

# NORMAL MATRIX POLYNOMIALS WITH NONSINGULAR LEADING COEFFICIENTS\*

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**Abstract.** In this paper, we introduce the notions of weakly normal and normal matrix polynomials, with nonsingular leading coefficients. We characterize these matrix polynomials, using orthonormal systems of eigenvectors and normal eigenvalues. We also study the conditioning of the eigenvalue problem of a normal matrix polynomial, constructing an appropriate Jordan canonical form.

**Key words.** Matrix polynomial, Normal eigenvalue, Weighted perturbation, Condition number.

**AMS subject classifications.** 15A18, 15A22, 65F15, 65F35.

**1. Introduction.** In pure and applied mathematics, normality of matrices (or operators) arises in many concrete problems. This is reflected to the fact that there are numerous ways to describe a normal matrix (or operator). A list of about ninety conditions on a square matrix equivalent to its being normal can be found in [5, 7].

The study of matrix polynomials has also a long history, especially in the context of their spectral analysis, which leads to the solutions of associated linear systems of higher order; see [6, 10, 11] and the references therein. Surprisingly, it seems that the notion of normality has been overlooked by people working in the area. The only exceptions are the works of Adam and Psarrakos [1] and Lancaster and Psarrakos [9].

Our present goal is to take a comprehensive look at normality of matrix polynomials. To avoid infinite eigenvalues, we restrict ourselves to matrix polynomials with nonsingular leading coefficients. The case of singular leading coefficients and infinite eigenvalues will be considered in a future work. The presentation of the article is organized as follows: In the next section, we provide the necessary theoretical background on the spectral analysis of matrix polynomials. In Section 3, we introduce the notions of weakly normal and normal matrix polynomials, and obtain necessary and sufficient conditions for a matrix polynomial to be weakly normal. In Section 4, we consider the normal eigenvalues of matrix polynomials and use them to provide sufficient conditions for a matrix polynomial to be normal. Finally, in Section

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5, we investigate the Jordan structure of a normal matrix polynomial, and study the conditioning of the associated eigenvalue problem.

**2. Spectral analysis of matrix polynomials.** Consider an  $n \times n$  matrix polynomial

$$(2.1) \quad P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_1 \lambda + A_0,$$

where  $\lambda$  is a complex variable and  $A_j \in \mathbb{C}^{n \times n}$  ( $j = 0, 1, \dots, m$ ) with  $\det A_m \neq 0$ . If the leading coefficient  $A_m$  coincides with the identity matrix  $I$ , then  $P(\lambda)$  is called *monic*. A scalar  $\lambda_0 \in \mathbb{C}$  is said to be an *eigenvalue* of  $P(\lambda)$  if  $P(\lambda_0)x_0 = 0$  for some nonzero  $x_0 \in \mathbb{C}^n$ . This vector  $x_0$  is known as a (*right*) *eigenvector* of  $P(\lambda)$  corresponding to  $\lambda_0$ . A nonzero vector  $y_0 \in \mathbb{C}^n$  that satisfies  $y_0^* P(\lambda_0) = 0$  is called a *left eigenvector* of  $P(\lambda)$  corresponding to  $\lambda_0$ .

The set of all eigenvalues of  $P(\lambda)$  is the *spectrum* of  $P(\lambda)$ , namely,  $\sigma(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$ , and since  $\det A_m \neq 0$ , it contains no more than  $nm$  distinct (finite) elements. The *algebraic multiplicity* of an eigenvalue  $\lambda_0 \in \sigma(P)$  is the multiplicity of  $\lambda_0$  as a zero of the (scalar) polynomial  $\det P(\lambda)$ , and it is always greater than or equal to the *geometric multiplicity* of  $\lambda_0$ , that is, the dimension of the null space of matrix  $P(\lambda_0)$ . A multiple eigenvalue of  $P(\lambda)$  is called *semisimple* if its algebraic multiplicity is equal to its geometric multiplicity.

Let  $\lambda_1, \lambda_2, \dots, \lambda_r \in \sigma(P)$  be the eigenvalues of  $P(\lambda)$ , where each  $\lambda_i$  appears  $k_i$  times if and only if the geometric multiplicity of  $\lambda_i$  is  $k_i$  ( $i = 1, 2, \dots, r$ ). Suppose also that for a  $\lambda_i \in \sigma(P)$ , there exist  $x_{i,1}, x_{i,2}, \dots, x_{i,s_i} \in \mathbb{C}^n$  with  $x_{i,1} \neq 0$ , such that

$$\begin{aligned} P(\lambda_i) x_{i,1} &= 0 \\ \frac{P'(\lambda_i)}{1!} x_{i,1} + P(\lambda_i) x_{i,2} &= 0 \\ &\vdots \\ \frac{P^{(s_i-1)}(\lambda_i)}{(s_i-1)!} x_{i,1} + \frac{P^{(s_i-2)}(\lambda_i)}{(s_i-2)!} x_{i,2} + \cdots + \frac{P'(\lambda_i)}{1!} x_{i,s_i-1} + P(\lambda_i) x_{i,s_i} &= 0, \end{aligned}$$

where the indices denote the derivatives of  $P(\lambda)$  and  $s_i$  cannot exceed the algebraic multiplicity of  $\lambda_i$ . Then the vector  $x_{i,1}$  is an eigenvector of  $\lambda_i$ , and the vectors  $x_{i,2}, x_{i,3}, \dots, x_{i,s_i}$  are known as *generalized eigenvectors*. The set  $\{x_{i,1}, x_{i,2}, \dots, x_{i,s_i}\}$  is called a *Jordan chain of length  $s_i$*  of  $P(\lambda)$  corresponding to the eigenvalue  $\lambda_i$ . Any eigenvalue of  $P(\lambda)$  of geometric multiplicity  $k$  has  $k$  maximal Jordan chains associated to  $k$  linearly independent eigenvectors, with total number of eigenvectors and generalized eigenvectors equal to the algebraic multiplicity of this eigenvalue.

