# On the continuity of Birkhoff-James $\epsilon$-orthogonality sets

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**Abstract**

Consider two matrices $A, B \in \mathbb{C}^{n\times m}$ with $B \neq 0$, a matrix norm $\|\cdot\|$, and a real parameter $\epsilon \in [0,1)$. The Birkhoff-James $\epsilon$-orthogonality set of $A$ with respect to $B$, $F^{\epsilon}_{\|\cdot\|}(A;B) = \left\{\mu \in \mathbb{C} : \|A - \lambda B\| \geq \sqrt{1-\epsilon^2}\,\|B\|\,|\mu - \lambda|, \forall\, \lambda \in \mathbb{C}\right\}$, is a compact and convex subset of the complex plane that has been recently introduced by the authors, as a natural generalization of the classical numerical range of square matrices. In this note, we derive the continuity of $F^{\epsilon}_{\|\cdot\|}(A;B)$ with respect to $A$ or $\epsilon$.

*Keywords:* Birkhoff-James $\epsilon$-orthogonality set, Hausdorff distance, continuity.

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## 1 Introduction

The *numerical range* (also known as the *field of values*) of a square complex matrix $A \in \mathbb{C}^{n\times n}$ is defined as $F(A) = \{x^*Ax \in \mathbb{C} : x \in \mathbb{C}^n, x^*x = 1\}$ [10]. It is a compact and convex subset of the complex plane that has been studied extensively for many decades, and it is useful in studying and understanding matrices and operators; see [3, 4, 9, 10, 14] and the references therein.

The numerical range $F(A)$ is also written in the form [4, 14],

$$\begin{aligned} F(A) &= \{\mu \in \mathbb{C} : \|A - \lambda I_n\|_2 \geq |\mu - \lambda|, \forall\, \lambda \in \mathbb{C}\} \\ &= \bigcap_{\lambda\in\mathbb{C}} \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda I_n\|_2\}, \end{aligned}$$

where $\|\cdot\|_2$ denotes the *spectral matrix norm* (i.e., that norm subordinate to the euclidean vector norm) and $I_n$ is the $n \times n$ identity matrix. Thus, $F(A)$ is an infinite intersection of closed disks $\mathcal{D}(\lambda, \|A - \lambda I_n\|_2) = \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|A - \lambda I_n\|_2\}$

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($\lambda \in \mathbb{C}$). Inspired by the above intersection property, Chorianopoulos, Karanasios and Psarrakos [6] proposed a definition of numerical range for rectangular complex matrices. In particular, for any $A, B \in \mathbb{C}^{n\times m}$ with $B \neq 0$, and any matrix norm $\|\cdot\|$, the *numerical range of* $A$ *with respect to* $B$ is defined as

$$\begin{aligned} F_{\|\cdot\|}(A;B) &= \{\mu \in \mathbb{C} : \|A-\lambda B\| \geq |\mu - \lambda|, \forall \lambda \in \mathbb{C}\} \\ &= \bigcap_{\lambda\in\mathbb{C}} \mathcal{D}\left(\lambda, \|A-\lambda B\|\right). \end{aligned}$$

This set is compact and convex, satisfies basic properties of the standard numerical range, and is nonempty if and only if $\|B\| \geq 1$ [6].

