The q-numerical range and the Davis-Wielandt shell of reducible 3-by-3 matrices

Mao-Ting Chien[1], Hiroshi Nakazato[2] and Panayiotis Psarrakos[3]

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Abstract

For a given $q \in [0,1]$, the q-numerical range of an $n \times n$ complex matrix A is defined by $F_q(A) = \{x^*Ay \in \mathbb{C} : x, y \in \mathbb{C}^n,\, x^*x = y^*y = 1,\, x^*y = q\}$, and it is closely related with the Davis-Wielandt shell of A, $W(A, A^*A) = \{(x^*Ax, x^*A^*Ax) \in \mathbb{C} \times \mathbb{R} : x \in \mathbb{C}^n,\, x^*x = 1\}$. In this paper, we investigate systematically the q-numerical range of the 3×3 matrix

$$A(\alpha) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad ; \quad \alpha > 0,$$

and obtain the equation of its boundary by taking advantage of the special shape of $W(A(\alpha), A(\alpha)^*A(\alpha))$. Furthermore, a parametric representation of $\partial F_q(A(\alpha))$ and the construction of a 4×4 matrix B_q such that $F_q(A(\alpha)) = F_1(B_q)$ are discussed. The q-numerical range of a certain normal operator on an infinite Hilbert space of complex valued (Lebesgue) measurable functions is also considered.

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1 Introduction

Let $\mathcal{M}_n$ be the algebra of all $n \times n$ complex matrices, and let $A \in \mathcal{M}_n$. For a real $q \in [0,1]$, the *q-numerical range* of A is denoted and defined by

$$F_q(A) = \{x^*Ay \in \mathbb{C} : x, y \in \mathbb{C}^n,\, x^*x = y^*y = 1,\, x^*y = q\}.$$

[1]Department of Mathematics, Soochow University, Taipei 11102, Taiwan (E-mail: mtchien@math.scu.edu.tw). The work of this author was supported in part by National Science Council of the Republic of China.

[2]Department of Mathematical System Science, Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan (E-mail: nakahr@cc.hirosaki-u.ac.jp).

[3]Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece (E-mail: ppsarr@math.ntua.gr).

The q-numerical range of a matrix is known to be a *compact* and *convex* subset of the complex plane [8, 18]. For $q = 1$, we have the classical *numerical range* (also known as *field of values*) of A, that is,

$$F(A) \equiv F_1(A) = \{x^* A x \in \mathbb{C} : x \in \mathbb{C}^n,\, x^* x = 1\}.$$

In the last decades, the q-numerical range of matrices has attracted attention, and several results have been obtained (see e.g., [1, 2, 4, 10, 11, 12, 14, 18]). These results are useful in investigating and understanding matrices and operators, and some of them have been generalized to matrix polynomials [4, 16, 17].

The *spectrum* of A, namely, $\sigma(A) = \{\lambda \in \mathbb{C} : \det(\lambda I_n - A) = 0\}$, always satisfies $q\,\sigma(A) \subseteq F_q(A)$, and therefore convex hull$\{q\,\sigma(A)\} \subseteq F_q(A)$. Moreover, the following properties are helpful in understanding the q-numerical range [8, 9, 11, 12].

($\mathbf{P}_1$) For any $a, b \in \mathbb{C}$, $F_q(aA + bI_n) = aF_q(A) + q\,b$.

($\mathbf{P}_2$) If $A = A_1 \oplus A_2$, then $F(A) =$ convex hull$\{F(A_1) \cup F(A_2)\}$. For $q < 1$, the q-numerical range of A does not satisfy this relation.

($\mathbf{P}_3$) For any unitary matrix $U \in \mathcal{M}_n$, $F_q(U^* A U) = F_q(A)$, i.e., $F_q(A)$ is invariant under unitary similarities.

($\mathbf{P}_4$) The corners of $F(A)$ ($\equiv F_1(A)$) are eigenvalues of A with special characteristics (see [8] for details). For $0 \leq q < 1$, the boundary $\partial F_q(A)$ is C^1-smooth (i.e., $F_q(A)$ has no corners) and contains no eigenvalues of A.

($\mathbf{P}_5$) If $R_A = \min\{\|A - \lambda I\| : \lambda \in \mathbb{C}\}$, where $\|\cdot\|$ denotes the spectral norm, then $F_0(A) = \{z \in \mathbb{C} : |z| \leq R_A\}$.

Despite the simplicity of the definition of the q-numerical range, there are many interesting unanswered questions and fundamental issues. Some are curiosity driven and some are driven by its ubiquitous nature in pure and applied mathematics. One of the most challenging problems is the characterization of the boundary $\partial F_q(A)$ and the construction of its equation [2, 3, 4, 8, 9]. Since $F_q(A)$ is a special C-numerical range (see [15, 18] for definitions and details), its boundary lies on an algebraic curve in the complex plane. In particular, if $F_q(A)$ is not a single point, then there exists a nonzero polynomial $\mathcal{P} \in \mathbb{R}[u, v]$, such that

(a) $\mathcal{P}(u, v) = 0$ for every boundary point $u + \mathrm{i}\,v$ ($u, v \in \mathbb{R}$) of $F_q(A)$, and

(b) the polynomial $\mathcal{P}$ is the product of real polynomials $\mathcal{P}_j$'s irreducible in $\mathbb{C}[u, v]$, and $\mathcal{P}_j(u, v) = 0$ for infinitely many boundary points $u + \mathrm{i}\,v$ of $F_q(A)$.

However, there are no currently known procedures for the construction of the polynomial $\mathcal{P}$, and the explicit equation of the boundary of $F_q(A)$ is known only when A is normal or 2×2 [4, 10, 12, 13, 14].

Consider now a 3×3 matrix of the form

$$A_0 = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}. \tag{1}$$

If $A_1 = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$, then by the first part of Property (P_2), $F(A_0)$ coincides with the convex hull of the union $F(A_1) \cup \{a_{3,3}\}$, where $F(A_1)$ is an elliptical disk with foci at the eigenvalues of A_1 [8]. On the other hand, by the second part of the same property, it is clear that we cannot have a similar conclusion for $F_q(A_0)$ when $0 \leq q < 1$. Hence, the computation of $F_q(A_0)$ and its boundary is a problem of special interest. Moreover, 3×3 matrices of the form (1), play an important role in the study of operator dilations and related numerical range inclusions [5].

By [8, Lemma 1.3.1], $A_1 - (1/2)\mathrm{trace}(A_1)I_2$ is unitarily similar to a matrix of the form $\begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix}$, where $\operatorname{Arg}\mu = \operatorname{Arg}\nu$. As a consequence, by Properties (P_1) and (P_3), without loss of generality, we may assume that A_0 is of the form

$$A(\alpha,\beta,\gamma) = \begin{pmatrix} 0 & 1+\gamma & 0 \\ 1-\gamma & 0 & 0 \\ 0 & 0 & \alpha + \mathrm{i}\,\beta \end{pmatrix} \quad (\alpha,\beta,\gamma \in \mathbb{R},\ 0 \leq \gamma \leq 1). \tag{2}$$

This matrix is unitarily similar to its transposed, and if $\gamma = 0$, then $A(\alpha,\beta,0)$ is normal and its eigenvalues are $1, -1, \alpha + \mathrm{i}\,\beta$. The range $F_q(A(\alpha,\beta,0))$ is exactly described in [10], and thus, in the remainder, we assume that γ is positive.

In this article, we study systematically the q-numerical range of $A(\alpha,0,1)$ for $\alpha > 0$, and obtain the equation of the boundary $\partial F_q(A(\alpha,0,1))$. Our main result is Theorem 1 in the next section. In Sections 3 and 4, the surface of the Davis-Wielandt shell of $A(\alpha,0,1)$ and the q-numerical range of a normal bounded linear operator acting on an infinite dimensional Hilbert space are fully described. A number of lemmas are contained in Section 6, and the proof of Theorem 1 is completed in Sections 5 and 7.

2 The main result

Consider the matrix $A(\alpha,\beta,\gamma)$ in (2) with $\alpha > 0$, $\beta \in \mathbb{R}$ and $0 < \gamma \leq 1$. Keeping in mind Property (P_2) and [8, Theorem 1.3.6], observe that

$$F(A(\alpha,\beta,\gamma)) = \text{convex hull}\left\{ \left\{ u + \mathrm{i}\,v : u, v \in \mathbb{R},\ u^2 + \frac{v^2}{\gamma^2} \leq 1 \right\} \cup \{\alpha + \mathrm{i}\,\beta\} \right\}.$$

If $\alpha^2 + \beta^2/\gamma^2 \leq 1$, then the q-numerical range of $A(\alpha,\beta,\gamma)$ $(0 \leq q \leq 1)$ is the elliptical disk [4, Theorem 6] (see also (12) below)

$$F_q(A(\alpha,\beta,\gamma)) = F_q\left(\begin{pmatrix} 0 & 1+\gamma \\ 1-\gamma & 0 \end{pmatrix} \right)$$

$$= \left\{ u + \mathrm{i}\, v : u, v \in \mathbb{R},\ \frac{u^2}{(1+\gamma\sqrt{1-q^2})^2} + \frac{v^2}{(\gamma+\sqrt{1-q^2})^2} \le 1 \right\}.$$

Suppose now that $\alpha^2 + \beta^2/\gamma^2 > 1$. If $\beta = 0$ and $\gamma = 1$, then $\alpha > 1$. In this case, we have the main result of the paper.

Theorem 1 *Let $q \in (0, 1]$ and consider the 3×3 matrix*

$$A(\alpha) \;=\; A(\alpha, 0, 1) \;=\; \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \quad ; \quad \alpha > 1. \tag{3}$$

(I) *If $1 < \alpha < 2$, and q satisfies*

$$q \;\le\; \frac{\alpha(2-\alpha)}{\alpha^2 - 2\alpha + 2}\,,$$

then the q-numerical range of $A(\alpha)$ is the circular disk

$$F_q(A(\alpha)) \;=\; \left\{ z \in \mathbb{C} : |z| \le 1 + \sqrt{1-q^2} \right\},$$

i.e., it coincides with the q-numerical range of $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

(II) *If $\alpha > 2$, and q satisfies*

$$q \;\le\; \frac{\alpha(\alpha-2)}{\alpha^2 - 2\alpha + 2}\,,$$

then the boundary of $F_q(A(\alpha))$ is parameterized in the form

$$\partial F_q(A(\alpha)) \;=\; \{ X(\vartheta) + \mathrm{i}\, Y(\vartheta) : 0 \le \vartheta \le 2\pi \}\,,$$

where

$$X(\vartheta) \;=\; \frac{2\alpha \cos\vartheta (2-\alpha^2) q^2 + [3\alpha^3 + \alpha(\alpha^2-1)\cos(2\vartheta) - 3\alpha]\, q - 2\alpha^3 \cos\vartheta}{4(\alpha^2-1)(1 - q\,\cos\vartheta)}$$
$$\frac{-\,\alpha\cos\vartheta\sqrt{2(1-q^2)}\sqrt{2\alpha^2 - (\alpha^2+1)q^2 - (\alpha^2-1)q^2\cos(2\vartheta)}}{} \tag{4}$$

and

$$Y(\vartheta) \;=\; \frac{2\alpha^3 \sin\vartheta\sqrt{1-q^2} - \alpha(\alpha^2-1)q\sqrt{1-q^2}\sin(2\vartheta)}{4(\alpha^2-1)(1-q\,\cos\vartheta)}$$
$$\frac{+\sqrt{2}\,\alpha\sin\vartheta\sqrt{2\alpha^2 - (\alpha^2+1)q^2 - (\alpha^2-1)q^2\cos(2\vartheta)}}{}\,. \tag{5}$$

Every boundary point $u + \mathrm{i}\,v$ *of* $F_q(A(\alpha))$ *(*$u, v \in \mathbb{R}$*) satisfies the equation* $\mathcal{L}(1, u, v) = 0$*, where*

$$\begin{aligned}\mathcal{L}(t,u,v) \quad = \quad & c_{6,0}\,u^6 + c_{4,2}\,u^4v^2 + c_{2,4}\,u^2v^4 + c_{0,6}\,v^6 + c_{5,0}\,t\,u^5 \\ & + c_{3,2}\,t\,u^3v^2 + c_{1,4}\,t\,u\,v^4 + c_{4,0}\,t^2u^4 + c_{2,2}\,t^2u^2v^2 \\ & + c_{0,4}\,t^2v^4 + c_{3,0}\,t^3u^3 + c_{1,2}\,t^3u\,v^2 + c_{2,0}\,t^4u^2 \\ & + c_{0,2}\,t^4v^2 + c_{1,0}\,t^5u + c_{0,0}\,t^6\end{aligned} \tag{6}$$

