Point equation of the boundary of the numerical range of a matrix polynomial

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Abstract

The numerical range of an $n \times n$ matrix polynomial

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A_1\lambda + A_0$$

is defined by

$$W(P) = \{\lambda \in \mathbb{C} : x^*P(\lambda)x = 0,\ x \in \mathbb{C}^n,\ x \neq 0\}.$$

For the linear pencil $P(\lambda) = I\lambda - A$, the range $W(P)$ coincides with the numerical range of matrix A, $F(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\}$. In this paper, we obtain necessary conditions for the origin to be a boundary point of $F(A)$. As a consequence, an algebraic curve of degree at most $2n(n-1)m$, which contains the boundary of $W(P)$, is constructed.

Keywords : matrix polynomial, numerical range, boundary, discriminant.

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1 Introduction

Consider a *matrix polynomial*

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A_1\lambda + A_0, \tag{1}$$

where $A_j \in \mathbb{C}^{n\times n}$ $(j = 0, 1, \ldots, m)$, $A_m \neq 0$, and λ is a complex variable. Matrix polynomials arise in many applications and their spectral analysis is very important when studying linear systems of ordinary differential equations with constant coefficients [4]. If all the coefficients of $P(\lambda)$ are Hermitian matrices,

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then $P(\lambda)$ is called *selfadjoint*. A scalar $\lambda_0 \in \mathbb{C}$ is said to be an *eigenvalue* of $P(\lambda)$ in (1) if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is the *spectrum* of $P(\lambda)$, namely,

$$\mathrm{sp}(P) = \{\lambda \in \mathbb{C} : \det P(\lambda) = 0\}.$$

The *numerical range* of $P(\lambda)$ in (1) is defined by

$$W(P) = \{\lambda \in \mathbb{C} : x^* P(\lambda) x = 0, \text{ for some nonzero } x \in \mathbb{C}^n\}. \quad (2)$$

Clearly, $W(P)$ is always closed and contains the spectrum $\mathrm{sp}(P)$. If $P(\lambda) = I\lambda - A$, then $W(P)$ coincides with the classical numerical range of matrix A,

$$F(A) = \{x^* A x : x \in \mathbb{C}^n, \ x^* x = 1\}.$$

The last decade, the numerical range of matrix polynomials has attracted attention, and several results have been obtained (see e.g., [1], [6], [7], [8] and [10]). The numerical range $W(P)$ in (2) is not always connected, and it is bounded if and only if $0 \notin F(A_m)$ [6]. Furthermore, if μ is a boundary point of $W(P)$, then the origin is also a boundary point of $F(P(\mu))$; in general, the corners of $W(P)$ are eigenvalues of $P(\lambda)$ [8].

F. D. Murnaghan [9] and R. Kippenhahn [5] independently showed that the boundary $\partial F(A)$ is the set of real points of the algebraic curve whose equation in line coordinates is

$$\det(uI + vA_h + wA_{sh}) = 0,$$

where the matrices

$$A_h = \frac{A + A^*}{2} \quad \text{and} \quad A_{sh} = \frac{A - A^*}{2i} \quad (3)$$

are Hermitian and satisfy $A = A_h + iA_{sh}$. Using this fact, R. Kippenhahn proved that the non-differentiable points of $\partial F(A)$ are eigenvalues of A. In the last fifty years, there were no results on the computation of the *point equation* of the curves $\partial F(A)$ and $\partial W(P)$. The only exception has been the work of M. Fiedler [2] on the the numerical range of complex matrices. Recently, in [1] and [10], it was proved that the boundary of the numerical range of a 2×2 matrix polynomial lies on an algebraic curve of degree at most $4m$, where m is the degree of the polynomial.

In this paper, we investigate the point equation of the boundary $\partial W(P)$ continuing the work in [1] and [10]. In Section 2, a necessary condition for the origin to be a boundary point of the numerical range of a fixed matrix is obtained. This result provides, in Section 3, a necessary condition for points in the complex plane to lie on the boundary of $W(P)$ in (2), and an algebraic curve of degree at most $2n(n-1)m$, which contains $\partial W(P)$ is formulated. The suggested reference on algebraic curves is [12]. Finally, we present examples to illustrate our results, and an open problem is stated.

