BIRKHOFF-JAMES ϵ-ORTHOGONALITY SETS IN NORMED LINEAR SPACES

MILTIADIS KARAMANLIS AND PANAYIOTIS J. PSARRAKOS

Dedicated to Professor Natalia Bebiano on the occasion of her 60th birthday

Abstract. Consider a complex normed linear space $(\mathcal{X}, \|\cdot\|)$, and let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$. Motivated by a recent work of Chorianopoulos and Psarrakos (2011) on rectangular matrices, we introduce the Birkhoff-James ϵ-orthogonality set of χ with respect to ψ, and explore its rich structure.

1. Introduction

The *numerical range* (also known as the *field of values*) of a square complex matrix $A \in \mathbb{C}^{n\times n}$ is defined as $F(A) = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n,\ x^*x = 1\}$ [8]. This range is a *non-empty*, *compact* and *convex* subset of $\mathbb{C}$, which has been studied extensively and is useful in understanding matrices and operators; see [2, 3, 8, 10] and the references therein. The numerical range $F(A)$ is also written in the form (see [3, 10]) $F(A) = \{\mu \in \mathbb{C} : \|A - \lambda I_n\|_2 \geq |\mu - \lambda|,\ \forall\, \lambda \in \mathbb{C}\}$, where $\|\cdot\|_2$ denotes the spectral matrix norm (i.e., that norm subordinate to the euclidean vector norm) and I_n is the $n \times n$ identity matrix. As a consequence, $F(A)$ is an infinite intersection of closed (circular) disks $\mathcal{D}(\lambda, \|A - \lambda I_n\|_2) = \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda I_n\|_2\}$ $(\lambda \in \mathbb{C})$, namely,

$$F(A) = \bigcap_{\lambda\in\mathbb{C}} \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda I_n\|_2\} = \bigcap_{\lambda\in\mathbb{C}} \mathcal{D}(\lambda, \|A - \lambda I_n\|_2). \tag{1.1}$$

For two elements χ and ψ of a complex normed linear space $(\mathcal{X}, \|\cdot\|)$, χ is said to be *Birkhoff-James orthogonal* to ψ, denoted by $\chi \perp_{BJ} \psi$, if $\|\chi + \lambda\psi\| \geq \|\chi\|$ for all $\lambda \in \mathbb{C}$ [1, 9]. This orthogonality is homogeneous, but it is neither symmetric nor additive [9]. Moreover, for any $\epsilon \in [0, 1)$, χ is called *Birkhoff-James ϵ-orthogonal* to ψ, denoted by $\chi \perp^{\epsilon}_{BJ} \psi$, if $\|\chi + \lambda\psi\| \geq \sqrt{1-\epsilon^2}\,\|\chi\|$ for

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all $\lambda \in \mathbb{C}$ [4, 7]. It is worth mentioning that this relation is also homogeneous. In an inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$, with the standard orthogonality relation $\perp$, a $\chi \in \mathcal{X}$ is called *ϵ-orthogonal* to a $\psi \in \mathcal{X}$, denoted by $\chi \perp^{\epsilon} \psi$, if $|\langle \chi, \psi \rangle| \leq \epsilon \|\chi\| \|\psi\|$. Furthermore, $\chi \perp \psi$ (resp., $\chi \perp^{\epsilon} \psi$) if and only if $\chi \perp_{BJ} \psi$ (resp., $\chi \perp_{BJ}^{\epsilon} \psi$) [4, 7].

Inspired by (1.1) and the above definition of Birkhoff-James ϵ-orthogonality, Chorianopoulos and Psarrakos [6] (see also [5] for a primer work) proposed the following definition for rectangular matrices: For any $A, B \in \mathbb{C}^{n\times m}$ with $B \neq 0$, any matrix norm $\|\cdot\|$, and any $\epsilon \in [0,1)$, the *Birkhoff-James ϵ-orthogonality set of A with respect to B* is defined as

$$\begin{aligned} F_{\|\cdot\|}^{\epsilon}(A;B) &= \{\mu \in \mathbb{C} : B \perp_{BJ}^{\epsilon} (A - \mu B)\} \\ &= \left\{\mu \in \mathbb{C} : \|A - \lambda B\| \geq \sqrt{1-\epsilon^2}\, \|B\|\, |\mu - \lambda|,\ \forall \lambda \in \mathbb{C}\right\} \\ &= \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \frac{\|A - \lambda B\|}{\sqrt{1-\epsilon^2}\, \|B\|}\right). \end{aligned} \tag{1.2}$$

The Birkhoff-James ϵ-orthogonality set is a direct generalization of the standard numerical range. In particular, for $n = m$, $\|\cdot\| = \|\cdot\|_2$, $B = I_n$ and $\epsilon = 0$, we have $F_{\|\cdot\|_2}^{0}(A; I_n) = F(A)$; see (1.1) and (1.2). Moreover, $F_{\|\cdot\|}^{\epsilon}(A;B)$ is a *non-empty*, *compact* and *convex* subset of $\mathbb{C}$ that lies in the closed disk $\mathcal{D}\left(0, \frac{\|A\|}{\sqrt{1-\epsilon^2}\, \|B\|}\right)$ and has interesting geometric properties [6].

In this note, we adopt ideas and techniques from [6] to introduce and study the Birkhoff-James ϵ-orthogonality set of elements of a complex normed linear space, generalizing results of [6]. In the next section, we give the definition of the set, and verify that it is always non-empty. In Section 3, we explore the growth of the set, and in Section 4, we derive characterizations of its interior and boundary. Finally, in Section 5, we describe the Birkhoff-James ϵ-orthogonality set when the norm is induced by an inner product.

