On the continuity of Birkhoff-James $\epsilon$-orthogonality sets

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Abstract

Consider two matrices $A, B \in \mathbb{C}^{n \times m}$ with $B \neq 0$, a matrix norm $\| \cdot \|$, and a real parameter $\epsilon \in [0, 1)$. The Birkhoff-James $\epsilon$-orthogonality set of $A$ with respect to $B$, $F_{\|\|}(A; B) = \{ \mu \in \mathbb{C} : \| A - \lambda B \| \geq \sqrt{1 - \epsilon^2} \| B \| \| \mu - \lambda \|, \forall \lambda \in \mathbb{C} \}$, is a compact and convex subset of the complex plane that has been recently introduced by the authors, as a natural generalization of the classical numerical range of square matrices. In this note, we derive the continuity of $F_{\|\|}(A; B)$ with respect to $A$ or $\epsilon$.

Keywords: Birkhoff-James $\epsilon$-orthogonality set, Hausdorff distance, continuity.

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1 Introduction

The numerical range (also known as the field of values) of a square complex matrix $A \in \mathbb{C}^{n \times n}$ is defined as $F(A) = \{ x^* Ax : x \in \mathbb{C}^n, x^* x = 1 \}$ [10]. It is a compact and convex subset of the complex plane that has been studied extensively for many decades, and it is useful in studying and understanding matrices and operators; see [3, 4, 9, 10, 14] and the references therein.

The numerical range $F(A)$ is also written in the form [4, 14],

$$F(A) = \left\{ \mu \in \mathbb{C} : \| A - \lambda I_n \|_2 \geq |\mu - \lambda|, \forall \lambda \in \mathbb{C} \right\} = \bigcap_{\lambda \in \mathbb{C}} \left\{ \mu \in \mathbb{C} : |\mu - \lambda| \leq \| A - \lambda I_n \|_2 \right\},$$

where $\| \cdot \|_2$ denotes the spectral matrix norm (i.e., that norm subordinate to the euclidean vector norm) and $I_n$ is the $n \times n$ identity matrix. Thus, $F(A)$ is an infinite intersection of closed disks $D(\lambda, \| A - \lambda I_n \|_2) = \{ \mu \in \mathbb{C} : |\mu - \lambda| \leq \| A - \lambda I_n \|_2 \}$

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(\lambda \in \mathbb{C}). Inspired by the above intersection property, Chorianopoulos, Karanasios and Psarrakos [6] proposed a definition of numerical range for rectangular complex matrices. In particular, for any $A, B \in \mathbb{C}^{n \times m}$ with $B \neq 0$, and any matrix norm $\| \cdot \|$, the numerical range of $A$ with respect to $B$ is defined as

$$F_{\| \cdot \|}(A; B) = \{ \mu \in \mathbb{C} : \|A - \lambda B\| \geq |\mu - \lambda|, \forall \lambda \in \mathbb{C} \} = \bigcap_{\lambda \in \mathbb{C}} D(\lambda, \|A - \lambda B\|).$$

This set is compact and convex, satisfies basic properties of the standard numerical range, and is nonempty if and only if $\|B\| \geq 1$ [6].

The analysis in [6] is based on the properties of matrix norms and the Birkhoff-James orthogonality [2, 11]; namely, for two elements $\chi$ and $\psi$ of a complex normed linear space $(\mathcal{X}, \| \cdot \|)$, $\chi$ is called Birkhoff-James orthogonal to $\psi$, denoted by $\chi \perp_{BJ} \psi$, if $\|\chi + \lambda \psi\| \geq \|\chi\|$ for all $\lambda \in \mathbb{C}$. This orthogonality is homogeneous, but it is neither symmetric nor additive [11]. Furthermore, for any $\epsilon \in [0, 1)$, we say that $\chi$ is Birkhoff-James $\epsilon$-orthogonal to $\psi$, denoted by $\chi \perp_{\epsilon, BJ} \psi$, if $\|\chi + \lambda \psi\| \geq \sqrt{1 - \epsilon^2}\|\chi\|$ for all $\lambda \in \mathbb{C}$. This $\epsilon$-orthogonality is well defined and denoted by $\mathcal{D}$

$$\mathcal{D}(\lambda, \|A - \lambda B\|).$$

If $\|B\| \geq 1$, then for $\epsilon_B = \sqrt{\|B\|^2 - 1} / \|B\|$, we have [7]

$$F_{\| \cdot \|}(A; B) = \{ \mu \in \mathbb{C} : B \perp_{\epsilon_B} (A - \mu B) \}.$$

As a consequence, the next definition (introduced by the authors in [7]) arises in a natural way.