2 Matrix Case

Suppose that A is an $n \times n$ $(n \geq 2)$ complex matrix with numerical range $F(A)$. In this section, we prove that if the origin is a boundary point of $F(A)$, then the polynomial $f(t) = \det(A + t\,A^*)$ has a multiple root of modulus one. The next lemma is necessary.

Lemma 1 *For any pair of $n \times n$ complex matrices A and B,*

$$\lim_{s \to 0} \frac{1}{s} \left(\det(2A + s\,A + s\,B) - \det(2A + s\,B) \right) \;=\; 2^{n-1}\, n \det A.$$

Proof Let $A = (a_{j,k})$ and $B = (b_{j,k})$ $(j, k = 1, 2, \dots, n)$ and denote

$$K(s) \;=\; \det(2A + s\,A + s\,B) - \det(2A + s\,B).$$

By the definition of the determinant, we have

$$\begin{aligned} K(s) \;\; &= \;\; \sum_{\sigma \in \mathbf{S}_n} \mathrm{sign}(\sigma) \prod_{j=1}^{n} (2\,a_{j,\sigma(j)} + s\,a_{j,\sigma(j)} + s\,b_{j,\sigma(j)}) \\ & \quad - \sum_{\sigma \in \mathbf{S}_n} \mathrm{sign}(\sigma) \prod_{j=1}^{n} (2\,a_{j,\sigma(j)} + s\,b_{j,\sigma(j)}). \end{aligned}$$

Straightforward computations yield

$$\begin{aligned} \lim_{s \to 0} \frac{1}{s} K(s) \;\; &= \;\; \sum_{\sigma \in \mathbf{S}_n} \mathrm{sign}(\sigma)\, a_{1,\sigma(1)}\, 2\,a_{2,\sigma(2)} \dots 2\,a_{n,\sigma(n)} \\ & \quad + \sum_{\sigma \in \mathbf{S}_n} \mathrm{sign}(\sigma)\, 2\,a_{1,\sigma(1)}\, a_{2,\sigma(2)} \dots 2\,a_{n,\sigma(n)} + \dots \\ & \quad + \sum_{\sigma \in \mathbf{S}_n} \mathrm{sign}(\sigma)\, 2\,a_{1,\sigma(1)}\, 2\,a_{2,\sigma(2)} \dots a_{n,\sigma(n)} \\ &= \;\; 2^{n-1}\, n \det A \end{aligned}$$

completing the proof. □

Our criterion, and main result of this section, is the following.

Theorem 2 *Let A be an $n \times n$ complex matrix with numerical range $F(A)$, and let $0 \in \partial F(A)$. Then the polynomial*

$$f(t) \;=\; \det(A + t\,A^*)$$

satisfies one of the following:

(i) *The polynomial $f(t)$ is identically zero, i.e., $f(t) = 0$ for every $t \in \mathbb{C}$.*

(ii) *$f(t)$ is a nonzero polynomial of degree at most n, it has a multiple root $t_0 = e^{i\theta_0}$, $\theta_0 \in [0, 2\pi]$, and the boundary $\partial F(A)$ has a unique tangent at the origin,*

$$\varepsilon_0 \;=\; \left\{ u + iv : (u,v) \in \mathbb{R}^2,\; u\cos\left(\frac{\theta_0}{2}\right) + v\sin\left(\frac{\theta_0}{2}\right) = 0 \right\}.$$

Proof If $0 \in \partial F(A) \cap \mathrm{sp}(A)$, then by Lemma 3.3 in [7], there is a unit vector $y \in \mathbb{C}^n$ such that $Ay = A^*y = 0$. Then the matrix $A + t\,A^*$ is singular for all complex t, and thus the polynomial $f(t) = \det(A + t\,A^*)$ is identically zero. Hence, we assume that the matrix A is nonsingular and we consider two cases.

