On the q-numerical range of matrices and matrix polynomials

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Abstract

The q-numerical range ($0 \le q \le 1$) of an $n \times n$ matrix polynomial $P(\lambda) = A_m \lambda^m + \cdots + A_1 \lambda + A_0$ is defined by

$$W_q(P) \,=\, \{\lambda \in \mathbb{C} : y^* P(\lambda) x = 0,\ x, y \in \mathbb{C}^n,\ x^* x = y^* y = 1,\ y^* x = q\}.$$

In this paper, we investigate the boundary and the shape of $W_q(P)$, using the notion of local dimension. We also obtain that the q-numerical range of first order matrix polynomials is always simply connected. Moreover, the special cases of 2×2 matrices and matrix polynomials are considered. In particular, the boundary of the q-numerical range of a 2×2 matrix polynomial of degree m lies on an algebraic curve of degree at most $8m$.

Keywords: boundary; connectedness; eigenvalue; ellipse; local dimension; matrix polynomial; q-numerical range

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1 Introduction

Consider a differential equation of the form

$$A_m g^{(m)}(t) + A_{m-1} g^{(m-1)}(t) + \cdots + A_1 g^{(1)}(t) + A_0 g(t) \,=\, f(t), \qquad (1)$$

where $A_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, m)$, $g(t) \in \mathbb{C}^n$ is the unknown vector function and $f(t) \in \mathbb{C}^n$ is piecewise continuous (the indices on $g(t)$ denote derivatives

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with respect to the independent variable t). Applying the Laplace transformation yields the associated *matrix polynomial*

$$P(\lambda) \,=\, A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \tag{2}$$

where λ is a complex variable. As a consequence, the spectral analysis of $P(\lambda)$ leads to the solutions of (1). The suggested references are [5, 16].

If all the coefficients of $P(\lambda)$ are Hermitian matrices, then $P(\lambda)$ is called *self-adjoint*. A scalar $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $P(\lambda)$ in (2) if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is the *spectrum* of $P(\lambda)$, namely, $\sigma(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}$. For a real $q \in [0, 1]$, the *q-numerical range* of $P(\lambda)$ is defined by

$$W_q(P) \,=\, \{\lambda \in \mathbb{C} : y^*P(\lambda)x = 0,\; x, y \in \mathbb{C}^n,\; x^*x = y^*y = 1,\; y^*x = q\}. \tag{3}$$

Clearly, $W_q(P)$ is always closed and contains the spectrum $\sigma(P)$. It is also easy to see that if $q = 0$, then $W_q(P) = \mathbb{C}$. For $q \in (0, 1]$ and $P(\lambda) = q^{-1}I\lambda - A$, $W_q(P)$ coincides with the q-numerical range of matrix A [1, 8, 9],

$$F_q(A) \,=\, \{y^*Ax : x, y \in \mathbb{C}^n,\; x^*x = y^*y = 1,\; y^*x = q\}.$$

Moreover, for $q = 1$, we have the (classical) *numerical range* of $P(\lambda)$, that is,

$$W(P) \,\equiv\, W_1(P) \,=\, \{\lambda \in \mathbb{C} : x^*P(\lambda)x = 0,\; x \in \mathbb{C}^n,\; x^*x = 1\},$$

and the numerical range (also known as field of values) of matrix A,

$$F(A) \,\equiv\, F_1(A) \,=\, \{x^*Ax : x \in \mathbb{C}^n,\; x^*x = 1\}.$$

During the last decade, the numerical range $W(P)$ has attracted attention, and several results have been obtained (see e.g., [2, 3, 6, 7, 10, 11, 13]). These results are helpful in studying and understanding matrix polynomials, and some of them have been generalized to the case of q-numerical range [14, 15]. If $q \in (0, 1]$, then the q-numerical range $W_q(P)$ in (3) is not always connected, and it is bounded if and only if $0 \notin F_q(A_m)$ [10, 14]. If $\mu \in \mathbb{C}$ is a boundary point of $W_q(P)$, then the origin is also a boundary point of the range $F_q(P(\mu))$ [11, 14]. Furthermore, for any $q \in [0, 1]$, $W(P) \subseteq W_q(P)$, and consequently, all the known applications of $W(P)$ (and its connected components) on the spectral analysis and the factorization of $P(\lambda)$ are also valid for $W_q(P)$ (see [15] and the references therein).

In this article, we continue the study of the q-numerical range $W_q(P)$, extending recent results on the boundary and the geometry of the classical numerical range of matrix polynomials [6, 13]. In Section 2, we investigate the boundary of $W_q(P)$ and obtain that, if the coefficients of $P(\lambda)$ are not all scalar multiples of the same matrix, then for $0 < q < 1$, $W_q(P)$ has a nonempty interior. In Section 3, it is proved that for $0 < q < 1$, every non-isolated point λ_0 of $W_q(P)$, which does not belong to the q-numerical range of the derivative $P^{(1)}(\lambda)$, has local dimension 2, i.e., there exists a sequence $\{\mu_k\}_{k\in\mathbb{N}} \in \operatorname{Int} W_q(P)$ converging to λ_0. Moreover, as it is shown in Section 4, the q-numerical range of a *linear pencil* $P(\lambda) = A\lambda - B$ is always simply connected. In Sections 5 and 6, we study the q-numerical range of 2×2 matrices and matrix polynomials. For a 2×2 general complex matrix A, we give an explicit formula for $F_q(A)$ and obtain a necessary algebraic condition for the origin to be a boundary point of $F_q(A)$. As a consequence, if the q-numerical range of a 2×2 matrix polynomial of degree m does not coincide with the complex plane, then its boundary lies on an algebraic curve of total degree at most $8m$.

2 Boundary

Consider an $n\times n$ matrix polynomial $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ and a real $q \in (0, 1]$. In the following theorem, we characterize the boundary $\partial W_q(P)$, extending a known result on $W(P)$.

Theorem 1 (For $q = 1$, see [6, Theorem 2])
Let $P(\lambda)$ be an $n\times n$ matrix polynomial and let $0 < q \le 1$.

(i) *If λ_0 is a boundary point of $W_q(P)$, then $0 \in \partial F_q(P(\lambda_0))$.*

(ii) *If $0 \in \partial F_q(P(\lambda_0))$, $P(\lambda_0) \neq 0$ and $0 \notin F_q(P^{(1)}(\lambda_0))$, then $\lambda_0 \in \partial W_q(P)$.*

Proof For statement (i), see [14, Theorem 2.2]. For the second statement, assume that $0 \in \partial F_q(P(\lambda_0))$ and $\lambda_0 \in \operatorname{Int} W_q(P)$. Then there is a real $\epsilon > 0$ such that for every $\mu \in \mathbb{C}$ with $|\mu - \lambda_0| \le \epsilon$, there exist two unit vectors $x_\mu, y_\mu \in \mathbb{C}^n$ satisfying $y_\mu^* P(\mu) x_\mu = 0$ and $y_\mu^* x_\mu = q$. Moreover,

$$y_\mu^* \{P(\lambda_0) + (\mu - \lambda_0) P^{(1)}(\lambda_0) + (\mu - \lambda_0) R(\mu, \lambda_0)\} x_\mu = 0,$$

where $\|R(\mu, \lambda_0)\| = o(1)$ as $|\mu - \lambda_0| \to 0$. The convexity of the q-numerical range of matrices and the arguments of the proof of [6, Theorem 2] imply that λ_0 is a boundary point of $W_q(P)$. $\square$

Remark 1 It is worth noting that the first statement of the above theorem holds also for continuous matrix functions and statement (ii) is true for analytic matrix functions. This follows readily from their proofs.