The analysis in [6] is based on the properties of matrix norms and the Birkhoff-James orthogonality [2, 11]; namely, for two elements $\chi$ and $\psi$ of a complex normed linear space $(\mathcal{X}, \|\cdot\|)$, $\chi$ is called *Birkhoff-James orthogonal* to $\psi$, denoted by $\chi \perp_{BJ} \psi$, if $\|\chi + \lambda\psi\| \geq \|\chi\|$ for all $\lambda \in \mathbb{C}$. This orthogonality is homogeneous, but it is neither symmetric nor additive [11]. Furthermore, for any $\epsilon \in [0,1)$, we say that $\chi$ is *Birkhoff-James $\epsilon$-orthogonal* to $\psi$, denoted by $\chi \perp^{\epsilon}_{BJ} \psi$, if $\|\chi + \lambda\psi\| \geq \sqrt{1-\epsilon^2}\,\|\chi\|$ for all $\lambda \in \mathbb{C}$. It is straightforward to see that this relation is also homogeneous. In an inner product space $(\mathcal{X}, \langle\cdot,\cdot\rangle)$, with the standard orthogonality relation $\perp$, a $\chi \in \mathcal{X}$ is called *$\epsilon$-orthogonal* to a $\psi \in \mathcal{X}$, denoted by $\chi \perp^{\epsilon} \psi$, if $|\langle\chi,\psi\rangle| \leq \epsilon\,\|\chi\|\,\|\psi\|$. Moreover, $\chi \perp \psi$ (resp., $\chi \perp^{\epsilon} \psi$) if and only if $\chi \perp_{BJ} \psi$ (resp., $\chi \perp^{\epsilon}_{BJ} \psi$) [5, 8].

If $\|B\| \geq 1$, then for $\epsilon_B = \sqrt{\|B\|^2 - 1}\,/\,\|B\|$, we have [7]

$$F_{\|\cdot\|}(A;B) = \left\{\mu \in \mathbb{C} : B \perp^{\epsilon_B}_{BJ} (A - \mu B)\right\}.$$

As a consequence, the next definition (introduced by the authors in [7]) arises in a natural way.

**Definition 1.** For any $A, B \in \mathbb{C}^{n\times m}$ with $B \neq 0$, any matrix norm $\|\cdot\|$, and any $\epsilon \in [0,1)$, the *Birkhoff-James $\epsilon$-orthogonality set of* $A$ *with respect to* $B$ is defined and denoted by

$$\begin{aligned} F^{\epsilon}_{\|\cdot\|}(A;B) &= \{\mu \in \mathbb{C} : B \perp^{\epsilon}_{BJ} (A-\mu B)\} \\ &= \left\{\mu \in \mathbb{C} : \|A-\lambda B\| \geq \sqrt{1-\epsilon^2}\,\|B\|\,|\mu-\lambda|, \forall \lambda \in \mathbb{C}\right\} \\ &= \bigcap_{\lambda\in\mathbb{C}} \mathcal{D}\left(\lambda, \frac{\|A-\lambda B\|}{\sqrt{1-\epsilon^2}\,\|B\|}\right). \end{aligned}$$

The Birkhoff-James $\epsilon$-orthogonality set $F^{\epsilon}_{\|\cdot\|}(A;B)$ is a nonempty, compact and convex subset of the complex plane that lies in the closed disk $\mathcal{D}\left(0, \frac{\|A\|}{\sqrt{1-\epsilon^2}\,\|B\|}\right)$ and is quite rich in structure [7]. In this note, we obtain the continuity of $F^{\epsilon}_{\|\cdot\|}(A;B)$ with respect to the matrix $A \in \mathbb{C}^{n\times m}$ (see Section 3), or to the real parameter $\epsilon \in [0,1)$ (see Section 4). The question of the continuity of $F^{\epsilon}_{\|\cdot\|}(A;B)$ with respect to matrix $B \in \mathbb{C}^{n\times m}$ is still open except the special case where the norm $\|\cdot\|$ is induced by an inner product of matrices (see Remark 8).

## 2 Preliminaries

We recall that for two compact subsets $\Omega_1$ and $\Omega_2$ of a metric space $(\mathcal{X},\rho)$, the Hausdorff distance between $\Omega_1$ and $\Omega_2$ is defined by

$$d_H(\Omega_1, \Omega_2) = \max\left\{\max_{x_1\in\Omega_1}\min_{x_2\in\Omega_2}\rho(x_1, x_2), \max_{x_2\in\Omega_2}\min_{x_1\in\Omega_1}\rho(x_1, x_2)\right\}.$$

For any $x_0 \in \mathcal{X}$ and $\delta > 0$, we define the closed ball $\mathcal{B}(x_0, \delta) = \{x \in \mathcal{X} : \rho(x_0, x) \leq \delta\}$.