**Definition 1.** For any $A, B \in \mathbb{C}^{n \times m}$ with $B \neq 0$, any matrix norm $\| \cdot \|$, and any $\epsilon \in [0, 1)$, the Birkhoff-James $\epsilon$-orthogonality set of $A$ with respect to $B$ is defined and denoted by

$$F^\epsilon_{\| \cdot \|}(A; B) = \{ \mu \in \mathbb{C} : B \perp_{\epsilon, BJ} (A - \mu B) \} = \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}(\lambda, \|A - \lambda B\| / \sqrt{1 - \epsilon^2}\|B\|).$$

The Birkhoff-James $\epsilon$-orthogonality set $F^\epsilon_{\| \cdot \|}(A; B)$ is a nonempty, compact and convex subset of the complex plane that lies in the closed disk $\mathcal{D}(0, \sqrt{\|A\|^2 / \sqrt{1 - \epsilon^2}\|B\|})$ and is quite rich in structure [7]. In this note, we obtain the continuity of $F^\epsilon_{\| \cdot \|}(A; B)$ with respect to the matrix $A \in \mathbb{C}^{n \times m}$ (see Section 3), or to the real parameter $\epsilon \in [0, 1)$ (see Section 4). The question of the continuity of $F^\epsilon_{\| \cdot \|}(A; B)$ with respect to matrix $B \in \mathbb{C}^{n \times m}$ is still open except the special case where the norm $\| \cdot \|$ is induced by an inner product of matrices (see Remark 8).
2 Preliminaries

We recall that for two compact subsets $\Omega_1$ and $\Omega_2$ of a metric space $(X, \rho)$, the Hausdorff distance between $\Omega_1$ and $\Omega_2$ is defined by

$$d_H(\Omega_1, \Omega_2) = \max \left\{ \max_{x_1 \in \Omega_1} \min_{x_2 \in \Omega_2} \rho(x_1, x_2), \max_{x_2 \in \Omega_2} \min_{x_1 \in \Omega_1} \rho(x_1, x_2) \right\}.$$

For any $x_0 \in X$ and $\delta > 0$, we define the closed ball $B(x_0, \delta) = \{x \in X : \rho(x_0, x) \leq \delta\}$.

**Definition 2.** [1] Suppose $(X, \rho_X)$ is a metric space and $(Y, \rho_Y)$ is a complete metric space. Consider a multi-valued mapping $F : X \mapsto Y$, and let $x_0 \in X$.

(i) $F$ is called $\delta$-upper semi-continuous at $x_0$ if for every $\delta > 0$, there is a neighborhood $N(x_0) \subset X$ of $x_0$ such that

$$F(x) \subseteq F(x_0) + B(0, \delta), \quad \forall \ x \in N(x_0).$$

(ii) $F$ is called $\delta$-lower semi-continuous at $x_0$ if for every $\delta > 0$, there is a neighborhood $N(x_0) \subset X$ of $x_0$ such that

$$F(x_0) \subseteq F(x) + B(0, \delta), \quad \forall \ x \in N(x_0).$$

(iii) $F$ is said to be $\delta$-continuous at $x_0$ if it is $\delta$-upper and $\delta$-lower semi-continuous.

(i') $F$ is called upper semi-continuous at $x_0$ if for every neighborhood $N(F(x_0)) \subset Y$ of the set $F(x_0)$, there is a neighborhood $N(x_0) \subset X$ of $x_0$ such that

$$F(x) \subseteq N(F(x_0)), \quad \forall \ x \in N(x_0).$$

(ii') $F$ is called lower semi-continuous at $x_0$ if for every $y_0 \in F(x_0)$ and every neighborhood $N(y_0) \subset Y$ of $y_0$, there exists a neighborhood $N(x_0) \subset X$ of $x_0$ such that

$$F(x) \cap N(y_0) \neq \emptyset, \quad \forall \ x \in N(x_0).$$

(iii') $F$ is said to be continuous at $x_0$ if it is upper and lower semi-continuous.