2. The definition

Consider a complex normed linear space $(\mathcal{X}, \|\cdot\|)$ (for simplicity, $\mathcal{X}$), and let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$. For any $\epsilon \in [0,1)$, the *Birkhoff-James ϵ-orthogonality set of χ with respect to ψ* is defined and denoted by

$$F_{\|\cdot\|}^{\epsilon}(\chi;\psi) = \{\mu \in \mathbb{C} : \psi \perp_{BJ}^{\epsilon} (\chi - \mu\psi)\}. \tag{2.1}$$

It is straightforward to see that

$$\begin{aligned}
F^{\epsilon}_{\|\cdot\|}(\chi;\psi) &= \left\{\mu\in\mathbb{C} : \|\psi-\lambda(\chi-\mu\psi)\| \ge \sqrt{1-\epsilon^2}\,\|\psi\|,\ \forall\,\lambda\in\mathbb{C}\right\}\\
&= \left\{\mu\in\mathbb{C} : \left\|\psi-\frac{1}{\lambda}(\chi-\mu\psi)\right\| \ge \sqrt{1-\epsilon^2}\,\|\psi\|,\ \forall\,\lambda\in\mathbb{C}\setminus\{0\}\right\}\\
&= \left\{\mu\in\mathbb{C} : \frac{1}{|\lambda|}\,\|\lambda\psi-(\chi-\mu\psi)\| \ge \sqrt{1-\epsilon^2}\,\|\psi\|,\ \forall\,\lambda\in\mathbb{C}\setminus\{0\}\right\}\\
&= \left\{\mu\in\mathbb{C} : \|\chi-(\mu-\lambda)\psi\| \ge \sqrt{1-\epsilon^2}\,\|\psi\|\,|\lambda|,\ \forall\,\lambda\in\mathbb{C}\right\}\\
&= \left\{\mu\in\mathbb{C} : \|\chi-\lambda\psi\| \ge \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu-\lambda|,\ \forall\,\lambda\in\mathbb{C}\right\} \qquad (2.2)\\
&= \bigcap_{\lambda\in\mathbb{C}} \mathcal{D}\left(\lambda, \frac{\|\chi-\lambda\psi\|}{\sqrt{1-\epsilon^2}\,\|\psi\|}\right). \qquad (2.3)
\end{aligned}$$

The defining formula (2.3) implies that $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ is a *compact* and *convex* subset of $\mathbb{C}$, which lies in the closed disk $\mathcal{D}\left(0, \frac{\|\chi\|}{\sqrt{1-\epsilon^2}\,\|\psi\|}\right)$. Furthermore, it is apparent that for any $0\le\epsilon_1<\epsilon_2<1$, $F^{\epsilon_1}_{\|\cdot\|}(\chi;\psi)\subseteq F^{\epsilon_2}_{\|\cdot\|}(\chi;\psi)$.

By Corollary 2.2 of [9], it follows that $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ is always *non-empty*. For clarity, we give a short proof, adopting arguments from the proofs of Theorem 2.1, Theorem 2.2 and Corollary 2.2 of [9].

Proposition 2.1. *For any $\chi,\psi\in\mathcal{X}$ with $\psi\ne 0$, and any $\epsilon\in[0,1)$, the Birkhoff-James ϵ-orthogonality set $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ is non-empty.*

Proof. Since $F^{0}_{\|\cdot\|}(\chi;\psi)\subseteq F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ for every $\epsilon\in[0,1)$, it is enough to prove that $F^{0}_{\|\cdot\|}(\chi;\psi)\ne\emptyset$. Applying the Hahn-Banach Theorem one can verify that for any nonzero $\psi\in\mathcal{X}$, there is a linear functional $T:\mathcal{X}\to\mathbb{C}$ such that $T(\psi)=\|T\|\,\|\psi\|$. As a consequence,

$$\|T\|\,\|\psi\| \;=\; |T(\psi)| \;=\; |T(\hat\chi+\psi)| \;\le\; \|T\|\,\|\hat\chi+\psi\|, \quad \forall\,\hat\chi\in\mathrm{Ker}(T),$$

and hence,

$$\psi\perp_{BJ}\hat\chi, \quad \forall\,\hat\chi\in\mathrm{Ker}(T). \qquad (2.4)$$

For the scalar $\mu=\frac{T(\chi)}{\|T\|\,\|\psi\|}$, we have that $T(\chi-\mu\psi)=0$, and thus, $\chi-\mu\psi\in\mathrm{Ker}(T)$. By (2.4), $\psi\perp_{BJ}(\chi-\mu\psi)$, and hence, $\mu\in F^{0}_{\|\cdot\|}(\chi;\psi)$. □

Next we derive some basic properties of the Birkhoff-James ϵ-orthogonality set.