We consider now the  $n \times nm$  matrix

$$X = [x_{1,1} \ x_{1,2} \ \cdots \ x_{1,s_1} \ x_{2,1} \ \cdots \ x_{r,1} \ x_{r,2} \ \cdots \ x_{r,s_r}]$$

formed by maximal Jordan chains of  $P(\lambda)$  and the associated  $nm \times nm$  Jordan matrix  $J = J_1 \oplus J_2 \oplus \cdots \oplus J_r$ , where each  $J_i$  is the Jordan block that corresponds to the

Jordan chain  $\{x_{i,1}, x_{i,2}, \dots, x_{i,s_i}\}$  of  $\lambda_i$ . Then the  $nm \times nm$  matrix  $Q = \begin{bmatrix} X \\ XJ \\ \vdots \\ XJ^{m-1} \end{bmatrix}$

is invertible, and we can define  $Y = Q^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ A_m^{-1} \end{bmatrix}$ . The set  $(X, J, Y)$  is called a

*Jordan triple* of  $P(\lambda)$ , and satisfies

$$(2.2) \quad P(\lambda)^{-1} = X(\lambda I - J)^{-1}Y; \quad \lambda \notin \sigma(P).$$

The set  $\{x_{1,1}, x_{1,2}, \dots, x_{1,s_1}, x_{2,1}, \dots, x_{r,1}, x_{r,2}, \dots, x_{r,s_r}\}$  is known as a *complete system* of eigenvectors and generalized eigenvectors of  $P(\lambda)$ .

**3. Weakly normal and normal matrix polynomials.** In [1], the term “normal matrix polynomial” has been used for the matrix polynomials that can be diagonalized by a unitary similarity. For matrix polynomials of degree  $m \geq 2$ , this definition does not ensure the semisimplicity of the eigenvalues, and hence, it is necessary to modify it. Consider, for example, the diagonal matrix polynomials

$$(3.1) \quad P(\lambda) = \begin{bmatrix} (\lambda - 2)(\lambda - 1) & 0 & 0 \\ 0 & \lambda(\lambda - 1) & 0 \\ 0 & 0 & (\lambda + 1)(\lambda + 2) \end{bmatrix}$$

and

$$(3.2) \quad R(\lambda) = \begin{bmatrix} (\lambda - 1)^2 & 0 & 0 \\ 0 & \lambda(\lambda - 2) & 0 \\ 0 & 0 & (\lambda + 1)(\lambda + 2) \end{bmatrix},$$

which have exactly the same eigenvalues (counting multiplicities),  $-2$ ,  $-1$ ,  $0$ ,  $1$  (double) and  $2$ . The eigenvalue  $\lambda = 1$  is semisimple as an eigenvalue of  $P(\lambda)$  with algebraic and geometric multiplicities equal to 2. On the other hand,  $\lambda = 1$  is not semisimple as an eigenvalue of  $R(\lambda)$  since its algebraic multiplicity is 2 and its geometric multiplicity is 1.

**DEFINITION 3.1.** The matrix polynomial  $P(\lambda)$  in (2.1) is called *weakly normal* if there is a unitary matrix  $U \in \mathbb{C}^{n \times n}$  such that  $U^*P(\lambda)U$  is diagonal for all  $\lambda \in \mathbb{C}$ . If,

in addition, every diagonal entry of  $U^*P(\lambda)U$  is a polynomial with exactly  $m$  distinct zeros, or equivalently, all the eigenvalues of  $P(\lambda)$  are semisimple, then  $P(\lambda)$  is called *normal*.

Clearly, the matrix polynomial  $P(\lambda)$  in (3.1) is normal, and the matrix polynomial  $R(\lambda)$  in (3.2) is weakly normal (but not normal). Note also that the notions of weakly normal and normal matrix polynomials coincide for matrices and for *linear pencils* of the form  $P(\lambda) = A_1\lambda + A_0$ .

The next two lemmas are necessary to characterize weakly normal matrix polynomials.

LEMMA 3.2. *Let  $A, B \in \mathbb{C}^{n \times n}$  be normal matrices such that  $AB^* = B^*A$ . Then the matrices  $A + B$  and  $AB$  are also normal.*

LEMMA 3.3. *Suppose that for every  $\mu \in \mathbb{C}$ , the matrix  $P(\mu)$  is normal. Then for every  $i, j = 0, 1, \dots, m$ , it holds that  $A_i A_j^* = A_j^* A_i$ . In particular, all coefficient matrices  $A_0, A_1, \dots, A_m$  are normal.*

*Proof.* Let  $P(\mu)$  be a normal matrix for every  $\mu \in \mathbb{C}$ . Then  $P(0) = A_0$  is normal, i.e.  $A_0 A_0^* = A_0^* A_0$ . From the proof of [14, Lemma 16], we have that  $A_0 A_i^* = A_i^* A_0$  for every  $i = 1, 2, \dots, m$ . Thus,  $P(\mu) A_0^* = A_0^* P(\mu)$  for every  $\mu \in \mathbb{C}$ . By Lemma 3.2, it follows that for the matrix polynomials  $P_0(\lambda) = A_m \lambda^{m-1} + \dots + A_2 \lambda + A_1$  and  $P(\lambda) - A_0 = \lambda P_0(\lambda)$ , the matrices  $P_0(\mu)$  and  $P(\mu) - A_0 = \mu P_0(\mu)$  are normal for every  $\mu \in \mathbb{C}$ .

Similarly, by [14, Lemma 16] and the fact that  $P_0(\mu)$  is normal for every  $\mu \in \mathbb{C}$ , we have that  $P_0(0) = A_1$  is also normal and  $A_1 A_i^* = A_i^* A_1$  for every  $i = 2, 3, \dots, m$ . Hence, as before,  $P_1(\mu) = A_m \mu^{m-2} + \dots + A_3 \mu + A_2$  and  $P_0(\mu) - A_1 = \mu P_1(\mu)$  are normal matrices for every  $\mu \in \mathbb{C}$ . Repeating the same process, we complete the proof.  $\square$

THEOREM 3.4. *The matrix polynomial  $P(\lambda)$  in (2.1) is weakly normal if and only if for every  $\mu \in \mathbb{C}$ , the matrix  $P(\mu)$  is normal.*

*Proof.* If the matrix polynomial  $P(\lambda)$  is weakly normal, then it is apparent that for every  $\mu \in \mathbb{C}$ , the matrix  $P(\mu)$  is normal.