The following three lemmas are crucial in our analysis.

**Lemma 3.** [1, Lemma 2.1] Suppose $(X, \rho_X)$ is a metric space and $(Y, \rho_Y)$ is a complete metric space. Consider a multi-valued mapping $F : X \mapsto Y$, and let $x_0 \in X$.

(i) If $F$ is upper semi-continuous at $x_0$, then it is $\delta$-upper semi-continuous at $x_0$. The converse is true when the set $F(x_0)$ is compact.

(ii) If $F$ is $\delta$-lower semi-continuous at $x_0$, then it is lower semi-continuous at $x_0$. The converse is true when the set $F(x_0)$ is compact.
Lemma 4. [7, Proposition 2] Let \( A, B \in \mathbb{C}^{n \times m} \) and \( 0 \leq \epsilon_1 < \epsilon_2 < 1 \). If the matrix \( A \) is not a scalar multiple of \( B \), then \( \mathcal{F}^\epsilon_{\| \cdot \|} (A; B) \) lies in the interior of \( \mathcal{F}^\epsilon_{\| \cdot \|} (A; B) \), \( \text{Int}[\mathcal{F}^\epsilon_{\| \cdot \|} (A; B)] \).

Lemma 5. [12] Let \( A, B \in \mathbb{C}^{n \times m} \) and \( \epsilon \in [0, 1) \), and suppose \( A \) is not a scalar multiple of \( B \). Then for every \( \delta > 0 \), there exist scalars \( \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C} \) such that

\[
\delta_H \left( \bigcap_{i=1}^{k} D \left( \lambda_i, \frac{\| A - \lambda_i B \|}{\sqrt{1 - \epsilon^2 \| B \|}} \right), F^\epsilon_{\| \cdot \|} (A; B) \right) \leq \delta.
\]

3 Continuity in \( A \)

In this section, we derive the continuity of the Birkhoff-James \( \epsilon \)-orthogonality set \( F^\epsilon_{\| \cdot \|} (A; B) \) with respect to matrix \( A \).

Theorem 6. Let \( A_0, B \in \mathbb{C}^{n \times m} \) (with \( B \neq 0 \)) and \( \epsilon \in [0, 1) \), and suppose that \( A_0 \) is not a scalar multiple of \( B \). Then, the mapping \( A \mapsto F^\epsilon_{\| \cdot \|} (A; B) \) is continuous at \( A_0 \).

Proof. We will first prove the upper semi-continuity of the mapping. Suppose that \( A_0 \in \mathbb{C}^{n \times m} \) is not a scalar multiple of \( B \), and let \( \delta > 0 \). By Lemma 5, there are \( \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C} \) such that

\[
\delta_H \left( G(A_0), F^\epsilon_{\| \cdot \|} (A_0; B) \right) \leq \frac{\delta}{2},
\]

where

\[
G(A_0) = \bigcap_{i=1}^{k} D \left( \lambda_i, \frac{\| A_0 - \lambda_i B \|}{\sqrt{1 - \epsilon^2 \| B \|}} \right).
\]

Moreover, for any \( E \in \mathbb{C}^{n \times m} \), we have

\[
\frac{\| A_0 - \lambda_i B \|}{\sqrt{1 - \epsilon^2 \| B \|}} = \frac{\| A_0 + E - \lambda_i B - E \|}{\sqrt{1 - \epsilon^2 \| B \|}} \leq \frac{\| A_0 + E - \lambda_i B \|}{\sqrt{1 - \epsilon^2 \| B \|}} + \frac{\| E \|}{\sqrt{1 - \epsilon^2 \| B \|}}
\]

for \( i = 1, 2, \ldots, k \). As a consequence, the set

\[
\Omega(A_0, E) = \bigcap_{i=1}^{k} D \left( \lambda_i, \frac{\| A_0 + E - \lambda_i B \|}{\sqrt{1 - \epsilon^2 \| B \|}} + \frac{\| E \|}{\sqrt{1 - \epsilon^2 \| B \|}} \right),
\]

contains

\[
F^\epsilon_{\| \cdot \|} (A_0 + E; B) = \bigcap_{\lambda \in \mathbb{C}} D \left( \lambda, \frac{\| A_0 + E - \lambda B \|}{\sqrt{1 - \epsilon^2 \| B \|}} \right).
\]