with

$$\begin{aligned}
c_{6,0} &= 16(\alpha^2-1)^2(\alpha^2+q^2-\alpha^2q^2)^2,\\
c_{4,2} &= 16(\alpha^2-1)^2(3\alpha^4+2\alpha^2q^2-4\alpha^4q^2+3q^4-4\alpha^2q^4+\alpha^4q^4),\\
c_{2,4} &= 16(\alpha^2-1)^2(3\alpha^4-2\alpha^2q^2-2\alpha^4q^2+3q^4-2\alpha^2q^4),\\
c_{0,6} &= 16(\alpha^2-1)^2(\alpha^2-q^2)^2,\\
c_{5,0} &= 32\alpha(\alpha^2-1)\,q\,(\alpha^2+q^2-\alpha^2q^2)(2\alpha^2-\alpha^4+3q^2-4\alpha^2q^2+\alpha^4q^2),\\
c_{3,2} &= 32\alpha(\alpha^2-1)\,q\,(4\alpha^4-2\alpha^6+2\alpha^2q^2-7\alpha^4q^2+3\alpha^6q^2+6q^4\\
&\quad -10\alpha^2q^4+5\alpha^4q^4-\alpha^6q^4),\\
c_{1,4} &= 32\alpha(\alpha^2-1)\,q\,(2\alpha^4-\alpha^6-3\alpha^2q^2+\alpha^6q^2+3q^4-3\alpha^2q^4+\alpha^4q^4),\\
c_{4,0} &= 8\alpha^2(-\alpha^6-\alpha^8+10\alpha^4q^2-13\alpha^6q^2+5\alpha^8q^2+39\alpha^2q^4-73\alpha^4q^4\\
&\quad +41\alpha^6q^4-7\alpha^8q^4+30q^6-81\alpha^2q^6+75\alpha^4q^6-27\alpha^6q^6+3\alpha^8q^6),\\
c_{2,2} &= 8\alpha^2(-2\alpha^6-2\alpha^8+16\alpha^4q^2-15\alpha^6q^2+7\alpha^8q^2-2\alpha^2q^4-27\alpha^4q^4\\
&\quad +33\alpha^6q^4-8\alpha^8q^4+36q^6-74\alpha^2q^6+55\alpha^4q^6-20\alpha^6q^6+3\alpha^8q^6),\\
c_{0,4} &= 8\alpha^2(-\alpha^6-\alpha^8+6\alpha^4q^2-2\alpha^6q^2+2\alpha^8q^2-9\alpha^2q^4+6\alpha^4q^4\\
&\quad -2\alpha^6q^4-\alpha^8q^4+6q^6-9\alpha^2q^6+6\alpha^4q^6-\alpha^6q^6),\\
c_{3,0} &= 8\alpha^3q(2\alpha^6+\alpha^8-2\alpha^4q^2+6\alpha^6q^2-3\alpha^8q^2-36\alpha^2q^4+51\alpha^4q^4\\
&\quad -22\alpha^6q^4+3\alpha^8q^4-40q^6+84\alpha^2q^6-57\alpha^4q^6+14\alpha^6q^6-\alpha^8q^6),\\
c_{1,2} &= 8\alpha^3q(2\alpha^6+\alpha^8-10\alpha^4q^2+3\alpha^6q^2-3\alpha^8q^2+12\alpha^2q^4+9\alpha^4q^4\\
&\quad -13\alpha^6q^4+3\alpha^8q^4-24q^6+36\alpha^2q^6-23\alpha^4q^6+8\alpha^6q^6-\alpha^8q^6),\\
c_{2,0} &= \alpha^4(\alpha^8-6\alpha^6q^2-4\alpha^8q^2-31\alpha^4q^4-14\alpha^6q^4+6\alpha^8q^4+112\alpha^2q^6\\
&\quad -122\alpha^4q^6+46\alpha^6q^6-4\alpha^8q^6+240q^8-368\alpha^2q^8+169\alpha^4q^4\\
&\quad -26\alpha^6q^8+\alpha^8q^8),\\
c_{0,2} &= \alpha^4(\alpha^8-10\alpha^6q^2-4\alpha^8q^2+33\alpha^4q^4+10\alpha^6q^4+6\alpha^8q^4-48\alpha^2q^6\\
&\quad -18\alpha^4q^6+10\alpha^6q^6-4\alpha^8q^6+48q^8-48\alpha^2q^8+33\alpha^4q^8\\
&\quad -10\alpha^6q^8+\alpha^8q^8),\\
c_{1,0} &= 2\alpha^5q^3(-\alpha^6+7\alpha^4q^2+3\alpha^6q^2+6\alpha^4q^4-3\alpha^6q^4-48q^6+48\alpha^2q^6\\
&\quad -13\alpha^4q^6+\alpha^6q^6),\\
c_{0,0} &= \alpha^6q^6(\alpha+2q+\alpha\,q)(\alpha+2q-\alpha\,q)(\alpha-2q+\alpha\,q)(\alpha-2q-\alpha\,q).
\end{aligned}$$

Moreover, $F_q(A(\alpha))$ *coincides with the (classical) numerical range of the*

4×4 *matrix*

$$B_q = \begin{pmatrix} b_{1,1} & 0 & 0 & b_{1,4} \\ 0 & b_{2,2} & b_{2,3} & b_{2,4} \\ 0 & -b_{2,3} & b_{3,3} & b_{3,4} \\ -b_{1,4} & -b_{2,4} & -b_{3,4} & b_{4,4} \end{pmatrix}, \tag{7}$$

where

$$b_{1,1} = \frac{\alpha^2 - 2\alpha\, q + \alpha^2 q}{2\alpha - 2}, \quad b_{1,4} = \frac{\alpha^2 \sqrt{1-q}\,\sqrt{\alpha - q + \alpha\, q}}{2(\alpha - 1)\sqrt{\alpha + 1}},$$

$$b_{2,2} = \frac{\alpha^2 + 2\alpha\, q + \alpha^2 q}{2\alpha + 2}, \quad b_{2,3} = \frac{\alpha^2 \sqrt{1-q}\,(1+q)}{2\sqrt{\alpha+1}\sqrt{\alpha - q + \alpha\, q}},$$

$$b_{2,4} = \frac{\alpha^2 \sqrt{(1-q)(1+2q)}}{2(\alpha+1)\sqrt{\alpha^2 + \alpha^2 q - q^2 + \alpha\, q^2}}, \quad b_{3,3} = \frac{-\alpha^2 - 2\alpha\, q + \alpha^2 q}{2\alpha - 2},$$

$$b_{3,4} = \frac{\alpha^2 \sqrt{2\alpha - 1}\,(1+q)}{2(\alpha-1)\sqrt{(\alpha+1)(\alpha+q)(\alpha - q + \alpha\, q)}}, \quad b_{4,4} = \frac{-\alpha^2 + 2\alpha\, q + \alpha^2 q}{2\alpha + 2}.$$

(III) *Suppose that*

$$0 < \frac{\alpha\,|\alpha - 2|}{\alpha^2 - 2\alpha + 2} < q < 1,$$

or equivalently, if $q = \cos\phi$ for some $\phi \in (0, \pi/2)$, then the quantity $\tau = \tan(\phi/2)$ satisfies $0 < \tau < \min\{\alpha - 1, 1/(\alpha - 1)\}$. Then the q-numerical range $F_q(A(\alpha))$ is the union of the convex sets,

$$\Delta_1 = \left\{ u + \mathrm{i}\, v : u, v \in \mathbb{R},\ u^2 + v^2 \le \frac{(\tau+1)^4}{(\tau^2+1)^2},\ u \le \frac{(\tau+1)[(\tau+1)^2 - \alpha^2\tau]}{\alpha(1-\tau)(\tau^2+1)} \right\} \tag{8}$$

and

$$\Delta_2 = F(B_q) \cap \left\{ u + \mathrm{i}\, v : u, v \in \mathbb{R},\ u \ge \frac{(\tau+1)[(\tau+1)^2 - \alpha^2\tau]}{\alpha(1-\tau)(\tau^2+1)} \right\}, \tag{9}$$

where B_q is the 4×4 matrix in (7).

We remark that in the last statement of this theorem, the convex sets Δ_1 and Δ_2 are not disjoint and their intersection is the line segment

$$\left\{ u + \mathrm{i}\, v : u, v \in \mathbb{R},\ u^2 + v^2 \le \frac{(\tau+1)^4}{(\tau^2+1)^2},\ u = \frac{(\tau+1)[(\tau+1)^2 - \alpha^2\tau]}{\alpha(1-\tau)(\tau^2+1)} \right\}$$

$$= F(B_q) \cap \left\{ u + \mathrm{i}\, v : u, v \in \mathbb{R},\ u = \frac{(\tau+1)[(\tau+1)^2 - \alpha^2\tau]}{\alpha(1-\tau)(\tau^2+1)} \right\}.$$

Hence, the claim that $F_q(A(\alpha))$ is the union of Δ_1 and Δ_2 correlates with the convexity of $F_q(A(\alpha))$ itself. Moreover, by straightforward computations one can see that in (8) and (9),

$$\frac{(\tau+1)^4}{(\tau^2+1)^2} = \left(1 + \sqrt{1 - q^2}\right)^2$$

and

$$\frac{(\tau+1)[(\tau+1)^2-\alpha^2\tau]}{\alpha(1-\tau)(\tau^2+1)} \;=\; \frac{(4-\alpha^2)(1+\sqrt{1-q^2})+(\alpha^2-2)q^2}{2\,\alpha\,q}\,.$$

The above theorem is illustrated in the following example.

Example 1 Let $\alpha = 3$ and $q = 0.5$. The conditions of Theorem 1 (II) hold, and the 0.5-numerical range of the matrix $A(3)$ is sketched in the left part of Figure 1 by using (4) and (5). The spectrum of $A(3)$ is obviously $\sigma(A(3)) = \{0, 3\}$,

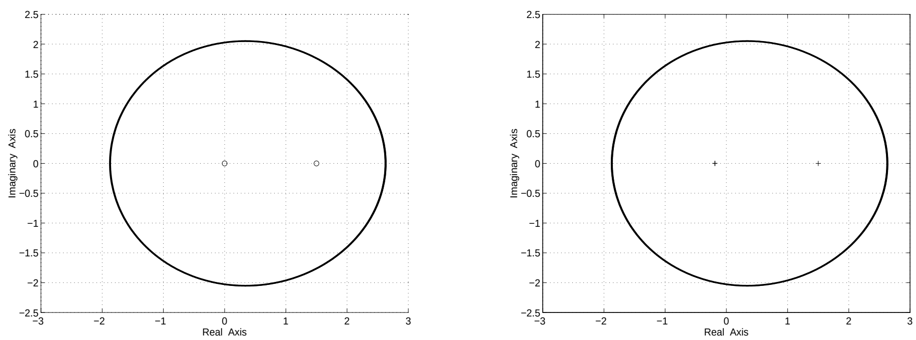

Figure 1: The ranges $F_{0.5}(A(3))$ and $F(B_{0.5})$.

and the numbers $0\,q = 0$ and $3\,q = 1.5$ are marked with o's. In the right part of the figure, the numerical range of the matrix

$$B_{0.5} \;=\; \begin{pmatrix} 2.6250 & 0 & 0 & 1.5910 \\ 0 & 2.0625 & 1.1932 & 0.3007 \\ 0 & -1.1932 & -1.8750 & 1.0085 \\ -1.5910 & -0.3007 & -1.0085 & -0.1875 \end{pmatrix}$$

(see (7)) is drawn by an algorithm of Horn and Johnson [8, pp. 33-39], and its eigenvalues -0.1875 and 1.5 are marked with $+$'s. Comparing $F_{0.5}(A(3))$ and $F(B_{0.5})$, we see that they are exactly the same, confirming the second part of Theorem 1.

3 Some geometry

It is known [10, 12] that the q-numerical range of a general matrix $A \in \mathcal{M}_n$ is strongly connected with the *Davis-Wielandt shell* of A, namely,

$$W(A, A^*A) \;=\; \{(x^*Ax, x^*A^*Ax) \in \mathbb{C}\times\mathbb{R} : x \in \mathbb{C}^n,\, x^*x = 1\}\,. \qquad (10)$$

For $n \geq 3$, the shell $W(A, A^*A)$ is always convex. If $n = 2$ and the affine hull of $W(A, A^*A)$ is (real) 3-dimensional, then it is the surface of an ellipsoid [1] (for more properties, see [6, 7]).

The Davis-Wielandt shell of A leads to a numerical approximation of $F_q(A)$ [12]. For each point $u+\mathrm{i}\,v$ ($u,v\in\mathbb{R}$) of the numerical range $F(A)$ ($\equiv F_1(A)$), we define

$$h(u+\mathrm{i}\,v) \;=\; \max\{w\in\mathbb{R} : (u+\mathrm{i}\,v,w)\in W(A,A^*A)\}. \tag{11}$$

This function and the function

$$\Phi(u+\mathrm{i}\,v) \;=\; \sqrt{h(u+\mathrm{i}\,v)-u^2-v^2}$$

are concave and upper semi-continuous on the convex set $F(A)$. Furthermore, h and Φ are continuous in the interior of $F(A)$ when $\mathrm{Int}F(A)\neq\emptyset$, and they are continuous on $F(A)$ when $F(A)$ has no interior. The set

$$\{(u+\mathrm{i}\,v, h(u+\mathrm{i}\,v)) : u,v\in\mathbb{R},\; u+\mathrm{i}\,v\in F(A)\}$$

is said to be the *upper surface* of the shell $W(A,A^*A)$, and by [12, 18],

$$F_q(A) \;=\{\, q\,(u{+}\mathrm{i}\,v){+}\sqrt{1-q^2}\,z\,\Phi(u{+}\mathrm{i}\,v) : u{+}\mathrm{i}\,v\in F(A),\; z\in\mathbb{C},\; |z|\le 1\}. \tag{12}$$

Consider now the 3×3 matrix $A(\alpha)$ in (3) for $\alpha>1$, and let $q\in[0,1]$. The Davis-Wielandt shell $W(A(\alpha),A(\alpha)^*A(\alpha))$ (see (10)) is the convex hull of the point $(\alpha+\mathrm{i}\,0,\alpha^2)$ and the Davis-Wielandt shell of $B=\begin{pmatrix}0&2\\0&0\end{pmatrix}$, that is,

$$W(B,B^*B) \;=\; \left\{(u+\mathrm{i}\,v,w) : u,v,w\in\mathbb{R},\; u^2+v^2+\frac{(w-2)^2}{4}=1\right\}. \tag{13}$$

As a consequence, the boundary of $W(A(\alpha),A(\alpha)^*A(\alpha))$ consists of two parts. The first part lies on the ellipsoid (13) and the second one lies on the cone

$$\{(u+\mathrm{i}\,v,w) : u,v,w\in\mathbb{R},\; \mathcal{Q}(u,v,w)=0\}\,, \tag{14}$$

where

$$\mathcal{Q}(u,v,w)=(\alpha^2{-}1)w^2{+}(-2\alpha^3{+}4\alpha)u\,w{+}(\alpha^4{-}4\alpha^2)u^2{+}\alpha^4v^2{+}4\alpha^3u{-}2\alpha^2w{-}\alpha^4.$$

This cone is a ruled surface and consists of a family of lines

$$\left\{\left(\alpha-\frac{s\,\alpha^2(\alpha-\cos\theta)}{\alpha^2-1}+\mathrm{i}\,\frac{s\,\alpha^2\sin\theta}{\alpha^2-1}\,,\;\alpha^2-\frac{s\,\alpha^3(\alpha^3-3\alpha+2\cos\theta)}{(\alpha^2-1)^2}\right) : s\in\mathbb{R}\right\} \tag{15}$$

($0\le\theta\le 2\pi$), where each of them is a tangent line of the ellipsoid in (13). This observation on the cone (14) helps us below to compute the boundary of $F_q(A(\alpha))$ (for $\alpha>1$). Setting $s=1$ implies that the lines in (15) pass through the (space) curve

$$\Gamma=\left\{\left(\frac{-\alpha+\alpha^2\cos\theta}{\alpha^2-1}+\mathrm{i}\,\frac{\alpha^2\sin\theta}{\alpha^2-1}\,,\;\frac{\alpha^2(1+\alpha^2-2\alpha\,\cos\theta)}{(\alpha^2-1)^2}\right) : 0\le\theta\le 2\pi\right\}, \tag{16}$$

which lies on the intersection of the plane

$$w \;=\; -\left(\frac{2\alpha^2}{\alpha^3-\alpha}\right)\, u + \frac{\alpha^2}{\alpha^2-1}$$

and the parabolic surface $w \;=\; u^2 + v^2$. Moreover, Γ is an ellipse and its projection on the (u,v)-plane coincides with the circle

$$\left(u + \frac{\alpha}{\alpha^2-1}\right)^2 + v^2 \;=\; \frac{\alpha^4}{(\alpha^2-1)^2}\,. \tag{17}$$

