BIRKHOFF-JAMES $\epsilon$-ORTHOGONALITY SETS IN NORMED LINEAR SPACES

MILTIADIS KARAMANLIS AND PANAYIOTIS J. PSARRAKOS

Abstract. Consider a complex normed linear space $(X, \| \cdot \|)$, and let $\chi, \psi \in X$ with $\psi \neq 0$. Motivated by a recent work of Chorianopoulos and Psarrakos (2011) on rectangular matrices, we introduce the Birkhoff-James $\epsilon$-orthogonality set of $\chi$ with respect to $\psi$, and explore its rich structure.

1. Introduction

The numerical range (also known as the field of values) of a square complex matrix $A \in \mathbb{C}^{n \times n}$ is defined as $F(A) = \{ x^* Ax \in \mathbb{C} : x \in \mathbb{C}^n, x^* x = 1 \}$ [8]. This range is a non-empty, compact and convex subset of $\mathbb{C}$, which has been studied extensively and is useful in understanding matrices and operators; see [2, 3, 8, 10] and the references therein. The numerical range $F(A)$ is also written in the form (see [3, 10])

$$F(A) = \{ \mu \in \mathbb{C} : \| A - \lambda I_n \|_2 \geq | \mu - \lambda |, \ \forall \lambda \in \mathbb{C} \},$$

where $\| \cdot \|_2$ denotes the spectral matrix norm (i.e., that norm subordinate to the euclidean vector norm) and $I_n$ is the $n \times n$ identity matrix. As a consequence, $F(A)$ is an infinite intersection of closed (circular) disks $D(\lambda, \| A - \lambda I_n \|_2) = \{ \mu \in \mathbb{C} : | \mu - \lambda | \leq \| A - \lambda I_n \|_2 \} (\lambda \in \mathbb{C})$, namely,

$$F(A) = \bigcap_{\lambda \in \mathbb{C}} D(\lambda, \| A - \lambda I_n \|_2) = \bigcap_{\lambda \in \mathbb{C}} D(\lambda, \| A - \lambda I_n \|_2).$$

For two elements $\chi$ and $\psi$ of a complex normed linear space $(X, \| \cdot \|)$, $\chi$ is said to be Birkhoff-James orthogonal to $\psi$, denoted by $\chi \perp_{BJ} \psi$, if $\| \chi + \lambda \psi \| \geq \| \chi \|$ for all $\lambda \in \mathbb{C}$ [1, 9]. This orthogonality is homogeneous, but it is neither symmetric nor additive [9]. Moreover, for any $\epsilon \in [0, 1)$, $\chi$ is called Birkhoff-James $\epsilon$-orthogonal to $\psi$, denoted by $\chi \perp_{BJ}^\epsilon \psi$, if $\| \chi + \lambda \psi \| \geq \sqrt{1 - \epsilon^2} \| \chi \|$ for
all \( \lambda \in \mathbb{C} \) \([4, 7]\). It is worth mentioning that this relation is also homogeneous. In an inner product space \((X, \langle \cdot, \cdot \rangle)\), with the standard orthogonality relation \( \perp \), a \( \chi \in X \) is called \( \epsilon \)-orthogonal to a \( \psi \in X \), denoted by \( \chi \perp \epsilon \psi \), if \( |\langle \chi, \psi \rangle| \leq \epsilon \| \chi \| \| \psi \| \). Furthermore, \( \chi \perp \epsilon \psi \) (resp., \( \chi \perp BJ \psi \)) if and only if \( \chi \perp BJ \psi \) \([4, 7]\).

Inspired by (1.1) and the above definition of Birkhoff-James \( \epsilon \)-orthogonality, Chorianopoulos and Psarrakos \([6]\) (see also \([5]\) for a primer work) proposed the following definition for rectangular matrices: For any \( A, B \in \mathbb{C}^{n \times m} \) with \( B \neq 0 \), any matrix norm \( \| \cdot \| \), and any \( \epsilon \in [0, 1) \), the Birkhoff-James \( \epsilon \)-orthogonality set of \( A \) with respect to \( B \) is defined as

\[
F_{\| \cdot \|}^\epsilon (A; B) = \{ \mu \in \mathbb{C} : B \perp_{BJ} (A - \mu B) \} = \{ \mu \in \mathbb{C} : \| A - \lambda B \| \geq \sqrt{1 - \epsilon^2} \| B \| \| \mu - \lambda \|, \forall \lambda \in \mathbb{C} \} = \bigcap_{\lambda \in \mathbb{C}} \mathcal{D} \left( \lambda, \frac{\| A - \lambda B \|}{\sqrt{1 - \epsilon^2 \| B \|}} \right).
\]

The Birkhoff-James \( \epsilon \)-orthogonality set is a direct generalization of the standard numerical range. In particular, for \( n = m \), \( \| \cdot \| = \| \cdot \|_2 \), \( B = I_n \) and \( \epsilon = 0 \), we have \( F_{\| \cdot \|}^0 (A; I_n) = F(A) \); see (1.1) and (1.2). Moreover, \( F_{\| \cdot \|}^\epsilon (A; B) \) is a non-empty, compact and convex subset of \( \mathbb{C} \) that lies in the closed disk \( \mathcal{D} \left( 0, \frac{\| A \|}{\sqrt{1 - \epsilon^2 \| B \|}} \right) \) and has interesting geometric properties \([6]\).