**Definition 2.** [1] Suppose $(\mathcal{X}, \rho_{\mathcal{X}})$ is a metric space and $(\mathcal{Y}, \rho_{\mathcal{Y}})$ is a complete metric space. Consider a multi-valued mapping $F : \mathcal{X} \mapsto \mathcal{Y}$, and let $x_0 \in \mathcal{X}$.

(i) $F$ is called *$\delta$-upper semi-continuous* at $x_0$ if for every $\delta > 0$, there is a neighborhood $\mathcal{N}(x_0) \subset \mathcal{X}$ of $x_0$ such that

$$F(x) \subseteq F(x_0) + \mathcal{B}(0, \delta), \quad \forall\, x \in \mathcal{N}(x_0).$$

(ii) $F$ is called *$\delta$-lower semi-continuous* at $x_0$ if for every $\delta > 0$, there is a neighborhood $\mathcal{N}(x_0) \subset \mathcal{X}$ of $x_0$ such that

$$F(x_0) \subseteq F(x) + \mathcal{B}(0, \delta), \quad \forall\, x \in \mathcal{N}(x_0).$$

(iii) $F$ is said to be *$\delta$-continuous* at $x_0$ if it is $\delta$-upper and $\delta$-lower semi-continuous.

(i′) $F$ is called *upper semi-continuous* at $x_0$ if for every neighborhood $\mathcal{N}(F(x_0)) \subset \mathcal{Y}$ of the set $F(x_0)$, there is a neighborhood $\mathcal{N}(x_0) \subset \mathcal{X}$ of $x_0$ such that

$$F(x) \subseteq \mathcal{N}(F(x_0)), \quad \forall\, x \in \mathcal{N}(x_0).$$

(ii′) $F$ is called *lower semi-continuous* at $x_0$ if for every $y_0 \in F(x_0)$ and every neighborhood $\mathcal{N}(y_0) \subset \mathcal{Y}$ of $y_0$, there exists a neighborhood $\mathcal{N}(x_0) \subset \mathcal{X}$ of $x_0$ such that

$$F(x) \cap \mathcal{N}(y_0) \neq \emptyset, \quad \forall\, x \in \mathcal{N}(x_0).$$

(iii′) $F$ is said to be *continuous* at $x_0$ if it is upper and lower semi-continuous.

The following three lemmas are crucial in our analysis.

**Lemma 3.** [1, Lemma 2.1] *Suppose $(\mathcal{X}, \rho_{\mathcal{X}})$ is a metric space and $(\mathcal{Y}, \rho_{\mathcal{Y}})$ is a complete metric space. Consider a multi-valued mapping $F : \mathcal{X} \mapsto \mathcal{Y}$, and let $x_0 \in \mathcal{X}$.*

**(i)** *If $F$ is upper semi-continuous at $x_0$, then it is $\delta$-upper semi-continuous at $x_0$. The converse is true when the set $F(x_0)$ is compact.*

**(ii)** *If $F$ is $\delta$-lower semi-continuous at $x_0$, then it is lower semi-continuous at $x_0$. The converse is true when the set $F(x_0)$ is compact.*

**Lemma 4.** [7, Proposition 2] *Let $A, B \in \mathbb{C}^{n\times m}$ and $0 \le \epsilon_1 < \epsilon_2 < 1$. If the matrix $A$ is not a scalar multiple of $B$, then $F^{\epsilon_1}_{\|\cdot\|}(A;B)$ lies in the interior of $F^{\epsilon_2}_{\|\cdot\|}(A;B)$,* $\mathrm{Int}[F^{\epsilon_2}_{\|\cdot\|}(A;B)]$.