In general, for a real $0 < q < 1$, the q-numerical range $W_q(P)$ has a nonempty interior. The next lemma is necessary and of independent interest.

Lemma 2 *Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n\times n$ matrix polynomial and let $q_1 \in (0, 1]$ such that for every $\mu \in W_{q_1}(P)$, $P(\mu)$ is a scalar matrix. Then there is an $n\times n$ matrix A with $0 \notin F_{q_1}(A)$ and a scalar polynomial $p(\lambda)$ such that $P(\lambda) = A\,p(\lambda)$. In particular, for every $q \in (0, 1]$, $W_q(P) = \{\lambda \in \mathbb{C} : p(\lambda) = 0\} = \sigma(P)$ when $0 \notin F_q(A)$, and $W_q(P) = \mathbb{C}$ when $0 \in F_q(A)$.*

Proof For any $\mu \in W_{q_1}(P)$, $P(\mu)$ is a scalar matrix such that $0 \in F_{q_1}(P(\mu))$. Hence, $\mu \in W_{q_1}(P)$ implies $P(\mu) = 0$. Since $P(\lambda)$ is a nonzero matrix polynomial of degree m, the equation $P(\lambda) = 0$ has no more than m roots. Consequently, the numerical range $W_{q_1}(P)$ is bounded (i.e., $0 \notin F_{q_1}(A_m)$) and consists of $s \leq m$ isolated points, $\lambda_1, \lambda_2, \ldots, \lambda_s$. Thus, by [15, Theorem 5] (see also [11, Theorem 2.1]), the matrix polynomial $P(\lambda)$ is written in the form $P(\lambda) = A\,p(\lambda)$, where $A = A_m$ and $p(\lambda) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_s)^{k_s}$ (with $k_1 + k_2 + \cdots + k_s = m$). □

Notice that if $P(\lambda) = A\,p(\lambda)$ for some $A \in \mathbb{C}^{n\times n}$ with $0 \in F_q(A)$ and a scalar polynomial $p(\lambda)$, then $W_q(P) = \mathbb{C}$ but $P(\lambda)$ is not necessarily a scalar matrix for every $\lambda \in \mathbb{C}$.

Theorem 3 *Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in (2) and let $0 < q < 1$. The q-numerical range $W_q(P)$ has no interior points if and only if there exist a matrix A and a scalar polynomial $p(\lambda)$ such that $P(\lambda) = A\,p(\lambda)$ and $0 \notin F_q(A)$.*

Proof Suppose that $P(\lambda)$ is written in the form $P(\lambda) = A\,p(\lambda)$, where A is an $n \times n$ matrix with $0 \notin F_q(A)$, and $p(\lambda)$ is a scalar polynomial. Then clearly, the q-numerical range of $P(\lambda)$ is

$$W_q(P) \,=\, \{\lambda \in \mathbb{C} : p(\lambda) = 0\} \,=\, \sigma(P),$$

and thus, it has no interior points.

Conversely, assume that $W_q(P)$ has no interior points and there exists a $\lambda_0 \in W_1(P)$ such that $P(\lambda_0)$ is not a scalar matrix. Since $q < 1$, by [8, Theorem 2.5], it follows that $0 \in qF_1(P(\lambda_0)) \subset \operatorname{Int} F_q(P(\lambda_0))$, i.e., the origin lies in the interior of $F_q(P(\lambda_0))$. Hence, by Theorem 1 (i) (see also [14, Theorem 2.2]), λ_0 is an interior point of $W_q(P)$. This is a contradiction, and consequently, $P(\lambda)$ satisfies the conditions of Lemma 2 for $q_1 = 1$. The proof is complete. □

3 Local dimension

Let Ω be a closed subset of $\mathbb{C}$ and let $\omega \in \Omega$. The *local dimension* of the point ω in Ω is defined by

$$\lim_{h\to 0^+} \dim\{\Omega \cap S(\omega, h)\} \qquad (h > 0),$$

where $\dim\{\cdot\}$ is the topological dimension and $S(\omega, h)$ denotes the closed circular disk with centre at ω and radius equal to h. The local dimension of $\omega \in \Omega$ takes the value 0 when ω is an isolated point of Ω, the value 2 when there exists a sequence $\{\mu_k\}_{k\in\mathbb{N}} \in \operatorname{Int}\Omega$ converging to ω, and the value 1 otherwise. It is important to remark that for $0 < q < 1$, all the points of the q-numerical range of a non-scalar matrix $A \in \mathbb{C}^{n\times n}$ have local dimension 2 (in $F_q(A)$) [8, 9].

On the other hand, the numerical range $F(A)$ $(\equiv F_1(A))$ is a line segment if and only if A is a normal matrix with colinear eigenvalues.

Consider a matrix polynomial $P(\lambda)$ as in (2) and a real $q \in (0,1)$. Then in general, the local dimension of any λ_0 in $W_q(P) \setminus W_q(P^{(1)})$ is 2. The only exception is the isolated points of $W_q(P)$ (if any).

Theorem 4 (For $q = 1$, see [13, Theorems 1 and 2])
Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial, $q \in (0,1)$ and let $\lambda_0 \in W_q(P) \setminus W_q(P^{(1)})$. Then either λ_0 is an isolated point of $W_q(P)$, or the local dimension of λ_0 in $W_q(P)$ is equal to 2.

Proof Assume that λ_0 is not an isolated point of $W_q(P)$ and the local dimension of λ_0 in $W_q(P)$ is 1. Obviously, λ_0 belongs to $\partial W_q(P)$ and there is a real $r_0 > 0$ such that $W_q(P) \cap S(\lambda_0, r_0)$ is a curve lying on $\partial W_q(P)$. By Theorem 1 (i), the origin is a boundary point of $F_q(P(\lambda_0))$. Since λ_0 is not an isolated point of the closed set $W_q(P)$, if $P(\lambda_0) = 0$, then the polynomial $P(\lambda)$ is written $P(\lambda) = (\lambda - \lambda_0)Q(\lambda)$, where $Q(\lambda)$ is a matrix polynomial of degree $m-1$ and $\lambda_0 \in W_q(Q)$. Consequently, $\lambda_0 \in W_q(P^{(1)})$; a contradiction. Hence, $P(\lambda_0) \neq 0$, $F_q(P(\lambda_0))$ is convex with a nonempty interior and 0 is a differentiable point of $\partial F_q(P(\lambda_0))$ [8, 9]. Thus, there exists a straight line ε_0 passing through the origin and defining two closed half planes $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $F_q(P(\lambda_0)) \subset \mathcal{H}_1$.