First we consider the case where $F(A)$ has no interior points. Since $0 \in F(A)$, it is clear that there exists an angle $\phi_0 \in [0, \pi]$ such that $A = i\,e^{i\phi_0} H$, where H is Hermitian. In this case, we can assume that

$$A \;=\; i\,e^{i\phi_0}\,\mathrm{diag}\{a_1, a_2, \ldots, a_n\},$$

$a_j \in \mathbb{R}$ $(j = 1, 2, \ldots, n)$, and hence the polynomial $f(t)$ is written

$$f(t) \;=\; \prod_{j=1}^{n} (i\,e^{i\phi_0}\,a_j - i\,t\,e^{-i\phi_0}\,a_j) \;=\; (-i\,e^{-i\phi_0})^n \left(\prod_{j=1}^{n} a_j \right) (t - e^{i2\phi_0})^n.$$

Since A is nonsingular, all the scalars $a_1, a_2, \ldots, a_n$ are nonzero. Then $f(t)$ has a multiple root $t_0 = e^{i2\phi_0}$. Note also that the origin is not an endpoint of the line segment $F(A)$, and $F(A)$ lies on the line

$$\varepsilon_0 \;=\; \left\{ u + iv \in \mathbb{C} : (u,v) \in \mathbb{R}^2,\; u\cos\phi_0 + v\sin\phi_0 = 0 \right\}.$$

Second we consider the case where $F(A)$ has a nonempty interior. Since A is nonsingular, the origin is not a corner of $F(A)$ [3], i.e., the boundary $\partial F(A)$ is differentiable sufficiently close to the origin. Thus, the supporting line of $F(A)$ at 0 is unique and coincides with the unique tangent of $\partial F(A)$ at 0. Moreover, there exists an angle $\phi_1 \in [0, 2\pi]$ such that the numerical range $F(e^{i\phi_1}A) = e^{i\phi_1}F(A)$ lies in the right closed half plane. Equivalently, the Hermitian matrix

$$\frac{e^{i\phi_1}A + e^{-i\phi_1}A^*}{2} \;=\; \cos\phi_1\,A_h + \sin\phi_1\,A_{sh},$$

where the matrices A_h and A_{sh} are defined in (3), is positive semidefinite. Consider the polynomial

$$\tilde{f}(t) \;=\; \det(e^{-i\phi_1}A + t\,e^{i\phi_1}A^*) \;=\; e^{-in\phi_1} f(e^{i2\phi_1}t).$$

Then $f(t)$ has a multiple root $t_0 = e^{i2\phi_1}$ if and only if $\tilde{f}(t)$ has a multiple root $t_1 = 1$. Hence, without loss of generality, we assume that the Hermitian matrix

A_h is singular and positive semidefinite, and it is enough to prove that $t_0 = 1$ is a multiple root of $f(t)$. Under our assumptions, it is clear that

$$f(1) \;=\; 2^n \det(A_h) \;=\; 0.$$

For the derivative, we have

$$\begin{aligned} f'(1) &= \lim_{t\to 0} \frac{1}{t}\,(f(1+t) - f(1)) \\ &= \lim_{t\to 0} \frac{1}{t}\,(\det(2A_h + t\,A_h - i\,t\,A_{sh}) - \det(2A_h - i\,t\,A_{sh})) \\ &\quad + \lim_{t\to 0} \frac{1}{t}\,(\det(2A_h - i\,t\,A_{sh}) - \det(2A_h)). \end{aligned}$$

Hence by Lemma 1,

$$\begin{aligned} f'(1) &= 2^{n-1}\,n \det(A_h) + \lim_{t\to 0} \frac{1}{t}\,(\det(2A_h - i\,t\,A_{sh}) - \det(2A_h)) \\ &= \lim_{t\to 0} \frac{1}{t}\,(\det(2A_h - i\,t\,A_{sh}) - \det(2A_h)). \end{aligned}$$

As the limit $\lim_{t\to 0} (1/t)\,(\det(2A_h - i\,t\,A_{sh}) - \det(2A_h))$ exists, we may evaluate it by restricting t pure imaginary. Consequently,

$$f'(1) \;=\; -\,i \lim_{s\to 0} \frac{1}{s}\,(\det(2A_h + s\,A_{sh}) - \det(2A_h)) \qquad (s \in \mathbb{R}). \tag{4}$$