**Lemma 5.** [12] *Let $A, B \in \mathbb{C}^{n\times m}$ and $\epsilon \in [0,1)$, and suppose $A$ is not a scalar multiple of $B$. Then for every $\delta > 0$, there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$ such that*

$$d_H\left(\bigcap_{i=1}^{k}\mathcal{D}\left(\lambda_i, \frac{\|A-\lambda_i B\|}{\sqrt{1-\epsilon^2}\,\|B\|}\right), F^{\epsilon}_{\|\cdot\|}(A;B)\right) \le \delta.$$

# 3 Continuity in $A$

In this section, we derive the continuity of the Birkhoff-James $\epsilon$-orthogonality set $F^{\epsilon}_{\|\cdot\|}(A;B)$ with respect to matrix $A$.

**Theorem 6.** *Let $A_0, B \in \mathbb{C}^{n\times m}$ (with $B \ne 0$) and $\epsilon \in [0,1)$, and suppose that $A_0$ is not a scalar multiple of $B$. Then, the mapping $A \mapsto F^{\epsilon}_{\|\cdot\|}(A;B)$ is continuous at $A_0$.*

*Proof.* We will first prove the upper semi-continuity of the mapping. Suppose that $A_0 \in \mathbb{C}^{n\times m}$ is not a scalar multiple of $B$, and let $\delta > 0$. By Lemma 5, there are $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$ such that

$$d_H\left(G(A_0), F^{\epsilon}_{\|\cdot\|}(A_0;B)\right) \le \frac{\delta}{2},$$

where

$$G(A_0) = \bigcap_{i=1}^{k}\mathcal{D}\left(\lambda_i, \frac{\|A_0-\lambda_i B\|}{\sqrt{1-\epsilon^2}\,\|B\|}\right).$$

Moreover, for any $E \in \mathbb{C}^{n\times m}$, we have

$$\frac{\|A_0-\lambda_i B\|}{\sqrt{1-\epsilon^2}\,\|B\|} = \frac{\|A_0+E-\lambda_i B-E\|}{\sqrt{1-\epsilon^2}\,\|B\|} \le \frac{\|A_0+E-\lambda_i B\|}{\sqrt{1-\epsilon^2}\,\|B\|} + \frac{\|E\|}{\sqrt{1-\epsilon^2}\,\|B\|}$$

for $i = 1, 2, \ldots, k$. As a consequence, the set

$$\Omega(A_0,E) = \bigcap_{i=1}^{k}\mathcal{D}\left(\lambda_i, \frac{\|A_0+E-\lambda_i B\|}{\sqrt{1-\epsilon^2}\,\|B\|} + \frac{\|E\|}{\sqrt{1-\epsilon^2}\,\|B\|}\right),$$

contains

$$F^{\epsilon}_{\|\cdot\|}(A_0+E;B) = \bigcap_{\lambda\in\mathbb{C}}\mathcal{D}\left(\lambda, \frac{\|A_0+E-\lambda B\|}{\sqrt{1-\epsilon^2}\,\|B\|}\right).$$

By [13, Theorem 1.7.3], there exists a $\gamma > 0$ such that for every $E \in \mathbb{C}^{n\times m}$ with $\|E\| \le \gamma$, $d_H(G(A_0), \Omega(A_0,E)) \le \delta/2$. Hence, for every $E \in \mathbb{C}^{n\times m}$ with $\|E\| \le \gamma$,

$$d_H(F^{\epsilon}_{\|\cdot\|}(A_0;B), \Omega(A_0,E)) \le d_H(F^{\epsilon}_{\|\cdot\|}(A_0;B), G(A_0)) + d_H(G(A_0), \Omega(A_0,E)) \le \delta.$$

This implies that

$$\Omega(A_0, E) \subseteq F^{\epsilon}_{\|\cdot\|}(A_0; B) + \mathcal{D}(0, \delta),$$

and thus,

$$F^{\epsilon}_{\|\cdot\|}(A_0 + E; B) \subseteq F^{\epsilon}_{\|\cdot\|}(A_0; B) + \mathcal{D}(0, \delta).$$

So, the mapping $A \mapsto F^{\epsilon}_{\|\cdot\|}(A; B)$ is $\delta$-upper semi-continuous at $A_0$, and by Lemma 3, it is also upper semi-continuous at $A_0$.