By [13, Theorem 1.7.3], there exists a \( \gamma > 0 \) such that for every \( E \in \mathbb{C}^{n \times m} \) with \( \| E \| \leq \gamma \), \( d_H(G(A_0), \Omega(A_0, E)) \leq \delta/2 \). Hence, for every \( E \in \mathbb{C}^{n \times m} \) with \( \| E \| \leq \gamma \),

\[
d_H(F^\epsilon_{\| \cdot \|} (A_0; B), \Omega(A_0, E)) \leq d_H(F^\epsilon_{\| \cdot \|} (A_0; B), G(A_0)) + d_H(G(A_0), \Omega(A_0, E)) \leq \delta.
\]
This implies that
\[ \Omega(A_0, E) \subseteq F^\epsilon_{\|\cdot\|}(A_0; B) + D(0, \delta), \]
and thus,
\[ F^\epsilon_{\|\cdot\|}(A_0 + E; B) \subseteq F^\epsilon_{\|\cdot\|}(A_0; B) + D(0, \delta). \]
So, the mapping \( A \mapsto F^\epsilon_{\|\cdot\|}(A; B) \) is \( \delta \)-upper semi-continuous at \( A_0 \), and by Lemma 3, it is also upper semi-continuous at \( A_0 \).

Next we derive the lower semi-continuity of the mapping. First we consider the case where \( \epsilon > 0 \). Since \( A_0 \) is not a scalar multiple of \( B \), Lemma 4 implies that \( \text{Int}[F^\epsilon_{\|\cdot\|}(A_0; B)] \neq \emptyset \) (see also Corollary 3 in [7]). Keeping in mind the convexity of \( F^\epsilon_{\|\cdot\|}(A; B) \), we have that for any \( \mu \in F^\epsilon_{\|\cdot\|}(A_0; B) \) and \( \delta > 0 \), the disc \( D(\mu, \delta) \) has a nonempty intersection with \( \text{Int}[F^\epsilon_{\|\cdot\|}(A_0; B)] \). Moreover, for any \( \mu_0 \in D(\mu, \delta) \cap \text{Int}[F^\epsilon_{\|\cdot\|}(A_0; B)] \), it holds that (see Proposition 16 in [7])
\[ \inf_{\lambda \in \mathbb{C}} \left\{ \|A_0 - \lambda B\| - |\lambda - \mu_0| \|B\| \sqrt{1 - \epsilon^2} \right\} = \xi > 0. \]
Thus, for every \( E \in \mathbb{C}^{n \times m} \) with \( \|E\| \leq \xi \), we have
\[ \|A_0 - \lambda B\| - \|E\| > |\lambda - \mu_0| \|B\| \sqrt{1 - \epsilon^2}, \quad \forall \lambda \in \mathbb{C}, \]
or
\[ \|A_0 + E - \lambda B\| > |\lambda - \mu_0| \|B\| \sqrt{1 - \epsilon^2}, \quad \forall \lambda \in \mathbb{C}. \]
As a consequence, \( \mu_0 \in F^\epsilon_{\|\cdot\|}(A_0 + E; B) \) for every \( E \in \mathbb{C}^{n \times m} \) with \( \|E\| \leq \xi \), and thus, \( D(\mu, \delta) \cap F^\epsilon_{\|\cdot\|}(A_0 + E; B) \neq \emptyset \). Hence, for \( \epsilon > 0 \), the mapping \( A \mapsto F^\epsilon_{\|\cdot\|}(A; B) \) is lower semi-continuous at \( A_0 \).