For the real variable s, consider the polynomial $g(s) = \det(A_h + s\,A_{sh})$. Then $g(0) = 0$ (recalling that $\det A_h = 0$), and by (4), it follows that $f'(1) = 0$ if and only if $g'(0) = 0$. By [11], sufficiently close to the origin, the polynomial $g(s)$ can be written in the form $g(s) = \prod_{j=1}^{n} \lambda_j(s)$, where the eigenvalues of the Hermitian matrix $A_h + s\,A_{sh}$, $\lambda_1(s), \lambda_2(s), \ldots, \lambda_n(s)$ depend analytically in $s \in \mathbb{R}$. Since $g(0) = 0$, we may assume that $\lambda_1(0) = 0$. Furthermore, for the sake of contradiction, assume that for every $j \in \{2, 3, \ldots, n\}$, $\lambda_j(0) \neq 0$. Since the Hermitian matrix A_h is positive semidefinite, all the eigenvalues $\lambda_2(0), \lambda_3(0), \ldots, \lambda_n(0)$ are positive. If $\lambda_1'(0) < 0$ (resp. $\lambda_1'(0) > 0$), then there exists a real $\delta > 0$ such that for $-\delta \leq s < 0$ (resp. $0 < s \leq \delta$),

$$\lambda_1(s) > 0 \quad \text{and} \quad \lambda_j(s) > 0 \;\; (j = 2, 3, \ldots, n).$$

Thus, the Hermitian matrix $A_h - \delta A_{sh}$ (resp. $A_h + \delta A_{sh}$) is positive definite. This means that for every unit vector $x_0 \in \mathbb{C}^n$ such that $x_0^* A_h x_0 = 0$, we have $x_0^* A_{sh} x_0 \neq 0$, which is a contradiction since $0 \in F(A)$. Hence $\lambda_1'(0) = 0$, and consequently, $g'(0) = 0$ and $f'(1) = 0$. The proof is complete. $\square$

The converse of the above theorem is not true. For example, if 0 is a normal eigenvalue of A (even in the interior of $F(A)$), then the polynomial $f(t)$ is always identically zero. Furthermore, for every complex diagonal matrix A with at least two nonzero real diagonal elements, the polynomial $f(t) = \det(A + t\,A^*)$ has a multiple root $t_0 = -1$, but the origin may be either an interior or an exterior point of $F(A)$.

Remark 1 One may ask if for every matrix A such that the polynomial $f(t) = \det(A + t\,A^*)$ is identically zero, 0 is necessarily a normal eigenvalue of A. This is not true as is illustrated in the following example. Consider the matrix $A = A_h + iA_{sh}$ with

$$A_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_{sh} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Then the polynomial $f(t) = \det(A + t\,A^*)$ is identically zero, or equivalently, for every $t \in \mathbb{C}$, the matrix

$$A_h + t\,A_{sh} = \begin{bmatrix} 0 & 2t & t \\ 2t & 4t-4 & 2t \\ t & 2t & t+1 \end{bmatrix}$$

is singular. It is easy to see that the eigenvector of $A_h + t\,A_{sh}$ corresponding to 0 is $y(t) = [\,-2,\ -t,\ 2t\,]^T$. The Hermitian matrices A_h and A_{sh} have no common eigenvector corresponding to 0. Consequently, the matrices A and A^* have also no common eigenvector corresponding to 0, and 0 is not a normal eigenvalue of A (see also Lemma 3.3 in [7]).

3 Matrix Polynomial Case

For the remainder of the paper, it is necessary to recall an algebraic criterion for a scalar polynomial to have a multiple root. Consider a polynomial of the form

$$g(t) = \alpha_l t^l + \alpha_{l-1} t^{l-1} + \ldots + \alpha_1 t + \alpha_0$$

and its derivative

$$g'(t) = l\alpha_l t^{l-1} + (l-1)\alpha_{l-1} t^{l-2} + \ldots + \alpha_1,$$

where $\alpha_0, \alpha_1, \ldots \alpha_l \in \mathbb{C}$ and t is a complex variable. We define $\tilde{D}_g$ the *resultant (Sylvester determinant)* of $g(t)$ and $g'(t)$, that is, $\tilde{D}_g = \det\Delta_g$, where Δ_g is the $(2l-1) \times (2l-1)$ matrix

$$\Delta_g = \begin{bmatrix} l\alpha_l & (l-1)\alpha_{l-1} & \ldots & \alpha_1 & 0 & 0 & \ldots & 0 \\ 0 & l\alpha_l & (l-1)\alpha_{l-1} & \ldots & \alpha_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & l\alpha_l & (l-1)\alpha_{l-1} & \ldots & \ldots & \alpha_1 \\ \alpha_l & \alpha_{l-1} & \ldots & \ldots & \alpha_0 & 0 & \ldots & 0 \\ 0 & \alpha_l & \alpha_{l-1} & \ldots & \ldots & \alpha_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \alpha_l & \alpha_{l-1} & \ldots & \ldots & \ldots & \alpha_0 \end{bmatrix}.$$