For every $r \in [0, r_0]$ and $\vartheta \in [0, 2\pi]$, either $\lambda_0 + re^{\mathrm{i}\vartheta} \notin W_q(P)$, or $\lambda_0 + re^{\mathrm{i}\vartheta} \in \partial W_q(P)$. Then by Theorem 1, for every $r \in [0, r_0]$ and $\vartheta \in [0, 2\pi]$, either $0 \notin F_q(P(\lambda_0 + re^{\mathrm{i}\vartheta}))$, or $0 \in \partial F_q(P(\lambda_0 + re^{\mathrm{i}\vartheta}))$. Moreover, the origin does not belong to the convex set $F_q(P^{(1)}(\lambda_0))$ and the matrix $P(\lambda_0 + re^{\mathrm{i}\vartheta})$ is written

$$P(\lambda_0 + re^{\mathrm{i}\vartheta}) \;=\; P(\lambda_0) + re^{\mathrm{i}\vartheta}P^{(1)}(\lambda_0) + re^{\mathrm{i}\vartheta}R(\lambda_0, r, \vartheta),$$

where $\|R(\lambda_0, r, \vartheta)\| = o(1)$ as $r \to 0$. Hence, for sufficiently small r, there is a cone $\mathcal{K}_{r,\lambda_0} = \{z \in \mathbb{C} : \varphi_1 \leq \mathrm{Arg} z \leq \varphi_2,\ 0 < \varphi_2 - \varphi_1 \leq \psi < \pi\}$ such that

$$F_q(P^{(1)}(\lambda_0) + R(\lambda_0, r, \vartheta)) \;\subset\; \mathcal{K}_{r,\lambda_0} \backslash \{0\}.$$

Following the steps of the proof of [13, Theorem 1], we obtain that the local dimension of the origin in $F_q(P(\lambda_0))$ is equal to 1. This is a contradiction and the proof is complete. $\square$

Remark 2 It is clear from its proof that the above theorem is also valid for analytic matrix functions.

4 Linear pencils

Consider a linear pencil $A\lambda - B$, where A and B are $n \times n$ complex matrices. By Theorem 4, if $q \in (0,1)$, then the q-numerical range $W_q(A\lambda - B)$ is either a singleton or all of its points have local dimension 2.

Recall that a bounded connected set $\Omega \subset \mathbb{C}$ is called *simply connected* if $\mathbb{C}\backslash\Omega$ is connected (in particular, Ω has no "holes"). If $\Omega \subset \mathbb{C}$ is unbounded, then we consider the set $\Omega \cup \{\infty\} \subset \mathbb{C} \cup \{\infty\}$ and say that $\Omega \cup \{\infty\}$ is *simply connected* if $(\mathbb{C} \cup \{\infty\})\backslash\Omega$ is connected. (Note that the two definitions coincide when Ω is a bounded subset of $\mathbb{C}$.) By [14], it is known that if $q \in (0,1)$, then $W_q(A\lambda - B)$ is always connected. Furthermore, we have the following result.

Theorem 5 (For $q = 1$, see [13, Theorems 4 and 5])
Let $A\lambda - B$ be a linear pencil, and let $q \in (0,1]$. If the q-numerical range $W_q(A\lambda - B)$ is bounded, then it is simply connected. If $W_q(A\lambda - B)$ is unbounded, then $W_q(A\lambda - B) \cup \{\infty\}$ is simply connected in the extended plane $\mathbb{C} \cup \{\infty\}$.

Proof Suppose that $W_q(A\lambda - B)$ is bounded and assume that it is not simply connected. Then $W_q(A\lambda - B)$ has a "hole", i.e., there is a complex number $\omega_0 \notin W_q(A\lambda - B)$ such that for every $\varphi \in [0, 2\pi]$, there exists a real $r_\varphi > 0$ satisfying

$$\omega_0 + r_\varphi e^{\mathrm{i}\varphi} \in W_q(A\lambda - B).$$

Since $W_q(A(\lambda+\mu) - B) = W_q(A\lambda - B) - \mu$ $(\mu \in \mathbb{C})$, without loss of generality, we may assume that $\omega_0 = 0$. Then $0 \notin W_q(A\lambda - B)$ and for every $\varphi \in [0, 2\pi]$, $r_\varphi e^{\mathrm{i}\varphi} \in W_q(A\lambda - B)$. Equivalently, $0 \notin F_q(B)$ and for every $\varphi \in [0, 2\pi]$, $0 \in F_q(Ar_\varphi e^{\mathrm{i}\varphi} - B)$. Since the origin does not belong to the convex sets $F_q(A)$ and $F_q(B)$, as in the proof of [13, Theorem 4], there exist an angle $\varphi_0 \in [0, 2\pi]$ and a cone $\mathcal{K}_0 = \{z \in \mathbb{C} : \vartheta_1 \le \mathrm{Arg} z \le \vartheta_2,\ 0 < \vartheta_2 - \vartheta_1 \le \psi < \pi\}$ such that

$$F_q(A(r_{\varphi_0} e^{\mathrm{i}\varphi_0}) - B) \subseteq r_{\varphi_0} e^{\mathrm{i}\varphi_0} F_q(A) + F_q(-B) \subset \mathrm{Int}\,\mathcal{K}_0.$$

Thus, $F_q(A(r_{\varphi_0} e^{\mathrm{i}\varphi_0}) - B)$ does not contain the origin; a contradiction. Hence, if $W_q(A\lambda - B)$ is bounded, then it is also simply connected.

Assume now that the q-numerical range $W_q(A\lambda - B)$ is unbounded, that is, $0 \in F_q(A)$ [14]. Since $\mathbb{C} \cup \{\infty\}\ (\cong S^2)$ is simply connected, we have nothing to prove when $W_q(A\lambda - B) = \mathbb{C}$. Suppose that there is a $\lambda_0 \notin W_q(A\lambda - B)$. Since $W_q(A(\lambda+\lambda_0) - B) = W_q(A\lambda - B) - \lambda_0$, $W_q(A\lambda - B) \cup \{\infty\}$ is homomorphic to the set $W_q(A\lambda - (B - A\lambda_0)) \cup \{\infty\}$. Hence, we can assume that $0 \notin W_q(A\lambda - B)$, or equivalently, $0 \notin F_q(B)$. Then one can verify that in the extended plane, $W_q(B\lambda - A) = \left\{\mu^{-1} : \mu \in W_q(A\lambda - B) \cup \{\infty\}\right\}$. The map $\Psi(\mu) = \mu^{-1}$ for $\mu \in W_q(A\lambda - B)$ and $\Psi(\infty) = 0$ is an homomorphism of $W_q(A\lambda - B) \cup \{\infty\}$ onto $W_q(B\lambda - A)$. By the first part of the proof, the bounded range $W_q(B\lambda - A)$ is simply connected, and since simply connectedness is a topological property, the proof is complete. □

5 Two by two matrix case

In the previous three sections, we investigated qualitative properties of the q-numerical range of matrix polynomials. However, we are also interested in the construction of explicit formulas for the boundary of the q-numerical range of matrices and matrix polynomials. Unfortunately, there are no currently known methods for the direct (analytical) computation of the boundary of the q-numerical range of $n \times n$ non-normal matrices for $n \geq 3$, and thus, we restrict ourselves to the 2×2 case.

For a 2×2 complex matrix A, consider the Hermitian matrices

$$A_h = \frac{A + A^*}{2} \quad \text{and} \quad A_{sh} = \frac{A - A^*}{2\,\mathrm{i}}, \tag{4}$$

and observe that $A = A_h + \mathrm{i}\, A_{sh}$. The q-numerical range $F_q(A)$ is a closed elliptical disk and it is explicitly known for special cases. In particular, we have the following result [9, 12].