Next we derive the lower semi-continuity of the mapping. First we consider the case where $\epsilon > 0$. Since $A_0$ is not a scalar multiple of $B$, Lemma 4 implies that $\mathrm{Int}[F^{\epsilon}_{\|\cdot\|}(A_0; B)] \neq \emptyset$ (see also Corollary 3 in [7]). Keeping in mind the convexity of $F^{\epsilon}_{\|\cdot\|}(A; B)$, we have that for any $\mu \in F^{\epsilon}_{\|\cdot\|}(A_0; B)$ and $\delta > 0$, the disc $\mathcal{D}(\mu, \delta)$ has a nonempty intersection with $\mathrm{Int}[F^{\epsilon}_{\|\cdot\|}(A_0; B)]$. Moreover, for any $\mu_0 \in \mathcal{D}(\mu, \delta) \cap \mathrm{Int}[F^{\epsilon}_{\|\cdot\|}(A_0; B)]$, it holds that (see Proposition 16 in [7])

$$\inf_{\lambda \in \mathbb{C}} \left\{ \|A_0 - \lambda B\| - |\lambda - \mu_0| \, \|B\| \sqrt{1 - \epsilon^2} \right\} = \xi > 0.$$

Thus, for every $E \in \mathbb{C}^{n \times m}$ with $\|E\| \leq \xi$, we have

$$\|A_0 - \lambda B\| - \|E\| > |\lambda - \mu_0| \, \|B\| \sqrt{1 - \epsilon^2}, \quad \forall\, \lambda \in \mathbb{C},$$

or

$$\|A_0 + E - \lambda B\| > |\lambda - \mu_0| \, \|B\| \sqrt{1 - \epsilon^2}, \quad \forall\, \lambda \in \mathbb{C}.$$

As a consequence, $\mu_0 \in F^{\epsilon}_{\|\cdot\|}(A_0 + E; B)$ for every $E \in \mathbb{C}^{n \times m}$ with $\|E\| \leq \xi$, and thus, $\mathcal{D}(\mu, \delta) \cap F^{\epsilon}_{\|\cdot\|}(A_0 + E; B) \neq \emptyset$. Hence, for $\epsilon > 0$, the mapping $A \mapsto F^{\epsilon}_{\|\cdot\|}(A; B)$ is lower semi-continuous at $A_0$.

Let now $\epsilon = 0$, and assume that the mapping $A \mapsto F^{0}_{\|\cdot\|}(A; B)$ is not lower semi-continuous at $A_0$. Then there exist a $\mu_0 \in F^{0}_{\|\cdot\|}(A; B)$ and a $\delta > 0$ such that for any $\xi > 0$, there is an $E \in \mathbb{C}^{n \times n}$ with $\|E\| \leq \xi$, which satisfies

$$F^{0}_{\|\cdot\|}(A_0 + E; B) \cap \mathcal{D}(\mu_0, \delta) = \emptyset.$$

Then, for every $\mu \in \mathcal{D}(\mu_0, \delta)$, there is a $\lambda_\mu \in \mathbb{C}$ (with $\lambda_\mu \neq \mu$) such that

$$\|A_0 + E - \lambda_\mu B\| < |\mu - \lambda_\mu| \, \|B\|.$$

Since this inequality is strict, the quantity $|\mu - \lambda_\mu| \, \|B\|$ is positive. Thus, for every $\mu \in \mathcal{D}(\mu_0, \delta)$, the number

$$\epsilon_\mu = \frac{1}{2} \sqrt{1 - \frac{\|A_0 + E - \lambda_\mu B\|}{|\mu - \lambda_\mu| \, \|B\|}} < \sqrt{1 - \frac{\|A_0 + E - \lambda_\mu B\|}{|\mu - \lambda_\mu| \, \|B\|}}$$

is positive and satisfies

$$\|A_0 + E - \lambda_\mu B\| < \sqrt{1 - \epsilon_\mu^2} \, |\mu - \lambda_\mu| \, \|B\|.$$