Proposition 2.2. *Let $\chi,\psi\in\mathcal{X}$ with $\psi\ne 0$, and let $\epsilon\in[0,1)$. Then, for any nonzero $b\in\mathbb{C}$, $F^{\epsilon}_{\|\cdot\|}(\chi;b\psi)=\frac{1}{b}\,F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$.*

Proof. By the defining formula (2.1) of the Birkhoff-James ϵ-orthogonality set $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ and the homogeneity of the Birkhoff-James ϵ-orthogonality, it is straightforward that $F^{\epsilon}_{\|\cdot\|}(\chi;b\psi)=\{\mu\in\mathbb{C}:\,\psi\perp^{\epsilon}_{BJ}(\chi-(b\mu)\psi)\}$. □

Proposition 2.3. *Let χ and ψ be two nonzero elements of $\mathcal{X}$. Then, for any $\epsilon\in[0,1)$,*

$$\left\{\mu^{-1}\in\mathbb{C}:\,\mu\in F^{\epsilon}_{\|\cdot\|}(\chi;\psi),\,|\mu|\geq\frac{\|\chi\|}{\|\psi\|}\right\}\subseteq F^{\epsilon}_{\|\cdot\|}(\psi;\chi).$$

Proof. Consider a $\mu\in F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ with $|\mu|\geq\frac{\|\chi\|}{\|\psi\|}$. Then, by (2.2), we have

$$|\lambda|\left\|\psi-\frac{1}{\lambda}\chi\right\|\geq\sqrt{1-\epsilon^2}\,\|\psi\|\,|\lambda|\left|\frac{\mu}{\lambda}-1\right|,\quad\forall\,\lambda\in\mathbb{C}\setminus\{0\},$$

or

$$\|\psi-\lambda\chi\|\geq\sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu|\,\left|\mu^{-1}-\lambda\right|\geq\sqrt{1-\epsilon^2}\,\|\chi\|\,\left|\mu^{-1}-\lambda\right|,\quad\forall\,\lambda\in\mathbb{C}.$$

Thus, μ^{-1} lies in $F^{\epsilon}_{\|\cdot\|}(\psi;\chi)$. □

Proposition 2.4. *Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two equivalent norms acting in $\mathcal{X}$, and suppose that for two real numbers $C,c>0$, $c\,\|\zeta\|_a\leq\|\zeta\|_b\leq C\,\|\zeta\|_a$ for all $\zeta\in\mathcal{X}$. Then, for any $\chi,\psi\in\mathcal{X}$ with $\psi\neq0$ and any $\epsilon\in[0,1)$, it holds that*

$$F^{\epsilon}_{\|\cdot\|_a}(\chi;\psi)\subseteq F^{\epsilon'}_{\|\cdot\|_b}(\chi;\psi),$$

where $\epsilon'=\sqrt{1-\frac{c^2(1-\epsilon^2)}{C^2}}$.

Proof. Suppose $\mu\in F^{\epsilon}_{\|\cdot\|_a}(\chi;\psi)$. Then, it follows readily that

$$\|\chi-\lambda\psi\|_a\geq\sqrt{1-\epsilon^2}\,\|\psi\|_a|\mu-\lambda|,\quad\forall\,\lambda\in\mathbb{C},$$

or

$$\|\chi-\lambda\psi\|_b\geq\sqrt{1-\epsilon^2}\,\frac{c}{C}\,\|\psi\|_b|\mu-\lambda|,\quad\forall\,\lambda\in\mathbb{C},$$

or

$$\|\chi-\lambda\psi\|_b\geq\sqrt{1-\sqrt{1-\frac{c^2(1-\epsilon^2)}{C^2}}^{\,2}}\;\|\psi\|_b|\mu-\lambda|,\quad\forall\,\lambda\in\mathbb{C},$$

and the proof is complete. □

For example, we consider the vectors $\chi=\begin{bmatrix}1\\2+\mathrm{i}\\-11\,\mathrm{i}\end{bmatrix}$, $\psi=\begin{bmatrix}1+\mathrm{i}\\2+\mathrm{i}\\\mathrm{i}\end{bmatrix}\in\mathbb{C}^3$, and recall that the (equivalent in $\mathbb{C}^3$) norms $\|\cdot\|_2$ and $\|\cdot\|_1$ satisfy $\|\zeta\|_2\leq\|\zeta\|_1\leq\sqrt{3}\,\|\zeta\|_2$ for all $\zeta\in\mathbb{C}^3$. The Birkhoff-James ϵ-orthogonality sets $F^{0.5}_{\|\cdot\|_2}(\chi;\psi)$, $F^{0.5}_{\|\cdot\|_1}(\chi;\psi)$ and $F^{\sqrt{0.75}}_{\|\cdot\|_1}(\chi;\psi)$ are estimated by the unshaded regions in the left, middle and right parts of Figure 1, respectively. Each estimation results from having drawn 2000 circles of the form $\{\mu \in \mathbb{C} : |\mu - \lambda| = \|\chi - \lambda\psi\|\}$; see (2.2) and (2.3). The compactness and the convexity of the sets are apparent, and since $\sqrt{0.75} = \sqrt{1 - \frac{1-0.5^2}{3}}$, Proposition 2.4 is also confirmed.

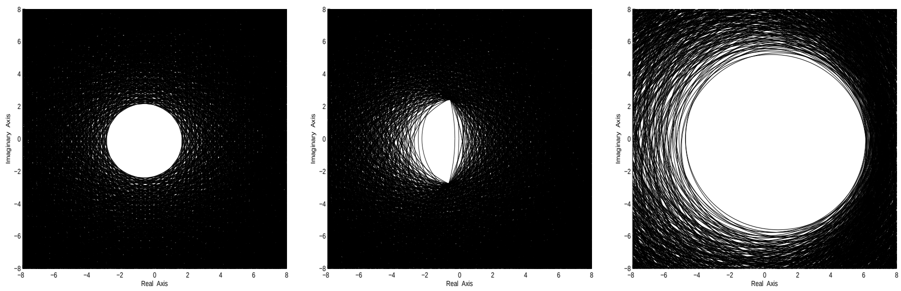

FIGURE 1. The sets $F^{0.5}_{\|\cdot\|_2}(\chi;\psi)$ (left), $F^{0.5}_{\|\cdot\|_1}(\chi;\psi)$ (middle), and $F^{\sqrt{0.75}}_{\|\cdot\|_1}(\chi;\psi)$ (right).