The ellipsoid and the cone part of $\partial W(A(\alpha), A(\alpha)^* A(\alpha))$ mentioned above meet on the plane

$$\{(u + \mathrm{i}\, v, w) : (u,v,w) \in \mathbb{R}^3,\, 4\alpha\, u + (\alpha^2-2)\, w - 2\alpha^2 = 0\}, \tag{18}$$

and their intersection is the (space) curve

$$\mathcal{C} \;=\; \{(u(t) + \mathrm{i}\, v(t), w(t)) \in \mathbb{C}\times\mathbb{R} : 0 \le t \le 2\pi\}, \tag{19}$$

where

$$\begin{aligned} u(t) &\;=\; \frac{4\alpha - \alpha^2(\alpha^2-2)\cos t}{\alpha^4+4}\,, \\ v(t) &\;=\; \frac{\alpha^2 \sin t}{\sqrt{\alpha^4+4}}\,, \\ w(t) &\;=\; \frac{4\alpha^2 + 2\alpha^4 + 4\alpha^3\cos t}{\alpha^4+4}\,. \end{aligned}$$

The projection $\mathcal{C}_{up}$ of the upper part of the curve $\mathcal{C}$ onto the (u,v)-plane is given by

$$\begin{aligned} \mathcal{C}_{up} &= \left\{ u + \mathrm{i}\, v : u = \frac{4\alpha + (2-\alpha^2)\sqrt{\alpha^4 - (\alpha^4+4)\, v^2}}{\alpha^4+4},\, -\frac{\sqrt{\alpha^2-1}}{\alpha} \le v \le \frac{\sqrt{\alpha^2-1}}{\alpha} \right\} \\ &= \left\{ u + \mathrm{i}\, v : (\alpha^4+4)\, u^2 - 8\alpha\, u + (\alpha^2-2)^2\, v^2 - \alpha^4 + 4\alpha^2 = 0,\, u \ge \frac{1}{\alpha} \right\} \end{aligned} \tag{20}$$

when $1 < \alpha < \sqrt{2}$, by

$$\mathcal{C}_{up} = \left\{ \frac{\sqrt{2}}{2} + \mathrm{i}\, v : -\frac{\sqrt{2}}{2} \le v \le \frac{\sqrt{2}}{2} \right\} \tag{21}$$

when $\alpha = \sqrt{2}$, and by

$$\mathcal{C}_{up} = \left\{ u + \mathrm{i}\, v : u = \frac{4\alpha - (\alpha^2-2)\sqrt{\alpha^4 - (\alpha^4+4)\, v^2}}{\alpha^4+4},\, -\frac{\alpha^2}{\sqrt{\alpha^4+4}} \le v \le \frac{\alpha^2}{\sqrt{\alpha^4+4}} \right\}$$

$$\cup \left\{ u + \mathrm{i}\, v : u = \frac{4\alpha - (\alpha^2 - 2)\sqrt{\alpha^4 - (\alpha^4 + 4)\, v^2}}{\alpha^4 + 4},\ \frac{\sqrt{\alpha^2 - 1}}{\alpha} \le |v| \le \frac{\alpha^2}{\sqrt{\alpha^4 + 4}} \right\}$$

$$= \left\{ u + \mathrm{i}\, v : (\alpha^4 + 4)\, u^2 - 8\alpha\, u + (\alpha^2 - 2)^2 v^2 - \alpha^4 + 4\alpha^2 = 0,\ u \le \frac{1}{\alpha} \right\} \quad (22)$$

when $\alpha > \sqrt{2}$. The endpoints of the arc $\mathcal{C}_{up}$ are $(1 \pm \mathrm{i}\sqrt{\alpha^2 - 1})\,/\,\alpha$. Furthermore, if $\alpha \neq \sqrt{2}$, then the ellipse

$$\{u + \mathrm{i}\, v : u, v \in \mathbb{R},\ (\alpha^4 + 4)\, u^2 - 8\alpha\, u + (\alpha^2 - 2)^2 v^2 - \alpha^4 + 4\alpha^2 = 0\}$$

is inscribed in the unit circle $\{u + \mathrm{i}\, v : u, v \in \mathbb{R},\ u^2 + v^2 = 1\}$ and the common points of these two curves are $(1 \pm \mathrm{i}\sqrt{\alpha^2 - 1})\,/\,\alpha$. At these points, the two curves have common tangents, which pass through α.

Note that the cone part of $W(A(\alpha), A(\alpha)^* A(\alpha))$ is the convex hull of the curve $\mathcal{C}$ in (19) and the point $(\alpha + \mathrm{i}\, 0, \alpha^2)$. Hence, since the ellipsoid part of

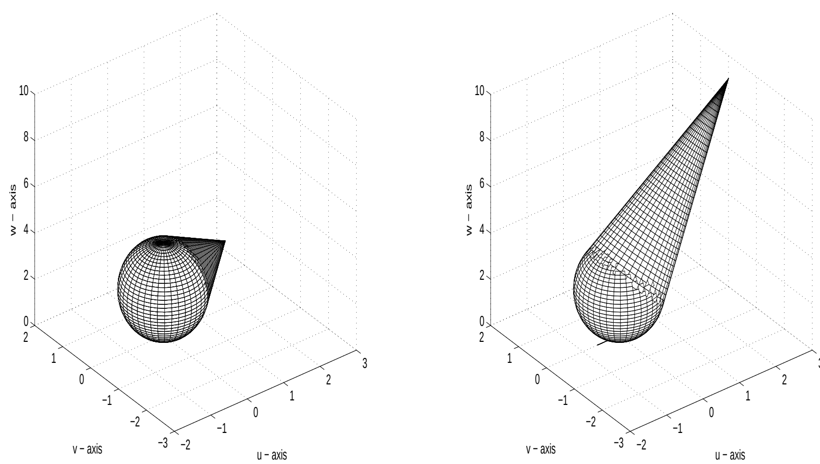

Figure 2: The shells $W(A(1.7), A(1.7)^* A(1.7))$ and $W(A(3), A(3)^* A(3))$.

$\partial W(A(\alpha), A(\alpha)^* A(\alpha))$ lies on the ellipsoid (13), the boundary of the Davis-Wielandt shell of $A(\alpha)$ can be easily drawn. Using this observation, the surface of $W(A(\alpha), A(\alpha)^* A(\alpha))$ is sketched in Figure 2 for $\alpha = 1.7, 3$. Notice the ellipsoid and the cone parts of the two surfaces, and observe the corner $(1.7 + \mathrm{i}\, 0, 2.89)$ of $W(A(1.7), A(1.7)^* A(1.7))$ in the left part of the figure and the corner $(3 + \mathrm{i}\, 0, 9)$ of $W(A(3), A(3)^* A(3))$ in the right part.

The projection of the shell $W(A(\alpha), A(\alpha)^* A(\alpha))$ onto the (u, v)-plane is the convex set

$$\mathcal{D} = F(A(\alpha)) = \text{convex hull} \big\{ \{u + \mathrm{i}\, v : u, v \in \mathbb{R},\ u^2 + v^2 = 1\} \cap \{\alpha\} \big\}. \quad (23)$$

With respect to the arc $\mathcal{C}_{up}$, we separate the interior of $\mathcal{D}$ as in the following:

$$\begin{aligned}\mathcal{D}_c &= \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ 1/\alpha<u<\alpha,\ |v|<(\alpha-u)/\sqrt{\alpha^2-1},\\ &\qquad (\alpha^4+4)u^2-8\alpha\,u+(\alpha^2-2)^2v^2-\alpha^4+4\alpha^2>0\},\end{aligned}$$

$$\begin{aligned}\mathcal{D}_e &= \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ u\le 1/\alpha,\ u^2+v^2<1\}\\ &\quad\cup\{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ u>1/\alpha,\\ &\qquad (\alpha^4+4)u^2-8\alpha\,u+(\alpha^2-2)^2v^2-\alpha^4+4\alpha^2<0\}\end{aligned}$$

when $1<\alpha<\sqrt{2}$,

$$\mathcal{D}_c = \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ 1/\sqrt{2}<u<\sqrt{2},\ |v|<\sqrt{2}-u\},$$

$$\mathcal{D}_e = \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ u<1/\sqrt{2},\ u^2+v^2<1\}$$

when $\alpha=\sqrt{2}$, and

$$\begin{aligned}\mathcal{D}_c &= \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ 1/\alpha<u<\alpha,\ |v|<(\alpha-u)/\sqrt{\alpha^2-1}\}\\ &\quad\cup\{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ u<1/\alpha,\\ &\qquad (\alpha^4+4)u^2-8\alpha\,u+(\alpha^2-2)^2v^2-\alpha^4+4\alpha^2<0\},\end{aligned}$$

$$\begin{aligned}\mathcal{D}_e &= \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ u<1/\alpha,\ u^2+v^2<1,\\ &\qquad (\alpha^4+4)u^2-8\alpha\,u+(\alpha^2-2)^2v^2-\alpha^4+4\alpha^2>0\}\end{aligned}$$

when $\alpha>\sqrt{2}$. We call $\mathcal{D}_c$ the *cone part* of $\mathcal{D}$, and $\mathcal{D}_e$ the *ellipsoid part* of $\mathcal{D}$.

For any $\theta\in[0,2\pi]$, the line segment with one endpoint $(u(\theta)+\mathrm{i}\,v(\theta),w(\theta))$ on the curve Γ in (16) and the other endpoint at $(\alpha+\mathrm{i}\,0,\alpha^2)$ is given by

$$\mathcal{E}_\theta = \{(\xi(s)+\mathrm{i}\,\eta(s),\zeta(s))\in\mathbb{C}\times\mathbb{R} : 0\le s\le 1\}, \tag{24}$$

where

$$\xi(s) = \alpha-\frac{s\,\alpha^2(\alpha-\cos\theta)}{\alpha^2-1}, \tag{25}$$

$$\eta(s) = \frac{s\,\alpha^2\sin\theta}{\alpha^2-1}, \tag{26}$$

$$\zeta(s) = \alpha^2-\frac{s\,\alpha^3(\alpha^3-3\alpha+2\cos\theta)}{(\alpha^2-1)^2}. \tag{27}$$

The line segment $\mathcal{E}_\theta$ in (24) intersects the plane (18) if and only if $4\alpha\,\xi(s)+(\alpha^2-2)\,\zeta(s)-2\alpha^2=0$. Hence, it is straightforward to see that for

$$s_\theta = \frac{(\alpha^2-1)^2}{\alpha^4-\alpha^2-2\alpha\,\cos\theta+2},$$

the point $(\xi(s_\theta) + \mathrm{i}\,\eta(s_\theta), \zeta(s_\theta)) \in \mathcal{E}_\theta$ lies on the curve $\mathcal{C}$ in (19). Moreover, a point $(\xi(s) + \mathrm{i}\,\eta(s), \zeta(s))) \in \mathcal{E}_\theta$ $(0 \le \theta \le 2\pi)$ lies on the upper surface of $W(A(\alpha), A(\alpha)^* A(\alpha))$ if and only if

$$0 \le s \le s_\theta \quad \text{and} \quad \arctan(\sqrt{\alpha^2 - 1}) \le \theta \le 2\,\pi - \arctan(\sqrt{\alpha^2 - 1})$$

or equivalently, if and only if

$$0 \le s \le s_\theta \quad \text{and} \quad -1 \le \cos\theta \le \frac{1}{\alpha}$$

(where $0 < \arctan(\sqrt{\alpha^2 - 1}) < \pi/2$). Using now (12), and the ellipsoid part and the cone part of the surface of $W(A(\alpha), A(\alpha)^* A(\alpha))$, the q-numerical range of $A(\alpha)$ can be written in the form

$$\begin{aligned} F_q(A(\alpha)) \quad = \quad & \{\, q\,(u + \mathrm{i}\,v) + \sqrt{1 - q^2}\, z \,\sqrt{w - (u^2 + v^2)} : u, v, w \in \mathbb{R}, \\ & \quad z \in \mathbb{C},\ |z| \le 1,\ u^2 + v^2 \le 1, \\ & \quad u \le (4\alpha - (\alpha^2 - 2)\sqrt{\alpha^4 - (\alpha^4 + 4)\, v^2}) \,/\, (\alpha^4 + 4) \\ & \quad w = 2 + 2\sqrt{1 - u^2 - v^2}\} \\ & \cup \{\, q\,(\xi(s) + \mathrm{i}\,\eta(s)) + \sqrt{1 - q^2}\, z \,\sqrt{\zeta(s) - (\xi(s)^2 + \eta(s)^2)} : \\ & \quad z \in \mathbb{C},\ |z| \le 1,\ -1 \le \cos\theta \le 1/\alpha, \\ & \quad 0 \le s \le (\alpha^4 - 2\alpha^2 + 1) \,/\, (\alpha^4 - \alpha^2 - 2\alpha \cos\theta + 2)\}, \end{aligned} \tag{28}$$

where $\xi(s)$, $\eta(s)$ and $\zeta(s)$ are given by (25), (26) and (27), respectively.