Hence, if we define $\hat{\epsilon} = \min\{\epsilon_\mu : \mu \in \mathcal{D}(\mu_0, \delta)\} > 0$, then it follows

$$\|A_0 + E - \lambda_\mu B\| < \sqrt{1 - \hat{\epsilon}^{\,2}}\, |\mu - \lambda_\mu|\, \|B\|,$$

and consequently,

$$F^{\hat{\epsilon}}_{\|\cdot\|}(A_0 + E; B) \cap \mathcal{D}(\mu_0, \delta) = \emptyset.$$

This means that the mapping $A \mapsto F^{\hat{\epsilon}}_{\|\cdot\|}(A; B)$ is not lower semi-continuous at $A_0$, which contradicts the result already proved for $\epsilon > 0$. $\square$

## 4 Continuity in $\epsilon$

In this section, we obtain the continuity of the Birkhoff-James $\epsilon$-orthogonality set $F^{\epsilon}_{\|\cdot\|}(A; B)$ with respect to the real parameter $\epsilon \in [0, 1)$.

**Theorem 7.** *Let $A, B \in \mathbb{C}^{n\times m}$ (with $B \neq 0$) and $\epsilon_0 \in [0, 1)$, and suppose that $A$ is not a scalar multiple of $B$. Then, the mapping $\epsilon \mapsto F^{\epsilon}_{\|\cdot\|}(A; B)$ is continuous at $\epsilon_0$.*

*Proof.* To obtain the upper semi-continuity of the mapping, it is enough to prove that for every $\delta > 0$, there is a neighborhood of $\epsilon_0$, say $\mathcal{N}(\epsilon_0)$, such that

$$F^{\hat{\epsilon}}_{\|\cdot\|}(A; B) \subseteq F^{\epsilon_0}_{\|\cdot\|}(A; B) + \mathcal{D}(0, \delta), \quad \forall\, \hat{\epsilon} \in \mathcal{N}(\epsilon_0).$$

For $\hat{\epsilon} < \epsilon_0$, Lemma 4 implies that $F^{\hat{\epsilon}}_{\|\cdot\|}(A; B) \subseteq F^{\epsilon_0}_{\|\cdot\|}(A; B)$, and the desired inclusion apparently holds.

We consider now the case where $\hat{\epsilon} > \epsilon_0$. As in the proof of Theorem 6, by Lemma 5, there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$ such that the Hausdorff distance between the set

$$G(\epsilon_0) = \bigcap_{i=1}^{k} \mathcal{D}\left(\lambda_i, \frac{\|A - \lambda_i B\|}{\sqrt{1 - \epsilon_0^2}\, \|B\|}\right)$$

and $F^{\epsilon_0}_{\|\cdot\|}(A; B)$ is less than or equal to $\delta/2$. By [13, Theorem 1.7.3], there is an $\hat{\epsilon}$ sufficiently close to $\epsilon_0$ such that $d_H(G(\epsilon_0), G(\hat{\epsilon})) \leq \delta/2$, where $G(\hat{\epsilon}) = \bigcap_{i=1}^{k} \mathcal{D}\left(\lambda_i, \frac{\|A - \lambda_i B\|}{\sqrt{1-\hat{\epsilon}^2}\, \|B\|}\right)$. Hence,

$$d_H(G(\hat{\epsilon}), F^{\epsilon_0}_{\|\cdot\|}(A; B)) \leq d_H(G(\hat{\epsilon}), G(\epsilon_0)) + d_H(G(\epsilon_0), F^{\epsilon_0}_{\|\cdot\|}(A; B)) \leq \delta.$$