3. ON THE GROWTH OF $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$

As mentioned before, for $0 \le \epsilon_1 < \epsilon_2 < 1$ and for any two elements χ and ψ of a complex normed linear space $\mathcal{X}$ with $\psi \neq 0$, it holds that $F^{\epsilon_1}_{\|\cdot\|}(\chi;\psi) \subseteq F^{\epsilon_2}_{\|\cdot\|}(\chi;\psi)$.

Theorem 3.1. (For matrices, see [6, Proposition 2].) *Let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$, and suppose that χ is not a scalar multiple of ψ. Then, for any $0 \le \epsilon_1 < \epsilon_2 < 1$, $F^{\epsilon_1}_{\|\cdot\|}(\chi;\psi)$ lies in the interior of $F^{\epsilon_2}_{\|\cdot\|}(\chi;\psi)$.*

PROOF. It is enough to prove that for any $\mu \in F^{\epsilon_1}_{\|\cdot\|}(\chi;\psi)$, there is a real $\rho_\mu > 0$ such that the disk $\mathcal{D}(\mu, \rho_\mu)$ lies in $F^{\epsilon_2}_{\|\cdot\|}(\chi;\psi)$. By the defining formula (2.2) of the Birkhoff-James ϵ-orthogonality set $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$, for any $\mu \in F^{\epsilon_1}_{\|\cdot\|}(\chi;\psi)$,

$$\|\chi - \lambda\psi\| \ \ge\ \sqrt{1-\epsilon_1^2}\,\|\psi\|\,|\mu - \lambda|, \quad \forall\, \lambda \in \mathbb{C},$$

or equivalently,

$$\|\chi - \mu\psi + (\mu - \lambda)\psi\| \ \ge\ \sqrt{1-\epsilon_1^2}\,\|\psi\|\,|\mu - \lambda|, \quad \forall\, \lambda \in \mathbb{C}.$$

As a consequence,

$$\|\chi - \mu\psi + \lambda\psi\| \,\geq\, \sqrt{1-\epsilon_1^2}\,\|\psi\|\,|\lambda| \,>\, \sqrt{1-\epsilon_2^2}\,\|\psi\|\,|\lambda|, \quad \forall\,\lambda \in \mathbb{C}.$$

Thus, for every complex number $\lambda \neq 0$,

$$\|\chi - \mu\psi + \lambda\psi\| - \sqrt{1-\epsilon_2^2}\,\|\psi\|\,|\lambda| \,\geq\, \left(\sqrt{1-\epsilon_1^2} - \sqrt{1-\epsilon_2^2}\right)\|\psi\|\,|\lambda| \,>\, 0.$$

Since χ is not a scalar multiple of ψ, it follows that $\|\chi - \mu\psi + \lambda\psi\| > 0$, and hence, the continuous function $f(\lambda) = \|\chi - \mu\psi + \lambda\psi\| - \sqrt{1-\epsilon_2^2}\,\|\psi\|\,|\lambda|$ takes only positive values in the disk $\mathcal{D}(0,1)$. Thus, $\inf\limits_{\lambda\in\mathcal{D}(0,1)} f(\lambda) = \min\limits_{\lambda\in\mathcal{D}(0,1)} f(\lambda) > 0$. This means that we can consider a real $\delta > 0$ such that

$$\delta \,\leq\, \min\left\{ \min_{\lambda\in\mathcal{D}(0,1)} f(\lambda), \left(\sqrt{1-\epsilon_1^2} - \sqrt{1-\epsilon_2^2}\right)\|\psi\| \right\}.$$

For every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, we have

$$\|\chi - \mu\psi + \lambda\psi\| - \sqrt{1-\epsilon_2^2}\,\|\psi\|\,|\lambda| \,\geq\, \left(\sqrt{1-\epsilon_1^2} - \sqrt{1-\epsilon_2^2}\right)\|\psi\| \,\geq\, \delta,$$

and thus,

$$\delta \,\leq\, \inf_{\lambda\in\mathbb{C}} \left\{ \|\chi - \mu\psi + \lambda\psi\| - \sqrt{1-\epsilon_2^2}\,\|\psi\|\,|\lambda| \right\}.$$

As a consequence, for every $\xi \in \mathcal{D}\left(0, \frac{\delta}{\|\psi\|}\right)$,

$$\|\chi - (\mu+\xi)\psi + \lambda\psi\| \,\geq\, \|\chi - \mu\psi + \lambda\psi\| - \|\xi\psi\| \,\geq\, \sqrt{1-\epsilon_2^2}\,\|\psi\|\,|\lambda|, \quad \forall\,\lambda \in \mathbb{C}.$$

Hence, $\mathcal{D}\left(0, \frac{\delta}{\|\psi\|}\right) \subseteq F^{\epsilon_2}_{\|\cdot\|}(\chi;\psi)$, and the proof is complete. □

Corollary 3.2. *Let $\chi, \psi \in X$ with $\psi \neq 0$, and suppose that χ is not a scalar multiple of ψ. Then, for any $\epsilon \in (0,1)$, $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ has a non-empty interior.*