In this note, we adopt ideas and techniques from \([6]\) to introduce and study the Birkhoff-James \( \epsilon \)-orthogonality set of elements of a complex normed linear space, generalizing results of \([6]\). In the next section, we give the definition of the set, and verify that it is always non-empty. In Section 3, we explore the growth of the set, and in Section 4, we derive characterizations of its interior and boundary. Finally, in Section 5, we describe the Birkhoff-James \( \epsilon \)-orthogonality set when the norm is induced by an inner product.

2. The definition

Consider a complex normed linear space \((X, \| \cdot \|)\) (for simplicity, \( X \)), and let \( \chi, \psi \in X \) with \( \psi \neq 0 \). For any \( \epsilon \in [0, 1) \), the Birkhoff-James \( \epsilon \)-orthogonality set of \( \chi \) with respect to \( \psi \) is defined and denoted by

\[
F_{\| \cdot \|}^\epsilon (\chi; \psi) = \{ \mu \in \mathbb{C} : \psi \perp_{BJ} (\chi - \mu \psi) \}.
\]
It is straightforward to see that  
\[ F_{\parallel}^\epsilon(\chi; \psi) = \left\{ \mu \in \mathbb{C} : \|\psi - \lambda(\chi - \mu \psi)\| \geq \sqrt{1 - \epsilon^2 \|\psi\|}, \forall \lambda \in \mathbb{C} \right\} \]

\[ = \left\{ \mu \in \mathbb{C} : \left\| \psi - \frac{1}{\lambda}(\chi - \mu \psi) \right\| \geq \sqrt{1 - \epsilon^2 \|\psi\|}, \forall \lambda \in \mathbb{C} \setminus \{0\} \right\} \]

\[ = \left\{ \mu \in \mathbb{C} : \left\| \lambda \psi - (\chi - \mu \psi) \right\| \geq \sqrt{1 - \epsilon^2 \|\psi\|}, \forall \lambda \in \mathbb{C} \setminus \{0\} \right\} \]

\[ = \left\{ \mu \in \mathbb{C} : \left\| \chi - (\mu - \lambda) \psi \right\| \geq \sqrt{1 - \epsilon^2 \|\psi\|} |\lambda|, \forall \lambda \in \mathbb{C} \right\} \]

(2.2) 
\[ \bigcap_{\lambda \in \mathbb{C}} D \left( \lambda, \frac{\|\chi - \lambda \psi\|}{\sqrt{1 - \epsilon^2 \|\psi\|}} \right). \]

The defining formula (2.3) implies that \( F_{\parallel}^\epsilon(\chi; \psi) \) is a compact and convex subset of \( \mathbb{C} \), which lies in the closed disk \( D \left( 0, \frac{\|\chi\|}{\sqrt{1 - \epsilon^2 \|\psi\|}} \right) \). Furthermore, it is apparent that for any \( 0 \leq \epsilon_1 < \epsilon_2 < 1 \), \( F_{\parallel}^{\epsilon_1}(\chi; \psi) \subseteq F_{\parallel}^{\epsilon_2}(\chi; \psi) \).

By Corollary 2.2 of [9], it follows that \( F_{\parallel}^\epsilon(\chi; \psi) \) is always non-empty. For clarity, we give a short proof, adopting arguments from the proofs of Theorems 2.1, 2.2 and Corollary 2.2 of [9].

**Proposition 2.1.** For any \( \chi, \psi \in \mathcal{X} \) with \( \psi \neq 0 \), and any \( \epsilon \in [0,1) \), the Birkhoff-James \( \epsilon \)-orthogonality set \( F_{\parallel}^\epsilon(\chi; \psi) \) is non-empty.

**Proof:** Since \( F_{\parallel}^0(\chi; \psi) \subseteq F_{\parallel}^\epsilon(\chi; \psi) \) for every \( \epsilon \in [0,1) \), it is enough to prove that \( F_{\parallel}^0(\chi; \psi) \neq \emptyset \). Applying the Hahn-Banach Theorem one can verify that for any nonzero \( \psi \in \mathcal{X} \), there is a linear functional \( T : \mathcal{X} \to \mathbb{C} \) such that \( T(\psi) = \|T\| \|\psi\| \). As a consequence,

\[ \|T\| \|\psi\| = |T(\psi)| = |T(\hat{\chi} + \psi)| \leq \|T\| \|\hat{\chi} + \psi\|, \quad \forall \hat{\chi} \in \text{Ker}(T), \]

and hence,

(2.4) 
\[ \psi \perp_{B J} \hat{\chi}, \quad \forall \hat{\chi} \in \text{Ker}(T). \]

For the scalar \( \mu = \frac{T(\chi)}{|T(\psi)|} \), we have that \( T(\chi - \mu \psi) = 0 \), and thus, \( \chi - \mu \psi \in \text{Ker}(T) \). By (2.4), \( \psi \perp_{B J} (\chi - \mu \psi) \), and hence, \( \mu \in F_{\parallel}^0(\chi; \psi) \). \( \square \)

Next we derive some basic properties of the Birkhoff-James \( \epsilon \)-orthogonality set.