Let now \( \epsilon = 0 \), and assume that the mapping \( A \mapsto F^0_{\|\cdot\|}(A; B) \) is not lower semi-continuous at \( A_0 \). Then there exist a \( \mu_0 \in F^0_{\|\cdot\|}(A; B) \) and a \( \delta > 0 \) such that for any \( \xi > 0 \), there is an \( E \in \mathbb{C}^{n \times n} \) with \( \|E\| \leq \xi \), which satisfies
\[ F^0_{\|\cdot\|}(A_0 + E; B) \cap D(\mu_0, \delta) = \emptyset. \]
Then, for every \( \mu \in D(\mu_0, \delta) \), there is a \( \lambda_\mu \in \mathbb{C} \) (with \( \lambda_\mu \neq \mu \)) such that
\[ \|A_0 + E - \lambda_\mu B\| < |\mu - \lambda_\mu| \|B\|. \]
Since this inequality is strict, the quantity \( |\mu - \lambda_\mu| \|B\| \) is positive. Thus, for every \( \mu \in D(\mu_0, \delta) \), the number
\[ \epsilon_\mu = \frac{1}{2} \sqrt{1 - \frac{\|A_0 + E - \lambda_\mu B\|}{|\mu - \lambda_\mu| \|B\|}} < \sqrt{1 - \frac{\|A_0 + E - \lambda_\mu B\|}{|\mu - \lambda_\mu| \|B\|}} \]
is positive and satisfies
\[ \|A_0 + E - \lambda_\mu B\| < \sqrt{1 - \epsilon_\mu^2} |\mu - \lambda_\mu| \|B\|. \]
Hence, if we define \( \hat{\epsilon} = \min \{ \epsilon_\mu : \mu \in \mathcal{D}(\mu_0, \delta) \} > 0 \), then it follows
\[
\| A_0 + E - \lambda_\mu B \| < \sqrt{1 - \hat{\epsilon}^2} |\mu - \lambda_\mu| \| B \|,
\]
and consequently,
\[
F^\hat{\epsilon}_{\| \cdot \|}(A_0 + E; B) \cap \mathcal{D}(\mu_0, \delta) = \emptyset.
\]
This means that the mapping \( A \mapsto F^\hat{\epsilon}_{\| \cdot \|}(A; B) \) is not lower semi-continuous at \( A_0 \), which contradicts the result already proved for \( \epsilon > 0 \). \( \square \)

4 Continuity in \( \epsilon \)

In this section, we obtain the continuity of the Birkhoff-James \( \epsilon \)-orthogonality set \( F^\epsilon_{\| \cdot \|}(A; B) \) with respect to the real parameter \( \epsilon \in [0, 1] \).

**Theorem 7.** Let \( A, B \in \mathbb{C}^{n \times m} \) (with \( B \neq 0 \)) and \( \epsilon_0 \in [0, 1] \), and suppose that \( A \) is not a scalar multiple of \( B \). Then, the mapping \( \epsilon \mapsto F^\epsilon_{\| \cdot \|}(A; B) \) is continuous at \( \epsilon_0 \).

**Proof.** To obtain the upper semi-continuity of the mapping, it is enough to prove that for every \( \delta > 0 \), there is a neighborhood of \( \epsilon_0 \), say \( \mathcal{N}(\epsilon_0) \), such that
\[
F^\hat{\epsilon}_{\| \cdot \|}(A; B) \subseteq F^{\epsilon_0}_{\| \cdot \|}(A; B) + \mathcal{D}(0, \delta), \quad \forall \hat{\epsilon} \in \mathcal{N}(\epsilon_0).
\]
For \( \hat{\epsilon} < \epsilon_0 \), Lemma 4 implies that \( F^\hat{\epsilon}_{\| \cdot \|}(A; B) \subseteq F^{\epsilon_0}_{\| \cdot \|}(A; B) \), and the desired inclusion apparently holds.

