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EIGENVALUE CHARACTERIZATION OF SOME STRUCTURED MATRIX PENCILS UNDER LINEAR PERTURBATION*

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Abstract. We study the effect of linear perturbations on three families of matrix pencils. The matrix pairs of the first two families are Hermitian/skew-Hermitian with special 3×3 block cases appeared in continuous-time control, and the matrix pairs of the third family are special 3×3 non-Hermitian block matrices appeared in discrete-time control. For the first family of matrix pencils and more general cases of the second family of matrix pencils, based on the properties of the involved matrices, we obtain some upper or lower bounds on the set of eigenvalues of linearly perturbed matrix pencils which are on the imaginary axis. Studying a special 3×3 block matrix pencil, which is associated with continuous-time control, leads us to some linear perturbation that do not preserve (properly) the structure of the matrices. This, in turn, leads to a numerical technique for finding the nearest Hermitian/skew-Hermitian matrix pencil which can satisfy conditions such that, for some nonzero real perturbation parameter, some or all of its eigenvalues lie on the imaginary axis. We also study the linearly perturbed matrix pencils, associated with discrete-time control, using an one-to-one equivalence between the matrix pencil of continuous-time problem and the matrix pencil of discrete-time problem.

Key words. Linear perturbation, Hermitian/skew-Hermitian matrix pencil, Continuous-time control, Discrete-time control.

AMS subject classifications. 15A22, 65F15, 47A55, 93C73.

1. Introduction and motivation. Linear perturbation theory is a useful tool for investigating the spectral sensitivity of standard and generalized eigenvalue problems [18, 30, 34]. It is also an effective way for estimating the distance of many problems from some desirable or undesirable situations. An important field where linear perturbation has specific applications is control theory [9, 12].

Our main motivation stems from the following two classical problems of optimal and robust control.

(A) The first problem is in continuous-time linear quadratic optimal control, and its objective is to find a control input $u(\tau)$ in linear constant coefficient dynamical system:

$$M_3\dot{x} = M_1 x + M_2 u, \quad x(\tau_0) = x^{(0)},$$

such that the closed loop system is asymptotically stable and

$$\int_{\tau_0}^{\infty} \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix}^T \begin{bmatrix} M_4 & M_5 \\ M_5^H & M_6 \end{bmatrix} \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix} d\tau,$$

is minimized, where $x(\tau) \in \mathbb{C}^{n_1}$ is the state, $x^{(0)}$ is an initial vector, $M_4 = M_4^H \in \mathbb{C}^{n_1 \times n_1}$, and $M_6 = M_6^H \in \mathbb{C}^{n_2 \times n_2}$. Here, X^H stands for the conjugate transpose of the complex matrix X. Using the maximum principle [25, 28], this problem can be related to the eigenvalue problem of the matrix pencil:

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$$A_c - \lambda B_c = \begin{bmatrix} 0 & M_1 & M_2 \\ M_1^H & M_4 & M_5 \\ M_2^H & M_5^H & M_6 \end{bmatrix} - \lambda \begin{bmatrix} 0 & M_3 & 0 \\ -M_3^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{1}$$

see also [20, 21, 33]. For the complex (resp., the real) matrices A_c and B_c in (1), the finite eigenvalues of $A_c - \lambda B_c$ are symmetric with respect to the imaginary axis (resp., the real axis). If M_3 is nonsingular, then under standard assumptions in control theory [25, 32, 33], such a matrix pencil has exactly n_1 (finite) eigenvalues in the left half-plane, n_1 (finite) eigenvalues in the right half-plane, and n_2 infinite eigenvalues. In this case, the pencil has a unique deflating subspace associated with the eigenvalues in the open left half-plane. The situation is more complicated when M_3 or M_6 are singular; see [9] and the references therein for more details. It is worth mentioning that the solution of the continuous-time optimal control problem becomes ill-conditioned when the eigenvalues are close to the imaginary axis, and it may not exist when the eigenvalues are on the imaginary axis [7, 13].