4 A normal operator

Let T be a bounded linear operator on an infinite Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For a real $q \in [0, 1]$, the q-numerical range of T is defined by

$$F_q(T) = \{\, \langle Ty, x \rangle \in \mathbb{C} : x, y \in \mathcal{H},\ \langle x, x \rangle = \langle y, y \rangle = 1,\ \langle y, x \rangle = q \}\,.$$

This subset of the complex plane is always bounded and convex, but it is not necessarily closed. The investigation of the q-numerical rage of the 3×3 reducible matrix $A(\alpha)$ in (3) leads to the construction of the q-numerical range of a normal bounded linear operator acting on an infinite dimensional Hilbert space. In particular, we consider the domain

$$\begin{aligned} \Delta_q \quad = \quad & \{\, q\,(u + \mathrm{i}\,v) + \sqrt{1 - q^2}\, z \,\sqrt{w - (u^2 + v^2)} : u, v, w \in \mathbb{R},\ z \in \mathbb{C},\ |z| \le 1, \\ & \quad u = \alpha - s\,\alpha^2(\alpha - \cos\theta) \,/\, (\alpha^2 - 1),\ v = s\,\alpha^2 \sin\theta \,/\, (\alpha^2 - 1), \\ & \quad w = \alpha^2 - s\,\alpha^3(-3\alpha + \alpha^3 + 2\cos\theta) \,/\, (\alpha^2 - 1)^2, \\ & \quad 0 \le s \le 1,\ \arctan(\sqrt{\alpha^2 - 1}) \le \theta \le 2\pi - \arctan(\sqrt{\alpha^2 - 1})\}. \end{aligned} \tag{29}$$

The unit disk $\{u + \mathrm{i}\,v : u, v \in \mathbb{R},\ u^2 + v^2 \le 1\}$ is contained in the region

$$\left\{ u + \mathrm{i}\,v : (u + \mathrm{i}\,v, w) \in \mathcal{E}_\theta,\ \arctan(\sqrt{\alpha^2 - 1}) \le \theta \le 2\pi - \arctan(\sqrt{\alpha^2 - 1}) \right\},$$

and the function $w = 2 + 2\sqrt{1 - u^2 - v^2}$ on the unit disk is dominated by the function defined by

$$(u + \mathrm{i}\, v) \mapsto w \quad \text{if and only if} \quad (u + \mathrm{i}\, v, w) \in \mathcal{E}_\theta. \tag{30}$$

As a consequence, by (28), the q-numerical range $F_q(A(\alpha))$ is contained in the set Δ_q. Moreover, the projection of the line $\mathcal{E}_\theta$ onto the (u, v)-plane for

$$\theta \in \left[-\arctan\left(\sqrt{\alpha^2 - 1}\right),\ \arctan\left(\sqrt{\alpha^2 - 1}\right)\right]$$

is contained in the projection of $\mathcal{E}_{\hat\theta}$ for some

$$\hat\theta \in \left[\arctan\left(\sqrt{\alpha^2 - 1}\right),\ 2\pi - \arctan\left(\sqrt{\alpha^2 - 1}\right)\right],$$

and the function (30) for θ is dominated by the same function for $\hat\theta$. Hence, we have

$$\begin{aligned} \Delta_q \quad = \quad & \{\, q\,(u + \mathrm{i}\, v) + \sqrt{1 - q^2}\, z\, \sqrt{w - (u^2 + v^2)} : u, v, w \in \mathbb{R}, \\ & z \in \mathbb{C},\ |z| \le 1,\ (u + \mathrm{i}\, v, w) \in \mathcal{E}_\theta,\ 0 \le \theta \le 2\pi\}. \end{aligned} \tag{31}$$

At this point, we introduce the separable infinite dimensional Hilbert space $L^2([0,1] \times \mathbf{T}^1 : ds\, d\theta)$, which consists of the complex valued (Lebesgue) measurable functions $f(s, \mathrm{e}^{\mathrm{i}\theta})$ that are square integrable with respect to the measure $ds\, d\theta$. The inner product of this space is defined by

$$\langle f, g\rangle \;=\; \int_0^{2\pi} \int_0^1 f(s, \mathrm{e}^{\mathrm{i}\theta})\, \overline{g(s, \mathrm{e}^{\mathrm{i}\theta})}\, ds\, d\theta\,.$$

Let T be the normal bounded linear operator on $L^2([0,1] \times \mathbf{T}^1 : ds\, d\theta)$ defined by

$$(T\, f)(s, \mathrm{e}^{\mathrm{i}\theta}) \;=\; \left(\alpha - s\, \frac{\alpha^3}{\alpha^2 - 1} + s\, \frac{\alpha^2}{\alpha^2 - 1}\, \mathrm{e}^{\mathrm{i}\theta}\right) f(s, \mathrm{e}^{\mathrm{i}\theta}) \tag{32}$$

$(0 \le s \le 1,\ 0 \le \theta \le 2\pi)$.

Proposition 2 *The domain Δ_q in (29) is the closure of the q-numerical range of the operator T in (32).*

Proof For any two compact subsets Ω_1, Ω_2 of the complex plane define the distance (standard Hausdorff metric)

$$\rho(\Omega_1, \Omega_2) \;=\; \max\left\{\max_{z_1 \in \Omega_1} \min_{z_2 \in \Omega_2} |z_1 - z_2|,\ \max_{z_2 \in \Omega_2} \min_{z_1 \in \Omega_1} |z_1 - z_2|\right\}.$$

If S and S' are bounded linear operators on a Hilbert space $\mathcal{H}$, satisfying the inequality $\|S - S'\| \le \epsilon$, then it follows that $\rho\left(F_q(S), F_q(S')\right) \le \epsilon$.

For any positive integer number N, consider now the mutually disjoint sets

$$R_{\mu,\nu} \;=\; \left(\frac{\mu-1}{N}\,,\;\frac{\mu}{N}\right] \times \left\{ \mathrm{e}^{\mathrm{i}\theta} : \frac{2(\nu-1)\,\pi}{N} < \theta \leq \frac{2\nu\,\pi}{N} \right\} \;;\;\; \mu,\nu = 1,2,\ldots,N$$

and define the linear bounded operator T_N acting on $L^2([0,1]\times\mathbf{T}^1 : ds\,d\theta)$ such that

$$(T_N\,f)(s,\mathrm{e}^{\mathrm{i}\theta}) \;=\; \left(\alpha - \frac{\mu}{N}\,\frac{\alpha^3}{\alpha^2-1} + \frac{\mu}{N}\,\frac{\alpha^2}{\alpha^2-1}\,\mathrm{e}^{\mathrm{i}\frac{2\nu\pi}{N}}\right) f(s,\mathrm{e}^{\mathrm{i}\theta})\;;\;\; (s,\mathrm{e}^{\mathrm{i}\theta}) \in R_{\mu,\nu}.$$

Then the q-numerical range of T_N is given by

$$F_q(T_N) \;=\; F_q\left(\mathrm{diag}\left\{\alpha - \frac{\mu}{N}\,\frac{\alpha^3}{\alpha^2-1} + \frac{\mu}{N}\,\frac{\alpha^2}{\alpha^2-1}\,\mathrm{e}^{\mathrm{i}\frac{2\nu\pi}{N}} : \mu,\nu = 1,2,\ldots,N\right\}\right),$$

and we have that

$$\lim_{N\to\infty} \rho\,(\mathrm{closure}\{F_q(T)\}, F_q(T_N)) \;=\; 0 \quad \text{and} \quad \lim_{N\to\infty} \rho\,(F_q(T_N),\Delta_q) \;=\; 0.$$

Hence, Δ_q coincides with the closure of $F_q(T)$. $\square$

By the above proposition, it is clear that the set Δ_q is compact and convex. Furthermore, the equation

$$\Delta_q \;=\; \bigcup \left\{ F_q(\mathrm{diag}\{\alpha, u+\mathrm{i}\,v\}) : u,v\in\mathbb{R},\; \left(u + \frac{\alpha}{\alpha^2-1}\right)^2 + v^2 = \frac{\alpha^4}{(\alpha^2-1)^2} \right\} \quad (33)$$

follows from the fact that the cone in (14) is a ruled surface and that the endpoint $(u+\mathrm{i}\,v, w)\in\Gamma$ of any line segment $\mathcal{E}_\theta$ $(0\leq\theta\leq 2\pi)$ satisfies the equation $u^2+v^2 = w$. Now we can prove the following theorem.

Theorem 3 *Let T be the normal operator on $L^2([0,1]\times\mathbf{T}^1 : ds\,d\theta)$ defined by (32), and let $0<q<1$. Then the closure Δ_q of the q-numerical range of T coincides with the numerical range $F(B_q)$ of the 4×4 matrix B_q in (7). Moreover, we have:*

(I) *The boundary of Δ_q is*

$$\partial\Delta_q \;=\; \partial F_q(T) \;=\; \{(X(\vartheta)+\mathrm{i}\,Y(\vartheta)) : 0\leq\vartheta\leq 2\pi\}\,, \quad (34)$$

where $X(\vartheta)$ and $Y(\vartheta)$ are given by (4) and (5). An irreducible sextic form $\mathcal{L}(t,u,v)$ satisfying $\mathcal{L}(1,X(\vartheta),Y(\vartheta))=0$ $(0\leq\vartheta\leq 2\pi)$ is given by (6).

(II) *The dual curve of the complex projective curve $\{[(t,u,v)]\in\mathbb{CP}^2 : \mathcal{L}(t,u,v) = 0\}$ is the quartic elliptic curve $\{[(t,u,v)]\in\mathbb{CP}^2 : \mathcal{K}(t,u,v)=0\}$, where*

$$\begin{aligned}\mathcal{K}(t,u,v) \;&=\; d_{4,0}\,u^4 + d_{2,2}\,u^2v^2 + d_{0,4}\,v^4 + d_{3,0}\,t\,u^3 + d_{1,2}\,t\,u\,v^2 \\ &\quad + d_{2,0}\,t^2u^2 + d_{0,2}\,t^2v^2 + d_{1,0}\,t^3u + d_{0,0}\,t^4 \end{aligned} \quad (35)$$

with

$$
\begin{array}{rcl}
d_{4,0} & = & \alpha^4(\alpha+2q+\alpha\,q)(\alpha+2q-\alpha\,q)(\alpha-2q+\alpha\,q)(\alpha-2q-\alpha\,q),\\
d_{2,2} & = & 2\alpha^6(\alpha^2-4q^2-2\alpha^2q^2-4q^4+\alpha^2\,q^4),\\
d_{0,4} & = & \alpha^8(1-q^2)^2,\\
d_{3,0} & = & 8\alpha^3q(-2\alpha^2-\alpha^4+8q^2-6\alpha^2q^2+\alpha^4q^2),\\
d_{1,2} & = & 8\alpha^5q(-2-\alpha^2-2q^2+\alpha^2q^2),\\
d_{2,0} & = & 8\alpha^2(-\alpha^2-\alpha^4+12q^2-13\alpha^2q^2+3\alpha^4q^2),\\
d_{0,2} & = & 8\alpha^4(-1-\alpha^2-q^2+\alpha^2q^2),\\
d_{1,0} & = & 32\alpha(\alpha^2-1)(\alpha^2-2)q,\\
d_{0,0} & = & 16(\alpha^2-1)^2.
\end{array}
$$

Furthermore, the matrix B_q in (7) satisfies the equation

$$
16(\alpha^2-1)^2\det(tI_4+(u/2)(B_q+B_q^*)-\mathrm{i}\,(v/2)(B_q-B_q^*)) \;=\; \mathcal{K}(t,u,v), \tag{36}
$$

i.e., the form $\mathcal{K}(t,u,v)$ is hyperbolic with respect to $(1,0,0)$.

Proof (I) Recall that the q-numerical range of a 2×2 diagonal matrix $\mathrm{diag}\{a,b\}$ ($a,b\in\mathbb{C}$) is the elliptical disk with foci at $q\,a$ and $q\,b$, and with eccentricity q [10, 13, 14], i.e., the boundary $\partial F_q(\mathrm{diag}\{a,b\})$ coincides with the ellipse

$$
\left\{\frac{q(a+b)}{2}+\frac{a-b}{2}\cos\theta+\frac{(-\mathrm{i})\sqrt{1-q^2}(a-b)}{2}\sin\theta : 0\le\theta\le 2\pi\right\}. \tag{37}
$$

By the equations (33) and (37), it follows that Δ_q is the convex hull of the set

$$
D_q \;=\; \{u(\theta,\phi)+\mathrm{i}\,v(\theta,\phi) : \theta,\phi\in[0,2\pi]\}\,, \tag{38}
$$

where

$$
\begin{array}{rcl}
u(\theta,\phi) & = & -\dfrac{\alpha}{2(\alpha^2-1)}\,(2q-\alpha^2q+\alpha^2\cos\phi-\alpha\,q\,\cos\theta\\
& & -\,\alpha\,\cos\theta\,\cos\phi-\alpha\sqrt{1-q^2}\sin\theta\,\sin\phi),\\
v(\theta,\phi) & = & \dfrac{\alpha^2}{2(\alpha^2-1)}\,(q\,\sin\theta+\alpha\sqrt{1-q^2}\sin\phi\\
& & -\sqrt{1-q^2}\cos\theta\,\sin\phi+\sin\theta\,\cos\phi).
\end{array}
$$

Notice also that $\partial\Delta_q$ is contained in D_q, and that for every $\phi\in[0,2\pi]$, we have

$$
\begin{array}{rcl}
u(\theta,\phi)+\mathrm{i}\,v(\theta,\phi) & = & \dfrac{-\alpha\,(2q-\alpha^2q+\alpha^2\cos\phi)}{2(\alpha^2-1)}+\mathrm{i}\,\dfrac{\alpha^3\sqrt{1-q^2}\,\sin\phi}{2(\alpha^2-1)}\\
& & +\,\dfrac{\alpha^2(q+\cos\phi)-\mathrm{i}\,\alpha^2\sqrt{1-q^2}\sin\phi)}{2(\alpha^2-1)}\,e^{\mathrm{i}\theta}
\end{array}
$$

and

$$|\alpha^2(q+\cos\phi) - \mathrm{i}\,\alpha^2\sqrt{1-q^2}\sin\phi| \;=\; \alpha^2(1+q\,\cos\phi).$$

As a consequence, the set D_q is represented as the union of a family of circles

$$D_q \;=\; \{X(t,\phi)+\mathrm{i}\,Y(t,\phi) : t,\,\phi\in[0,2\pi]\}\,,$$

where

$$\begin{aligned} X(t,\phi) &= \frac{\alpha^2(1+q\,\cos\phi)}{2(\alpha^2-1)}\,\cos t + \frac{\alpha\,(-2q+\alpha^2 q-\alpha^2\cos\phi)}{2(\alpha^2-1)}\,,\\ Y(t,\phi) &= \frac{\alpha^2(1+q\,\cos\phi)}{2(\alpha^2-1)}\,\sin t + \frac{\alpha^3\sqrt{1-q^2}\sin\phi}{2(\alpha^2-1)}\,. \end{aligned}$$

We remark that the centers of these circles lie on the ellipse

$$\left\{X_0(\phi)+\mathrm{i}\,Y_0(\phi) \;=\; \frac{\alpha^3(q-\cos\phi)-2q\,\alpha}{2(\alpha^2-1)} + \mathrm{i}\,\frac{\alpha^3\sqrt{1-q^2}\sin\phi}{2(\alpha^2-1)} : 0\le\phi\le 2\pi\right\}.$$

The boundary of Δ_q is the envelope of this 1-parameter family of circles and next we obtain a parametric representation of this boundary by a geometric method. The radii

$$r(\phi) \;=\; \frac{\alpha^2(1+q\,\cos\phi)}{2(\alpha^2-1)} \;\;;\quad 0\;\le\;\phi\;\le\;2\pi$$

of the above circles with centers at $X_0(\phi)+\mathrm{i}\,Y_0(\phi)$ $(\phi\in[0,2\pi])$ satisfy

$$\begin{aligned} |r(\phi_1)-r(\phi_2)| &= \frac{q\,\alpha^2}{2(\alpha^2-1)}\,|\cos\phi_1-\cos\phi_2|\\ &< \frac{\alpha^3}{2(\alpha^2-1)}\,|\cos\phi_1-\cos\phi_2| = |X_0(\phi_1)-X_0(\phi_2)|\\ &\le |(X_0(\phi_1)+\mathrm{i}\,Y_0(\phi_1))-(X_0(\phi_2)+\mathrm{i}\,Y_0(\phi_2))| \end{aligned}$$

for $0\le\phi_1<\phi_2\le\pi$ or $\pi\le\phi_1<\phi_2\le 2\pi$.