The polynomial $\tilde{D}_g = \det\Delta_g$ is homogeneous in variables $\alpha_0, \alpha_1, \ldots \alpha_l$, of degree at most $2l-1$, and contains a factor α_l. If $\alpha_l \neq 0$, then the quantity $D_g = \tilde{D}_g/\alpha_l$ is said to be the *discriminant* of the polynomial $g(t)$, and it is a homogeneous polynomial in $\alpha_0, \alpha_1, \ldots \alpha_l$ of degree at most $2l-2$. Furthermore, $D_g = 0$ if and only if $g(t)$ has a multiple root [12]. If $\alpha_l = 0$, then D_g is assumed to be zero.

For an $n \times n$ matrix $A = (a_{j,k})$, consider the polynomial

$$f(t) \,=\, \det(A + t\,A^*) \,=\, \alpha_n t^n + \alpha_{n-1}t^{n-1} + \ldots + \alpha_1 t + \alpha_0,$$

where α_n is allowed to be zero. Then every coefficient α_q $(q = 0, 1, \ldots, n)$ is a homogeneous polynomial in $a_{j,k}$ and $\overline{a}_{j,k}$ $(j, k = 1, 2, \ldots, n)$ of degree at most n. Hence, the discriminant D_f of the polynomial $f(t) = \det(A + t\,A^*)$ can be viewed as a homogeneous polynomial in $a_{j,k}$ and $\overline{a}_{j,k}$ $(j, k = 1, 2, \ldots, n)$ of degree at most $2n(n-1)$.

Let $P(\lambda) = A_m\lambda^m + \ldots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial as in (1) with numerical range $W(P)$ as in (2). In the following, we estimate the point equation of the curve $\partial W(P)$ generalizing somehow results in [1], [5], [9] and [10].

Theorem 3 *Consider a matrix polynomial $P(\lambda)$ as above. Then the boundary of the numerical range $W(P)$ lies on the algebraic curve*

$$\{u + iv \in \mathbb{C} : (u, v) \in \mathbb{R}^2,\ D_P(u, v) \,=\, 0\},$$

where $D_P(u, v)$ is the discriminant of the polynomial

$$G_P(t; u, v) \,=\, \det(P(u + iv) + t\,[P(u + iv)]^*), \tag{5}$$

with respect to variable t. This discriminant is a polynomial (not necessarily with real coefficients) in $u, v \in \mathbb{R}$ of total degree at most $2n(n-1)m$.

Proof By Theorem 1.1 in [8], every boundary point μ of $W(P)$ satisfies the condition $0 \in \partial F(P(\mu))$. Thus, by Theorem 2, the discriminant $D_P(u, v)$ of the polynomial (with respect to t),

$$G_P(t; u, v) \,=\, \det(P(u + iv) + t\,[P(u + iv)]^*),$$

where $(u, v) \in \mathbb{R}^2$ and $t \in \mathbb{C}$, satisfies $D_P(u, v) = 0$ when $u + iv \in \partial W(P)$. The matrix polynomial $P(\lambda)$ can also be written in the form

$$P(\lambda) \,=\, \begin{bmatrix} p_{1,1}(\lambda) & p_{1,2}(\lambda) & \ldots & p_{1,n}(\lambda) \\ p_{2,1}(\lambda) & p_{2,2}(\lambda) & \ldots & p_{2,n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,1}(\lambda) & p_{n,2}(\lambda) & \ldots & p_{n,n}(\lambda) \end{bmatrix},$$

where the entries $p_{j,k}(\lambda) = p_{j,k}(u + iv)$ $(j, k = 1, 2, \ldots, n)$ are polynomials in $u, v \in \mathbb{R}$ of degree at most m. Therefore, the discriminant $D_P(u, v)$ of the polynomial $G_P(t;u,v)$ in (5) is a polynomial in $u,v\in\mathbb{R}$ of total degree at most $2n(n-1)m$. $\square$

It is worth noting that for the polynomial $G_P(t;u,v)$ in (5), we have

$$t^n\,\overline{G_P(\bar{t}^{-1};u,v)} \;=\; \det(t\,[P(u+iv)]^* + P(u+iv)) \;=\; G_P(t;u,v).$$

Hence, t_0 is a nonzero root of $G_P(t;u,v)$ with multiplicity k if and only if $\bar{t}_0^{-1}$ is a root of $G_P(t;u,v)$ with the same multiplicity. Moreover, if the polynomial $G_P(t;u,v)$ is quadratic with respect to t, and for some $(u_0,v_0)\in\mathbb{R}^2$, $G_P(t;u_0,v_0)$ has a double root t_0, then clearly $|t_0|=1$ (see Examples 1 and 3 below).

