{q_\nu-1}A_{m-1} + B_{q_\nu-2}A_m)\,(\lambda^{q_\nu-2}u_m + \cdots + u_{m+q_\nu-2}) \\
& \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
& -\,(B_{q_\nu}A_{m-q_\nu} + B_{q_\nu-1}A_{m-q_\nu+1} + B_0A_m)\,u_m \\
= \quad & 0\,.
\end{aligned}
$$

Thus, by (12), we have $\beta_{pol}(\lambda) = u_0$. Moreover, applying the inverse Z-transformation on the equation (7), it follows that

$$
u_j \;=\; Z^{-1}\{\beta_{pol}(\lambda) + \beta_{sp}(\lambda)\} \;=\; Z^{-1}\{\beta_{pol}(\lambda)\} + Z^{-1}\{\beta_{sp}(\lambda)\}. \tag{13}
$$

By the equation (9) and the properties of the Z-transformation and its inverse [2], it is clear that

$$\begin{aligned} Z^{-1}\{\beta_{sp}(\lambda)\} &= Z^{-1}\{\gamma_1\lambda^{-1}+\gamma_2\lambda^{-2}+\gamma_3\lambda^{-3}+\cdots\} \\ &= \gamma_1\,\delta_1(j)+\gamma_2\,\delta_2(j)+\gamma_3\,\delta_3(j)+\cdots \end{aligned} \tag{14}$$

and

$$Z^{-1}\{\beta_{pol}(\lambda)\} \;=\; Z^{-1}\{u_0\} \;=\; u_0\,\delta_0(j), \tag{15}$$

where $\delta_s(j)$ is the Kronecker delta defined by (4) and the coefficients $\gamma_1, \gamma_2, \gamma_3, \ldots$ are given by (10). Hence, for appropriate vectors $u_0, u_1, \ldots, u_{m+q_\nu}$, the solution of the difference equation (2) depends only on the strictly proper part of $Z\{u_j\}$, that is, $\beta_{sp}(\lambda)$ in (8).

Let $u_0, u_1, \ldots, u_{m-1}$ be any choice of initial conditions. Then by (10), (13), (14) and (15), we can construct the corresponding solution $\mathbf{u} = \{u_0, u_1, u_2, \ldots\}$ of the equation (2). As a consequence, the above discussion yields the following result.

Theorem 1 *Consider the homogeneous difference equation (2) and the matrix polynomial* $L(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$. *Let* $S^{\infty}_{L(\lambda)}$ *be the Smith-McMillan form of* $L(\lambda)$ *as in (5), and assume that* $L(\lambda)$ *has at least one zero at* ∞, *i.e.,* $q_\nu \geq 1$. *Then for every choice of initial conditions* $u_0, u_1, \ldots, u_{m-1}$, *the corresponding solution* $\boldsymbol{u} = \{u_0, u_1, u_2, \ldots\}$ *of (2) does not depend on the zeros of* $L(\lambda)$ *at infinity. In particular, the vectors* $u_0, u_1, \ldots, u_{m-1}$ *are arbitrary, and for every* $j = m, m+1, m+2, \ldots$, *the vector* u_j *coincides with* γ_j *in (10).*

Next we present an illustrative example.

Example
Let

$$L(\lambda) \;=\; \begin{bmatrix} \lambda+1 & \lambda^3 \\ 0 & \lambda+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\lambda^3 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\lambda + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

be the matrix polynomial, which corresponds to the difference equation

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u_{j+3} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_{j+1} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_j \;=\; 0 \quad (j = 0, 1, 2, \ldots). \tag{16}$$

Then one can verify that $q_\nu = 1$ and

$$\begin{aligned} L(\lambda)^{-1} &= \begin{bmatrix} (\lambda+1)^{-1} & -\lambda^3(\lambda+1)^{-2} \\ 0 & (\lambda+1)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\lambda + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\lambda^{-1} + \cdots. \end{aligned}$$

Hence, for the vectors

$$u_0 \;=\; \begin{bmatrix} u_{0,1} \\ u_{0,2} \end{bmatrix}, \quad u_1 \;=\; \begin{bmatrix} u_{1,1} \\ u_{1,2} \end{bmatrix} \quad \text{and} \quad u_2 \;=\; \begin{bmatrix} u_{2,1} \\ u_{2,2} \end{bmatrix},$$

we have

$$\begin{aligned}
\beta_{pol} &= \begin{bmatrix} 1 & \lambda-2 & 0 & \lambda-1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{0,1} \\ u_{0,2} \\ u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{bmatrix} \\
&= \begin{bmatrix} (u_{0,2}+u_{1,2})\lambda + u_{0,1} - 2u_{0,2} - u_{1,2} + u_{2,2} \\ u_{0,2} \end{bmatrix}.
\end{aligned}$$

It is clear that for every solution $\{u_0, u_1, u_2, \ldots\}$ of the difference equation (16), the vectors u_0, u_1, u_2 satisfy $u_{0,2}+u_{1,2}=0$ and $u_{2,2}-u_{0,2}=0$, and thus,

$$\begin{aligned}
Z^{-1}\{\beta_{pol}(\lambda)\} &= \begin{bmatrix} (u_{0,2}+u_{1,2})\delta_1(j) + (u_{0,1} - 2u_{0,2} - u_{1,2} + u_{2,2})\delta_0(j) \\ u_{0,2}\delta_0(j) \end{bmatrix} \\
&= \begin{bmatrix} u_{0,2}+u_{1,2} \\ 0 \end{bmatrix} \delta_{-1}(j) + \begin{bmatrix} u_{0,1} - 2u_{0,2} - u_{1,2} + u_{2,2} \\ u_{0,2} \end{bmatrix} \delta_0(j) \\
&= \begin{bmatrix} u_{0,1} \\ u_{0,2} \end{bmatrix} \delta_0(j) \;=\; u_0\, \delta_0(j).
\end{aligned}$$

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