For a moment, identify the Gaussian plane $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$, denote $r_1=r(\phi_1)$ and $r_2=r(\phi_2)$, and assume that the two circles

$$(u-u_1)^2+(v-v_1)^2 \;=\; r_1^2 \quad\text{and}\quad (u-u_2)^2+(v-v_2)^2 \;=\; r_2^2$$

have an intersection with a nonempty interior, i.e.,

$$|r_1-r_2| \;<\; \sqrt{(u_1-u_2)^2+(v_1-v_2)^2} \;<\; r_1+r_2.$$

Then they meet at two distinct (real) points (u_0,v_0) and (u_0',v_0'), which lie on the straight line

$$2\,(u_2-u_1)\,u+(v_2-v_1)\,v+(u_1^2-u_2^2-r_1^2+v_1^2-v_2^2+r_2^2) \;=\; 0.$$

The (real) coordinates u_0 and u_0' are roots of the quadratic equation (with respect to u)

$$\begin{aligned} 0 \;=\; & 4[(u_1-u_2)^2+(v_1-v_2)^2]\,u^2+4\,[-u_1^3-u_2^3+u_1^2u_2+u_1u_2^2 \\ & -(u_1+u_2)(v_1-v_1)^2+(r_1^2-r_2^2)(u_1-u_2)]\,u+(u_1^2-u_2^2)^2 \\ & +2(u_1^2+u_2^2)(v_1^2+v_2^2)-4(u_1^2+u_2^2+v_1^2+v_2^2)\,v_1\,v_2+v_1^4+6v_1^2v_2^2 \\ & +v_2^4+2(r_2^2-r_1^2)(u_1^2-u_2^2)-2(r_1^2+r_2^2)(v_1-v_2)^2+(r_1^2-r_2^2)^2. \end{aligned}$$

The coordinates v_0, v_0' also satisfy a similar equation. By using these equations, one can determine the two points

$$X(\phi_1,\phi_2)+\mathrm{i}\,Y(\phi_1,\phi_2) \quad \text{and} \quad \tilde{X}(\phi_1,\phi_2)+\mathrm{i}\,\tilde{Y}(\phi_1,\phi_2),$$

where the two circles

$$\big\{X+\mathrm{i}\,Y : X,Y\in\mathbb{R},\, (X-X_0(\phi_1))^2+(Y-Y_0(\phi_1))^2=r(\phi_1)^2\big\}$$

and

$$\big\{X+\mathrm{i}\,Y : X,Y\in\mathbb{R},\, (X-X_0(\phi_2))^2+(Y-Y_0(\phi_2))^2=r(\phi_2)^2\big\}$$

$(\phi_1\neq\phi_2)$ meet. Consider now the limits of these two points for $\phi_1\to\vartheta$ and $\phi_2\to\vartheta$,

$$X(\vartheta)+\mathrm{i}\,Y(\vartheta) \quad \text{and} \quad \tilde{X}(\vartheta)+\mathrm{i}\,\tilde{Y}(\vartheta).$$

One can see that $X(\vartheta)$ and $Y(\vartheta)$ are given by (4) and (5), respectively, and that $\tilde{X}(\vartheta)$ and $\tilde{Y}(\vartheta)$ are given by

$$\tilde{X}(\vartheta) \;=\; \frac{2\alpha\,(2-\alpha^2)\,q^2\cos\vartheta+[3\alpha^3+\alpha\,(\alpha^2-1)\cos(2\vartheta)-3\alpha]\,q-2\alpha^3\cos\vartheta}{4(\alpha^2-1)(1-q\cos\vartheta)} \\ \underline{+\sqrt{2}\,\alpha\,\cos\vartheta\sqrt{1-q^2}\sqrt{2\alpha^2-(\alpha^2+1)\,q^2-(\alpha^2-1)\,q^2\cos(2\vartheta)}}$$

and

$$\tilde{Y}(\vartheta) \;=\; \frac{2\alpha^3\sin\vartheta\sqrt{1-q^2}\,-\alpha\,(\alpha^2-1)\,q\,\sqrt{1-q^2}\sin(2\vartheta)}{4\,(\alpha^2-1)(1-q\,\cos\vartheta)} \\ \underline{-\sqrt{2}\,\alpha\,\sin\vartheta\,\sqrt{2\alpha^2-(\alpha^2+1)\,q^2-(\alpha^2-1)\,q^2\cos(2\vartheta)}}\,.$$

We take the outer envelope $X(\vartheta)+\mathrm{i}\,Y(\vartheta)$ of the 1-parameter family of the circles. Thus, the boundary of the q-numerical range $F_q(T)$ is parameterized by (34). For the polynomial $\mathcal{L}(t,u,v)$ in (6), the equation $\mathcal{L}(1,u,v)=0$ for $u=X(\vartheta)$ and $v=Y(\vartheta)$ can be obtained by the elimination of the variable ϑ, but there is also an alternative method for the construction of $\mathcal{L}(t,u,v)$. Indeed, for every θ, the variables $u=u(\theta,\phi)$ and $v=v(\theta,\phi)$ in (38) are roots of the polynomial

$$
\begin{aligned}
&M(u,v;\theta) \;=\\
&\alpha^2(\alpha^2+4\alpha^4+\alpha^6)-(4+6\alpha^2+2\alpha^6)q^2+(2+9\alpha^2-6\alpha^4+\alpha^6)q^4\\
&+4\alpha(\alpha^2-1)\,q\,(-2-2\alpha^2+\alpha^4+q^2+3\alpha^2q^2-\alpha^4q^2)u\\
&+2(\alpha^2-1)^2(-2-2\alpha^2+q^2+2\alpha^2q^2)u^2+2(\alpha^2-1)^2(-2-2\alpha^2+q^2)v^2\\
&+4\alpha[\alpha^2(1-q^2)(-\alpha^2-\alpha^4+3q^2-\alpha^2q^2)-\alpha(\alpha^2-1)(\alpha^2-5)(1-q^2)q\,u\\
&+2(\alpha^2-1)^2(1-q^2)u^2+2(\alpha^2-1)^2v^2]\cos\theta\\
&+4\alpha(\alpha^2-1)\,q\,[\alpha(1+\alpha^2-3q^2+\alpha^2q^2)v+2q(1-\alpha^2)\,u\,v]\sin\theta\\
&+[2\alpha^2(\alpha^2-q^2)^2-4\alpha(\alpha^2-1)(\alpha^2-q^2)\,q\,u+2q^2(\alpha^2-1)^2(u^2-v^2)]\cos(2\theta)\\
&+4(\alpha^2-1)\,q\,[-\alpha(\alpha^2-q^2)v+q(\alpha^2-1)u\,v]\sin(2\theta).
\end{aligned}
$$

If we let $\mu=\tan(\theta/2)$, then

$$\cos\theta \;=\; \frac{1-\mu^2}{1+\mu^2} \quad \text{and} \quad \sin\theta \;=\; \frac{2\mu}{1+\mu^2}\,,$$

and the polynomial

$$(1+\mu^2)^2\,M(u,v;\theta)$$

is a quartic polynomial in μ. To obtain the equation of the envelope of the 1-parameter family of ellipses $\{M(u,v;\theta)=0 : 0\le\theta\le 2\pi\}$, we take the discriminant of the polynomial $(1+\mu^2)^2M(u,v;\theta)$ in μ [3, 19]. Then it follows that this discriminant is a constant multiple of the polynomial

$$(\alpha^2-1)^{12}(1-q^2)^2[(u-\alpha\,q)^2+v^2]^3\mathcal{L}(1,u,v),$$

where the point $\alpha\,q\in\mathbb{C}$ is a focus of the ellipses.

(II) The equation $\mathcal{K}(t,u,v)=0$ of the dual curve of the curve $\mathcal{L}(t,u,v)=0$ is obtained by considering the dual curve $G(u,v;\theta)=0$ of the curve $M(u,v;\theta)=0$. The 1-parameter family of the polynomials $G(u,v;\theta)$ is given by

$$
\begin{aligned}
G(u,v;\theta) \;=\;& 4(\alpha^2-1)^2+4q\,\alpha(\alpha^2-1)(\alpha^2-2)u+\alpha^2[-\alpha^2-\alpha^4\\
&+(4-3\alpha^2+\alpha^4)q^2]u^2-(\alpha^4+\alpha^6)(1-q^2)v^2\\
&+2\alpha^2[2q(\alpha^2-1)u+\alpha(\alpha^2-2q^2+\alpha^2q^2)u^2\\
&+\alpha^3(1-q^2)v^2]\cos\theta+4\alpha^2(\alpha^2-1)\,q\,(v+\alpha\,q\,u\,v)\sin\theta.
\end{aligned}
$$

As above, let $\mu=\tan(\theta/2)$. Then the polynomial $(1+\mu^2)G(u,v;\theta)$ is quadratic in μ, and its discriminant (with respect to μ) is $-4(\alpha^2-1)^2\mathcal{K}(1,u,v)$, where $\mathcal{K}(t,u,v)$ is given by (35). A straightforward computation implies equation (36) for the matrix B_q in (7). By this equation and [9], it follows that $\Delta_q=F(B_q)$, completing the proof of the theorem. □

We remark that the polynomial $\mathcal{K}(t,u,v)$ in (35) is irreducible in $\mathbb{C}[t,u,v]$. The complex projective curve $\mathcal{K}(t,u,v)=0$ has a pair of simple cusps

$$\left(1,\,-\frac{1}{\alpha\,q}\,,\,\pm\,\frac{\mathrm{i}}{\alpha\,q}\right),$$

and has no singular points other than these two points. Hence, $\mathcal{K}(t,u,v)=0$ is a quartic elliptic curve.

5 The proof of Theorem 1 (II)

In this section, we present a proof of the second part of Theorem 1. Suppose that

$$\alpha > 2 \quad \text{and} \quad 0 < q \le \frac{\alpha(\alpha-2)}{\alpha^2-2\alpha+2}.$$

First consider the q-numerical range of a 2×2 diagonal matrix,

$$\begin{aligned} F_q(\mathrm{diag}\{a,b\}) &= \{\, q\,(u+\mathrm{i}\,v)+\sqrt{1-q^2}\,z\,\sqrt{w-u^2-v^2}\,: \\ &\qquad (u+\mathrm{i}\,v,w)\in W(\mathrm{diag}\{a,b\},\mathrm{diag}\{|a|^2,|b|^2\}),\ z\in\mathbb{C},\ |z|\le 1\} \\ &= \{\, q\,[(1-s)\,a+s\,b] \\ &\qquad +\sqrt{1-q^2}\,z\,\sqrt{(1-s)\,|a|^2+s\,|b|^2-|(1-s)\,a+s\,b|^2}\,: \\ &\qquad z\in\mathbb{C},\ |z|\le 1,\ 0\le s\le 1\}. \end{aligned}$$

This set is the union of a family of (circular) disks C_s $(s\in[0,1])$ whose centers $q\,[(1-s)\,a+s\,b]$ lie on the line segment $[a,b]$. The boundary of $F_q(\mathrm{diag}\{a,b\})$ is an ellipse with foci at $q\,a$ and $q\,b$, and with eccentricity q [10, 13, 14]. If $(1+q)/2\le s\le 1$ or $0\le s\le (1-q)/2$, then the disk C_s does not intersect $\partial F_q(\mathrm{diag}\{a,b\})$. As a consequence,

$$\begin{aligned} F_q(\mathrm{diag}\{a,b\}) &= \{\, q\,[(1-s)\,a+s\,b] \\ &\qquad +\sqrt{1-q^2}\,z\,\sqrt{(1-s)\,|a|^2+s\,|b|^2-|(1-s)\,a+s\,b|^2}\,: \\ &\qquad z\in\mathbb{C},\ |z|\le 1,\ (1-q)/2\le s\le (1+q)/2\}. \end{aligned}$$

Thus, for every real s_1 and s_2 such that

$$0 \le s_1 \le \frac{1-q}{2} \quad \text{and} \quad \frac{1+q}{2} \le s_2 \le 1,$$

it is clear that

$$\begin{aligned} F_q(\mathrm{diag}\{a,b\}) &= \{\, q\,[(1-s)a+s\,b] \\ &\qquad +\sqrt{1-q^2}\,z\,\sqrt{(1-s)|a|^2+s\,|b|^2-|(1-s)\,a+s\,b|^2}\,: \\ &\qquad z\in\mathbb{C},\ |z|\le 1,\ s_1\le s\le s_2\}. \end{aligned}$$

As it is already mentioned in Section 3, any point $(\xi(s)+\mathrm{i}\,\eta(s),\zeta(s))$ of a line segment $\mathcal{E}_\theta$ $(0\le\theta\le 2\pi)$ defined by (24)-(27) lies on the cone part of the upper surface of the Davis-Wielandt shell $W(A(\alpha),A(\alpha)^*A(\alpha))$ if and only if

$$0 \le s \le \frac{(\alpha^2-1)^2}{\alpha^4-\alpha^2-2\alpha\cos\theta+2} \quad (=s_\theta)$$

and $\theta \in [\arctan(\sqrt{\alpha^2-1}), 2\pi - \arctan(\sqrt{\alpha^2-1})]$. For $\theta = \pi$, the quantity s_θ attains its minimum

$$s_\pi = \frac{\alpha^2 - 2\alpha + 1}{\alpha^2 - 2\alpha + 2} = \frac{(\alpha-1)^2}{(\alpha-1)^2+1} > \frac{1}{2}\,.$$

By the assumption

$$0 < q \le \frac{\alpha(\alpha-2)}{\alpha^2 - 2\alpha + 2} = 2s_\pi - 1,$$

it follows that $s_\pi \ge (1+q)/2$. Hence, for any $\theta \in [\arctan(\sqrt{\alpha^2-1}), 2\pi - \arctan(\sqrt{\alpha^2-1})]$, all the points

$$(\xi(s) + \mathrm{i}\,\eta(s), \zeta(s)) \quad ; \quad 0 \le s \le \frac{1+q}{2}$$

lie on the cone part of the upper surface of the Davis-Wielandt shell of $A(\alpha)$. The ellipsoid part of the upper surface of $W(A(\alpha), A(\alpha)^* A(\alpha))$ is dominated by the cone part, and keeping in mind the above discussion on the q-numerical range of a 2×2 diagonal matrix (see also (33)), we can relax the condition

$$0 \le s \le \frac{\alpha^4 - 2\alpha^2 + 1}{\alpha^4 - \alpha^2 - 2\alpha\cos\theta + 2}$$

in (28) by the condition

$$0 \le s \le 1.$$

Hence, $F_q(A(\alpha))$ coincides with the domain Δ_q in (29). By the first part of Theorem 3, the proof of Theorem 1 (II) is complete. $\square$

Note that if $q = \cos\phi$ for some $\phi \in (0, \pi/2)$, and if $\tau = \tan(\phi/2)$, then $0 < \tau < 1$ and the condition

$$0 < q \le \frac{\alpha(\alpha-2)}{\alpha^2 - 2\alpha + 2}$$

can be written as

$$\tau \ge \frac{1}{\alpha - 1}\,.$$

6 Some lemmas

Let $A(\alpha)$ $(\alpha > 1)$ be the 3×3 reducible matrix in (3), and let $0 < q < 1$. Next we obtain five somewhat technical lemmas, which are necessary for the proof of Theorem 1 (I),(III). These results describe the function h in (11) and the function $\Phi(u + \mathrm{i}\,v) = \sqrt{h(u + \mathrm{i}\,v) - u^2 - v^2}$.