**Proposition 2.2.** Let \( \chi, \psi \in \mathcal{X} \) with \( \psi \neq 0 \), and let \( \epsilon \in [0,1) \). Then, for any nonzero \( b \in \mathbb{C} \), \( F_{\parallel}^\epsilon(\chi; b\psi) = \frac{1}{b} F_{\parallel}^\epsilon(\chi; \psi) \).
Proposition 2.4. Let \( \epsilon \in \mathbb{C} \) and suppose that for two real numbers \( \zeta \in \mathbb{C} \), \( \epsilon \) \( \| \zeta \|_{a} \leq \| \zeta \|_{b} \leq C \| \zeta \|_{a} \), \( \forall \zeta \in \mathbb{C} \). Then, for any \( \chi, \psi \in \mathcal{X} \) with \( \psi \neq 0 \) and any \( \epsilon \in [0,1) \), it holds that

\[
F^{\epsilon}_{\| \cdot \|_{a}}(\chi; \psi) \subseteq F^{\epsilon}_{\| \cdot \|_{b}}(\chi; \psi),
\]

where \( \epsilon' = \sqrt{1 - \frac{c^2(1-c^2)}{C^2}} \).

Proof. Suppose \( \mu \in F^{\epsilon}_{\| \cdot \|_{a}}(\chi; \psi) \). Then, it follows readily that

\[
\| \chi - \lambda \psi \|_{a} \geq \sqrt{1 - \epsilon^2} \| \psi \|_{a} | \mu - \lambda |, \quad \forall \lambda \in \mathbb{C},
\]

or

\[
\| \chi - \lambda \psi \|_{b} \geq \sqrt{1 - \epsilon^2} \frac{c}{\| \psi \|_{b}} | \mu - \lambda |, \quad \forall \lambda \in \mathbb{C},
\]

and

\[
\| \chi - \lambda \psi \|_{b} \geq \sqrt{1 - \frac{c^2(1-c^2)}{C^2}} \| \psi \|_{b} | \mu - \lambda |, \quad \forall \lambda \in \mathbb{C},
\]

and the proof is complete. \( \square \)

For example, we consider the vectors \( \chi = \begin{bmatrix} 1 + i \\ 2 + i \\ -1 - i \end{bmatrix} \in \mathbb{C}^3 \), and recall that the (equivalent in \( \mathbb{C}^3 \)) norms \( \| \cdot \|_2 \) and \( \| \cdot \|_1 \) satisfy \( \| \zeta \|_2 \leq \| \zeta \|_1 \leq \sqrt{3} \| \zeta \|_2 \) for all \( \zeta \in \mathbb{C}^3 \). The Birkhoff-James \( \epsilon \)-orthogonality sets \( F^{0.5}_{\| \cdot \|_{1}}(\chi; \psi) \), \( F^{0.5}_{\| \cdot \|_{2}}(\chi; \psi) \) and \( F^{0.5}_{\| \cdot \|_{\frac{3}{2}}}(\chi; \psi) \) are estimated by the unshaded regions in the left,
middle and right parts of Figure 1, respectively. Each estimation results from having drawn 2000 circles of the form \( \{ \mu \in \mathbb{C} : |\mu - \lambda| = \| \chi - \lambda \psi \| \} \); see (2.2) and (2.3). The compactness and the convexity of the sets are apparent, and since \( \sqrt{0.75} = \sqrt{1 - \frac{1-0.5^2}{4}} \), Proposition 2.4 is also confirmed.

![Figure 1](image)

**Figure 1.** The sets \( F_{\|\cdot\|_2}^{0.5}(\chi; \psi) \) (left), \( F_{\|\cdot\|_1}^{0.5}(\chi; \psi) \) (middle), and \( F_{\|\cdot\|_1}^{\sqrt{0.75}}(\chi; \psi) \) (right).

### 3. On the growth of \( F_{\|\cdot\|_2}^{\varepsilon}(\chi; \psi) \)

As mentioned before, for \( 0 \leq \epsilon_1 < \epsilon_2 < 1 \) and for any two elements \( \chi \) and \( \psi \) of a complex normed linear space \( \mathcal{X} \) with \( \psi \neq 0 \), it holds that \( F_{\|\cdot\|_2}^{\epsilon_2}(\chi; \psi) \subseteq F_{\|\cdot\|_2}^{\epsilon_1}(\chi; \psi) \).