Lemma 6 *Suppose that $q \in [0,1]$, A is a 2×2 complex matrix with zero trace and the Hermitian matrices A_h and A_{sh} in (4) satisfy* $\mathrm{trace}(A_h A_{sh}) = 0$. *Then the q-numerical range of A is given by*

$$F_q(A) = \{u + \mathrm{i}\, v \in \mathbb{C} : u, v \in \mathbb{R},\, H_0(u,v) \leq 1\},$$

where

$$H_0(u,v) = 2\left(\frac{u}{\sqrt{\mathrm{trace}(A_h^2)} + \sqrt{(1-q^2)\,\mathrm{trace}(A_{sh}^2)}}\right)^2 + 2\left(\frac{v}{\sqrt{\mathrm{trace}(A_{sh}^2)} + \sqrt{(1-q^2)\,\mathrm{trace}(A_h^2)}}\right)^2.$$

This lemma allows us to describe the q-numerical range of a 2×2 matrix A with zero trace (see also [1, 9]). To facilitate the presentation, we give the methodology next. If $\mathrm{trace}(A_h A_{sh}) = 0$, then Lemma 6 is directly applicable. If $\mathrm{trace}(A_h A_{sh}) \neq 0$, then we have to rotate the principal axes of the elliptical disk $F_q(A)$. For any angle $\theta \in [-\pi, \pi]$, we consider the matrices

$$B(\theta) = e^{-\mathrm{i}\theta} A = \cos\theta\, A - \mathrm{i}\sin\theta\, A,$$

$$B_h(\theta) = \frac{B(\theta) + B(\theta)^*}{2} = \cos\theta\, A_h + \sin\theta\, A_{sh}$$

and

$$B_{sh}(\theta) = \frac{B(\theta) - B(\theta)^*}{2\,\mathrm{i}} = -\sin\theta\, A_h + \cos\theta\, A_{sh}.$$

Then one can verify that

$$\mathrm{trace}(B_h(\theta) B_{sh}(\theta)) = -\frac{1}{2}\sin(2\theta)\left(\mathrm{trace}(A_h^2) - \mathrm{trace}(A_{sh}^2)\right) + \cos(2\theta)\,\mathrm{trace}(A_h A_{sh}).$$

If $\operatorname{trace}(A_h^2) = \operatorname{trace}(A_{sh}^2)$, then for $\theta = \pi/4$,

$$\operatorname{trace}(B_h(\pi/4)B_{sh}(\pi/4)) \;=\; -\frac{1}{2}\left(\operatorname{trace}(A_h^2) - \operatorname{trace}(A_{sh}^2)\right) \;=\; 0.$$

If $\operatorname{trace}(A_h^2) \neq \operatorname{trace}(A_{sh}^2)$, then there is exactly one angle $\theta_0 \in (-\pi/4, \pi/4)$ such that

$$\tan(2\theta_0) \;=\; \frac{2\operatorname{trace}(A_h A_{sh})}{\operatorname{trace}(A_h^2) - \operatorname{trace}(A_{sh}^2)}\,. \tag{5}$$

For this angle, $\operatorname{trace}(B_h(\theta_0)B_{sh}(\theta_0)) = 0$. Note also that

$$\cos(2\theta_0) \;=\; \frac{1}{\sqrt{1+\tan^2(2\theta_0)}} \quad \text{and} \quad \sin(2\theta_0) \;=\; \frac{\tan(2\theta_0)}{\sqrt{1+\tan^2(2\theta_0)}}\,. \tag{6}$$

As a consequence,

$$\begin{aligned}\operatorname{trace}(B_h(\theta_0)^2) &= \operatorname{trace}((\cos\theta_0\, A_h + \sin\theta_0\, A_{sh})^2)\\ &= \frac{1}{2}\left(1 + \frac{1}{\cos(2\theta_0)}\right)\operatorname{trace}(A_h^2)\\ &\quad + \frac{1}{2}\left(1 - \frac{1}{\cos(2\theta_0)}\right)\operatorname{trace}(A_{sh}^2)\end{aligned} \tag{7}$$

and

$$\begin{aligned}\operatorname{trace}(B_{sh}(\theta_0)^2) &= \operatorname{trace}((\cos\theta_0\, A_h + \sin\theta_0\, A_{sh})^2)\\ &= \frac{1}{2}\left(1 - \frac{1}{\cos(2\theta_0)}\right)\operatorname{trace}(A_h^2)\\ &\quad + \frac{1}{2}\left(1 + \frac{1}{\cos(2\theta_0)}\right)\operatorname{trace}(A_{sh}^2).\end{aligned} \tag{8}$$

By the condition $F_q(A) = e^{\mathrm{i}\theta_0}F_q(B(\theta_0))$ and Lemma 6, it is clear that

$F_q(A) \;=\; \{u + \mathrm{i}\,v \in \mathbb{C} : u, v \in \mathbb{R},\; H_0(\cos\theta_0\, u + \sin\theta_0\, v, -\sin\theta_0\, u + \cos\theta_0\, v) \le 1\}$.

Thus, the boundary of $F_q(A)$ is the ellipse

$$H_0(\cos\theta_0\, u + \sin\theta_0\, v, -\sin\theta_0\, u + \cos\theta_0\, v) \;=\; 1.$$

Moreover, straightforward computations yield

$$\begin{aligned}H_0(\cos\theta_0\, u + \sin\theta_0\, v, -\sin\theta_0\, u + \cos\theta_0\, v) \;&=\; M_q\,(u^2+v^2)\\ &+ N_q\,[\cos(2\theta_0)\,(u^2 - v^2) + 2\sin(2\theta_0)\,u\,v],\end{aligned} \tag{9}$$

where

$$\begin{aligned}M_q \;&=\; \left(\frac{1}{\sqrt{\operatorname{trace}(B_h(\theta_0)^2)} + \sqrt{(1-q^2)\operatorname{trace}(B_{sh}(\theta_0)^2)}}\right)^2\\ &+ \left(\frac{1}{\sqrt{\operatorname{trace}(B_{sh}(\theta_0)^2)} + \sqrt{(1-q^2)\operatorname{trace}(B_h(\theta_0)^2)}}\right)^2\end{aligned}$$

and

$$N_q = \left(\frac{1}{\sqrt{\mathrm{trace}(B_h(\theta_0)^2)} + \sqrt{(1-q^2)\,\mathrm{trace}(B_{sh}(\theta_0)^2)}} \right)^2 - \left(\frac{1}{\sqrt{\mathrm{trace}(B_{sh}(\theta_0)^2)} + \sqrt{(1-q^2)\,\mathrm{trace}(B_h(\theta_0)^2)}} \right)^2 .$$

Suppose now that A is a 2×2 general complex matrix. At this point and for the remainder of the section, it is convenient to define

$$\begin{aligned} a &= \mathrm{trace}([A_h - (1/2)\mathrm{trace}(A_h)I_2]^2) \\ &= \mathrm{trace}(A_h^2) - (1/2)\mathrm{trace}(A_h)^2, \\ b &= \mathrm{trace}([A_{sh} - (1/2)\mathrm{trace}(A_{sh})I_2]^2) \\ &= \mathrm{trace}(A_{sh}^2) - (1/2)\mathrm{trace}(A_{sh})^2, \\ c &= \mathrm{trace}([A_h - (1/2)\mathrm{trace}(A_h)I_2][A_{sh} - (1/2)\mathrm{trace}(A_{sh})I_2]) \\ &= \mathrm{trace}(A_h A_{sh}) - (1/2)\mathrm{trace}(A_h)\,\mathrm{trace}(A_{sh}). \end{aligned} \tag{10}$$