Remark 2 If u_0+iv_0 $(u_0,v_0\in\mathbb{R})$ is an isolated point of $W(P)$, then by Theorem 2.1 in [8], $P(u_0+iv_0)=0$ and consequently, $G_P(t;u_0,v_0)\equiv 0$.

Our method is applicable to a "generic" matrix A or matrix polynomial $P(\lambda)$ to compute the point equation of the boundary $\partial F(A)$ or $\partial W(P)$ as it is illustrated in the following three examples. Notice that in the first example we consider a known special case of matrices verifying Theorem 3.

Example 1 Let A be a 2×2 complex matrix. By Properties 1.2.3 and 1.2.4 in [3], we may assume that

$$A \;=\; \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix},$$

where $a\in\mathbb{R}$ and $b\in\mathbb{C}$. Then for the linear pencil $L(\lambda)=I\lambda-A$, we have

$$\begin{aligned} G_L(t;u,v) &= \det(L(u+iv)+t\,[L(u+iv)]^*) \\ &= ((u-iv)^2-a^2)\,t^2+(2u^2+2v^2-2a^2-|b|^2)\,t \\ &\quad +(u+iv)^2-a^2. \end{aligned}$$

The discriminant of $G_L(t;u,v)$ is

$$D_L(u,v) \;=\; 4|b|^2u^2+(16a^2+4|b|^2)v^2-|b|^4-4a^2|b|^2,$$

and for $b\neq 0$, the algebraic curve $D_L(u,v)=0$ is the ellipsis

$$\partial F(A) \;=\; \left\{u+iv\in\mathbb{C}:(u,v)\in\mathbb{R}^2,\;\frac{4u^2}{4a^2+|b|^2}+\frac{4v^2}{|b|^2}=1\right\}$$

as in Theorem 1.3.6 in [3].

Example 2 For the 3×3 matrix

$$A \;=\; \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

consider the linear pencil $L(\lambda) = A\lambda + I$. Then the numerical range of $L(\lambda)$ is

$$\begin{aligned} W(L) &= \{\, \lambda^{-1} \in \mathbb{C} : \lambda \in F(-A),\ \lambda \neq 0 \} \\ &= \mathbb{C} \setminus \{\, \lambda \in \mathbb{C} : |\lambda| < 2 \}. \end{aligned}$$

The polynomial

$$\begin{aligned} G_L(t;u,v) &= \det(L(u+iv) + t\,[L(u+iv)]^*) \\ &= (t+1)^3 - (t+1)\,|u+iv|^2 t \\ &= (t+1)(t^2 + (2-u^2-v^2)\,t + 1) \end{aligned}$$

has a multiple root (which always has modulus equal to one) if and only if

$$u^2 + v^2 \;=\; 4 \qquad \text{or} \qquad u^2 + v^2 \;=\; 0.$$

Hence, if $D_L(u,v)$ is the disciminant of $G_L(t;u,v)$, then the algebraic curve $D_L(u,v) = 0$ is the union of the origin and the circle

$$\partial W(L) \;=\; \{u + iv \in \mathbb{C} : (u,v) \in \mathbb{R}^2,\ u^2 + v^2 = 4\}.$$

Example 3 Consider the quadratic matrix polynomial

$$P(\lambda) \;=\; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \lambda^2 - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The numerical range $W(P)$ consists of two unbounded connected components, and a few thousands of (mostly interior) points of $W(P)$ are sketched in Figure 1. It is obvious that we cannot have a clear picture of $W(P)$ and its boundary, yet. The polynomial $G_P(t;u,v)$ is given by

$$\begin{aligned} G_P(t;u,v) &= \det(P(u+iv) + t\,[P(u+iv)]^*) \\ &= \det\left(\begin{bmatrix} (u^2-v^2-1+i2uv) + (u^2-v^2-1-i2uv)\,t & -1 \\ -t & -1-t \end{bmatrix}\right) \\ &= -\,(u^2-v^2-1-i2uv)\,t^2 - (2u^2-2v^2-1)\,t \\ &\quad -\,(u^2-v^2-1+i2uv). \end{aligned}$$