**Theorem 3.1.** (For matrices, see [6, Proposition 2].) Let \( \chi, \psi \in \mathcal{X} \) with \( \psi \neq 0 \), and suppose that \( \chi \) is not a scalar multiple of \( \psi \). Then, for any \( 0 \leq \epsilon_1 < \epsilon_2 < 1 \), \( F_{\|\cdot\|_2}^{\epsilon_1}(\chi; \psi) \) lies in the interior of \( F_{\|\cdot\|_2}^{\epsilon_2}(\chi; \psi) \).

**Proof.** It is enough to prove that for any \( \mu \in F_{\|\cdot\|_2}^{\epsilon_1}(\chi; \psi) \), there is a real \( \rho_\mu > 0 \) such that the disk \( D(\mu, \rho_\mu) \) lies in \( F_{\|\cdot\|_2}^{\epsilon_2}(\chi; \psi) \). By the defining formula (2.2) of the Birkhoff-James \( \epsilon \)-orthogonality set \( F_{\|\cdot\|_2}^{\epsilon}(\chi; \psi) \), for any \( \mu \in F_{\|\cdot\|_2}^{\epsilon_1}(\chi; \psi) \),

\[
\| \chi - \lambda \psi \| \geq \sqrt{1 - \epsilon_2^2 \| \psi \| \| \mu - \lambda \|}, \quad \forall \lambda \in \mathbb{C},
\]

or equivalently,

\[
\| \chi - \mu \psi + (\mu - \lambda)\psi \| \geq \sqrt{1 - \epsilon_2^2 \| \psi \| \| \mu - \lambda \|}, \quad \forall \lambda \in \mathbb{C}.
\]
As a consequence,
\[ \|\chi - \mu\psi + \lambda\psi\| \geq \sqrt{1 - \epsilon_1^2} \|\psi\| |\lambda| > \sqrt{1 - \epsilon_2^2} \|\psi\| |\lambda|, \quad \forall \lambda \in \mathbb{C}. \]

Thus, for every complex number \(\lambda \neq 0\),
\[ \|\chi - \mu\psi + \lambda\psi\| - \sqrt{1 - \epsilon_2^2} \|\psi\| |\lambda| \geq \left( \sqrt{1 - \epsilon_1^2} - \sqrt{1 - \epsilon_2^2} \right) \|\psi\| |\lambda| > 0. \]

Since \(\chi\) is not a scalar multiple of \(\psi\), it follows that \(\|\chi - \mu\psi + \lambda\psi\| > 0\), and hence, the continuous function \(f(\lambda) = \|\chi - \mu\psi + \lambda\psi\| - \sqrt{1 - \epsilon_2^2} \|\psi\| |\lambda|\) takes only positive values in the disk \(D(0, 1)\). Thus, \(\inf_{\lambda \in D(0, 1)} f(\lambda) = \min_{\lambda \in D(0, 1)} f(\lambda) > 0\).

This means that we can consider a real \(\delta > 0\) such that
\[ \delta \leq \min_{\lambda \in D(0, 1)} \left\{ \min_{\lambda \in D(0, 1)} f(\lambda), \left( \sqrt{1 - \epsilon_1^2} - \sqrt{1 - \epsilon_2^2} \right) \|\psi\| \right\} \cdot \]

For every \(\lambda \in \mathbb{C}\) with \(|\lambda| > 1\), we have
\[ \|\chi - \mu\psi + \lambda\psi\| - \sqrt{1 - \epsilon_2^2} \|\psi\| |\lambda| \geq \left( \sqrt{1 - \epsilon_1^2} - \sqrt{1 - \epsilon_2^2} \right) \|\psi\| \geq \delta, \]

and thus,
\[ \delta \leq \min_{\lambda \in \mathbb{C}} \left\{ \|\chi - \mu\psi + \lambda\psi\| - \sqrt{1 - \epsilon_2^2} \|\psi\| |\lambda| \right\}. \]

As a consequence, for every \(\xi \in D \left( 0, \frac{\delta}{\|\psi\|} \right)\),
\[ \|\chi - (\mu + \xi)\psi + \lambda\psi\| \geq \|\chi - \mu\psi + \lambda\psi\| - \|\xi\psi\| \geq \sqrt{1 - \epsilon_2^2} \|\psi\| |\lambda|, \quad \forall \lambda \in \mathbb{C}. \]

Hence, \(D \left( 0, \frac{\delta}{\|\psi\|} \right) \subseteq F^\epsilon_{\|\psi\|}(\chi; \psi)\), and the proof is complete. \(\square\)

**Corollary 3.2.** Let \(\chi, \psi \in X\) with \(\psi \neq 0\), and suppose that \(\chi\) is not a scalar multiple of \(\psi\). Then, for any \(\epsilon \in (0, 1)\), \(F^\epsilon_{\|\psi\|}(\chi; \psi)\) has a non-empty interior.