Proposition 3.3. *Let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$. Then, $\chi = a\psi$ for some $a \in \mathbb{C}$ if and only if $F^{\epsilon}_{\|\cdot\|}(\chi;\psi) = \{a\}$ for every $\epsilon \in [0,1)$.*

Proof. If $\chi = a\psi$ for some $a \in \mathbb{C}$, then (2.2) yields

$$\begin{aligned} F^{\epsilon}_{\|\cdot\|}(\chi;\psi) &= F^{\epsilon}_{\|\cdot\|}(a\psi;\psi) \\ &= \left\{ \mu \in \mathbb{C} : \|(a-\lambda)\psi\| \geq \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu-\lambda|,\ \forall\,\lambda \in \mathbb{C} \right\} \\ &= \left\{ \mu \in \mathbb{C} : |a-\lambda| \geq \sqrt{1-\epsilon^2}\,|\mu-\lambda|,\ \forall\,\lambda \in \mathbb{C} \right\}. \end{aligned}$$

It is apparent that $a \in F^{\epsilon}_{\|\cdot\|}(a\chi;\psi)$. Furthermore, for any $\mu \neq a$ and $\lambda = a$, we have $0 = |a-\lambda| < \sqrt{1-\epsilon^2}\,|\mu-\lambda|$, i.e., $\mu \notin F^{\epsilon}_{\|\cdot\|}(a\psi;\psi)$.

For the converse, suppose that $F^{\epsilon}_{\|\cdot\|}(\chi;\psi) = \{a\}$ for an $\epsilon \in (0,1)$. Then, by Corollary 3.2, χ is a scalar multiple of ψ, i.e., there is a $b \in \mathbb{C}$ such that $\chi = b\psi$. As a consequence,

$$|a-\lambda| \;\geq\; \sqrt{1-\epsilon^2}\,|b-\lambda|, \quad \forall\,\lambda \in \mathbb{C}.$$

For $\lambda = a$, it follows that $|b-a| = 0$, and the proof is complete. □

By the proof of the previous proposition, it is clear that if $F^{\epsilon}_{\|\cdot\|}(\chi;\psi) = \{a\}$ for an $\epsilon \in (0,1)$, then $\chi = a\psi$, and consequently, $F^{\epsilon}_{\|\cdot\|}(\chi;\psi) = \{a\}$ for all $\epsilon \in [0,1)$.

Proposition 3.4. *Let $\chi,\psi \in \mathcal{X}$ with $\psi \neq 0$, and let $\epsilon \in [0,1)$. Then, for any $a,b \in \mathbb{C}$, $F^{\epsilon}_{\|\cdot\|}(a\chi + b\psi;\psi) = a\,F^{\epsilon}_{\|\cdot\|}(\chi;\psi) + b$.*

Proof. If $a = 0$, then Proposition 3.3 yields $F^{\epsilon}_{\|\cdot\|}(a\chi;\psi) = \{0\} = 0\,F^{\epsilon}_{\|\cdot\|}(\psi;\psi)$. If $a \neq 0$, then

$$\begin{aligned}
F^{\epsilon}_{\|\cdot\|}(a\chi;\psi) &= \left\{\mu \in \mathbb{C} :\ \|a\chi - \lambda\psi\| \geq \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu-\lambda|,\ \forall\,\lambda \in \mathbb{C}\right\} \\
&= \left\{\mu \in \mathbb{C} :\ \left\|\chi - \frac{\lambda}{a}\,\psi\right\| \geq \sqrt{1-\epsilon^2}\,\|\psi\|\,\left|\frac{\mu}{a} - \frac{\lambda}{a}\right|,\ \forall\,\lambda \in \mathbb{C}\right\} \\
&= \left\{\mu \in \mathbb{C} :\ \|\chi - \lambda\psi\| \geq \sqrt{1-\epsilon^2}\,\|\psi\|\,\left|\frac{\mu}{a} - \lambda\right|,\ \forall\,\lambda \in \mathbb{C}\right\} \\
&= a\,F^{\epsilon}_{\|\cdot\|}(\chi;\psi).
\end{aligned}$$

Furthermore, for any $a,b \in \mathbb{C}$,

$$\begin{aligned}
F^{\epsilon}_{\|\cdot\|}(a\chi + b\psi;\psi) &= \left\{\mu \in \mathbb{C} :\ \|a\chi + (b-\lambda)\psi\| \geq \sqrt{1-\epsilon^2}\|\psi\|\,|\mu-\lambda|,\ \forall\,\lambda \in \mathbb{C}\right\} \\
&= \left\{\mu \in \mathbb{C} :\ \|a\chi - \lambda\psi\| \geq \sqrt{1-\epsilon^2}\,\|\psi\|\,|(\mu-b)-\lambda),\ \forall\,\lambda \in \mathbb{C}\right\} \\
&= \left\{\mu \in \mathbb{C} :\ \mu - b \in F^{\epsilon}_{\|\cdot\|}(a\chi;\psi)\right\},
\end{aligned}$$

and the proof is complete. □

If we allow the value $\epsilon = 1$, then (2.2) implies that $F^{1}_{\|\cdot\|}(A;B) = \mathbb{C}$. Furthermore, if χ is not a scalar multiple of ψ, then $F^{\epsilon}_{\|\cdot\|}(A;B)$ can be arbitrarily large for ϵ sufficiently close to 1.