Lemma 4 *Consider the function h in (11) defined by $h(u + \mathrm{i}\,v) = w$, where $(u + \mathrm{i}\,v, w)$ is a point of the upper surface of $W(A(\alpha), A(\alpha)^* A(\alpha))$. This function is continuously differentiable on the interior of $F(A(\alpha))$, and the cone part and the ellipsoid part of the function $w = h(u + \mathrm{i}\,v)$ have common partial derivatives on the arc $\mathcal{C}_{up}$ in (20)-(22).*

Proof The intersection of the ellipsoid and the cone part of the surface of the Davis-Wielandt shell $W(A(\alpha), A(\alpha)^*A(\alpha))$ is the curve $\mathcal{C}$ in (19). The projection $\mathcal{C}_{up}$ of the upper part of this curve onto the (u,v)-plane is given by (20)-(22). By the discussion in Section 3, this arc has a parameter representation

$$u \;=\; \xi(s;\theta) \quad \text{and} \quad v \;=\; \eta(s;\theta)$$

given by (25) and (26), respectively, when the real s takes values sufficiently close to

$$s_\theta \;=\; \frac{(\alpha^2-1)^2}{\alpha^4-\alpha^2-2\alpha\,\cos\theta+2}\,.$$

Let θ take values in the interval $(\theta_0, 2\pi-\theta_0)$, where $0<\theta_0<\pi/2$ and $\cos\theta_0 = 1/\alpha$. Its Jacobian $\partial(u,v)\,/\,\partial(s,\theta)$ on the arc $\mathcal{C}_{up}$ is given by

$$\begin{aligned}
\frac{\partial(u,v)}{\partial(s,\theta)} &= \frac{\alpha^4\,(1-\alpha\,\cos\theta)\,s}{(\alpha^2-1)^2} \\
&= \frac{\alpha^4(1-\alpha\,\cos\theta)}{\alpha^4-\alpha^2-2\alpha\,\cos\theta+2} \;>\; 0.
\end{aligned}$$

On the ellipsoid part of the upper surface of $W(A(\alpha), A(\alpha)^*A(\alpha))$, the function h coincides with

$$\begin{aligned}
g(s,\theta) &= 2+2\sqrt{1-u^2-v^2} \\
&= 2+\frac{2\sqrt{\alpha^2(1-s)-1}}{(\alpha^2-1)}\,\sqrt{-\alpha^4(1-s)-2\alpha^3 s\cos\theta+\alpha^2(2+s)-1}\,,
\end{aligned}$$

and its partial derivatives are given by

$$\frac{\partial g}{\partial s} \;=\; -\frac{2\alpha^3}{\alpha^2-1}\,\frac{-\alpha^3(1-s)+\alpha^2(1-2s)\cos\theta+\alpha(1+s)-\cos\theta}{\sqrt{\alpha^2(1-s)-1}\sqrt{-\alpha^4(1-s)-2\alpha^3 s\cos\theta+\alpha^2(2+s)-1}}\,,$$

and

$$\frac{\partial g}{\partial\theta} \;=\; \frac{2\alpha^3 s\,\sin\theta}{\alpha^2-1}\;\frac{\sqrt{\alpha^2(1-s)-1}}{\sqrt{-\alpha^4(1-s)-2\alpha^3 s\cos\theta+\alpha^2(2+s)-1}}\,.$$

On the cone part, the function h is written

$$f(s,\theta) \;=\; \alpha^2-\frac{\alpha^3 s\,(\alpha^3-3\alpha+2\cos\theta)}{(\alpha^2-1)^2}\,.$$

Its partial derivatives are given by

$$\frac{\partial f}{\partial s} \;=\; -\frac{\alpha^3}{(\alpha^2-1)^2}\,(\alpha^3-3\alpha+2\cos\theta) \quad \text{and} \quad \frac{\partial f}{\partial\theta} \;=\; \frac{2\alpha^3 s\,\sin\theta}{(\alpha^2-1)^2}\,.$$

Under the condition $s=s_\theta$, we have $\partial f\,/\,\partial s = \partial g\,/\,\partial s$ and $\partial f\,/\,\partial\theta = \partial g\,/\,\partial\theta$. These equations are deduced from the relations

$$\frac{2(\alpha^2-1)}{\alpha^3-3\alpha+2\cos\theta}\,[-\alpha^3(1-s)+\alpha^2(1-2s)\cos\theta+\alpha(1+s)-\cos\theta]$$

$$= \frac{2(\alpha^2-1)^2(1-\alpha\cos\theta)}{\alpha^4-\alpha^2-2\alpha\cos\theta+2} > 0,$$

$$[\alpha^2(1-s)-1]\,[-\alpha^4(1-s)-2\alpha^3 s\cos\theta+\alpha^2(2+s)-1]$$
$$= 4\,\frac{(\alpha^2-1)^2(1-\alpha\cos\theta)^2}{(\alpha^4-\alpha^2-2\alpha\cos\theta+2)^2} > 0$$

and

$$\frac{\alpha^2(1-s)-1}{-\alpha^4(1-s)-2\alpha^3 s\cos\theta+\alpha^2(2+s)-1} = \frac{1}{(\alpha^2-1)^2} > 0$$

(always under the assumption that $s = s_\theta$).

Notice that the ellipsoid part

$$w = 2+2\sqrt{1-u^2-v^2}\ ;\quad u^2+v^2\le 1$$

of the upper surface of $W(A(\alpha), A(\alpha)^*A(\alpha))$ is continuously differentiable on the interior of the unit disk. It is also clear that the cone part of the upper surface of the shell defines a continuously differentiable function on the domain surrounded by the curve $\mathcal{C}_{up}$ and the two lines

$$\left\{u+\mathrm{i}\,v : \frac{1}{\alpha}\le u\le\alpha,\ v=\pm\frac{u-\alpha}{\sqrt{\alpha^2-1}}\right\}.$$

The proof is complete. $\square$

Lemma 5 *Consider the family of line segments*

$$\left\{\mathcal{E}_\theta : \arctan(\sqrt{\alpha^2-1})\le\theta\le 2\pi-\arctan(\sqrt{\alpha^2-1})\right\}$$

defined by (24)-(27), and let Ω be the convex hull of the circle (17) and the point α. Suppose that h_1 is the concave function on Ω defined by $h_1(\xi+\mathrm{i}\,\eta)=\zeta$, where $(\xi+\mathrm{i}\,\eta,\zeta)\in\mathcal{E}_\theta$ for some $0<\theta<2\pi$ satisfying $-1\le\cos\theta\le 1/\alpha$. Let also Φ_1 be the concave function defined by

$$\Phi_1(\xi+\mathrm{i}\,\eta) = \sqrt{h_1(\xi+\mathrm{i}\,\eta)-\xi^2-\eta^2}.$$

(i) *The function Φ_1 is continuously differentiable in the interior of Ω, and the norm of its gradient is given by*

$$\begin{aligned}\|\mathrm{grad}\Phi_1(\xi(s;\theta)+\mathrm{i}\,\eta(s;\theta))\|^2 &= (\Phi_\xi^2+\Phi_\eta^2)(\xi(s;\theta)+\mathrm{i}\,\eta(s;\theta))\\ &= \frac{[1-4s(1-s)](1-\alpha\cos\theta)^2+\alpha^2\sin^2\theta}{4s(1-s)(1-\alpha\cos\theta)^2}.\end{aligned}\tag{39}$$

If we denote by $\Psi(s;\cos\theta)$ the right-hand part of (39), then

$$\frac{\partial\Psi(s;\psi)}{\partial s} = \frac{(2s-1)(\alpha^2-2\alpha\,\psi+1)}{4s^2(1-s)^2(1-\alpha\,\psi)^2} > 0\tag{40}$$

when $1/2 < s < 1$ *and* $-1 \le \psi < 1/\alpha$, *and*

$$\frac{\partial \Psi(s;\psi)}{\partial \psi} \;=\; \frac{\alpha^2(\alpha-\psi)}{2s\,(1-s)\,(1-\alpha\,\psi)^3} \;>\; 0 \tag{41}$$

when $0 < s < 1$ *and* $-1 \le \psi < 1/\alpha$.

(ii) *Any point* $\xi(s;\theta)+\mathrm{i}\,\eta(s;\theta) \in \mathrm{Int}\Omega$ *satisfying* $\|\mathrm{grad}\Phi_1(\xi(s;\theta)+\mathrm{i}\,\eta(s;\theta))\|^2 = q^2/(1-q^2)$ *is characterized by the equation*

$$\begin{aligned} M_q(s;\theta) \;&=\; 4\alpha^2 s(1-s)\cos^2\theta + [2\alpha\,(1-q^2) - 8\alpha\,s(1-s)]\cos\theta \\ &\quad -(\alpha^2+1)(1-q^2) + 4s\,(1-s) \;=\; 0, \end{aligned} \tag{42}$$

which has a solution

$$\cos\theta \;=\; \frac{q^2-1+4s(1-s) - \sqrt{1-q^2}\sqrt{1-q^2+4(\alpha^2-1)\,s\,(1-s)}}{4\alpha\,s(1-s)} \tag{43}$$

for $(1-q)/2 \le s \le (1+q)/2$. *Moreover, if* $\mathcal{C}_q$ *is the curve consisting of the points* $\xi(s;\theta)+\mathrm{i}\,\eta(s;\theta)$, *which satisfy (42), then for any* $0 < q_1 < q_2 < 1$, $\mathcal{C}_{q_1}$ *is contained in the open set surrounded by* $\mathcal{C}_{q_2}$.

(iii) *If* $\xi_0 + \mathrm{i}\,\eta_0$ *is a boundary point of the domain* Ω, *then there exists an interior point* $\xi_1 + \mathrm{i}\,\eta_1$ *of* Ω *such that*

$$\Phi_1(\xi_1+\mathrm{i}\,\eta_1) - \Phi_1(\xi_0+\mathrm{i}\,\eta_0) \;>\; \frac{1}{\sqrt{1-q^2}}\,|(\xi_1+\mathrm{i}\,\eta_1)-(\xi_0+\mathrm{i}\,\eta_0)|. \tag{44}$$

Proof (i),(ii) We substitute $w = u^2+v^2+z^2$ into the equation $\mathcal{G}(u,v,w)=0$ of the cone (14). Then we get an implicit expression of the function $z = \Phi_1(u,v)$,

$$\begin{aligned} \mathcal{G}_0(u,v,z) \;&=\; (\alpha^2-1)\,z^4 + 2[(u-\alpha)((\alpha^2-1)u+\alpha) + (\alpha^2-1)v^2]z^2 \\ &\quad +[(u-\alpha)^2+v^2]\,[(\alpha^2-1)(u^2+v^2)+2\alpha\,u-\alpha^2] \;=\; 0. \end{aligned}$$

This equation has 2 positive roots in the domain $\mathcal{D}$ in (23). The function $\Phi_1(u+\mathrm{i}\,v)$ is the greatest root, and hence,

$$\Phi_1(u+\mathrm{i}\,v) \;=\; \sqrt{\frac{\alpha^2+\alpha\,(\alpha^2-2)\,u-(\alpha^2-1)(u^2+v^2)+\alpha^2\sqrt{(u-\alpha)^2-(\alpha^2-1)\,v^2}}{\alpha^2-1}}\,. \tag{45}$$

Straightforward computations imply that $\|\mathrm{grad}\Phi_1(u+\mathrm{i}\,v)\|^2$ equals

$$\frac{\Psi_0(u,v)}{4(\alpha^2-1)(\alpha^2+\alpha\,(\alpha^2-2)\,u-(\alpha^2-1)(u^2+v^2)+\alpha^2\sqrt{(u-\alpha)^2-(\alpha^2-1)\,v^2})}\,,$$

where

$$\Psi_0(u,v) \;=\; v^2\left[-2(\alpha^2-1) - \frac{\alpha^2(\alpha^2-1)}{\sqrt{(u-\alpha)^2-(\alpha^2-1)v^2}}\right]^2$$

$$+\left[2\alpha-\alpha^3+2(\alpha^2-1)\,u+\frac{\alpha^2(\alpha-u)}{\sqrt{(u-\alpha)^2-(\alpha^2-1)\,v^2}}\right]^2.$$

Substituting now (see (25) and (26))

$$u\;=\;\xi(s;\theta)\;=\;\alpha-\frac{s\,\alpha^2(\alpha-\cos\theta)}{\alpha^2-1}\quad\text{ and }\quad v\;=\;\eta(s;\theta)\;=\;\frac{s\,\alpha^2(\sin\theta)}{\alpha^2-1},$$

we obtain (39), and multiplying $\|\mathrm{grad}\Phi_1(\xi(s;\theta)+\mathrm{i}\,\eta(s;\theta))\|^2-q^2/(1-q^2)$ by $-4(1-q^2)\,s\,(1-s)\,(1-\alpha\,\cos\theta)^2$, we obtain (42). The relations (40) and (41) are also obtained by straightforward computations. Furthermore, the solution $\chi\;=\;\cos\theta$ of the equation (42) is given by (43) and attains its maximum at $s=1/2$. Since $-1\leq\cos\theta$, the solution of the equation (42) exists if and only if $(1-q)/2\leq s\leq(1+q)/2$. Finally, by (40) and (41), it follows that for any $0<q_1<q_2<1$, the curve $\mathcal{C}_{q_1}$ is contained in the open set surrounded by the curve $\mathcal{C}_{q_2}$.