**Proposition 3.3.** Let \(\chi, \psi \in X\) with \(\psi \neq 0\). Then, \(\chi = a\psi\) for some \(a \in \mathbb{C}\) if and only if \(F^\epsilon_{\|\psi\|}(\chi; \psi) = \{a\}\) for every \(\epsilon \in (0, 1)\).

**Proof.** If \(\chi = a\psi\) for some \(a \in \mathbb{C}\), then (2.2) yields
\[ F^\epsilon_{\|\psi\|}(\chi; \psi) = F^\epsilon_{\|\psi\|}(a\psi; \psi) = \left\{ \mu \in \mathbb{C} : \|a - \lambda\| \geq \sqrt{1 - \epsilon^2} \|\psi\| |\mu - \lambda|, \quad \forall \lambda \in \mathbb{C} \right\}. \]

It is apparent that \(a \in F^\epsilon_{\|\psi\|}(a\chi; \psi)\). Furthermore, for any \(\mu \neq a\) and \(\lambda = a\), we have \(0 = |a - \lambda| < \sqrt{1 - \epsilon^2} |\mu - \lambda|\), i.e., \(\mu \notin F^\epsilon_{\|\psi\|}(a\psi; \psi)\).
For the converse, suppose that $F^*_\parallel(\chi; \psi) = \{a\}$ for an $\epsilon \in (0, 1)$. Then, by Corollary 3.2, $\chi$ is a scalar multiple of $\psi$, i.e., there is a $b \in \mathbb{C}$ such that $\chi = b\psi$.

As a consequence,

$$|a - \lambda| \geq \sqrt{1 - \epsilon^2}|b - \lambda|, \quad \forall \lambda \in \mathbb{C}.$$ 

For $\lambda = a$, it follows that $|b - a| = 0$, and the proof is complete.

By the proof of the previous proposition, it is clear that if $F^*_{\parallel}(\chi; \psi) = \{a\}$ for an $\epsilon \in (0, 1)$, then $\chi = a\psi$, and consequently, $F^*_{\parallel}(\chi; \psi) = \{a\}$ for all $\epsilon \in [0, 1)$.

**Proposition 3.4.** Let $\chi, \psi \in X$ with $\psi \neq 0$, and let $\epsilon \in [0, 1)$. Then, for any $a, b \in \mathbb{C}$, $F^*_{\parallel}(a\chi + b\psi; \psi) = a F^*_{\parallel}(\chi; \psi) + b$.

**Proof.** If $a = 0$, then Proposition 3.3 yields $F^*_{\parallel}(a\chi; \psi) = \{0\} = 0 F^*_{\parallel}(\chi; \psi)$.

If $a \neq 0$, then

$$F^*_{\parallel}(a\chi; \psi) = \left\{ \mu \in \mathbb{C} : \|a\chi - \lambda\psi\| \geq \sqrt{1 - \epsilon^2}\|\psi\| |\mu - \lambda|, \quad \forall \lambda \in \mathbb{C} \right\}$$

$$= \left\{ \mu \in \mathbb{C} : \left|\frac{\chi}{a} - \frac{\lambda}{a}\psi\right| \geq \sqrt{1 - \epsilon^2}\|\psi\| \left|\frac{\mu}{a} - \frac{\lambda}{a}\right|, \quad \forall \lambda \in \mathbb{C} \right\}$$

$$= \left\{ \mu \in \mathbb{C} : \|\chi - \lambda\psi\| \geq \sqrt{1 - \epsilon^2}\|\psi\| \left|\frac{\mu}{a} - \frac{\lambda}{a}\right|, \quad \forall \lambda \in \mathbb{C} \right\}$$

$$= a F^*_{\parallel}(\chi; \psi).$$

Furthermore, for any $a, b \in \mathbb{C}$,

$$F^*_{\parallel}(a\chi + b\psi; \psi) = \left\{ \mu \in \mathbb{C} : \|a\chi + (b - \lambda)\psi\| \geq \sqrt{1 - \epsilon^2}\|\psi\| |\mu - \lambda|, \quad \forall \lambda \in \mathbb{C} \right\}$$

$$= \left\{ \mu \in \mathbb{C} : \|a\chi - \lambda\psi\| \geq \sqrt{1 - \epsilon^2}\|\psi\| |(\mu - b) - \lambda|, \quad \forall \lambda \in \mathbb{C} \right\}$$

$$= \left\{ \mu \in \mathbb{C} : \mu - b \in F^*_{\parallel}(a\chi; \psi) \right\},$$

and the proof is complete.

If we allow the value $\epsilon = 1$, then (2.2) implies that $F^*_\parallel(A; B) = C$. Furthermore, if $\chi$ is not a scalar multiple of $\psi$, then $F^*_\parallel(A; B)$ can be arbitrarily large for $\epsilon$ sufficiently close to 1.