For the converse, suppose that for every  $\mu \in \mathbb{C}$ , the matrix  $P(\mu)$  is normal. The next assumption is necessary.

*Assumption.* Suppose that there is a coefficient matrix  $A_i$  with  $s \geq 2$  distinct eigenvalues. Then without loss of generality, we may assume that

$$A_i = \lambda_{i1} I_{k_1} \oplus \lambda_{i2} I_{k_2} \oplus \dots \oplus \lambda_{is} I_{k_s}$$

and

$$A_j = A_{j1} \oplus A_{j2} \oplus \cdots \oplus A_{js}; \quad j \neq i,$$

where the eigenvalues  $\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{is}$  of  $A_i$  are distinct and nonzero, with multiplicities  $k_1, k_2, \dots, k_s$ , respectively, and  $A_{j1} \in \mathbb{C}^{k_1 \times k_1}$ ,  $A_{j2} \in \mathbb{C}^{k_2 \times k_2}$ ,  $\dots$ ,  $A_{js} \in \mathbb{C}^{k_s \times k_s}$ .

*Proof of Assumption.* Since  $A_i$  is normal, there is a unitary matrix  $U \in \mathbb{C}^{n \times n}$  such that

$$U^* A_i U = \lambda_{i1} I_{k_1} \oplus \lambda_{i2} I_{k_2} \oplus \cdots \oplus \lambda_{is} I_{k_s}.$$

We observe that for any  $\mu, a \in \mathbb{C}$ , the matrix  $P(\mu)$  is normal if and only if the matrix  $U^* P(\mu) U + a I \mu^i$  is normal. Thus, without loss of generality, we may assume that all  $\lambda_{il}$ 's are nonzero. By Lemma 3.3, it follows that for every  $j \neq i$ ,

$$A_i A_j^* = A_j^* A_i,$$

or equivalently,

$$A_j^* = A_i^{-1} A_j^* A_i.$$

By straightforward calculations, we complete the proof of the assumption.

We proceed now with the proof of the converse, which is by induction with respect to the order  $n$  of  $P(\lambda)$ . Clearly, for  $n = 1$ , there is nothing to prove.

If  $n = 2$ , and there is a coefficient matrix with distinct eigenvalues, then by Assumption, all  $A_0, A_1, \dots, A_m$  are diagonal. If there is no coefficient matrix of  $P(\lambda)$  with distinct eigenvalues, then each  $A_i$  ( $i = 0, 1, \dots, m$ ) is normal with a double eigenvalue, and hence, it is scalar, i.e.,  $A_i = a_i I$ . As a consequence,  $P(\lambda)$  is diagonal.

Assume now that for any  $n = 3, 4, \dots, k-1$ , every  $n \times n$  matrix polynomial  $P(\lambda)$  such that the matrix  $P(\mu)$  is normal for any  $\mu \in \mathbb{C}$ , is weakly normal.

Let  $P(\lambda) = A_m \lambda^m + A_{m-1} \lambda^{m-1} + \cdots + A_1 \lambda + A_0$  be a  $k \times k$  matrix polynomial, and suppose that there is a  $A_i$  with  $s \geq 2$  distinct eigenvalues. By Assumption,

$$A_i = \lambda_{i1} I_{k_1} \oplus \lambda_{i2} I_{k_2} \oplus \cdots \oplus \lambda_{is} I_{k_s},$$

and for every  $j \neq i$ ,

$$A_j = A_{j1} \oplus A_{j2} \oplus \cdots \oplus A_{js}.$$

Then

$$P(\lambda) = P_1(\lambda) \oplus P_2(\lambda) \oplus \cdots \oplus P_s(\lambda),$$

where the matrix polynomials  $P_1(\lambda), P_2(\lambda), \dots, P_s(\lambda)$  are weakly normal. Hence, there are unitary matrices  $U_t \in \mathbb{C}^{k_t \times k_t}$ ,  $t = 1, 2, \dots, s$ , such that  $U_t^* P_t(\lambda) U_t$ ,  $t = 1, 2, \dots, s$ , are diagonal. Thus, for the  $k \times k$  unitary matrix  $U = U_1 \oplus U_2 \oplus \dots \oplus U_s$ , the matrix polynomial  $U^* P(\lambda) U$  is diagonal.

Suppose that there is no  $A_i$  with at least two distinct eigenvalues. Then each coefficient matrix  $A_i$  is normal with exactly one eigenvalue (of algebraic multiplicity  $k$ ), and hence, it is scalar, i.e.,  $A_i = a_i I$ . As a consequence,  $P(\lambda)$  is diagonal.  $\square$

By this theorem, it follows that the matrix polynomial  $P(\lambda)$  is weakly normal if and only if  $P(\lambda)[P(\lambda)]^* - [P(\lambda)]^* P(\lambda) = 0$  for every  $\lambda \in \mathbb{C}$ . We observe that each entry of the matrix function  $P(\lambda)[P(\lambda)]^* - [P(\lambda)]^* P(\lambda)$  is of the form  $\chi(\alpha, \beta) + i\psi(\alpha, \beta)$ , where  $\alpha$  and  $\beta$  are the real and imaginary parts of variable  $\lambda$ , respectively, and  $\chi(\alpha, \beta)$  and  $\psi(\alpha, \beta)$  are real polynomials in  $\alpha, \beta \in \mathbb{R}$  of total degree at most  $2m$ . As a consequence, Lemma 3.1 of [12] yields the following.