The discriminant of $G_P(t;u,v)$ is

$$D_P(u,v) \;=\; -16u^2v^2 + 4u^2 - 4v^2 - 3,$$

and the real algebraic curve $D_P(u,v) = 0$ coincides with the boundary of $W(P)$ and has two branches,

$$\left\{ u + iv \in \mathbb{C} : (u,v) \in \mathbb{R}^2,\ u \geq \frac{\sqrt{3}}{2},\ v = \pm\sqrt{\frac{4u^2-3}{4+16u^2}} \right\}$$

and

$$\left\{u+iv \in \mathbb{C} : (u,v) \in \mathbb{R}^2,\ u \leq -\frac{\sqrt{3}}{2},\ v = \pm\sqrt{\frac{4u^2-3}{4+16u^2}}\right\}$$

(see Figure 1). Notice that for every point $(u_0, v_0) \in \mathbb{R}^2$ such that $D_P(u_0, v_0) = 0$, the polynomial $G_P(t; u_0, v_0)$ has a double root

$$t_0 \;=\; -\,\frac{2u_0^2 - 2v_0^2 - 1}{2u_0^2 - 2v_0^2 - 2 - i4u_0v_0}$$

with $|t_0| = 1$. This explains why the curve $D_P(u,v) = 0$ is exactly the same with the boundary $\partial W(P)$.

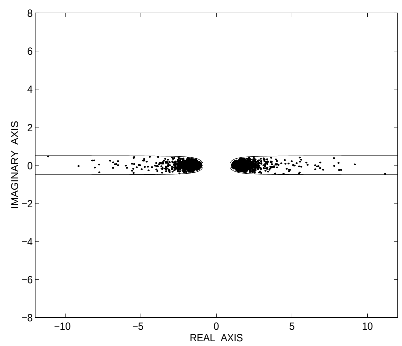

Figure 1: An unbounded numerical range and its boundary.

Remark 3 The disciminant $D_P(u,v)$ in Theorem 3 is a polynomial in real variables u, v (of total degree at most $2n(n-1)m$) but its coefficients may be non-real. Then we can write

$$D_P(u,v) \;=\; \mathrm{Re}D_P(u,v) + i\,\mathrm{Im}D_P(u,v),$$

where $\mathrm{Re}D_P(u,v)$ and $\mathrm{Im}D_P(u,v)$ are real polynomials. As a consequence, if $\Gamma(u,v)$ is the greatest common divisor of the polynomials $\mathrm{Re}D_P(u,v)$ and $\mathrm{Im}D_P(u,v)$, then the non-isolated part of the algebraic curve

$$\{(u,v) \in \mathbb{R}^2 : D_P(u,v) \;=\; 0\}$$

coincides with the non-isolated part of the *real* algebraic curve of total degree at most $2n(n-1)m$,

$$\{(u,v) \in \mathbb{R}^2 : \Gamma(u,v) \;=\; 0\}.$$

(By the non-isolated part of an algebraic curve we mean the curve without its isolated points).

4 An Open Problem

Let $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \ldots + A_1\lambda + A_0$ be an $n \times n$ matrix polynomial with numerical range $W(P) \neq \mathbb{C}$. Suppose now that the polynomial $G_P(t; u, v)$ in (5) is not identically zero and has the following irreducible form:

$$G_P(t; u, v) \;=\; \prod_{j=1}^{\rho_1} [G_{1,j}(u, v)]^{n_j} \prod_{j=1}^{\rho_2} [G_{2,j}(t, u, v)]^{m_j}, \tag{6}$$

where $G_{1,j}(u, v)$'s and $G_{2,j}(t, u, v)$'s are mutually distinct (up to constant multiple) irreducible polynomials in the polynomial ring $\mathbb{C}[t, u, v]$ with at least one nonzero power of t in every $G_{2,j}(t, u, v)$ ($j = 1, 2, \ldots, \rho_2$), and n_j's and m_j's are positive integers. If $m_j \geq 2$ for some $j \in \{1, 2, \ldots, \rho_2\}$, then the discriminant $D_P(u, v)$ of the polynomial $G_P(t; u, v)$ (with respect to t) is identically zero providing no information.