Lemma 7 *Consider a 2×2 complex matrix A, and let a, b and c as defined in (10). Then $a\,b - c^2 \geq 0$ and equality holds if and only if A is a normal matrix.*

Proof The matrix A is unitarily similar to a matrix with diagonal elements equal to $(1/2)\,\mathrm{trace}(A)$ [4]. Since $\mathrm{trace}(A - (1/2)\mathrm{trace}(A)I_2) = 0$, it follows $\mathrm{trace}(A_h - (1/2)\mathrm{trace}(A_h)I_2) = \mathrm{trace}(A_{sh} - (1/2)\mathrm{trace}(A_{sh})I_2) = 0$, and we may assume that

$$A = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix} ; \qquad \alpha, \beta \in \mathbb{C}.$$

Then we can see that

$$a = \mathrm{trace}(A_h^2) = \frac{1}{2}\,|\alpha + \overline{\beta}|^2 \geq 0,$$

$$b = \mathrm{trace}(A_{sh}^2) = \frac{1}{2}\,|\alpha - \overline{\beta}|^2 \geq 0$$

and

$$c = \mathrm{trace}(A_h A_{sh}) = \frac{\alpha\beta - \overline{\alpha\beta}}{2\,\mathrm{i}} = \mathrm{Im}(\alpha\beta).$$

Hence, it is clear that

$$\begin{aligned} a\,b - c^2 &= \mathrm{trace}(A_h^2)\mathrm{trace}(A_{sh}^2) - \mathrm{trace}(A_h A_{sh})^2 \\ &= \frac{1}{4}\left[|(\alpha + \overline{\beta})(\alpha - \overline{\beta})|^2 + (\alpha\beta - \overline{\alpha\beta})^2\right] \\ &= \frac{1}{4}\left(|\alpha|^4 + |\beta|^4 - 2\,|\alpha|^2|\beta|^2\right) \\ &= \frac{1}{4}\,(|\alpha|^2 - |\beta|^2)^2 \geq 0. \end{aligned}$$

Moreover, the equality $a\,b - c^2 = 0$ holds if and only if $|\alpha| = |\beta|$, or equivalently, if and only if the matrix A is normal. $\square$

The above lemma yields a strong connection between the classical numerical range and the q-numerical range of 2×2 matrices.

Theorem 8 *Let $q \in [0,1)$ and A be a 2×2 complex matrix, and let a, b and c as defined in (10). Then the quantities*

$$\begin{aligned} \alpha_q &= a + (1-q^2)\,b + 2\sqrt{(1-q^2)(a\,b - c^2)} \\ \beta_q &= (1-q^2)\,a + b + 2\sqrt{(1-q^2)(a\,b - c^2)} \\ \gamma_q &= c\,q^2 \end{aligned} \tag{11}$$

satisfy the inequality $\alpha_q\,\beta_q \geq \gamma_q^2$ and there exists a 2×2 complex matrix B_q such that

$$\begin{aligned} \mathrm{trace}(B_q) &= q\,\mathrm{trace}(A), \\ \alpha_q &= \mathrm{trace}([B_q - (1/2)\mathrm{trace}(B_q)I_2]_h^2), \\ \beta_q &= \mathrm{trace}([B_q - (1/2)\mathrm{trace}(B_q)I_2]_{sh}^2), \\ \gamma_q &= \mathrm{trace}([B_q - (1/2)\mathrm{trace}(B_q)I_2]_h\,[B_q - (1/2)\mathrm{trace}(B_q)I_2]_{sh}) \end{aligned}$$

and

$$F_q(A) = F(B_q).$$

Proof By [4], A is unitarily similar to a matrix with diagonal elements equal to $(1/2)\,\mathrm{trace}(A)$. Moreover, by [4, Property 1.2.3],

$$F_q\left(A + \frac{1}{2}\,\mathrm{trace}(A)I_2\right) = F_q(A) + \frac{1}{2}\,q\,\mathrm{trace}(A).$$

As in the proof of Lemma 7 and without loss of generality, we may assume that

$$A = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}; \qquad \alpha, \beta \in \mathbb{C}$$

with

$$a = \mathrm{trace}(A_h^2) = \frac{1}{2}\,|\alpha + \overline{\beta}|^2 \geq 0,$$

$$b = \mathrm{trace}(A_{sh}^2) = \frac{1}{2}\,|\alpha - \overline{\beta}|^2 \geq 0$$

and

$$c = \mathrm{trace}(A_h A_{sh}) = \frac{\alpha\beta - \overline{\alpha\beta}}{2\,\mathrm{i}} = \mathrm{Im}(\alpha\beta).$$

Define the angle $\theta_0 = \pi/4$ when $a = b$, and $\theta_0 \in (-\pi/4, \pi/4)$ such that

$$\tan(2\theta_0) = \frac{2\,c}{a-b} = \frac{\mathrm{Im}(\alpha\beta)}{\mathrm{Re}(\alpha\beta)}$$

when $a \neq b$ (see (5)). Then by (6), it follows

$$\cos(2\theta_0) = \frac{a-b}{\sqrt{(a-b)^2+4\,c^2}} \quad \text{and} \quad \sin(2\theta_0) = \frac{2\,c}{\sqrt{(a-b)^2+4\,c^2}}\,.$$

Moreover, (7) and (8) are written

$$\operatorname{trace}(B_h(\theta_0)^2) = \frac{a+b+\sqrt{(a-b)^2+4\,c^2}}{2}$$

and

$$\operatorname{trace}(B_{sh}(\theta_0)^2) = \frac{a+b-\sqrt{(a-b)^2+4\,c^2}}{2}\,,$$

respectively. Thus, by straightforward computations, (9) implies that the boundary of the q-numerical range $F_q(A)$ is the curve

$$\{u+\mathrm{i}\,v \in \mathbb{C} : u, v \in \mathbb{R},\; H(u,v) = 0\},$$

where

$$\begin{aligned} H(u,v) \;=\;& (2\,b + 4\sqrt{(1-q^2)(a\,b-c^2)} + 2\,a\,(1-q^2))\,u^2 \\ & + (2\,a + 4\sqrt{(1-q^2)(a\,b-c^2)} + 2\,b\,(1-q^2))\,v^2 \\ & - 4\,q^2\,c\,u\,v \;-\; 2\,(a+b)(2-q^2)\sqrt{(1-q^2)(a\,b-c^2)} \\ & + (1-q^2)(2\,c^2 - a^2 - b^2 - 4\,a\,b) \\ & - (2-2\,q^2+q^4)(a\,b-c^2). \end{aligned} \tag{12}$$

For the quantities α_q, β_q and γ_q in (11), by Lemma 7, it follows that $\alpha_q \geq 0$, $\beta_q \geq 0$ and

$$\begin{aligned} \alpha_q\,\beta_q - \gamma_q^2 \;=\;& a\,b - c^2 + 2\,(a+b)(2-q^2)\sqrt{(1-q^2)(a\,b-c^2)} \\ & + (1-q^2)\,[(a+b)^2 + 2\,(a\,b-c^2)] \\ \geq\;& 0. \end{aligned}$$