Theorem 3.5. (For matrices, see [6, Proposition 4].) *Let $\chi,\psi \in \mathcal{X}$ with $\psi \neq 0$, and suppose that χ is not a scalar multiple of ψ. Then, for any bounded region $\Omega \subset \mathbb{C}$, there is an $\epsilon_\Omega \in [0,1)$ such that $\Omega \subseteq F^{\epsilon_\Omega}_{\|\cdot\|}(\chi;\psi)$.*

Proof. Without loss of generality, we may assume that the region Ω is compact. For the sake of contradiction, we also assume that for every $\epsilon \in [0,1)$, there is scalar $\mu_\epsilon \in \mathbb{C}$ such that $\mu_\epsilon \notin F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$. Then, there exist two sequences

$\{\epsilon_n\}_{n\in\mathbb{N}} \subset [0,1)$ and $\{\mu_n\}_{n\in\mathbb{N}} \subset \Omega$ such that $\epsilon_n \longrightarrow 1^-$ and $\mu_n \notin F^{\epsilon_n}_{\|\cdot\|}(\chi;\psi)$ for all $n \in \mathbb{N}$. By the compactness of Ω, it follows that $\{\mu_n\}_{n\in\mathbb{N}}$ has a converging subsequence, say $\{\mu_{k_n}\}_{n\in\mathbb{N}} \subset \Omega$, which converges to a $\mu \in \Omega$.

If $\mu \in F^{\hat\epsilon}_{\|\cdot\|}(\chi;\psi)$ for some $\hat\epsilon \in [0,1)$, then by Theorem 3.1, and without loss of generality, we may assume that μ lies in the interior of $F^{\hat\epsilon}_{\|\cdot\|}(\chi;\psi)$. Then there is an $n' \in \mathbb{N}$ such that $\mu_{k_n} \in F^{\hat\epsilon}_{\|\cdot\|}(\chi;\psi)$ for every $n \geq n'$. Moreover, there is an $n'' \in \mathbb{N}$ such that $\epsilon_{k_n} > \hat\epsilon$ for every $n \geq n''$. As a consequence, for every $n \geq \max\{n', n''\}$, $\mu_{k_n} \in F^{\hat\epsilon}_{\|\cdot\|}(\chi;\psi) \subseteq F^{\epsilon_{k_n}}_{\|\cdot\|}(\chi;\psi)$; this a a contradiction. So, for every $\epsilon \in [0,1)$, $\mu \notin F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$. Thus, for every $\epsilon_n = \sqrt{1-\frac{1}{n^2}}$, $n \in \mathbb{N}$, there is a scalar $\lambda_n \in \mathbb{C}$ such that

$$\|\chi - (\mu - \lambda_n)\psi\| \;<\; \sqrt{1-\left(\sqrt{1-\frac{1}{n^2}}\right)^2}\,\|\psi\|\,|\lambda_n| \;=\; \frac{1}{n}\,\|\psi\|\,|\lambda_n|,$$

or

$$\big|\,\|\lambda_n\psi\| - \|\chi - \mu\psi\|\,\big| \;\leq\; \|\lambda_n\psi - \chi - \mu\psi\| \;<\; \frac{1}{n}\,\|\psi\|\,|\lambda_n|, \tag{3.1}$$

or

$$|\lambda_n|\;\|\psi\|\left(1-\frac{1}{n}\right) \;<\; \|\chi - \mu\psi\|\,.$$

Hence, for every $n \geq 2$,

$$|\lambda_n| \;<\; \frac{\|\chi-\mu\psi\|}{\|\psi\|\left(1-\frac{1}{n}\right)} \;\leq\; 2\,\frac{\|\chi-\mu\psi\|}{\|\psi\|}\,.$$

The bounded sequence $\lambda_2, \lambda_3, \ldots$ has a converging subsequence $\{\lambda_{k_n}\}_{n\in\mathbb{N}}$ which converges to a scalar $\lambda_0 \in \mathbb{C}$. By (3.1), it follows

$$\|\lambda_{k_n}\psi - \chi - \mu\psi\| \;<\; \frac{1}{k_n}\,\|\psi\|\,|\lambda_{k_n}|,$$

and as $n \longrightarrow +\infty$,

$$\|\lambda_0\psi - \chi - \mu\psi\| \;=\; 0.$$

This is a contradiction because χ is not a scalar multiple of ψ. □

Corollary 3.6. *Let $\chi, \psi \in X$ with $\psi \neq 0$. If χ is not a scalar multiple of ψ, then*

$$\mathbb{C} \;=\; \bigcup_{n\in\mathbb{N}} F^{1-\frac{1}{n}}_{\|\cdot\|}(\chi;\psi).$$

4. The interior and the boundary of $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$

Consider the Birkhoff-James ϵ-orthogonality set $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$, and denote its interior by Int $\left[F^{\epsilon}_{\|\cdot\|}(\chi;\psi)\right]$, and its boundary by $\partial F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$.