**COROLLARY 3.5.** *The matrix polynomial  $P(\lambda)$  in (2.1) is weakly normal if and only if for any distinct real numbers  $s_1, s_2, \dots, s_{4m^2+2m+1}$ , the matrices  $P(s_j + i s_j^{2m+1})$  ( $j = 1, 2, \dots, 4m^2 + 2m + 1$ ) are normal.*

By Theorem 3.4, Corollary 3.5 and [16] (see also the references therein), the next corollary follows readily.

**COROLLARY 3.6.** *Let  $P(\lambda) = A_m \lambda^m + \dots + A_1 \lambda + A_0$  be an  $n \times n$  matrix polynomial as in (2.1). Then the following are equivalent:*

- (i) *The matrix polynomial  $P(\lambda)$  is weakly normal.*
- (ii) *For every  $\mu \in \mathbb{C}$ , the matrix  $P(\mu)$  is normal.*
- (iii) *For any distinct real numbers  $s_1, s_2, \dots, s_{4m^2+2m+1}$ , the matrices  $P(s_j + i s_j^{2m+1})$  ( $j = 1, 2, \dots, 4m^2 + 2m + 1$ ) are normal.*
- (iv) *The coefficient matrices  $A_0, A_1, \dots, A_m$  are normal and mutually commuting, i.e.,  $A_i A_j = A_j A_i$  for  $i \neq j$ .*
- (v) *All the linear combinations of the coefficient matrices  $A_0, A_1, \dots, A_m$  are normal matrices.*
- (vi) *The coefficient matrices  $A_0, A_1, \dots, A_m$  are normal and satisfy property L, that is, there exists an ordering of the eigenvalues  $\lambda_1^{(j)}, \lambda_2^{(j)}, \dots, \lambda_n^{(j)}$  of  $A_j$  ( $j = 0, 1, \dots, m$ ) such that for all scalars  $t_0, t_1, \dots, t_m \in \mathbb{C}$ , the eigenvalues of  $t_0 A_0 + t_1 A_1 + \dots + t_m A_m$  are  $t_0 \lambda_i^{(0)} + t_1 \lambda_i^{(1)} + \dots + t_m \lambda_i^{(m)}$  ( $i = 1, 2, \dots, n$ ).*
- (vii) *There exists a unitary matrix  $U \in \mathbb{C}^{n \times n}$  such that  $U^* A_j U$  is diagonal for every  $j = 0, 1, \dots, m$ .*

**4. Normal eigenvalues.** In the matrix case, it is well known that normality (or diagonalizability) is equivalent to the orthogonality (resp., linear independence) of eigenvectors. In the matrix polynomial case, it is clear (by definition) that any  $n \times n$

normal matrix polynomial of degree  $m$  has an orthogonal system of  $n$  eigenvectors, such that each one of these eigenvectors corresponds to exactly  $m$  distinct eigenvalues.

**PROPOSITION 4.1.** *Consider a matrix polynomial  $P(\lambda)$  as in (2.1), with all its eigenvalues semisimple. Suppose also that  $P(\lambda)$  has a complete system of eigenvectors where each vector of a basis  $\{g_1, g_2, \dots, g_n\}$  of  $\mathbb{C}^n$  appears exactly  $m$  times. Then there exists a diagonal matrix polynomial  $D(\lambda)$  such that*

$$P(\lambda) = A_m G D(\lambda) G^{-1},$$

where  $G = [g_1 \ g_2 \ \cdots \ g_n] \in \mathbb{C}^{n \times n}$ .

*Proof.* Each vector  $g_i$  ( $i = 1, 2, \dots, n$ ) appears exactly  $m$  times as an eigenvector of  $m$  distinct eigenvalues of  $P(\lambda)$ , say  $\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{im}$ . By [8, Theorem 1], we have that

$$P(\lambda) g_i = \prod_{j=1}^m (\lambda - \lambda_{ij}) g_i; \quad i = 1, 2, \dots, n.$$

Thus,

$$\begin{aligned} P(\lambda) [g_1 \ g_2 \ \cdots \ g_n] &= A_m \left[ \prod_{j=1}^m (\lambda - \lambda_{1j}) g_1 \quad \prod_{j=1}^m (\lambda - \lambda_{2j}) g_2 \quad \cdots \quad \prod_{j=1}^m (\lambda - \lambda_{nj}) g_n \right] \\ &= A_m [g_1 \ g_2 \ \cdots \ g_n] \text{diag} \left\{ \prod_{j=1}^m (\lambda - \lambda_{1j}) g_1, \prod_{j=1}^m (\lambda - \lambda_{2j}) g_2, \dots, \prod_{j=1}^m (\lambda - \lambda_{nj}) g_n \right\}. \end{aligned}$$

Consequently,

$$P(\lambda) = A_m G \text{diag} \left\{ \prod_{j=1}^m (\lambda - \lambda_{1j}), \prod_{j=1}^m (\lambda - \lambda_{2j}), \dots, \prod_{j=1}^m (\lambda - \lambda_{nj}) \right\} G^{-1},$$

and the proof is complete.  $\square$

**COROLLARY 4.2.** *Under the hypothesis of Proposition 4.1, it holds that:*

- (i) *If the matrix  $G^{-1} A_m G$  is diagonal, then the matrix polynomial  $G^{-1} P(\lambda) G$  is diagonal.*
- (ii) *If  $G$  is unitary, then there exists a diagonal matrix polynomial  $D(\lambda)$  such that  $P(\lambda) = A_m G D(\lambda) G^*$ .*
- (iii) *If  $G$  is unitary and the matrix  $G^* A_m G$  is diagonal, then the matrix polynomial  $G^* P(\lambda) G$  is diagonal, i.e.,  $P(\lambda)$  is normal.*

Note that if  $P(\lambda) = I\lambda - A$ , then in Corollary 4.2,  $G^{-1}A_mG = G^{-1}IG = I$  for nonsingular  $G$ , and  $G^*A_mG = G^*IG = I$  for unitary  $G$ . This means that all the parts of the corollary are direct generalizations of standard results on matrices.

The following definition was originally introduced in [9].