**Theorem 3.5.** (For matrices, see [6, Proposition 4] ) Let $\chi, \psi \in X$ with $\psi \neq 0$, and suppose that $\chi$ is not a scalar multiple of $\psi$. Then, for any bounded region $\Omega \subseteq \mathbb{C}$, there is an $\epsilon_\Omega \in [0, 1)$ such that $\Omega \subseteq F^*_\parallel(\chi; \psi)$.

**Proof.** Without loss of generality, we may assume that the region $\Omega$ is compact. For the sake of contradiction, we also assume that for every $\epsilon \in [0, 1)$, there is scalar $\mu_\epsilon \in \mathbb{C}$ such that $\mu_\epsilon \notin F^*_{\parallel}(\chi; \psi)$. Then, there exist two sequences...
\[ \{ \epsilon_n \}_{n \in \mathbb{N}} \subset [0,1) \text{ and } \{ \mu_n \}_{n \in \mathbb{N}} \subset \Omega \text{ such that } \epsilon_n \rightarrow 1^+ \text{ and } \mu_n \notin F^\epsilon_n(\chi; \psi) \text{ for all } n \in \mathbb{N}. \] By the compactness of \( \Omega \), it follows that \( \{ \mu_n \}_{n \in \mathbb{N}} \) has a converging subsequence, say \( \{ \mu_{k_n} \}_{n \in \mathbb{N}} \subset \Omega \), which converges to a \( \mu \in \Omega \).

If \( \mu \in F^\epsilon(\chi; \psi) \) for some \( \epsilon \in [0,1) \), then by Theorem 3.1, and without loss of generality, we may assume that \( \mu \) lies in the interior of \( F^\epsilon(\chi; \psi) \). Then there is an \( n' \in \mathbb{N} \) such that \( \mu_{k_n} \in F^\epsilon(\chi; \psi) \) for every \( n \geq n' \). Moreover, there is an \( n'' \in \mathbb{N} \) such that \( \epsilon_{k_n} > \epsilon \) for every \( n \geq n'' \). As a consequence, for every \( n \geq \max\{n', n''\} \), \( \mu_{k_n} \in F^\epsilon(\chi; \psi) \subseteq F^\epsilon_n(\chi; \psi) \); this is a contradiction. So, for every \( \epsilon \in [0,1) \), \( \mu \notin F^\epsilon(\chi; \psi) \).

Thus, for every \( \epsilon_n = \sqrt{1 - \frac{1}{n^2}}, \ n \in \mathbb{N} \), there is a scalar \( \lambda_n \in \mathbb{C} \) such that

\[
\| \chi - (\mu - \lambda_n)\psi \| < \sqrt{1 - \left(\frac{1}{n^2}\right)} \| \psi \| |\lambda_n| = \frac{1}{n} \| \psi \| |\lambda_n|,
\]
or

\[
\| \lambda_n \psi - \chi - \mu \psi \| \leq \lambda_n \| \psi - \mu \psi \| < \frac{1}{n} \| \psi \| |\lambda_n|,
\]
or

\[
|\lambda_n| \| \psi \| \left(1 - \frac{1}{n}\right) < \| \chi - \mu \psi \|.
\]

Hence, for every \( n \geq 2 \),

\[
|\lambda_n| < \frac{\| \chi - \mu \psi \|}{\| \psi \| \left(1 - \frac{1}{n}\right)} \leq 2 \frac{\| \chi - \mu \psi \|}{\| \psi \|}.
\]

The bounded sequence \( \lambda_2, \lambda_3, \ldots \) has a converging subsequence \( \{ \lambda_{k_n} \}_{n \in \mathbb{N}} \) which converges to a scalar \( \lambda_0 \in \mathbb{C} \). By (3.1), it follows

\[
\| \lambda_{k_n} \psi - \chi - \mu \psi \| < \frac{1}{k_n} \| \psi \| |\lambda_{k_n}|,
\]
and as \( n \rightarrow +\infty \),

\[
\| \lambda_0 \psi - \chi - \mu \psi \| = 0.
\]
This is a contradiction because \( \chi \) is not a scalar multiple of \( \psi \). \( \Box \)

**Corollary 3.6.** Let \( \chi, \psi \in X \) with \( \psi \neq 0 \). If \( \chi \) is not a scalar multiple of \( \psi \), then

\[
\mathcal{C} = \bigcup_{n \in \mathbb{N}} F^{1 - \frac{1}{n}}(\chi; \psi).
\]
4. The interior and the boundary of \( F_{\parallel}^{\epsilon} (\chi; \psi) \)

Consider the Birkhoff-James \( \epsilon \)-orthogonality set \( F_{\parallel}^{\epsilon} (\chi; \psi) \), and denote its interior by \( \text{Int} \left[ F_{\parallel}^{\epsilon} (\chi; \psi) \right] \), and its boundary by \( \partial F_{\parallel}^{\epsilon} (\chi; \psi) \).