When we study the perturbation theory for the eigenvalue problem (1), usually two main types of perturbations are considered, the perturbations that do not preserve the structure and the perturbations that preserve the structure [30]. In this paper, first we analyze the structure-preserving linear perturbations for more general matrix pencils (for brevity, we call them *MG matrix pencils*) including non-block matrix pencils and block matrix pencils whose structure are very close to (1) with the difference that their first diagonal block is not necessarily zero. What we actually do is to provide sufficient conditions under which, for these kind of matrix pencils and for some $z = i\gamma$ ($\gamma \in \mathbb{R}$), there exists at least one nonzero real perturbation parameter t such that the determinant of the perturbed matrix pencil becomes zero. These sufficient conditions depend on the properties of the involved perturbation matrices and provide us with lower or upper bounds on $\gamma \in \mathbb{R}$ of the catched $z = i\gamma$ on the imaginary axis. Applying such methods leads us to the following two achievements:

(a_1) We give a solution for the problem: Find a small enough (the smallest if possible) nonzero real perturbation parameter t such that the structured linear perturbation of an MG matrix pencil has one or more eigenvalues on the imaginary axis.

(a_2) We get very close to a solution of the problem: For any given MG matrix pencil with one or more eigenvalues on the imaginary axis, find a small real perturbation parameter t for the linearly perturbed version of this MG matrix pencil that removes all the eigenvalues from the imaginary axis.

The first problem has an important partner in the sensitivity analysis of the optimal problem [9], and the second problem has partners which are closely related to structured eigenvalue/eigenvector backward errors of matrix pencils arising in optimal control [24] and to the problem of computing nearest stable matrix pairs [15] (we remark that there are also some research works which study backward errors for eigenvalues and eigenvectors of structured matrix pencils [1], structured nonhomogeneous matrix polynomials [2], and structured homogeneous matrix polynomials [3, 4]).

It is impossible to answer the question in (a_1) for the matrix pencils of type (1). Therefore, we will suggest a two-phase numerical method to find the closest block MG matrix pencil to a matrix pencil of type (1) for which there exists the smallest nonzero real number t, which moves at least one of the eigenvalues of the perturbed matrix to the imaginary axis. This by itself suggests some specific linear perturbations that do not preserve the structure of the problem (1).

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(B) The second problem our motivation comes from is in discrete-time linear quadratic optimal control problem, and its objective can be related to the eigenvalue problem of the matrix pencil:

$$A_d - \lambda B_d = \begin{bmatrix} 0 & M_1 & M_2 \\ -M_3^H & M_4 & M_5 \\ 0 & M_5^H & M_6 \end{bmatrix} - \lambda \begin{bmatrix} 0 & M_3 & 0 \\ -M_1^H & 0 & 0 \\ -M_2^H & 0 & 0 \end{bmatrix}. \tag{2}$$

The finite eigenvalues of this problem are in pairs λ, $\dfrac{1}{\lambda}$ in the case of complex matrices, or in quadruples λ, $\bar{\lambda}$, $\dfrac{1}{\lambda}$, $\dfrac{1}{\bar{\lambda}}$ in the case of real matrices. For the discrete-time case, the important problem that arises is the study of small perturbations which lead to eigenvalues on the unit circle. These are the perturbations that may disturb the spectral symmetry and the uniqueness of the deflating subspace associated with the eigenvalues inside the open unit circle. As we will see, like the continuous-time problem, it is not possible to find the above answer for the discrete-time problem. For this reason, a three-phase technique will be proposed to find the answer to the above question for the nearest discrete-time problem. We achieve this goal by exploiting an equivalence relation between discrete-time problem and continuous-time problem.

One major problem in both continuous-time control and discrete-time control is to provide more customized and reliable methods for distinguishing the sensitivity of the problems. Any progress in this regard can be helpful in designing more efficient numerical methods for solving both continuous-time and discrete-time optimal and robust control problems. Therefore, our main aim in this work is to provide some easy-to-use reliable methods for quantifying the sensitivity of the both problems.

For the classification of the problems that we study here, let $t \in \mathbb{C}$, and suppose that the matrices $A, B \in \mathbb{C}^{n\times n}$ of the matrix pencil $P(z) = A - zB$ are perturbed by the matrices $t\Delta A, t\Delta B \in \mathbb{C}^{n\times n}$ ($\Delta A, \Delta B \in \mathbb{C}^{n\times n}$), respectively, and consider the perturbed matrix pencil:

$$P(z,t) = (A + t\Delta A) - z(B + t\Delta B). \tag{3}$$

The following arrangement of (3)

$$P(z,t) = A - zB + t(\Delta A - z\Delta B) = P(z) + t\Delta P(z), \tag{4}$$

is also helpful in what follows.