**DEFINITION 4.3.** Let  $P(\lambda)$  be an  $n \times n$  matrix polynomial as in (2.1). An eigenvalue  $\lambda_0 \in \sigma(P)$  of algebraic multiplicity  $k$  is said to be *normal* if there exists a unitary matrix  $U \in \mathbb{C}^{n \times n}$  such that

$$U^*P(\lambda)U = [(\lambda - \lambda_0)D(\lambda)] \oplus Q(\lambda),$$

where the matrix polynomial  $D(\lambda)$  is  $k \times k$  diagonal and  $\lambda_0 \notin \sigma(D) \cup \sigma(Q)$ .

By this definition and Definition 3.1, it is obvious that every normal matrix polynomial has all its eigenvalues normal. In the sequel, we obtain the converse.

**PROPOSITION 4.4.** *Suppose that all the eigenvalues of  $P(\lambda)$  in (2.1) are semisimple, and that  $\lambda_1, \lambda_2, \dots, \lambda_s$  are normal eigenvalues of  $P(\lambda)$  with multiplicities  $m_1, m_2, \dots, m_s$ , respectively, such that  $m_1 + m_2 + \dots + m_s = n$ . If  $P(\lambda)$  satisfies*

$$U_j^*P(\lambda)U_j = [(\lambda - \lambda_j)D_j(\lambda)] \oplus Q_j(\lambda); \quad j = 1, 2, \dots, s,$$

where for each  $j$ , the matrix  $U_j \in \mathbb{C}^{n \times n}$  is unitary, the matrix polynomial  $D_j(\lambda)$  is  $m_j \times m_j$  diagonal and  $\lambda_i \in \sigma(Q_j) \setminus \sigma(D_j)$  ( $i = j+1, \dots, s$ ), then the matrix polynomial  $P(\lambda)$  is normal.

*Proof.* For the eigenvalue  $\lambda_1$ , by hypothesis, we have that

$$U_1^*P(\lambda)U_1 = [(\lambda - \lambda_1)D_1(\lambda)] \oplus Q_1(\lambda),$$

where  $U_1$  is unitary and  $D_1(\lambda)$  is  $m_1 \times m_1$  diagonal. The first  $m_1$  columns of  $U_1$  are an orthonormal system of eigenvectors of  $\lambda_1$ . From hypothesis we also have that  $\det A_m \neq 0$  and all the eigenvalues of  $P(\lambda)$  are semisimple. As a consequence, each one of the  $m_1$  eigenvectors of  $\lambda_1$  is an eigenvector for  $m$  exactly eigenvalues of  $P(\lambda)$  (counting  $\lambda_1$ ). Moreover, since  $\lambda_i \in \sigma(Q_1) \setminus \sigma(D_1)$ ,  $i = 2, 3, \dots, s$ , these  $m_1$  eigenvectors of  $\lambda_1$ , are orthogonal to the eigenspaces of the eigenvalues  $\lambda_2, \lambda_3, \dots, \lambda_s$ .

Similarly, for the eigenvalue  $\lambda_2$ , we have

$$U_2^*P(\lambda)U_2 = [(\lambda - \lambda_2)D_2(\lambda)] \oplus Q_2(\lambda),$$

where  $U_2$  is unitary and  $D_2(\lambda)$  is  $m_2 \times m_2$  diagonal. As before, the first  $m_2$  columns of  $U_2$  are an orthonormal system of eigenvectors of  $\lambda_2$ . In addition, each one of these  $m_2$  eigenvectors of  $\lambda_2$  is an eigenvector for  $m$  exactly eigenvalues of  $P(\lambda)$  (counting  $\lambda_2$ ). Since  $\lambda_i \in \sigma(Q_2) \setminus \sigma(D_2)$ ,  $i = 3, 4, \dots, s$ , these  $m_2$  eigenvectors of  $\lambda_2$  are orthogonal to the eigenspaces of the eigenvalues  $\lambda_3, \lambda_4, \dots, \lambda_s$ .



Repeating this process for the eigenvalues  $\lambda_3, \lambda_4, \dots, \lambda_s$ , we construct an orthonormal basis of  $\mathbb{C}^n$ ,

$$\underbrace{u_1, u_2, \dots, u_{m_1}}_{\text{of } \lambda_1}, \underbrace{u_{m_1+1}, u_{m_1+2}, \dots, u_{m_1+m_2}}_{\text{of } \lambda_2}, \dots, \underbrace{u_{n-m_s+1}, u_{n-m_s+2}, \dots, u_n}_{\text{of } \lambda_s},$$

where each vector is an eigenvector for  $m$  distinct eigenvalues of  $P(\lambda)$  and an eigenvector of the leading coefficient  $A_m$ . By Corollary 4.2,  $P(\lambda)$  is a normal matrix polynomial.  $\square$

The next lemma follows readily and it is necessary for our discussion.

LEMMA 4.5. *If an  $n \times n$  matrix polynomial  $P(\lambda)$  has a normal eigenvalue of multiplicity  $n$  or  $n - 1$ , then it is weakly normal.*

THEOREM 4.6. *Consider a matrix polynomial  $P(\lambda)$  as in (2.1). If all its eigenvalues are normal, then  $P(\lambda)$  is normal.*

*Proof.* Let  $\lambda_1, \lambda_2, \dots, \lambda_s$  be the distinct eigenvalues of  $P(\lambda)$  with corresponding multiplicities  $k_1, k_2, \dots, k_s$  ( $k_1 + k_2 + \dots + k_s = nm$ ), and suppose that they are normal. It is easy to see that  $s \geq m$ . If  $s = m$  then all the eigenvalues have multiplicity  $n$  and by Lemma 4.5, the theorem follows.