(iii) Assume that $\xi_0+\mathrm{i}\,\eta_0=\xi(s_0;\theta_0)+\mathrm{i}\,\eta(s_0;\theta_0)\;\in\;\partial\Omega$, where $0<s_0<1$ and $\sin\theta_0=\sqrt{\alpha^2-1}\,/\,\alpha$. Then $0<\xi_0<\alpha$ and $\eta_0=(\alpha-\xi_0)\,/\,\sqrt{\alpha^2-1}$. We choose $\xi_1=\xi_0$ and $\eta_1=(1-\epsilon)(\alpha-\xi_0)\,/\,\sqrt{\alpha^2-1}$ for some $\epsilon>0$. Then by (45), we have

$$\Phi_1(\xi_0+\mathrm{i}\,\eta_0)\;=\;\sqrt{\frac{\alpha^2\xi_0\,(\alpha-\xi_0)}{\alpha^2-1}}\,.$$

The function

$$k(\xi,\eta)\;=\;\alpha^2+\alpha\,(\alpha^2-2)\,\xi-(\alpha^2-1)(\xi^2+\eta^2)+\alpha^2\sqrt{(\xi-\alpha)^2-(\alpha^2-1)\,\eta^2}$$

satisfies

$$k(\xi_0,\eta_1)-k(\xi_0,\eta_0)\;=\;\alpha^2\sqrt{2\,\epsilon}\,(\alpha-\xi_0)+\{\text{higher order terms of }\epsilon\},$$

and the equation

$$\begin{aligned}\sqrt{\alpha^2\xi_0\,(\alpha-\xi_0)+\delta}\quad&=\quad\sqrt{\alpha^2\xi_0\,(\alpha-\xi_0)}+\frac{\delta}{2\,\sqrt{\alpha^2\xi_0\,(\alpha-\xi_0)}}\\&\qquad+\{\text{higher order terms of }\delta\}\end{aligned}$$

holds for $\delta>0$. Moreover, the distance between the points $\xi_0+\mathrm{i}\,\eta_0$ and $\xi_0+\mathrm{i}\,\eta_1$ is $\epsilon\,(\alpha-\xi_0)\,/\,\sqrt{\alpha^2-1}$, and thus, (44) holds for sufficiently small $\epsilon>0$. The case $\sin\theta_0=-\sqrt{\alpha^2-1}\,/\,\alpha$ can be treated similarly.

Finally, suppose that $\xi_0+\mathrm{i}\,\eta_0=\xi(1,\theta)+\mathrm{i}\,\eta(1,\theta)\;\in\;\partial\Omega\;(0\leq\theta\leq2\pi)$. In this case, $\Phi_1(\xi_0+\mathrm{i}\,\eta_0)=0$, and we choose

$$\xi_1\;=\;\xi(1-\epsilon;\theta)\quad\text{ and }\quad\eta_1\;=\;\eta(1-\epsilon;\theta)$$

for some $\epsilon>0$. Then the distance between the points $\xi_0+\mathrm{i}\,\eta_0$ and $\xi_1+\mathrm{i}\,\eta_1$ is

$$|(\xi_0+\mathrm{i}\,\eta_0)-(\xi_1+\mathrm{i}\,\eta_1)|\;=\;\frac{\epsilon\,\alpha^2}{\alpha^2-1}\,\sqrt{(\alpha-\cos\theta)^2+\sin^2\theta}\,.$$

Note also that for $s = 1-\epsilon$, the last square root of the function (see (45))

$$\Phi_1(\xi + \mathrm{i}\,\eta) \;=\; \frac{\alpha^2}{\alpha^2-1}\,\sqrt{\alpha^2 - 2\alpha\,\cos\theta + 1}\,\sqrt{s(1-s)}$$

satisfies

$$\sqrt{(1-\epsilon)(1-(1-\epsilon))} \;=\; \sqrt{\epsilon} + \{\text{higher order terms of } \epsilon\}.$$

Hence, the relation (44) holds for sufficiently small $\epsilon > 0$. The case $s = 0$ can be treated similarly. The proof is complete. $\square$

Corollary 6 *Suppose that Ω is the convex hull of the circle (17) and the point α, and consider the (connected and closed) curve*

$$\mathcal{C}_q \;=\; \left\{ \xi(s;\theta) + \mathrm{i}\,\eta(s;\theta) : \frac{1-q}{2} \le s \le \frac{1+q}{2}\,,\ \cos\theta \ \textit{ in (43)} \right\}.$$

Then the compact convex set Δ_q defined by (29) satisfies the relation

$$\partial\Delta_q \subset \{\, q\,(u+\mathrm{i}\,v) + \sqrt{1-q^2}\,z\,\Phi_1(u+\mathrm{i}\,v) : u+\mathrm{i}\,v \in \mathcal{C}_q,\ z\in\mathbb{C},\ |z| = 1\}. \quad (46)$$

Proof By (iii) of Lemma 5, it follows that for every boundary point $u_0 + \mathrm{i}\,v_0$ of Ω, we can choose an interior point $u_1 + \mathrm{i}\,v_1$ of Ω such the circular disk

$$\left\{ q(u_0+\mathrm{i}\,v_0) + \sqrt{1-q^2}\,z\,\Phi_1(u_0+\mathrm{i}\,v_0) : z\in\mathbb{C},\ |z|\le 1\right\}$$

is a subset of the disk

$$\left\{ q(u_1+\mathrm{i}\,v_1) + \sqrt{1-q^2}\,z\,\Phi_1(u_1+\mathrm{i}\,v_1) : z\in\mathbb{C}, |z|\le 1\right\}.$$

Thus, the corollary follows from Lemma 5 (i), (ii) and [2, Theorem 2]. $\square$

Lemma 7 *Let* $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, *and let h_B be the function on the unit disk $F(B) = \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ u^2+v^2 \le 1\}$ defined by the relation $h_B(u+\mathrm{i}\,v) = w$, where $(u+\mathrm{i}\,v, w)$ is a point of the upper part of the shell $W(B, B^*B)$, i.e., $h_B(u+\mathrm{i}\,v) = 2 + 2\sqrt{1-u^2-v^2}$. Then the function*

$$\Phi_B(u+\mathrm{i}\,v) \;=\; \sqrt{h_B(u+\mathrm{i}\,v) - u^2 - v^2}$$

satisfies

$$\|\mathrm{grad}\Phi_B(u+\mathrm{i}\,v)\|^2 \;=\; \frac{u^2+v^2}{1-u^2-v^2} \quad (47)$$

for every $u+\mathrm{i}\,v \in \mathrm{Int}F(B)$. If $\tilde{\mathcal{C}}_q$ $(0<q<1)$ is the curve consisting of the points $u+\mathrm{i}\,v\in\mathrm{Int}F(B)$, which satisfy $\|\mathrm{grad}\Phi_B(u+\mathrm{i}\,v)\|^2 = q^2/(1-q^2)$, then

$$\tilde{\mathcal{C}}_q \;=\; \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\ u^2+v^2 = q^2\}.$$

Furthermore, for every boundary point $u_0 + \mathrm{i}\, v_0$ *of* $F(B)$, *we have* $\Phi_B(u_0 + \mathrm{i}\, v_0) = 1$, *and the inequality*

$$\Phi_B((1-\epsilon)(u_0 + \mathrm{i}\, v_0)) - \Phi_B(u_0 + \mathrm{i}\, v_0) \; > \; \frac{\epsilon}{\sqrt{1-q^2}} \tag{48}$$

holds for any $q \in (0,1)$ *and for sufficiently small positive* ϵ.

Proof The function Φ_B is given by

$$\Phi_B(u + \mathrm{i}\, v) \; = \; \sqrt{2 + 2\sqrt{1 - u^2 - v^2} - u^2 - v^2},$$

and its partial derivatives satisfy

$$\left(\frac{\partial \Phi_B(u + \mathrm{i}\, v)}{\partial u}\right)^2 \; = \; \frac{u^2}{1 - u^2 - v^2} \quad \text{and} \quad \left(\frac{\partial \Phi_B(u + \mathrm{i}\, v)}{\partial v}\right)^2 \; = \; \frac{v^2}{1 - u^2 - v^2}\,.$$

In this way, we obtain the equation (47), which implies that $\tilde{\mathcal{C}}_q$ coincides with the circle $\{u + \mathrm{i}\, v : u, v \in \mathbb{R},\ u^2 + v^2 = q^2\}$. Furthermore, it is easy to see that for every boundary point $u_0 + \mathrm{i}\, v_0$ of the unit disk $F(B)$,

$$\Phi_B(u_0 + \mathrm{i}\, v_0) \; = \; 1 \quad \text{and} \quad \Phi_B((1-\epsilon)(u_0 + \mathrm{i}\, v_0) \; = \; 1 + \sqrt{2\epsilon - \epsilon^2}$$

$(0 < \epsilon < 1)$. Hence, (48) holds for sufficiently small $\epsilon > 0$. $\square$

Lemma 8 *Let* $\mathcal{D}_c$ *and* $\mathcal{D}_e$ *be the cone part and the ellipsoid part of the domain* $\mathcal{D}$ *in (23), respectively, and let* $\mathcal{C}_{up}$ *be the projection of the upper part of the curve* $\mathcal{C}$ *in (19) onto the* (u, v)*-plane.*

(i) *Suppose that* $\alpha > 1$ *and* $\alpha \neq \sqrt{2}$, *and let*

$$\begin{aligned} \hat{\mathcal{C}} \; &= \; \left\{u + \mathrm{i}\, v : (\alpha^4 + 4)\, u^2 - 8\alpha\, u + (\alpha^2 - 2)^2 v^2 - \alpha^4 + 4\alpha^2 = 0\right\} \\ &= \; \left\{\frac{4\alpha + \alpha^2(2 - \alpha^2)\cos\theta}{\alpha^4 + 4} + \mathrm{i}\, \frac{\alpha^2 \sin\theta}{\sqrt{\alpha^4 + 4}} \; : \; 0 \le \theta \le 2\pi\right\} \end{aligned}$$

be the ellipse, which contains the arc $\mathcal{C}_{up}$ *(in (20) or (22)). On this ellipse, define the function*

$$\Theta(\theta) \; = \; \hat{\Theta}(u, v) \; = \; u^2 + v^2 \; = \; \left(\frac{4\alpha + \alpha^2(2 - \alpha^2)\cos\theta}{\alpha^4 + 4}\right)^2 + \left(\frac{\alpha^2 \sin\theta}{\sqrt{\alpha^4 + 4}}\right)^2.$$

Then the stationary points and the stationary values of Θ *are given by*

$$\hat{\Theta}\left(\frac{1}{\alpha}\,,\, \pm\frac{\sqrt{\alpha^2 - 1}}{\alpha}\right) \; = \; 1\,, \quad \hat{\Theta}\left(\frac{\alpha(2 - \alpha)}{\alpha^2 - 2\alpha + 2}\,,\, 0\right) \; = \; \frac{\alpha^2(2 - \alpha)^2}{(\alpha^2 - 2\alpha + 2)^2} \; < \; 1$$

$$\text{and} \quad \hat{\Theta}\left(\frac{\alpha(2 + \alpha)}{\alpha^2 + 2\alpha + 2}\,,\, 0\right) \; = \; \frac{\alpha^2(2 + \alpha)^2}{(\alpha^2 + 2\alpha + 2)^2} \; < \; 1.$$

(ii) *If* $1 < \alpha < 2$, *then the minimum of the function* $|u + \mathrm{i}\, v| = \sqrt{u^2 + v^2}$ *on the closure of* $\mathcal{D}_c$ *is attained at the point* $u_0 = \alpha(2-\alpha) \,/\, (\alpha^2 - 2\alpha + 2)$ *of* $\mathcal{C}_{up}$, *and if* $\alpha > 2$, *then the minimum of the function* $|u + \mathrm{i}\, v| = \sqrt{u^2 + v^2}$ *on the closure of* $\mathcal{D}_e$ *is attained at the point* $u_0 = \alpha(2-\alpha) \,/\, (\alpha^2 - 2\alpha + 2)$ *of* $\mathcal{C}_{up}$.

Proof (i) The function Θ is written

$$\Theta(\theta) \;=\; \frac{\alpha^2[-4\alpha^4 \cos^2\theta + (-8\alpha^3 + 16\alpha)\cos\theta + \alpha^6 + 4\alpha^2 + 16]}{(\alpha^4 + 4)^2}\,,$$

and thus, its derivative is

$$\Theta'(\theta) \;=\; \frac{8\alpha^3(\alpha^3 \cos\theta + \alpha^2 - 2)\sin\theta}{(\alpha^4 + 4)^2}\,.$$

Hence, the stationary points satisfy

$$\cos\theta \;=\; \frac{2-\alpha^2}{\alpha^3}\,, \quad \text{or} \quad \theta \;=\; 0, \quad \text{or} \quad \theta \;=\; \pi,$$

and the proof of (i) follows readily.

(ii) The quantity $|u + \mathrm{i}\, v|$ is the distance between the point $u + \mathrm{i}\, v$ and the origin. If $1 < \alpha < 2$, then the origin does not belong to the closure of $\mathcal{D}_c$. Thus, for every point $u + \mathrm{i}\, v \in \mathcal{D}_c$, the line segment $[0, u + \mathrm{i}\, v]$ meets the arc $\mathcal{C}_{up}$. Consequently, the minimum of the function $|u + \mathrm{i}\, v| = \sqrt{u^2 + v^2}$ on the closure of $\mathcal{D}_c$ is attained at a point of $\mathcal{C}_{up}$. If $\alpha > 2$, then we can similarly see that the minimum of the function $|u + \mathrm{i}\, v| = \sqrt{u^2 + v^2}$ on the closure of $\mathcal{D}_e$ is attained at a point of $\mathcal{C}_{up}$. Moreover, in both cases, the point $u_0 = \alpha(2-\alpha) \,/\, (\alpha^2 - 2\alpha + 2)$ lies on $\mathcal{C}_{up}$, and using (i), we obtain that u_0 is the closest to origin point of the arc $\mathcal{C}_{up}$ (even for $\alpha = \sqrt{2}$). $\square$

In Lemmas 5 and 7, we considered the cone part and the ellipsoid part of the surface of $W(A(\alpha), A(\alpha)^* A(\alpha))$, respectively. The combination of these two results yields the last lemma of the section.