We consider now the case where \( \hat{\epsilon} > \epsilon_0 \). As in the proof of Theorem 6, by Lemma 5, there exist scalars \( \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C} \) such that the Hausdorff distance between the set
\[
G(\epsilon_0) = \bigcap_{i=1}^k \mathcal{D} \left( \lambda_i, \frac{\| A - \lambda_i B \|}{\sqrt{1 - \epsilon_0^2} \| B \|} \right)
\]
and \( F^{\epsilon_0}_{\| \cdot \|}(A; B) \) is less than or equal to \( \delta/2 \). By [13, Theorem 1.7.3], there is an \( \hat{\epsilon} \) sufficiently close to \( \epsilon_0 \) such that \( d_H(G(\epsilon_0), G(\hat{\epsilon})) \leq \delta/2 \), where \( G(\hat{\epsilon}) = \bigcap_{i=1}^k \mathcal{D} \left( \lambda_i, \frac{\| A - \lambda_i B \|}{\sqrt{1 - \hat{\epsilon}^2} \| B \|} \right) \).

Hence,
\[
d_H(G(\hat{\epsilon}), F^{\epsilon_0}_{\| \cdot \|}(A; B)) \leq d_H(G(\hat{\epsilon}), G(\epsilon_0)) + d_H(G(\epsilon_0), F^{\epsilon_0}_{\| \cdot \|}(A; B)) \leq \delta.
\]
As a consequence,
\[
G(\hat{\epsilon}) \subseteq F^{\epsilon_0}_{\| \cdot \|}(A; B) + \mathcal{D}(0, \delta).
\]
Since \( F^\hat{\epsilon}_{\| \cdot \|}(A; B) \subseteq G(\hat{\epsilon}) \), it follows
\[
F^\hat{\epsilon}_{\| \cdot \|}(A; B) \subseteq F^{\epsilon_0}_{\| \cdot \|}(A; B) + \mathcal{D}(0, \delta),
\]
which means that the mapping \( \epsilon \mapsto F^\epsilon_{\| \cdot \|}(A; B) \) is \( \delta \)-upper semi-continuous at \( \epsilon_0 \), and by Lemma 3, it is upper semi-continuous at \( \epsilon_0 \).
Next we prove the lower semi-continuity of the mapping. If \( \epsilon_0 = 0 \), then the result follows readily from Lemma 4. Let \( \epsilon_0 > 0 \) and \( \mu \in F^c_{\|\cdot\|}(A; B) \). We will see that for any \( \delta > 0 \), there exists an open interval \( \mathcal{N}(\epsilon_0) = (\epsilon_0 - \gamma, \epsilon_0 + \gamma) \), \( \gamma > 0 \), such that

\[
F^c_{\|\cdot\|}(A; B) \cap \mathcal{D}(\mu, \delta) \neq \emptyset, \quad \forall \, \epsilon \in (\epsilon_0 - \gamma, \epsilon_0 + \gamma).
\]

By Lemma 4, for any \( \hat{\epsilon} \) and \( \epsilon \) of contradiction, and without loss of generality, we may assume that there exists a scalar \( \delta \) such that

\[
F^c_{\|\cdot\|}(A; B) \cap \mathcal{D}(\mu, \delta) = \emptyset, \quad \forall \, \epsilon \in (\epsilon_0 - \gamma, \epsilon_0 + \gamma).
\]

Thus, the sequence \( \mu^k \) follows readily from Lemma 4. Let \( \mu^k \rightarrow \mu \) and since \( \|\cdot\| \in \mathcal{D}(\mu, \delta) \neq \emptyset \), then by Lemma 4, we can set \( \gamma = \epsilon_0 - \hat{\epsilon} \). Thus, for the sake of contradiction, and without loss of generality, we may assume that there exists a \( \delta \) such that \( F^c_{\|\cdot\|}(A; B) \cap \mathcal{D}(\mu, \delta) = \emptyset \) for all nonnegative \( \hat{\epsilon} < \epsilon_0 \). Then, choosing \( \delta \) sufficiently small, there is a \( \theta \in [0, 2\pi] \) such that

\[
\mu + \delta e^{i\theta} \in \text{Int}[F^c_{\|\cdot\|}(A; B)]
\]

and

\[
\mu + \delta e^{i\theta} \notin F^c_{\|\cdot\|}(A; B), \quad \forall \, \hat{\epsilon} \in [0, \epsilon_0).
\]