Suppose that  $s > m$  and for every  $j = 0, 1, \dots, s$ , there is a unitary matrix  $U_j = [u_{j1} \ u_{j2} \ \dots \ u_{jn}]$  such that

$$U_j^* P(\lambda) U_j = [(\lambda - \lambda_j) D_j(\lambda)] \oplus Q_j(\lambda),$$

where  $D_j(\lambda)$  is  $k_j \times k_j$  diagonal and  $\lambda_j \notin \sigma(D_j) \cup \sigma(Q_j)$ . Then the first  $k_j$  columns of  $U_j$  ( $j = 1, 2, \dots, s$ ) are right and left eigenvectors of  $P(\lambda)$ , and also of  $A_0, A_1, \dots, A_m$ . The set of all vectors

$$u_{1,1}, u_{1,2}, \dots, u_{1,k_1}, u_{2,1}, u_{2,2}, \dots, u_{2,k_2}, \dots, u_{s,1}, u_{s,2}, \dots, u_{s,k_s}$$

form a complete system of eigenvectors of  $P(\lambda)$ . So, by [13], there is a basis of  $\mathbb{C}^n$

$$\{u_1, u_2, \dots, u_n\} \subseteq \{u_{11}, u_{12}, \dots, u_{1k_1}, u_{21}, u_{22}, \dots, u_{2k_2}, \dots, u_{s1}, u_{s2}, \dots, u_{sk_s}\}.$$

We also observe that the vectors  $u_1, u_2, \dots, u_n$  are linearly independent right and left common eigenvectors of the coefficient matrices  $A_0, A_1, \dots, A_m$ . Keeping in mind that any left and right eigenvectors (of the same matrix) corresponding to distinct eigenvalues are orthogonal, [7, Condition 13] implies that all  $A_j$ 's are normal. Moreover, any two vectors  $u_i, u_j$  ( $i \neq j$ ) that correspond to distinct eigenvalues of a coefficient matrix are also orthogonal. Hence, it is straightforward to see that there exists a unitary matrix  $U$  such that all  $U^* A_j U$ 's are diagonal. As a consequence, the matrix polynomial  $P(\lambda)$  is weakly normal, and since all its eigenvalues are semisimple,  $P(\lambda)$  is normal.  $\square$

**5. Weighted perturbations and condition numbers.** Let  $P(\lambda)$  be a matrix polynomial as in (2.1). We are interested in perturbations of  $P(\lambda)$  of the form

$$(5.1) \quad Q(\lambda) = P(\lambda) + \Delta(\lambda) = \sum_{j=0}^m (A_j + \Delta_j) \lambda^j,$$

where the matrices  $\Delta_0, \Delta_1, \dots, \Delta_m \in \mathbb{C}^{n \times n}$  are arbitrary. For a given parameter  $\varepsilon > 0$  and a given set of nonnegative weights  $w = \{w_0, w_1, \dots, w_m\}$  with  $w_0 > 0$ , we define the set of admissible perturbed matrix polynomials

$$(5.2) \quad \mathcal{B}(P, \varepsilon, w) = \{Q(\lambda) \text{ as in (5.1)} : \|\Delta_j\| \leq \varepsilon w_j, j = 0, 1, \dots, m\},$$

where  $\|\cdot\|$  denotes the spectral matrix norm (i.e., that norm subordinate to the euclidean vector norm). The weights  $w_0, w_1, \dots, w_m$  allow freedom in how perturbations are measured, and the set  $\mathcal{B}(P, \varepsilon, w)$  is convex and compact [3] with respect to the max norm  $\|P(\lambda)\|_\infty = \max_{0 \leq j \leq m} \|A_j\|$ .

In [15], motivated by (2.2) and the work of Chu [4], for a Jordan triple  $(X, J, Y)$  of  $P(\lambda)$ , the authors introduced the *condition number of the eigenproblem of  $P(\lambda)$* , that is,<sup>1</sup>

$$k(P) = \|X\| \|Y\|.$$

Furthermore, they applied the Bauer-Fike technique [2, 4] and used  $k(P)$ , to bound eigenvalues of perturbations of  $P(\lambda)$ . Denoting  $w(\lambda) = w_m \lambda^m + \dots + w_1 \lambda + w_0$ , one of the results of [15] is the following.

**PROPOSITION 5.1.** *Let  $(X, J, Y)$  be a Jordan triple of  $P(\lambda)$ , and let  $Q(\lambda) \in \mathcal{B}(P, \varepsilon, w)$  for some  $\varepsilon > 0$ . If the Jordan matrix  $J$  is diagonal, then for any  $\mu \in \sigma(Q) \setminus \sigma(P)$ ,*

$$\min_{\lambda \in \sigma(P)} |\mu - \lambda| \leq k(P) \varepsilon w(|\mu|).$$

As in the matrix case, we say that a matrix polynomial eigenvalue problem is *well-conditioned* (or *ill-conditioned*) if its condition number is sufficiently small (resp., sufficiently large).

In the remainder of this section, we confine our discussion to normal matrix polynomials. Recall that for an  $n \times n$  normal matrix polynomial  $P(\lambda)$ , there is a unitary matrix  $U \in \mathbb{C}^{n \times n}$  such that  $U^* P(\lambda) U$  is diagonal and all its diagonal entries

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<sup>1</sup>Note that the definition of  $k(P)$  clearly depends on the choice of the triple  $(X, J, Y)$ , but to keep things simple, the Jordan triple will not appear explicitly in the notation.

are polynomials of degree exactly  $m$ , with distinct zeros. Moreover, for any Jordan triple  $(X, J, Y)$  of  $P(\lambda)$ , the Jordan matrix  $J$  is diagonal. In the sequel, we derive some bounds for the condition number  $k(P)$ .

The following lemma is a simple exercise.

LEMMA 5.2. *Let  $\lambda_1, \lambda_2, \dots, \lambda_m$  be  $m$  distinct scalars. Then it holds that*

$$\frac{1}{\prod_{j=1}^m (\lambda - \lambda_j)} = \sum_{j=1}^m \frac{1}{(\lambda - \lambda_j) \prod_{i \neq j} (\lambda_i - \lambda_j)}.$$

Next we compute a Jordan triple of a monic normal matrix polynomial  $P(\lambda)$  and the associated condition number  $k(P)$ .