Define now the matrix B_q by

$$B_q = \begin{bmatrix} 0 & (1+\sqrt{1-q^2})\,\alpha \\ (1-\sqrt{1-q^2})\,\beta & 0 \end{bmatrix}$$

when $|\alpha| \geq |\beta|$, and

$$B_q = \begin{bmatrix} 0 & (1-\sqrt{1-q^2})\,\alpha \\ (1+\sqrt{1-q^2})\,\beta & 0 \end{bmatrix}$$

when $|\alpha| < |\beta|$. Then $\operatorname{trace}(B_q) = 0$, and we can verify that $\operatorname{trace}(B_{q,h}^2) = \alpha_q$, $\operatorname{trace}(B_{q,sh}^2) = \beta_q$ and $\operatorname{trace}(B_{q,h}B_{q,sh}) = \gamma_q$. Furthermore, by [4, Theorem

1.3.6], a point $u+\mathrm{i}\,v \in \mathbb{C}$ $(u,v\in\mathbb{R})$ belongs to the boundary of the numerical range $F(B_q)$ if and only if

$$2\,\beta_q\,u^2 + 2\,\alpha_q\,v^2 - 4\,\gamma_q\,u\,v - \alpha_q\,\beta_q + \gamma_q^2 \;=\; 0. \tag{13}$$

Substituting (11) into (13) yields $H(u,v)=0$ for the polynomial $H(u,v)$ in (12). Consequently, $\partial F_q(A) = \partial F(B_q)$, completing the proof. □

Next we describe the boundary of $F_q(A)$ by a quartic equation without use of any radical expressions such as $\sqrt{a\,b-c^2}$, and give a necessary algebraic condition for the origin to be a boundary point of $F_q(A)$.

Theorem 9 *Suppose that $q\in[0,1]$ and A is a 2×2 complex matrix. Let*

$$f_A \;=\; \frac{1}{2}\,\mathrm{Re}(\mathrm{trace}(A)) \quad \textit{and} \quad g_A = \frac{1}{2}\,\mathrm{Im}(\mathrm{trace}(A)),$$

and let a, b and c as defined in (10). If the origin belongs to the boundary of the q-numerical range $F_q(A)$, then

$$\begin{aligned} G_q(A) \;&=\; q^4(c_{4,0}\,f_A^4 + c_{0,4}\,g_A^4 + c_{3,1}\,f_A^3\,g_A + c_{1,3}\,f_A\,g_A^3 + c_{2,2}\,f_A^2\,g_A^2) \\ &\quad + q^2(c_{2,0}\,f_A^2 + c_{0,2}\,g_A^2 + c_{1,1}\,f_A\,g_A) + c_{0,0} \;=\; 0, \end{aligned}$$

where

$$\begin{aligned}
c_{4,0} \;&=\; 4\,((1-q^2)^2a^2 - 2\,(1-q^2)a\,b + b^2 + 4\,(1-q^2)c^2),\\
c_{0,4} \;&=\; 4\,((1-q^2)^2b^2 - 2\,(1-q^2)a\,b + a^2 + 4\,(1-q^2)c^2),\\
c_{3,1} \;&=\; -\,16\,q^2c\,((1-q^2)a+b),\\
c_{1,3} \;&=\; -\,16\,q^2c\,((1-q^2)b+a),\\
c_{2,2} \;&=\; 8\,((1-q^2)(a^2+b^2) + (4-4\,q^2+2\,q^4)c^2 + (-2+2\,q^2+q^4)a\,b),\\
c_{2,0} \;&=\; 4\,[-(1-q^2)^2a^3 - (1-q^2)b^3 + (1+q^2-3\,q^4+q^6)a^2b\\
&\quad + (1-4\,q^2+2\,q^4)a\,b^2 + (-4+4\,q^2+q^4-q^6)a\,c^2 + (-4+8\,q^2-3\,q^4)b\,c^2],\\
c_{0,2} \;&=\; 4\,[-(1-q^2)^2b^3 - (1-q^2)a^3 + (1+q^2-3\,q^4+q^6)a\,b^2\\
&\quad + (1-4\,q^2+2\,q^4)a^2b + (-4+4\,q^2+q^4-q^6)b\,c^2 + (-4+8\,q^2-3\,q^4)a\,c^2],\\
c_{1,1} \;&=\; 8\,c\,q^2[(1-q^2)(a^2+b^2) + (6-6\,q^2+q^4)a\,b + (-4+4\,q^2-q^4)c^2],\\
c_{0,0} \;&=\; [(1-q^2)(a^2+b^2) + (-2+2\,q^2-q^4)a\,b + (4-4\,q^2+q^4)c^2]^2.
\end{aligned}$$

Proof Consider the matrix $A_0 = A - \mathrm{trace}(A)I_2$ and notice that the origin is a boundary point of $F_q(A) = F_q(A_0) + (1/2)\,q\,\mathrm{trace}(A)$ if and only if the point

$$u_0 + \mathrm{i}\,v_0 \;=\; -\,\frac{q}{2}\,\mathrm{trace}(A_h) - \mathrm{i}\,\frac{q}{2}\,\mathrm{trace}(A_{sh})$$

belongs to $\partial F_q(A_0)$. By the proof of Theorem 8, a point $u+\mathrm{i}\,v\in\mathbb{C}$ $(u,v\in\mathbb{R})$ belongs to the boundary $\partial F_q(A_0)$ if and only if

$$H(u,v) \;=\; 0,$$

where the polynomial $H(u,v)$ is defined in (12) (recalling that $\mathrm{trace}(A_0)=0$). Hence, $u_0+\mathrm{i}\,v_0\in\partial F_q(A_0)$ if and only if

$$\begin{aligned} 0 \;&=\; (2\,b+4\sqrt{(1-q^2)(a\,b-c^2)}+2\,a\,(1-q^2))\,u_0^2 \\ &\quad +(2\,a+4\sqrt{(1-q^2)(a\,b-c^2)}+2\,b\,(1-q^2))\,v_0^2 \\ &\quad -4\,q^2\,c\,u_0\,v_0 \;-\; 2\,(a+b)(2-q^2)\sqrt{(1-q^2)(a\,b-c^2)} \\ &\quad +(1-q^2)(2\,c^2-a^2-b^2-4\,a\,b) \\ &\quad -(2-2\,q^2+q^4)(a\,b-c^2), \end{aligned}$$

where $a=\mathrm{trace}(A_{0,h}^2)$, $b=\mathrm{trace}(A_{0,sh}^2)$ and $c=\mathrm{trace}(A_{0,h}A_{0,sh})$, or equivalently, if and only if

$$\begin{aligned} &2\sqrt{(1-q^2)(a\,b-c^2)}((a+b)(2-q^2)-2u_0^2-2v_0^2) \\ &\qquad =(2\,b+2\,a\,(1-q^2))\,u_0^2+\,(2\,a+2\,b\,(1-q^2))\,v_0^2 \\ &\qquad\quad -4\,q^2\,c\,u_0\,v_0+(1-q^2)(2\,c^2-a^2-b^2-4\,a\,b) \\ &\qquad\quad -(2-2\,q^2+q^4)(a\,b-c^2). \end{aligned}$$