Consider now a sequence \( \{\epsilon_k\}_{k \in \mathbb{N} \setminus \{0\}} \subset [0, \epsilon_0) \) that converges to \( \epsilon_0 \). Then for every \( k = 1, 2, \ldots \), there exists a scalar \( \lambda_k(\mu, \theta) \) such that

\[
|\mu + \delta e^{i\theta} - \lambda_k(\mu, \theta)| > \frac{\|A - \lambda_k(\mu, \theta)B\|}{\sqrt{1 - \epsilon_k^2 \|B\|}}, \quad (1)
\]

or

\[
|\mu + \delta e^{i\theta}| + |\lambda_k(\mu, \theta)| > \frac{1}{\sqrt{1 - \epsilon_k^2 \|B\|}} \left( \|A\| - |\lambda_k(\mu, \theta)| \|B\| \right).
\]

If \( |\lambda_k(\mu, \theta)| \|B\| < \|A\| \), then we have \( |\lambda_k(\mu, \theta)| < \|A\| / \|B\| \). If not, then

\[
|\lambda_k(\mu, \theta)| \|B\| - \|A\| < \sqrt{1 - \epsilon_k^2 \|B\|} \left( |\mu + \delta e^{i\theta}| + |\lambda_k(\mu, \theta)| \right),
\]

and since \( \epsilon_k > 0 \), it follows

\[
|\lambda_k(\mu, \theta)| < \frac{\|A\| + \sqrt{1 - \epsilon_k^2 \|B\|} \|\mu + \delta e^{i\theta}\|}{\|B\| \left( 1 - \sqrt{1 - \epsilon_k^2} \right)}.
\]

Thus, the sequence \( \lambda_k(\mu, \theta) \) \((k = 1, 2, \ldots)\) is bounded, and hence, it has a converging subsequence \( \lambda_{k_t}(\mu, \theta) \) \((t = 1, 2, \ldots)\). If \( \lambda_0 = \lim_{k_t \to \infty} \lambda_{k_t}(\mu, \theta) \), then \( (1) \) yields

\[
\lim_{k_t \to \infty} |\mu + \delta e^{i\theta} - \lambda_{k_t}(\mu, \theta)| \geq \lim_{k_t \to \infty} \frac{\|A - \lambda_{k_t}(\mu, \theta)B\|}{\sqrt{1 - \epsilon_{k_t}^2 \|B\|}},
\]

or

\[
|\mu + \delta e^{i\theta} - \lambda_0| \geq \frac{\|A - \lambda_0 B\|}{\sqrt{1 - \epsilon_0^2 \|B\|}}.
\]

This contradicts to Property \((P_3)\) and Proposition 16 in [7], because \( \mu + \delta e^{i\theta} \) is an interior point of \( F^c_{\|\cdot\|}(A; B) \). Consequently, the mapping \( \epsilon \mapsto F^c_{\|\cdot\|}(A; B) \) is lower semi-continuous at \( \epsilon_0 \), and the proof is complete. \( \square \)
Remark 8. Let \( A, B \in \mathbb{C}^{n \times m} \) (with \( B \neq 0 \)) and \( \epsilon \in [0,1) \), and suppose that \( A \) is not a scalar multiple of \( B \). Suppose also that the matrix norm \( \| \cdot \| \) is induced by an inner product of matrices, say \( \langle \cdot, \cdot \rangle \). Then the Birkhoff-James \( \epsilon \)-orthogonality set of \( A \) with respect to \( B \) is a closed disk \([6, 7] \), namely, 
\[
F_{\| \cdot \|}^\epsilon (A; B) = \mathcal{D} \left( \frac{\langle A, B \rangle}{\| B \|^2}, \frac{A - \langle A, B \rangle \| B \|^2}{\| B \|^2} B \frac{\epsilon}{\sqrt{1 - \epsilon^2 \| B \|^2}} \right).
\]
By the continuity of the inner product and the norm, the continuity of \( F_{\| \cdot \|}^\epsilon (A; B) \) with respect to \( A, B \) or \( \epsilon \) is readily verified. In general, it is not known to the authors whether the mapping \( B \mapsto F_{\| \cdot \|}^\epsilon (A; B) \) is always continuous (i.e., for all matrix norms) or not.

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