As a consequence,

$$G(\hat{\epsilon}) \subseteq F^{\epsilon_0}_{\|\cdot\|}(A; B) + \mathcal{D}(0, \delta).$$

Since $F^{\hat{\epsilon}}_{\|\cdot\|}(A; B) \subseteq G(\hat{\epsilon})$, it follows

$$F^{\hat{\epsilon}}_{\|\cdot\|}(A; B) \subseteq F^{\epsilon_0}_{\|\cdot\|}(A; B) + \mathcal{D}(0, \delta),$$

which means that the mapping $\epsilon \mapsto F^{\epsilon}_{\|\cdot\|}(A; B)$ is $\delta$-upper semi-continuous at $\epsilon_0$, and by Lemma 3, it is upper semi-continuous at $\epsilon_0$.

Next we prove the lower semi-continuity of the mapping. If $\epsilon_0 = 0$, then the result follows readily from Lemma 4. Let $\epsilon_0 > 0$ and $\mu \in F^{\epsilon_0}_{\|\cdot\|}(A;B)$. We will see that for any $\delta > 0$, there exists an open interval $\mathcal{N}(\epsilon_0) = (\epsilon_0 - \gamma, \epsilon_0 + \gamma)$, $\gamma > 0$, such that

$$F^{\hat{\epsilon}}_{\|\cdot\|}(A;B) \cap \mathcal{D}(\mu,\delta) \neq \emptyset, \quad \forall\, \hat{\epsilon} \in (\epsilon_0 - \gamma, \epsilon_0 + \gamma).$$

By Lemma 4, for any $\hat{\epsilon} \in [\epsilon_0, \epsilon_0 + \gamma)$, we have that $\mu \in F^{\hat{\epsilon}}_{\|\cdot\|}(A;B)$. Thus, it suffices to examine the case $\hat{\epsilon} \in (\epsilon_0 - \gamma, \epsilon_0)$. Moreover, if there is an $\hat{\epsilon}$ less than $\epsilon_0$ such that $F^{\hat{\epsilon}}_{\|\cdot\|}(A;B) \cap \mathcal{D}(\mu,\delta) \neq \emptyset$, then by Lemma 4, we can set $\gamma = \epsilon_0 - \hat{\epsilon}$. Thus, for the sake of contradiction, and without loss of generality, we may assume that there exists a $\delta_\mu > 0$ such that $F^{\hat{\epsilon}}_{\|\cdot\|}(A;B) \cap \mathcal{D}(\mu,\delta_\mu) = \emptyset$ for all nonnegative $\hat{\epsilon} < \epsilon_0$. Then, choosing $\delta_\mu$ sufficiently small, there is a $\theta \in [0, 2\pi]$ such that

$$\mu + \delta_\mu e^{\mathrm{i}\theta} \in \mathrm{Int}[F^{\epsilon_0}_{\|\cdot\|}(A;B)]$$

and

$$\mu + \delta_\mu e^{\mathrm{i}\theta} \notin F^{\hat{\epsilon}}_{\|\cdot\|}(A;B), \quad \forall\, \hat{\epsilon} \in [0, \epsilon_0).$$

Consider now a sequence $\{\epsilon_k\}_{k\in\mathbb{N}\setminus\{0\}} \subset [0, \epsilon_0)$ that converges to $\epsilon_0$. Then for every $k = 1, 2, \ldots$, there exists a scalar $\lambda_k(\mu,\theta)$ such that

$$|\mu + \delta_\mu e^{\mathrm{i}\theta} - \lambda_k(\mu,\theta)| > \frac{\|A - \lambda_k(\mu,\theta)B\|}{\sqrt{1-\epsilon_k^2}\,\|B\|}, \tag{1}$$

or

$$|\mu + \delta_\mu e^{\mathrm{i}\theta}| + |\lambda_k(\mu,\theta)| > \frac{1}{\sqrt{1-\epsilon_k^2}\,\|B\|}\,\big|\,\|A\| - |\lambda_k(\mu,\theta)|\,\|B\|\,\big|\,.$$