PROPOSITION 5.3. *Let  $P(\lambda)$  be an  $n \times n$  monic normal matrix polynomial of degree  $m$ , and let  $U^* P(\lambda) U = (I\lambda - J_1)(I\lambda - J_2) \cdots (I\lambda - J_m)$  for some unitary  $U \in \mathbb{C}^{n \times n}$  and diagonal matrices  $J_1, J_2, \dots, J_m$ . Then a Jordan triple  $(X, J, Y)$  of  $P(\lambda)$  is given by*

$$X = U \left[ I \quad (J_2 - J_1)^{-1} \quad \cdots \quad \prod_{i=1}^{m-2} (J_{m-1} - J_i)^{-1} \quad \prod_{i=1}^{m-1} (J_m - J_i)^{-1} \right],$$

$$J = J_1 \oplus J_2 \oplus \cdots \oplus J_m \quad \text{and} \quad Y = \begin{bmatrix} \prod_{i=2}^m (J_1 - J_i)^{-1} \\ \prod_{i=3}^m (J_2 - J_i)^{-1} \\ \vdots \\ (J_{m-1} - J_m)^{-1} \\ I \end{bmatrix} U^*.$$

*Proof.* <sup>2</sup>By Lemma 5.2 and straightforward calculations, we see that

$$\prod_{j=1}^m (I\lambda - J_j)^{-1} = \sum_{j=1}^m \left[ (I\lambda - J_j) \prod_{i \neq j} (J_i - J_j) \right]^{-1},$$

or equivalently,

$$P(\lambda)^{-1} = X(I\lambda - J)^{-1}Y.$$

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<sup>2</sup>Originally, we have derived another (constructive) proof of the proposition, which explains the choice of matrices  $X$  and  $Y$ . That proof is inductive with respect to the degree  $m$  of  $P(\lambda)$ , and uses Corollary 3.3 of [6]. It also requires some computations, and hence, we decided to present this short proof.

Theorem 2.6 of [6] completes the proof.  $\square$

**THEOREM 5.4.** *Let  $P(\lambda)$  be an  $n \times n$  monic normal matrix polynomial of degree  $m$ , and let  $(X, J, Y)$  be the Jordan triple given by Proposition 5.3. If we denote*

$$J_i = \text{diag}\{\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_n^{(i)}\}; \quad i = 1, 2, \dots, m,$$

then the condition number of the eigenproblem of  $P(\lambda)$  is

$$\begin{aligned} k(P) &= \left( 1 + \max_{s=1,2,\dots,n} \left\{ \frac{1}{|\lambda_s^{(2)} - \lambda_s^{(1)}|^2} + \dots + \prod_{i=1}^{m-1} \frac{1}{|\lambda_s^{(m)} - \lambda_s^{(i)}|^2} \right\} \right)^{1/2} \\ &\quad \times \left( 1 + \max_{s=1,2,\dots,n} \left\{ \frac{1}{|\lambda_s^{(m)} - \lambda_s^{(m-1)}|^2} + \dots + \prod_{i=2}^m \frac{1}{|\lambda_s^{(i)} - \lambda_s^{(1)}|^2} \right\} \right)^{1/2}. \end{aligned}$$

*Proof.* Since  $P(\lambda)$  is normal, it follows  $\lambda_s^{(i)} \neq \lambda_s^{(j)}$ ,  $i \neq j$ ,  $s = 1, 2, \dots, n$ . Recall that

$$X = U \left[ I \quad (J_2 - J_1)^{-1} \quad \dots \quad \prod_{i=1}^{m-2} (J_{m-1} - J_i)^{-1} \quad \prod_{i=1}^{m-1} (J_m - J_i)^{-1} \right],$$

and observe that

$$XX^* = U \left( I + (J_2 - J_1)^{-1} \overline{(J_2 - J_1)^{-1}} + \dots + \prod_{i=1}^{m-1} (J_m - J_i)^{-1} \overline{(J_m - J_i)^{-1}} \right) U^*.$$

If we denote by  $\lambda_{\max}(\cdot)$  the largest eigenvalue of a square matrix, then

$$\|X\|^2 = \lambda_{\max}(XX^*) = 1 + \max_{s=1,2,\dots,n} \left\{ \frac{1}{|\lambda_s^{(2)} - \lambda_s^{(1)}|^2} + \dots + \prod_{i=1}^{m-1} \frac{1}{|\lambda_s^{(m)} - \lambda_s^{(i)}|^2} \right\}.$$

Similarly, we verify that

$$\|Y\|^2 = \lambda_{\max}(Y^*Y) = 1 + \max_{s=1,2,\dots,n} \left\{ \frac{1}{|\lambda_s^{(m)} - \lambda_s^{(m-1)}|^2} + \dots + \prod_{i=2}^m \frac{1}{|\lambda_s^{(i)} - \lambda_s^{(1)}|^2} \right\},$$

and the proof is complete.  $\square$

It is worth noting that since a (monic) normal matrix polynomial is ‘‘essentially diagonal’’, the condition number of its eigenproblem depends on the eigenvalues and not on the eigenvectors. Furthermore, by the above theorem, it is apparent that if  $m \geq 2$  and the mutual distances of the eigenvalues of the monic matrix polynomial  $P(\lambda)$  are sufficiently large, then the condition number  $k(P)$  is relatively close to 1, i.e., the eigenproblem of  $P(\lambda)$  is well-conditioned. On the other hand, if  $m \geq 2$  and

the mutual distances of the eigenvalues are sufficiently small, then  $k(P)$  is relatively large, i.e., the eigenproblem of  $P(\lambda)$  is ill-conditioned.

Theorem 5.4 implies practical lower and upper bounds for the condition number  $k(P)$ . For convenience, we denote

$$\Theta = \max_{\substack{\lambda, \hat{\lambda} \in \sigma(P) \\ \lambda \neq \hat{\lambda}}} |\lambda - \hat{\lambda}| \quad \text{and} \quad \theta = \min_{\substack{\lambda, \hat{\lambda} \in \sigma(P) \\ \lambda \neq \hat{\lambda}}} |\lambda - \hat{\lambda}|,$$

and assume that  $\Theta, \theta \neq 1$ .