**Proposition 4.1.** Let \( \chi, \psi \in X \), with \( \psi \neq 0 \). Then, for any \( \epsilon \in [0,1) \),

\[
\text{Int} \left[ F_{\parallel}^{\epsilon} (\chi; \psi) \right] \subseteq \left\{ \mu \in \mathbb{C} : \| \chi - \lambda \psi \| > \sqrt{1 - \epsilon^2} \| \psi \| |\mu - \lambda|, \forall \lambda \in \mathbb{C} \right\}.
\]

**Proof.** If \( \mu \in \text{Int} \left[ F_{\parallel}^{\epsilon} (\chi; \psi) \right] \), then there is a real \( \rho > 0 \) such that \( \mu + \rho e^{i\theta} \in F_{\parallel}^{\epsilon} (\chi; \psi) \) for every \( \theta \in [0, 2\pi] \). Hence, for every \( \lambda \in \mathbb{C} \),

\[
\| \chi - \lambda \psi \| \geq \sqrt{1 - \epsilon^2} \| \psi \| |\mu + \rho e^{i\theta} - \lambda|, \forall \theta \in [0, 2\pi].
\]

Setting \( \theta_\lambda = \arg(\mu - \lambda) \), we observe that

\[
\| \chi - \lambda \psi \| \geq \sqrt{1 - \epsilon^2} \| \psi \| |\mu + \rho e^{i\theta_\lambda} - \lambda| > \sqrt{1 - \epsilon^2} \| \psi \| |\mu - \lambda|,
\]

completing the proof. \( \square \)

**Theorem 4.2.** (For matrices, see [6, Proposition 16].) Let \( \chi, \psi \in X \) with \( \psi \neq 0 \), and let \( \epsilon \in [0,1) \). Suppose also that \( \mu_0 \in F_{\parallel}^{\epsilon} (\chi; \psi) \).

(i): The scalar \( \mu_0 \) lies on the boundary \( \partial F_{\parallel}^{\epsilon} (\chi; \psi) \) if and only if

\[
\inf_{\lambda \in \mathbb{C}} \left\{ \| \chi - \lambda \psi \| - \sqrt{1 - \epsilon^2} \| \psi \| \right\} = 0.
\]

(ii): If \( \epsilon > 0 \), then \( \mu_0 \in \partial F_{\parallel}^{\epsilon} (\chi; \psi) \) if and only if

\[
\min_{\lambda \in \mathbb{C}} \left\{ \| \chi - \lambda \psi \| - \sqrt{1 - \epsilon^2} \| \psi \| \right\} = 0,
\]

or equivalently, if and only if \( \| \chi - \lambda_0 \psi \| = \sqrt{1 - \epsilon^2} \| \psi \| \) for some \( \lambda_0 \in \mathbb{C} \).

**Proof.** (i) Suppose that \( \mu_0 \) is a boundary point of the Birkhoff-James \( \epsilon \)-orthogonality set (recall (2.3))

\[
F_{\parallel}^{\epsilon} (\chi; \psi) = \bigcap_{\lambda \in \mathbb{C}} \mathcal{D} \left( \lambda, \frac{\| \chi - \lambda \psi \|}{\sqrt{1 - \epsilon^2} \| \psi \|} \right).
\]

Then, for any \( \delta > 0 \), there is a \( \lambda_\delta \in \mathbb{C} \) such that

\[
\| \chi - \lambda_\delta \psi \| < \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda_\delta| + \delta.
\]

Since the quantity \( \| \chi - \lambda_\delta \psi \| - \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda_\delta| \) is nonnegative, as \( \delta \to 0^+ \), it follows that \( \inf_{\lambda \in \mathbb{C}} \left\{ \| \chi - \lambda \psi \| - \sqrt{1 - \epsilon^2} \| \psi \| \right\} = 0. \)
For the converse, we assume that $\inf_{\lambda \in \mathbb{C}} \left\{ \| \chi - \lambda \psi \| - \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda| \right\} = 0$ and $\mu_0 \in \text{Int} \left[ F^\epsilon_{\| \cdot \|} (\chi; \psi) \right]$. Then, by (2.3), there exists a real $\rho > 0$ such that

$$D(\mu_0, \rho) \subseteq \text{Int} \left[ D \left( \lambda, \frac{\| \chi - \lambda \psi \|}{\sqrt{1 - \epsilon^2} \| \psi \|} \right) \right], \quad \forall \lambda \in \mathbb{C}.$$ 

As a consequence,

$$\| \chi - \lambda \psi \| - \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda| > \sqrt{1 - \epsilon^2} \| \psi \| \rho > 0, \quad \forall \lambda \in \mathbb{C}.$$ 

This means that

$$\inf_{\lambda \in \mathbb{C}} \left\{ \| \chi - \lambda \psi \| - \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda| \right\} > 0$$

which is a contradiction.