Proposition 4.1. *Let $\chi,\psi\in\mathcal{X}$, with $\psi\neq 0$. Then, for any $\epsilon\in[0,1)$,*

$$\mathrm{Int}\left[F^{\epsilon}_{\|\cdot\|}(\chi;\psi)\right] \subseteq \left\{\mu\in\mathbb{C}:\ \|\chi-\lambda\psi\| > \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu-\lambda|,\ \forall\,\lambda\in\mathbb{C}\right\}.$$

Proof. If $\mu\in\mathrm{Int}\left[F^{\epsilon}_{\|\cdot\|}(\chi;\psi)\right]$, then there is a real $\rho>0$ such that $\mu+\rho\,e^{\mathrm{i}\theta}\in F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ for every $\theta\in[0,2\,\pi]$. Hence, for every $\lambda\in\mathbb{C}$,

$$\|\chi-\lambda\psi\| \ \geq\ \sqrt{1-\epsilon^2}\ \|\psi\|\ |\mu+\rho\,e^{\mathrm{i}\theta}-\lambda|, \quad \forall\,\theta\in[0,2\,\pi].$$

Setting $\theta_\lambda=\arg(\mu-\lambda)$, we observe that

$$\|\chi-\lambda\psi\| \ \geq\ \sqrt{1-\epsilon^2}\ \|\psi\|\ |\mu+\rho\,e^{\mathrm{i}\theta_\lambda}-\lambda| \ >\ \sqrt{1-\epsilon^2}\ \|\psi\|\ |\mu-\lambda|,$$

completing the proof. □

Theorem 4.2. (For matrices, see [6, Proposition 16].) *Let $\chi,\psi\in\mathcal{X}$ with $\psi\neq 0$, and let $\epsilon\in[0,1)$. Suppose also that $\mu_0\in F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$.*

(i): *The scalar μ_0 lies on the boundary $\partial F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ if and only if*

$$\inf_{\lambda\in\mathbb{C}}\left\{\|\chi-\lambda\psi\|-\sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0-\lambda|\right\} \ =\ 0.$$

(ii): *If $\epsilon>0$, then $\mu_0\in\partial F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ if and only if*

$$\min_{\lambda\in\mathbb{C}}\left\{\|\chi-\lambda\psi\|-\sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0-\lambda|\right\} \ =\ 0,$$

or equivalently, if and only if $\|\chi-\lambda_0\psi\|=\sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0-\lambda_0|$ for some $\lambda_0\in\mathbb{C}$.

Proof. (i) Suppose that μ_0 is a boundary point of the Birkhoff-James ϵ-orthogonality set (recall (2.3))

$$F^{\epsilon}_{\|\cdot\|}(\chi;\psi) \ =\ \bigcap_{\lambda\in\mathbb{C}}\mathcal{D}\left(\lambda,\frac{\|\chi-\lambda\psi\|}{\sqrt{1-\epsilon^2}\,\|\psi\|}\right).$$

Then, for any $\delta>0$, there is a $\lambda_\delta\in\mathbb{C}$ such that

$$\|\chi-\lambda_\delta\psi\| \ <\ \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0-\lambda_\delta|+\delta.$$

Since the quantity $\|\chi-\lambda_\delta\psi\|-\sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0-\lambda_\delta|$ is nonnegative, as $\delta\longrightarrow 0^+$, it follows that $\inf_{\lambda\in\mathbb{C}}\left\{\|\chi-\lambda\psi\|-\sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0-\lambda|\right\}=0$.

For the converse, we assume that $\inf_{\lambda\in\mathbb{C}} \left\{\|\chi - \lambda\psi\| - \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0 - \lambda|\right\} = 0$ and $\mu_0 \in \mathrm{Int}\left[F^{\epsilon}_{\|\cdot\|}(\chi;\psi)\right]$. Then, by (2.3), there exists a real $\rho > 0$ such that

$$\mathcal{D}(\mu_0,\rho) \subseteq \mathrm{Int}\left[\mathcal{D}\left(\lambda, \frac{\|\chi - \lambda\psi\|}{\sqrt{1-\epsilon^2}\,\|\psi\|}\right)\right], \quad \forall\,\lambda \in \mathbb{C}.$$

As a consequence,

$$\|\chi - \lambda\psi\| - \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0 - \lambda| \;>\; \sqrt{1-\epsilon^2}\,\|\psi\|\,\rho \;>\; 0, \quad \forall\,\lambda \in \mathbb{C}.$$

This means that

$$\inf_{\lambda\in\mathbb{C}} \left\{\|\chi - \lambda\psi\| - \sqrt{1-\epsilon^2}\,\|\psi\|\;|\mu_0 - \lambda|\right\} \;>\; 0$$

which is a contradiction.

(ii) For every $\delta_n = \frac{1}{n}$ $(n \in \mathbb{N})$, there is a $\lambda_n \in \mathbb{C}$ such that

$$\|\chi - \lambda_n\psi\| \;<\; \sqrt{1-\epsilon^2}\,\|\psi\|\;|\mu_0 - \lambda_n| + \delta_n,$$

or

$$|\|\chi\| - \|\lambda_n\psi\|| \;<\; \sqrt{1-\epsilon^2}\,\|\psi\|\;|\mu_0 - \lambda_n| + \frac{1}{n}\,,$$

or

$$|\lambda_n|\,\|\psi\| - \|\chi\| \;<\; \sqrt{1-\epsilon^2}\,\|\psi\|\;(\|\mu_0\| + \|\lambda_n\|) + \frac{1}{n}\,.$$

Since $\epsilon > 0$, one can verify that

$$|\lambda_n| \;<\; \frac{\|\chi\| + \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu_0| + 1}{\|\psi\|\left(1 - \sqrt{1-\epsilon^2}\right)}\,,$$

i.e., the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is bounded and has a converging subsequence $\lambda_{k_n} \longrightarrow \lambda_0$. As a consequence,

$$\|\chi - \lambda_{k_n}\psi\| \;<\; \sqrt{1-\epsilon^2}\,\|\psi\|\;|\mu_0 - \lambda_{k_n}| + \frac{1}{k_n}, \quad \forall\,n \in \mathbb{N},$$

and as $n \longrightarrow +\infty$,

$$\|\chi - \lambda_0\psi\| \;\leq\; \sqrt{1-\epsilon^2}\,\|\psi\|\;|\mu_0 - \lambda_0|\,.$$

This inequality can hold only as an equality because $\mu \in F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$, and the proof is complete. ☐

Proposition 4.1 and Theorem 4.2 yield readily the following.