If $|\lambda_k(\mu,\theta)|\,\|B\| < \|A\|$, then we have $|\lambda_k(\mu,\theta)| < \|A\| \,/\, \|B\|$. If not, then

$$|\lambda_k(\mu,\theta)|\,\|B\| - \|A\| < \sqrt{1-\epsilon_k^2}\,\|B\|\,(|\mu + \delta_\mu e^{\mathrm{i}\theta}| + |\lambda_k(\mu,\theta)|),$$

and since $\epsilon_k > 0$, it follows

$$|\lambda_k(\mu,\theta)| < \frac{\|A\| + \sqrt{1-\epsilon_k^2}\,\|B\|\,|\mu + \delta_\mu e^{\mathrm{i}\theta}|}{\|B\|\,(1 - \sqrt{1-\epsilon_k^2})}.$$

Thus, the sequence $\lambda_k(\mu,\theta)$ $(k = 1, 2, \ldots)$ is bounded, and hence, it has a converging subsequence $\lambda_{k_t}(\mu,\theta)$ $(t = 1, 2, \ldots)$. If $\lambda_0 = \lim_{k_t\to\infty} \lambda_{k_t}(\mu,\theta)$, then (1) yields

$$\lim_{k_t\to\infty} |\mu + \delta_\mu e^{\mathrm{i}\theta} - \lambda_{k_t}(\mu,\theta)| \geq \lim_{k_t\to\infty} \frac{\|A - \lambda_{k_t}(\mu,\theta)B\|}{\sqrt{1-\epsilon_{k_t}^2}\,\|B\|},$$

or

$$|\mu + \delta_\mu e^{\mathrm{i}\theta} - \lambda_0| \geq \frac{\|A - \lambda_0 B\|}{\sqrt{1-\epsilon_0^2}\,\|B\|}.$$

This contradicts to Property ($P_5$) and Proposition 16 in [7], because $\mu + \delta_\mu e^{\mathrm{i}\theta}$ is an interior point of $F^{\epsilon_0}_{\|\cdot\|}(A;B)]$. Consequently, the mapping $\epsilon \mapsto F^{\epsilon}_{\|\cdot\|}(A;B)$ is lower semi-continuous at $\epsilon_0$, and the proof is complete. □

**Remark 8.** Let $A, B \in \mathbb{C}^{n\times m}$ (with $B \neq 0$) and $\epsilon \in [0,1)$, and suppose that $A$ is not a scalar multiple of $B$. Suppose also that the matrix norm $\|\cdot\|$ is induced by an inner product of matrices, say $\langle\cdot,\cdot\rangle$. Then the Birkhoff-James $\epsilon$-orthogonality set of $A$ with respect to $B$ is a closed disk [6, 7], namely,

$$F^{\epsilon}_{\|\cdot\|}(A;B) = \mathcal{D}\left(\frac{\langle A,B\rangle}{\|B\|^2}\,,\,\left\|A-\frac{\langle A,B\rangle}{\|B\|^2}\,B\right\|\frac{\epsilon}{\sqrt{1-\epsilon^2}\,\|B\|}\right).$$

By the continuity of the inner product and the norm, the continuity of $F^{\epsilon}_{\|\cdot\|}(A;B)$ with respect to $A$, $B$ or $\epsilon$ is readily verified. In general, it is not known to the authors whether the mapping $B \mapsto F^{\epsilon}_{\|\cdot\|}(A;B)$ is always continuous (i.e., for all matrix norms) or not.

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