Lemma 9 *Let* h *be the function in (11) defined by the relation* $h(u + \mathrm{i}\, v) = w$, *where* $(u + \mathrm{i}\, v, w)$ *is a point of the upper surface of the Davis-Wielandt shell of* $A(\alpha)$, *and suppose* $\mathcal{C}_q^{\wedge}$ *is the curve consisting of points* $u + \mathrm{i}\, v \in \mathrm{Int}F(A(\alpha))$ *satisfying*

$$\|\mathrm{grad}\Phi(u + \mathrm{i}\, v)\|^2 \;=\; \frac{q^2}{1 - q^2}\,,$$

where

$$\Phi(u + \mathrm{i}\, v) \;=\; \sqrt{h(u + \mathrm{i}\, v) - u^2 - v^2}.$$

(i) *The curve* $\mathcal{C}_q^{\wedge}$ *is connected, and it is a simply closed curve in* $\mathrm{Int}F(A(\alpha))$. *Moreover, the boundary* $\partial F_q(A(\alpha))$ *lies in the set*

$$\{\, q\,(u + \mathrm{i}\, v) + \sqrt{1 - q^2}\, z\, \Phi(u + \mathrm{i}\, v) : u + \mathrm{i}\, v \in \mathcal{C}_q^{\wedge},\ z \in \mathbb{C},\ |z| = 1\}. \tag{49}$$

(This means that $\partial F_q(A(\alpha))$ *is the outer envelope of the circles in (49).)*

(ii) *The curve $\mathcal{C}_q^\wedge$ meets the (u,v)-projection $\mathcal{C}_{up}$ of the upper part of the curve $\mathcal{C}$ (see (19)-(22)) if and only if*

$$\frac{\alpha\,|\alpha-2|}{\alpha^2-2\alpha+2} \;\le\; q \;<\; 1.$$

Proof (i) First observe that the unique zero point of the function $\|\mathrm{grad}\Phi_1\|^2$ in Lemma 5 is given by

$$u+\mathrm{i}\,v \;=\; \xi(1/2;\pi)+\mathrm{i}\,\eta(1/2;\pi) \;=\; \frac{\alpha(\alpha-2)}{2(\alpha-1)}\,.$$

We compare this point with the real point $u_0=\alpha(2-\alpha)\,/\,(\alpha^2-2\alpha+2)$ of the curve $\mathcal{C}_{up}$, and we have

$$\begin{array}{lll} \xi(1/2;\pi)<u_0 & \text{if} & 1<\alpha<2,\\ \xi(1/2;\pi)=u_0 & \text{if} & \alpha=2,\\ \xi(1/2;\pi)>u_0 & \text{if} & \alpha>2. \end{array}$$

The unique zero point of the function $\|\mathrm{grad}\Phi_B\|^2$ in Lemma 7 is the origin, and

$$\begin{array}{lll} 0>u_0 & \text{if} & 1<\alpha<2,\\ u_0=0 & \text{if} & \alpha=2,\\ u_0<0 & \text{if} & \alpha>2. \end{array}$$

Thus, the unique zero point of the function $\|\mathrm{grad}\Phi\|^2$ is equal to $\xi(1/2;\pi)$ when $\alpha\ge 2$, and it is equal to 0 when $1<\alpha\le 2$.

By the definition of $\mathcal{C}_q^\wedge$, it follows that

$$\mathcal{C}_q^\wedge \;=\; \{u+\mathrm{i}\,v : u,v\in\mathbb{R},\; u+\mathrm{i}\,v\in\mathcal{C}_q\cap\mathcal{D}_c\}\cup\{u+\mathrm{i}\,v : u,v\in\mathbb{R},\; u+\mathrm{i}\,v\in\tilde{\mathcal{C}}_q\cap\mathcal{D}_e\},$$

where $\mathcal{D}_c$ and $\mathcal{D}_e$ are the cone part and the ellipsoid part of the convex set $\mathcal{D}$ in (23), respectively. By Lemma 5, the curve $\mathcal{C}_q$ is connected, closed and symmetric with respect to the real axis. Moreover, by (43), we verify that the real point $\xi((1+q)/2,\pi)=\alpha[(1-q)\alpha-2]\,/\,(2\alpha-2)$ belongs to $\mathcal{C}_q$. By Lemma 7, the curve $\tilde{\mathcal{C}}_q$ coincides with the circle $\{u+\mathrm{i}\,v : u,v\in\mathbb{R},\; u^2+v^2=q^2\}$. If $\mathcal{C}_q^\wedge$ has a point on $\mathcal{C}_{up}$, then the connectedness of $\mathcal{C}_q^\wedge$ follows readily from Lemmas 5 and 7. Therefore, for the proof of the connectedness of $\mathcal{C}_q^\wedge$, it is enough to prove that if $\mathcal{C}_q^\wedge\subset\mathcal{D}_c\cup\mathcal{D}_e$, then $\mathcal{C}_q^\wedge$ lies either in $\mathcal{D}_c$, or in $\mathcal{D}_e$. In particular, we have the following two cases.

Case (a) Suppose that $\mathcal{C}_q^\wedge$ has a point in $\mathcal{D}_c$ and that $\mathcal{C}_q$ does not meet $\mathcal{C}_{up}$. Since $\mathcal{C}_q\cap\mathcal{C}_{up}=\emptyset$, the inequality

$$u_0 \;=\; \frac{\alpha(2-\alpha)}{\alpha^2-2\alpha+2} \;<\; \xi((1+q)/2),\pi) \;<\; \xi(1/2,\pi)$$

holds (recall that s_θ attains its minimum at $\theta = \pi$, and that (43) holds for $(1-q)/2 \le s \le (1+q)/2$), and thus, $\alpha > 2$. Furthermore, the positive quantity $\xi((1+q)/2), \pi) - u_0$ equals to

$$\frac{\alpha^2[\alpha(\alpha-2) - (\alpha^2 - 2\alpha + 2)\,q]}{2(\alpha-1)(\alpha^2 - 2\alpha + 2)}\,,$$

and hence, $q < \alpha(\alpha-2)\,/\,(\alpha^2 - 2\alpha + 2)$. Since $\alpha > 2$, by Lemma 8, the minimum of $u^2 + v^2$ for $u + \mathrm{i}\, v \in \mathcal{D}_e$ is attained at the point

$$u + \mathrm{i}\, v \;=\; u_0 \;=\; \frac{\alpha(2-\alpha)}{\alpha^2 - 2\alpha + 2} \;\in\; \mathcal{C}_{up}.$$

Thus, for every $u + \mathrm{i}\, v \in \mathcal{D}_e$,

$$u^2 + v^2 \;\ge\; \frac{\alpha^2(\alpha-2)^2}{(\alpha^2 - 2\alpha + 2)^2} \;>\; q^2$$

and consequently, $\mathcal{C}_q^{\wedge} = \{u + \mathrm{i}\, v : u, v \in \mathbb{R},\; u + \mathrm{i}\, v \in \mathcal{C}_q \cap \mathcal{D}_c\}$.

Case (b) Assume that $\mathcal{C}_q^{\wedge}$ has a point in $\mathcal{D}_e$ and that $\tilde{\mathcal{C}}_q$ does not meet $\mathcal{C}_{up}$. Then the origin lies in $\mathcal{D}_e$ and $u_0 > 0$. Hence, the inequalities

$$1 \,<\, \alpha \,<\, 2 \quad \text{and} \quad u_0 \;=\; \frac{\alpha(2-\alpha)}{\alpha^2 - 2\alpha + 2} \;>\; q$$

hold. Moreover one can see that the set $\{u + \mathrm{i}\, v : u, v \in \mathbb{R},\; u + \mathrm{i}\, v \in \mathcal{C}_q \cap \mathcal{D}_c\}$ is empty, and thus, $\mathcal{C}_q^{\wedge} = \{u + \mathrm{i}\, v : u, v \in \mathbb{R},\; u + \mathrm{i}\, v \in \tilde{\mathcal{C}}_q \cap \mathcal{D}_e\}$.

By [2, Theorem 2], the boundary of $F_q(A(\alpha))$ lies in the set

$$\{\, q\,(u + \mathrm{i}\, v) + \sqrt{1-q^2}\, z\, \Phi(u + \mathrm{i}\, v) : u + \mathrm{i}\, v \in \mathcal{C}_q^{\wedge},\; z \in \mathbb{C},\; |z| = 1\},$$

and the proof of (i) is complete.

(ii) Consider the cone part $\mathcal{D}_c$ and the ellipsoid part $\mathcal{D}_e$ of $\mathcal{D}$, and suppose that

$$\alpha \,>\, 2 \quad \text{and} \quad 0 \,<\, q \,<\, \frac{\alpha(\alpha-2)}{\alpha^2 - 2\alpha + 2}\,.$$

Then every point $u + \mathrm{i}\, v \in F(A(\alpha))$ of the closure of $\mathcal{D}_e$ satisfies

$$|u + \mathrm{i}\, v| \;\ge\; \frac{\alpha(\alpha-2)}{\alpha^2 - 2\alpha + 2} \;\; (= -u_0) \;>\; q.$$

Thus, $\mathcal{C}_q^{\wedge}$ coincides with the curve $\mathcal{C}_q$ in Lemma 5, and is contained in $\mathcal{D}_c$. As a consequence, $\mathcal{C}_q^{\wedge}$ does not meet $\mathcal{C}_{up}$.

Let us now assume that

$$1 \,<\, \alpha \,<\, 2 \quad \text{and} \quad 0 \,<\, q \,<\, \frac{\alpha(2-\alpha)}{\alpha^2 - 2\alpha + 2}\,.$$

Then for every point $u + \mathrm{i}\, v$ of $\mathcal{D}_c$, we have

$$|u + \mathrm{i}\, v| \;\geq\; \frac{\alpha(2-\alpha)}{\alpha^2 - 2\alpha + 2} \;(= u_0)\; > \; q.$$

Hence, the curve $\mathcal{C}_q^{\wedge}$ is the same as $\tilde{\mathcal{C}}_q$ in Lemma 7, and lies in $\mathcal{D}_e$. Thus, $\mathcal{C}_q^{\wedge}$ does not meet $\mathcal{C}_{up}$.

Suppose that

$$1 \;<\; \alpha \;<\; 2 \quad \text{and} \quad q \;=\; \frac{\alpha(2-\alpha)}{\alpha^2 - 2\alpha + 2}\,,$$

or

$$\alpha \;>\; 2 \quad \text{and} \quad q \;=\; \frac{\alpha(\alpha-2)}{\alpha^2 - 2\alpha + 2}\,.$$

Then $\mathcal{C}_q^{\wedge}$ lies in the closure of $\mathcal{D}_e$ and meets $\mathcal{C}_{up}$ at the point $\alpha(2-\alpha)/(\alpha^2-2\alpha+2)$, or lies in the closure of $\mathcal{D}_c$ and meets $\mathcal{C}_{up}$ at the point $\alpha(\alpha-2)/(\alpha^2-2\alpha+2)$, respectively.

Finally, suppose that

$$1 \;<\; \alpha \;<\; \infty \quad \text{and} \quad \frac{\alpha\,|\alpha-2|}{\alpha^2 - 2\alpha + 2} \;<\; q \;<\; 1.$$

Then the intersection of $\mathcal{C}_{up}$ and the circle $\{u + \mathrm{i}\, v \,:\, u, v \in \mathbb{R},\, u^2 + v^2 \,=\, q^2\}$ consists of two points $u_c \pm \mathrm{i}\, v_c$, where

$$u_c \;=\; \frac{1}{\alpha} + \frac{(2-\alpha^2)\sqrt{1-q^2}}{2\alpha}$$

and

$$v_c \;=\; \frac{\sqrt{-8 + 4\alpha^2 - \alpha^4 + (\alpha^4 + 4)q + 4(\alpha^2 - 2)\sqrt{1-q^2}}}{2\alpha}\,.$$

In this case, both $\mathcal{C}_q^{\wedge} \cap \mathcal{D}_c$ and $\mathcal{C}_q^{\wedge} \cap \mathcal{D}_e$ have infinitely many points. $\square$

7 The proof of Theorem 1 (I),(III)

In this section, we complete the proof of the main result of the paper.

First, we consider the case (III). In other words, we assume that $q = \cos\phi$ $(0 < \phi < \pi/2)$ and $\alpha > 1$ satisfy the inequality

$$\frac{\alpha\,|\alpha-2|}{\alpha^2 - 2\alpha + 2} \;<\; q \;<\; 1,$$

or equivalently, the quantity $\tau = \tan(\phi/2)$ satisfies

$$0 \;<\; \tau \;<\; \min\left\{\alpha - 1, \frac{1}{\alpha-1}\right\}.$$

By Lemma 9, the arc $\mathcal{C}_{up}$ meets the circle $\{u + \mathrm{i}\, v : u, v \in \mathbb{R},\, u^2 + v^2 = q^2\}$ at two points $u_c \pm \mathrm{i}\, v_c$. Moreover, the part of the curve $\mathcal{C}_q^{\wedge}$ in $\mathcal{D}_e$ is the circular arc $\{u + \mathrm{i}\, v : u, v \in \mathbb{R},\, u^2 + v^2 = q^2,\, u \leq u_c\}$, and the corresponding envelope of the family of circles in (49) is the circular arc

$$\left\{ u + \mathrm{i}\, v : u, v \in \mathbb{R},\, u^2 + v^2 = (1 + \sqrt{1 - q^2})^2,\, u \leq \frac{u_c\,(1 + \sqrt{1 - q^2})}{q} \right\}.$$

The quantity $u_c\,(1 + \sqrt{1 - q^2})\,/\,q$ is equal to

$$\frac{(\tau + 1)[(\tau + 1)^2 - \alpha^2 \tau]}{\alpha(1 - \tau)(\tau^2 + 1)} \;=\; \frac{(4 - \alpha^2)(1 + \sqrt{1 - q^2}) + (\alpha^2 - 2)\, q^2}{2\alpha\, q}\;.$$

Furthermore, the envelope of the family of circles in (49) with centers in the cone part lies on the boundary of $F(B_q)$ (see Theorem 1 (II), Theorem 3 and their proofs). The proof of (III) is complete.

Finally, we treat the first part of the theorem, that is, we assume that

$$1 \,<\, \alpha \,<\, 2 \quad \text{and} \quad 0 \,<\, q \,\leq\, \frac{\alpha^2(2 - \alpha)}{(\alpha^2 - 2\alpha + 2)}\;.$$

Then, as it has been already mentioned in the proof of Lemma 9, the curve $\mathcal{C}_q^{\wedge}$ coincides with the circle $\{u + \mathrm{i}\, v \,:\, u, v \,\in\, \mathbb{R},\, u^2 \,+\, v^2 \,=\, q^2\}$. Hence, the corresponding envelope of the family of circles in (49) is the circle

$$\left\{ u + \mathrm{i}\, v : u, v \in \mathbb{R},\, u^2 + v^2 = (1 + \sqrt{1 - q^2})^2 \right\}.$$

The proof of Theorem 1 is now complete. $\square$

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