By straightforward computations and keeping in mind that

$$u_0^2 \;=\; \frac{q^2}{4}\,\mathrm{trace}(A_h)^2 \;=\; q^2\,f_A^2,$$

$$v_0^2 \;=\; \frac{q^2}{4}\,\mathrm{trace}(A_{sh})^2 \;=\; q^2\,g_A^2$$

and

$$u_0\,v_0 \;=\; \frac{q^2}{4}\,\mathrm{trace}(A_h)\,\mathrm{trace}(A_{sh}) \;=\; q^2\,f_A\,g_A,$$

the proof is completed. □

6 Two by two matrix polynomial case

Let $P(\lambda)$ be an $n\times n$ matrix polynomial as in (2), and let $x,y\in\mathbb{C}^n$ be two unit vectors with $y^*x=q\in(0,1]$ (for $q=1$, $x=y$). Then there exist an $n\times 2$ matrix T and two vectors $w_x,w_y\in\mathbb{C}^2$ such that $T^*T=I_2$, $x=Tw_x$ and $y=Tw_y$. Moreover, we have $w_x^*w_x=w_x^*(T^*T)w_x=x^*x=1$, $w_y^*w_y=w_y^*(T^*T)w_y=y^*y=1$ and $w_y^*w_x=w_y^*(T^*T)w_x=y^*x=q$. Hence, by [14, Proposition 1.1 (iv)], it follows

$$W_q(P) \;=\; \bigcup_{T\in\mathbb{C}^{n\times 2},\,T^*T=I_2} W_q(T^*P(\lambda)T).$$

Similarly, we can see that for any $s\in\{2,3,\ldots,n-1\}$,

$$W_q(P) \;=\; \bigcup_{T\in\mathbb{C}^{n\times s},\,T^*T=I_s} W_q(T^*P(\lambda)T).$$

As a consequence, investigating the boundary of the q-numerical range of a 2×2 matrix polynomial is of interest.

Consider now a 2×2 matrix polynomial $L(\lambda)$ of degree m. As it has been shown in [13], the boundary of the numerical range $W(L)$ ($\equiv W_1(L)$) lies on an algebraic curve of degree at most $4m$. In this section, we apply the results of Section 5 to obtain that the boundary of $W_q(L)$ $(0 < q < 1)$ lies on an algebraic curve of degree at most $8m$.

The matrix polynomial $L(\lambda)$ can be written in the form

$$L(\lambda) = \begin{bmatrix} p_{1,1}(\lambda) & p_{1,2}(\lambda) \\ p_{2,1}(\lambda) & p_{2,2}(\lambda) \end{bmatrix},$$

where $p_{i,j}(\lambda)$ $(i,j = 1,2)$ are scalar polynomials of degree at most m. Similarly to the matrix case, define the matrix polynomial

$$Q(\lambda) = L(\lambda) - \frac{1}{2}\left(p_{1,1}(\lambda) + p_{2,2}(\lambda)\right) I_2$$

and the scalar polynomials

$$\begin{aligned}
f_L(\lambda) &= \frac{1}{2}\operatorname{Re}(p_{1,1}(\lambda) + p_{2,2}(\lambda)), \\
g_L(\lambda) &= \frac{1}{2}\operatorname{Im}(p_{1,1}(\lambda) + p_{2,2}(\lambda)), \\
a(\lambda) &= \operatorname{trace}(Q(\lambda)_h^2) = \frac{1}{2}\left[\operatorname{Re}\left(p_{1,1}(\lambda) + p_{2,2}(\lambda)\right)\right]^2 + \frac{1}{2}\left|p_{1,2}(\lambda) + \overline{p_{2,1}(\lambda)}\right|^2, \\
b(\lambda) &= \operatorname{trace}(Q(\lambda)_{sh}^2) = \frac{1}{2}\left[\operatorname{Im}\left(p_{1,1}(\lambda) - p_{2,2}(\lambda)\right)\right]^2 + \frac{1}{2}\left|p_{1,2}(\lambda) - \overline{p_{2,1}(\lambda)}\right|^2, \\
c(\lambda) &= \operatorname{trace}(Q(\lambda)_h\, Q(\lambda)_{sh}) \\
&= \frac{1}{2}\left[\operatorname{Re}\left(p_{1,1}(\lambda) - p_{2,2}(\lambda)\right)\right]\left[\operatorname{Im}\left(p_{1,1}(\lambda) - p_{2,2}(\lambda)\right)\right] + \operatorname{Im}\left(p_{1,2}(\lambda)p_{2,1}(\lambda)\right).
\end{aligned}$$

Theorem 10 *Suppose that $q \in (0,1)$ and $L(\lambda)$ is a 2×2 matrix polynomial as above with $W_q(L) \neq \mathbb{C}$. Consider the polynomial*

$$\begin{aligned}
G_{q,L}(u,v) = \; & q^4(c_{4,0}(u+\mathrm{i}\,v)\, f_L(u+\mathrm{i}\,v)^4 + c_{0,4}(u+\mathrm{i}\,v)\, g_L(u+\mathrm{i}\,v)^4 \\
& + c_{3,1}(u+\mathrm{i}\,v)\, f_L(u+\mathrm{i}\,v)^3\, g_L(u+\mathrm{i}\,v) \\
& + c_{1,3}(u+\mathrm{i}\,v)\, f_L(u+\mathrm{i}\,v)\, g_L(u+\mathrm{i}\,v)^3 \\
& + c_{2,2}(u+\mathrm{i}\,v)\, f_L(u+\mathrm{i}\,v)^2\, g_L(u+\mathrm{i}\,v)^2) \\
& + q^2(c_{2,0}(u+\mathrm{i}\,v)\, f_L(u+\mathrm{i}\,v)^2 + c_{0,2}(u+\mathrm{i}\,v)\, g_L(u+\mathrm{i}\,v)^2 \\
& + c_{1,1}(u+\mathrm{i}\,v)\, f_L(u+\mathrm{i}\,v)\, g_L(u+\mathrm{i}\,v)) + c_{0,0}(u+\mathrm{i}\,v)
\end{aligned}$$

in $u, v \in \mathbb{R}$ of degree at most $8m$, where

$$\begin{aligned}
c_{4,0}(\lambda) = \; & 4\left((1-q^2)^2 a(\lambda)^2 - 2\,(1-q^2) a(\lambda)\, b(\lambda) + b(\lambda)^2\right. \\
& \left. + 4\,(1-q^2) c(\lambda)^2\right),
\end{aligned}$$