**COROLLARY 5.5.** *Let  $P(\lambda)$  be an  $n \times n$  monic normal matrix polynomial of degree  $m$ , and let  $(X, J, Y)$  be the Jordan triple given by Proposition 5.3. Then the condition number  $k(P)$  satisfies*

$$\frac{\Theta^{2m} - 1}{\Theta^{2m} - \Theta^{2(m-1)}} \leq k(P) \leq \frac{\theta^{2m} - 1}{\theta^{2m} - \theta^{2(m-1)}}.$$

Consider the matrix polynomial  $P(\lambda)$  in (2.1), and recall that its leading coefficient  $A_m$  is nonsingular. By [6, 10],  $(X, J, Y)$  is a Jordan triple of the monic matrix polynomial  $\hat{P}(\lambda) = A_m^{-1}P(\lambda) = I\lambda^m + A_m^{-1}A_{m-1}\lambda^{m-1} + \dots + A_m^{-1}A_1\lambda + A_m^{-1}A_0$  if and only if  $(X, J, YA_m^{-1})$  is a Jordan triple of  $P(\lambda)$ . This observation and the proof of Theorem 5.4 yield the next results.

**COROLLARY 5.6.** *Let  $P(\lambda) = A_1\lambda + A_0$  be an  $n \times n$  normal linear pencil, and let  $U^*P(\lambda)U$  be diagonal for some unitary  $U \in \mathbb{C}^{n \times n}$ . Then a Jordan triple of  $P(\lambda)$  is  $(X, J, Y) = (U, J, UA_1^{-1})$ , and  $k(P) = \|A_1^{-1}\|$ .*

**THEOREM 5.7.** *Let  $P(\lambda)$  in (2.1) be normal, and let  $(X, J, Y)$  be the Jordan triple of the monic matrix polynomial  $\hat{P}(\lambda) = A_m^{-1}P(\lambda)$  given by Proposition 5.3. Then for the condition number  $k(P) = \|X\| \|YA_m^{-1}\|$ , we have*

$$\|A_m\|^{-1} \frac{\Theta^{2m} - 1}{\Theta^{2m} - \Theta^{2(m-1)}} \leq k(P) \leq \|A_m^{-1}\| \frac{\theta^{2m} - 1}{\theta^{2m} - \theta^{2(m-1)}}.$$

*Proof.* As mentioned above,  $(X, J, Y)$  is a Jordan triple of the monic matrix polynomial  $\hat{P}(\lambda)$  if and only if  $(X, J, YA_m^{-1})$  is a Jordan triple of  $P(\lambda)$ . Notice also that  $\hat{P}(\lambda)$  is normal, and by the proof of Theorem 5.4,

$$\sqrt{\frac{\Theta^{2m} - 1}{\Theta^{2m} - \Theta^{2(m-1)}}} \leq \|X\|, \|Y\| \leq \sqrt{\frac{\theta^{2m} - 1}{\theta^{2m} - \theta^{2(m-1)}}}.$$

Furthermore, there is an  $n \times n$  unitary matrix  $U$  such that  $D(\lambda) = U^*P(\lambda)U$  and  $D_m = U^*A_mU$  are diagonal. As a consequence,  $\|A_m\| = \|D_m\|$  and  $\|A_m^{-1}\| = \|D_m^{-1}\|$ .

Since the matrix  $YU \in \mathbb{C}^{nm \times n}$  is a block-column of  $m$  diagonal matrices of order  $n$ , it is straightforward to see that

$$\|YA_m^{-1}\| = \|YUD_m^{-1}U^*\| = \|YUD_m^{-1}\| = \|YD_m^{-1}\|.$$

The matrix  $Y^*Y$  is also diagonal, and thus,

$$\|YD_m^{-1}\| = \left\| (D_m^{-1})^* Y^* Y D_m^{-1} \right\|^{1/2} = \left\| Y^* Y (D_m^{-1})^* D_m^{-1} \right\|^{1/2}.$$

Hence, it follows

$$\|Y\| \|D_m\|^{-1} \leq \|YD_m^{-1}\| \leq \|Y\| \|D_m^{-1}\|.$$

By Corollary 5.5, the proof is completed.  $\square$

Proposition 5.1 implies directly the following.

**COROLLARY 5.8.** *Let  $P(\lambda)$  in (2.1) be normal, and let  $Q(\lambda) \in \mathcal{B}(P, \varepsilon, w)$  for some  $\varepsilon > 0$ . Then for any  $\mu \in \sigma(Q) \setminus \sigma(P)$ , it holds that*

$$\min_{\lambda \in \sigma(P)} |\mu - \lambda| \leq \varepsilon w(|\mu|) \|A_m^{-1}\| \frac{\theta^{2m} - 1}{\theta^{2m} - \theta^{2(m-1)}}.$$

**REMARK 5.9.** In the construction of the above bounds, we have assumed that  $\Theta$  and  $\theta$  are different than 1. Suppose that this assumption fails for a normal matrix polynomial  $P(\lambda)$ . Then, keeping in mind the Jordan triple  $(X, J, Y)$  of  $\hat{P}(\lambda) = A_m^{-1}P(\lambda)$  given by Proposition 5.3 and the proofs of Theorems 5.4 and 5.7, one can easily see that  $k(P) \geq m \|A_m\|^{-1}$  when  $\Theta = 1$ , and  $k(P) \leq m \|A_m^{-1}\|$  when  $\theta = 1$ .

Finally, as an example, recall the monic normal matrix polynomial  $P(\lambda)$  in (3.1). For the weights  $w_0 = w_1 = w_2 = 1$ , Theorem 5.4 yields  $k(P) = 2$ , i.e., the eigenproblem of  $P(\lambda)$  is well-conditioned. Note also that  $\theta = 1$  and the value 2 coincides with the upper bound given by Remark 5.9.

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