(ii) For every $\delta_n = \frac{1}{n}$ ($n \in \mathbb{N}$), there is a $\lambda_n \in \mathbb{C}$ such that

$$\| \chi - \lambda_n \psi \| < \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda_n| + \delta_n,$$

or

$$\| \| \chi \| - \| \lambda_n \psi \| \| < \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda_n| + \frac{1}{n},$$

or

$$|\lambda_n| \| \psi \| - \| \chi \| < \sqrt{1 - \epsilon^2} \| \psi \| (|\mu_0| + |\lambda_n|) + \frac{1}{n}.$$ 

Since $\epsilon > 0$, one can verify that

$$|\lambda_n| < \frac{\| \chi \| + \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0| + 1}{\| \psi \| (1 - \sqrt{1 - \epsilon^2})},$$

i.e., the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ is bounded and has a converging subsequence $\lambda_{k_n} \longrightarrow \lambda_0$. As a consequence,

$$\| \chi - \lambda_{k_n} \psi \| < \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda_{k_n}| + \frac{1}{k_n}, \quad \forall n \in \mathbb{N},$$

and as $n \longrightarrow +\infty$,

$$\| \chi - \lambda_0 \psi \| \leq \sqrt{1 - \epsilon^2} \| \psi \| |\mu_0 - \lambda_0|.$$ 

This inequality can hold only as an equality because $\mu \in F^\epsilon_{\| \cdot \|} (\chi; \psi)$, and the proof is complete. \hfill \Box

**Proposition 4.1** and **Theorem 4.2** yield readily the following.

**Corollary 4.3.** Let $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$. Then, for any $\epsilon \in (0, 1)$,

$$\text{Int} \left[ F^\epsilon_{\| \cdot \|} (\chi; \psi) \right] = \left\{ \mu \in \mathbb{C} : \| \chi - \lambda \psi \| > \sqrt{1 - \epsilon^2} \| \psi \| |\mu - \lambda|, \quad \forall \lambda \in \mathbb{C} \right\}.$$
5. The Case of Norms Induced by Inner Products

In the special case of norms induced by inner products, we can fully describe the Birkhoff-James $\epsilon$-orthogonality set $F_{\|\cdot\|}(\chi; \psi)$. In particular, $F_{\|\cdot\|}(\chi; \psi)$ is always a closed disk; this is the case for $F_{0,1}(\chi; \psi)$ in the left part of Figure 1.

**Theorem 5.1.** (For matrices, see [6, Section 5].) Let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$ and $\epsilon \in [0,1)$, and suppose that the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot \rangle$. Then the Birkhoff-James $\epsilon$-orthogonality set of $\chi$ with respect to $\psi$ is the closed disk

$$F_{\|\cdot\|}(\chi; \psi) = D\left(\frac{\langle \chi, \psi \rangle}{\|\psi\|^2}, \left\| \chi - \frac{\langle \chi, \psi \rangle}{\|\psi\|^2} \psi \right\| \sqrt{1 - \epsilon^2} \|\psi\| \right).$$

**Proof.** A scalar $\mu \in \mathbb{C}$ lies in $F_{\|\cdot\|}(\chi; \psi)$ if and only if [4, 7]

$$\psi \perp \epsilon (\chi - \mu \psi),$$

or equivalently, if and only if

$$|\langle \psi, \chi - \mu \psi \rangle| \leq \epsilon \|\psi\| \|\chi - \mu \psi\|,$$

or equivalently, if and only if

$$\langle \psi, \chi - \mu \psi \rangle \langle \chi - \mu \psi, \psi \rangle \leq \epsilon^2 \|\psi\|^2 \langle \chi - \mu \psi, \chi - \mu \psi \rangle,$$

or equivalently, if and only if

$$\frac{|\langle \chi, \psi \rangle|^2}{\|\psi\|^4} - \frac{\langle \psi, \chi \rangle}{\|\psi\|^2} - \frac{\langle \chi, \psi \rangle}{\|\psi\|^2} |\mu|^2 \leq \epsilon^2 \left(\frac{\|\chi\|^2}{\|\psi\|^2} - \frac{\langle \psi, \chi \rangle}{\|\psi\|^2} - \frac{\langle \chi, \psi \rangle}{\|\psi\|^2} + |\mu|^2\right),$$

or equivalently, if and only if

$$\left| \mu - \frac{\langle \chi, \psi \rangle}{\|\psi\|^2} \right|^2 (1 - \epsilon^2) \leq \frac{\epsilon^2}{\|\psi\|^2} \left\| \chi - \frac{\langle \chi, \psi \rangle}{\|\psi\|^2} \psi \right\|^2.$$

The proof is complete. □

**References**


Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece

E-mail address: kararemilt@gmail.com, ppsarr@math.ntua.gr