Suppose that there is a factor $[G_{2,j}(t, u, v)]^{m_j}$ in (6) with $m_j \geq 2$. If for every $(u, v) \in \mathbb{R}^2$, $G_{2,j}(t, u, v)$ has no root on the unit circle of the complex plane, then by the second part of Theorem 2, we can just take away this factor and compute the disciminant of $G_P(t; u, v)/[G_{2,j}(t, u, v)]^{m_j}$. Otherwise, we consider the polynomial

$$G_P^{\#}(t; u, v) \;=\; \prod_{j=1}^{\rho_1} [G_{1,j}(u, v)]^{n_j} \prod_{j=1}^{\rho_2} [G_{2,j}(t, u, v)].$$

The discriminant $D_P^{\#}(u, v)$ of $G_P^{\#}(t; u, v)$ (with respect to t) is a nonzero polynomial in $u, v \in \mathbb{R}$, and it seems that the boundary of $W(P)$ lies on the algebraic curve $D_P^{\#}(u, v) = 0$, as in the Examples 4 and 5 bellow. Until now, there is no proof known to the authors.

Example 4 For an $n \times n$ matrix polynomial of the form $P(\lambda) = p(\lambda)I_n$, where $p(\lambda)$ is a scalar polynomial of degree m, it is easy to see that $W(P)$ coincides with the set of the roots of $p(\lambda)$ (with at most m elements). The polynomial $G_P(t; u, v) = \det(P(u + iv) + t\,[P(u + iv)]^*)$ is of the form

$$G_P(t; u, v) \;=\; (p(u + iv) + t\,\overline{p(u + iv)})^n.$$

Hence, the *square free* polynomial, with respect to t,

$$G_P^{\#}(t; u, v) \;=\; p(u + iv) + t\,\overline{p(u + iv)}$$

is identically zero when $p(u + iv) = 0$, and it has a single root

$$t_1(u, v) \;=\; -\frac{p(u + iv)}{\overline{p(u + iv)}}$$

when $p(u + iv) \neq 0$.

Example 5 Consider the linear pencil $L(\lambda) = I\lambda - A$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the polynomial $G_L(t; u, v) = \det(L(u + iv) + t\,[L(u + iv)]^*)$ is given by

$$\begin{aligned} G_L(t; u, v) &= (u - iv)^4 t^4 + (u - iv)^2 (4u^2 + 4v^2 - 3)\, t^3 \\ &\quad + 6(u^2 + v^2)(u^2 + v^2 - 1)\, t^2 \\ &\quad + (u + iv)^2 (4u^2 + 4v^2 - 3)\, t + (u + iv)^4 \\ &= [u + iv + (u - iv)\, t\,]^2\, [(u - iv)^2 t^2 \\ &\quad + (2u^2 + 2v^2 - 3)\, t + (u + iv)^2], \end{aligned}$$

and its discriminant $D_L(u, v)$ (with respect to t) is identically zero. Note also that for every $(u, v) \in \mathbb{R}^2 \setminus (0, 0)$, $G_L(t; u, v)$ has a multiple root

$$t(u, v) = -\frac{u + iv}{u - iv}$$

with $|t(u, v)| = 1$. By the above discussion, we consider again the square free polynomial

$$G_L^{\#}(t; u, v) = [u + iv + (u - iv)\, t\,]\, [(u - iv)^2 t^2 + (2u^2 + 2v^2 - 3)\, t + (u + iv)^2].$$

The discriminant of $G_L^{\#}(t; u, v)$ (with respect to t) is

$$D_L^{\#}(u, v) = 27(u^2 + v^2)(4u^2 + 4v^2 - 3),$$

and the algebraic curve $D_L^{\#}(u, v) = 0$ coincides with the union of the origin and the circle $\{u + iv \in \mathbb{C} : (u, v) \in \mathbb{R}^2,\ u^2 + v^2 = 3/4\}$. One can verify that the numerical range $W(L) \equiv F(A)$ is the circular disk

$$\{u + iv \in \mathbb{C} : (u, v) \in \mathbb{R}^2,\ u^2 + v^2 \leq 3/4\}.$$

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