$$
\begin{aligned}
c_{0,4}(\lambda) &= 4\,((1-q^2)^2 b(\lambda)^2 - 2\,(1-q^2)a(\lambda)\,b(\lambda) + a(\lambda)^2 \\
&\quad + 4\,(1-q^2)c(\lambda)^2), \\
c_{3,1}(\lambda) &= -\,16\,q^2 c(\lambda)\,((1-q^2)a(\lambda) + b(\lambda)), \\
c_{1,3}(\lambda) &= -\,16\,q^2 c(\lambda)\,((1-q^2)b(\lambda) + a(\lambda)), \\
c_{2,2}(\lambda) &= 8\,[(1-q^2)(a(\lambda)^2 + b(\lambda)^2) + (4 - 4\,q^2 + 2\,q^4)c(\lambda)^2 \\
&\quad + (-2 + 2\,q^2 + q^4)a(\lambda)\,b(\lambda)], \\
c_{2,0}(\lambda) &= 4\,[-(1-q^2)^2 a(\lambda)^3 - (1-q^2)b(\lambda)^3 \\
&\quad + (1 + q^2 - 3\,q^4 + q^6)a(\lambda)^2 b(\lambda) + (1 - 4\,q^2 + 2\,q^4)a(\lambda)\,b(\lambda)^2 \\
&\quad + (-4 + 4\,q^2 + q^4 - q^6)a(\lambda)\,c(\lambda)^2 + (-4 + 8\,q^2 - 3\,q^4)b(\lambda)\,c(\lambda)^2], \\
c_{0,2}(\lambda) &= 4\,[-(1-q^2)^2 b(\lambda)^3 - (1-q^2)a(\lambda)^3 \\
&\quad + (1 + q^2 - 3\,q^4 + q^6)a(\lambda)\,b(\lambda)^2 + (1 - 4\,q^2 + 2\,q^4)a(\lambda)^2 b(\lambda), \\
&\quad + (-4 + 4\,q^2 + q^4 - q^6)b(\lambda)\,c(\lambda)^2 + (-4 + 8\,q^2 - 3\,q^4)a(\lambda)\,c(\lambda)^2], \\
c_{1,1}(\lambda) &= 8\,c(\lambda)\,q^2[(1-q^2)(a(\lambda)^2 + b(\lambda)^2) + (6 - 6\,q^2 + q^4)a(\lambda)\,b(\lambda) \\
&\quad + (-4 + 4\,q^2 - q^4)c(\lambda)^2], \\
c_{0,0}(\lambda) &= [(1-q^2)(a(\lambda)^2 + b(\lambda)^2) + (-2 + 2\,q^2 - q^4)a(\lambda)\,b(\lambda) \\
&\quad + (4 - 4\,q^2 + q^4)c(\lambda)^2]^2.
\end{aligned}
$$

Then the boundary of $W_q(L)$ lies on the curve

$$\{u + \mathrm{i}\,v \in \mathbb{C} : u, v \in \mathbb{R},\, G_{q,L}(u,v) = 0\}.$$

Proof By Theorem 1 (i), every boundary point μ of $W_q(L)$ satisfies the condition $0 \in \partial F_q(L(\mu))$. Then the result follows readily from Theorem 9. □

Finally, we present an illustrative example (see also [15]).

Example Consider the 2×2 self-adjoint matrix polynomial

$$L(\lambda) \,=\, I_2 \lambda^2 + A_1 \lambda + A_0,$$

where

$$A_1 \,=\, \begin{bmatrix} 0 & \mathrm{i}\,14/5 \\ -\,\mathrm{i}\,14/5 & 0 \end{bmatrix} \quad \text{and} \quad A_0 \,=\, \begin{bmatrix} 3/2 & 1 \\ 1 & 3/2 \end{bmatrix}.$$

Notice that for all unit vectors $x, y \in \mathbb{C}^2$, $\overline{y^* L(\lambda) x} = x^* L(\bar{\lambda}) y$ and $\overline{y^* x} = x^* y$. Hence, for every $q \in [0,1]$, the q-numerical range of $L(\lambda)$ is symmetric with respect to the real axis (this is true of all self-adjoint matrix polynomials).

For $q \in (0,1]$ sufficiently close to 1, the boundary of $W_q(L)$ is given by

$$\{u \pm \mathrm{i}\,v \in \mathbb{C} : u, v \in \mathbb{R},\, v \geq 0,\, G_{q,L}(u,v) = 0\},$$

where

$$G_{q,L}(u,v) \;=\; 2500 - 8125\,q^2 + 5625\,q^4 + \sqrt{1-q^2}(28000 - 45500\,q^2)v$$

$$
\begin{aligned}
&+ (39200 - 90800\, q^2 + 51600\, q^4) u^2 \\
&+ (117600 - 154200\, q^2 + 12100\, q^4) v^2 \\
&+ \sqrt{1-q^2}(219520 - 151760\, q^2) u^2 v \\
&+ \sqrt{1-q^2}(219520 - 67760\, q^2) v^3 \\
&+ (153664 - 214964\, q^2 + 61300\, q^4) u^4 \\
&+ (307328 - 312328\, q^2 + 53800\, q^4) u^2 v^2 \\
&+ (153664 - 97364\, q^2 + 2500\, q^4) v^4 \\
&- \sqrt{1-q^2}\, 2800\, q^2 u^2 v^3 \\
&- 14000\, q^2 \sqrt{1-q^2}(u^4 v + v^5) + 19600(q^4 - q^2) u^6 \\
&- (58800\, q^2 - 39200\, q^4) u^4 v^2 \\
&- (58800\, q^2 - 19600\, q^4) u^2 v^4 - 19600\, q^2 v^6.
\end{aligned}
$$

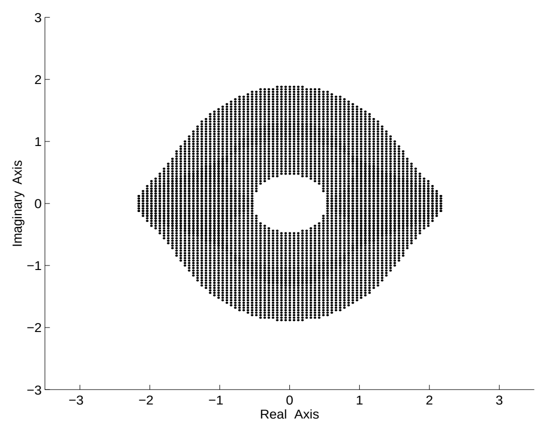

Figure 1: A connected q-numerical range.

If we choose $q_1 = 99/101$, then $G_{q_1,L}(u,v)$ is a scalar multiple of the polynomial

$$
\begin{aligned}
\hat{G}_{q_1,L}(u,v) \;=\;& 118512500 + 3238413500\, v + 423409600\, u^2 \\
&+ 20170918245\, v^2 - 15188859952\, u^2 v - 31819196752\, v^3 \\
&- 386690065\, u^4 - 59221277000\, u^2 v^2 - 64960336444\, v^4 \\
&+ 5543445600\, u^2 v^3 + 2771722800\, u^4 v + 2771722800\, v^5 \\
&+ 768398400\, u^6 + 21132876996\, u^4 v^2 + 39960558792\, u^2 v^4 \\
&+ 19596080\, v^6.
\end{aligned}
$$

The q_1-numerical range

$$
W_{q_1}(L) \;=\; \{u \pm \mathrm{i}\, v \in \mathbb{C} : u, v \in \mathbb{R},\, v \geq 0,\, \hat{G}_{q_1,L}(u,v) \leq 0\}
$$

is sketched in Figure 1; its boundary intersects the real axis (i.e., $v = 0$) at four points. By the equation

$$\hat{G}_{q_1,L}(u,0) \;=\; 4\,(196\,u^2 + 25)(980100\,u^4 - 5057284\,u^2 + 1185125) \;=\; 0,$$

it follows that these points are $(u,v) \cong (\pm 2.2167, 0)$, $(\pm 0.4961, 0)$. □

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