Corollary 4.3. *Let $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$. Then, for any $\epsilon \in (0,1)$,*

$$\mathrm{Int}\left[F^{\epsilon}_{\|\cdot\|}(\chi;\psi)\right] \;=\; \left\{\mu \in \mathbb{C} :\; \|\chi - \lambda\psi\| > \sqrt{1-\epsilon^2}\,\|\psi\|\,|\mu - \lambda|,\ \forall\,\lambda \in \mathbb{C}\right\}.$$

5. The case of norms induced by inner products

In the special case of norms induced by inner products, we can fully describe the Birkhoff-James ϵ-orthogonality set $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$. In particular, $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ is always a closed disk; this is the case for $F^{0.5}_{\|\cdot\|_2}(\chi;\psi)$ in the left part of Figure 1.

Theorem 5.1. (For matrices, see [6, Section 5].) *Let $\chi,\psi \in \mathcal{X}$ with $\psi \neq 0$ and $\epsilon \in [0,1)$, and suppose that the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$. Then the Birkhoff-James ϵ-orthogonality set of χ with respect to ψ is the closed disk*

$$F^{\epsilon}_{\|\cdot\|}(\chi;\psi) = \mathcal{D}\left(\frac{\langle\chi,\psi\rangle}{\|\psi\|^2}\,,\ \left\|\chi - \frac{\langle\chi,\psi\rangle}{\|\psi\|^2}\,\psi\right\| \frac{\epsilon}{\sqrt{1-\epsilon^2}\,\|\psi\|}\right).$$

Proof. A scalar $\mu \in \mathbb{C}$ lies in $F^{\epsilon}_{\|\cdot\|}(\chi;\psi)$ if and only if [4, 7]

$$\psi \perp^{\epsilon} (\chi - \mu\psi),$$

or equivalently, if and only if

$$|\langle\psi,\chi-\mu\psi\rangle| \le \epsilon\,\|\psi\|\,\|\chi-\mu\psi\|,$$

or equivalently, if and only if

$$\langle\psi,\chi-\mu\psi\rangle\,\langle\chi-\mu\psi,\psi\rangle \le \epsilon^2\|\psi\|^2\langle\chi-\mu\psi,\chi-\mu\psi\rangle,$$

or equivalently, if and only if

$$\frac{|\langle\chi,\psi\rangle|^2}{\|\psi\|^4} - \mu\,\frac{\langle\psi,\chi\rangle}{\|\psi\|^2} - \overline{\mu}\,\frac{\langle\chi,\psi\rangle}{\|\psi\|^2} + |\mu|^2 \le \epsilon^2\left(\frac{\|\chi\|^2}{\|\psi\|^2} - \mu\,\frac{\langle\psi,\chi\rangle}{\|\psi\|^2} - \overline{\mu}\,\frac{\langle\chi,\psi\rangle}{\|\psi\|^2} + |\mu|^2\right),$$

or equivalently, if and only if

$$\left|\mu - \frac{\langle\chi,\psi\rangle}{\|\psi\|^2}\right|^2 (1-\epsilon^2) \le \frac{\epsilon^2}{\|\psi\|^2}\left\|\chi - \frac{\langle\chi,\psi\rangle}{\|\psi\|^2}\,\psi\right\|^2.$$

The proof is complete. □

References

[1] G. Birkhoff, Orthogonality in linear metric spaces, *Duke Math. J.*, **1** (1935), 169–172.

[2] F.F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Mathematical Society Lecture Note Series, Cambridge University Press, New York, 1971.

[3] F.F. Bonsall and J. Duncan, *Numerical Ranges II*, London Mathematical Society Lecture Notes Series, Cambridge University Press, New York, 1973.

[4] J. Chmielinski, On an ε-Birkhoff orthogonality, *J. Ineq. Pure and Appl. Math.*, **6** (2005), Article 79.

[5] Ch. Chorianopoulos, S. Karanasios and P. Psarrakos, A definition of numerical range of rectangular matrices, *Linear Multilinear Algebra*, **57** (2009), 459–475.

[6] Ch. Chorianopoulos and P. Psarrakos, Birkhoff-James approximate orthogonality sets and numerical ranges, *Linear Algebra Appl.*, **434** (2011), 2089–2108.

[7] S.S. Dragomir, On approximation of continuous linear functionals in normed linear spaces, *An. Univ. Timişoara Ser. Ştiinţ. Mat.*, **29** (1991), 51–58.

[8] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[9] R.C. James, Orthogonality and linear functionals in normed linear spaces, *Trans. Amer. Math. Soc.*, **61** (1947), 265–292.

[10] J.G. Stampfli and J.P. Williams, Growth conditions and the numerical range in a Banach algebra, *Tôhoku Math. Journ.*, **20** (1968), 417–424.

Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece

E-mail address: `kararemilt@gmail.com, ppsarr@math.ntua.gr`