In this paper, we characterize the eigenvalues of some families of matrix pencils under linear perturbation, where the involved matrix pencils $P(z) = A - zB$ and perturbation matrix pencils $\Delta P(z) = \Delta A - z\Delta B$ satisfy one of the following conditions:

I. Both the matrix pencil $P(z)$ and the perturbation matrix pencil $\Delta P(z)$ are $*$-even (H-even) matrix pencils [23], i.e., $P(-z)^* = P(z)$ and $\Delta P(-z)^* = \Delta P(z)$. In this case, A and ΔA are Hermitian, and B and ΔB are skew-Hermitian. Also one of A or ΔA is either positive or negative definite.
II. Both the matrix pencil $P(z)$ and the perturbation matrix pencil $\Delta P(z)$ are $*$-even (H-even) matrix pencils [23]. A special 3×3 block structure of the matrix pencil $P(z)$ appears in continuous-time control [9, 35].
III. All matrices A, B, ΔA, and ΔB are special non-Hermitian 3×3 block matrices where the associated block non-Hermitian/non-Hermitian matrix pencil occurs in discrete-time control [9, 35].

It is worth mentioning that for each of the problems in the category I and for each of the MG matrix pencils in the category II, we first introduce different conditions on $z = i\gamma$ ($\gamma \in \mathbb{R}$) and on the involved

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matrices which make the matrix pencils $P(z)$ and $\Delta P(z)$ in (4) Hermitian and at least one of them positive or negative definite. In particular, we provide conditions on the matrices and on the set of perturbation parameter (which here is $z = i\gamma$, $\gamma \in \mathbb{R}$) such that the perturbed (or deviated) matrix pair $(P(z), -\Delta P(z))$ has at least one real and nonzero eigenvalue, t. As the value of z satisfying the sufficient conditions can belong to a very large subset of the imaginary axis, we may call this homotopic deviation of matrix pair $(A, -\Delta A)$ using homotopic deviation pair $(B, -\Delta B)$ and the deviation parameter z [5].

Our approaches for studying and treating all of these problems, to the best of our knowledge, are new and provide us with new insights into the problem of the smallest value of nonzero real t and into the notion of distance to the boundary of a desired or undesired set [9].

The paper is organized as follows. Section 2 includes general notation, theory, and assumptions which will be used in the next sections. In Section 3, we study the perturbed matrix pencils $P(z, t)$ of the category I and obtain the conditions which guarantee that for some nonzero real values t and some $z = i\gamma$ ($\gamma \in \mathbb{R}$), it holds that $\det(P(z, t)) = 0$. Then, we specify the case(s) with minimum $|t|$. In Section 4, we investigate a perturbed MG matrix pencil $P(z, t)$ of the category II and characterize the purely imaginary $z = i\gamma$ ($\gamma \in \mathbb{R}$) which may belong to the spectrum of $P(z, t)$ for some nonzero real t. Having such information, we can specify the case(s) with minimum $|t|$. Then, we characterize the eigenvalues of some 3×3 block Hermitian/skew-Hermitian matrix pencils, and explain that a special case of these block matrix pencils arisen in continuous-time control cannot satisfy necessary conditions under which its eigenvalues, for small enough real perturbation parameter t, go onto the imaginary axis. Therefore, a practical way of finding the nearest problem having the necessary conditions will be suggested and implemented. In Section 5, we consider the perturbed matrix pencils $P(z, t)$ whose matrices are in the category III. We discuss their lackness of necessary conditions under which their eigenvalues, for nonzero real perturbation parameter t, go onto the unit circle. For this category of problems, we use an one-to-one relationship between the block matrix pencils of continuous-time problem and the block matrix pencils of discrete-time problem, to find the nearest matrix pencil to the matrix pencil of discrete-time problem for which we can characterize the complex number $z \in \mathbb{C}$ with $|z| = 1$ that may belong to the spectrum of $P(z, t)$ for some nonzero real t. Finally, in Appendix A, we obtain sufficient conditions under which, for a subset of z on the unit circle, all the eigenvalues of the matrix pair $(P(z), -\Delta P(z))$, associated with the discrete-time problem, are real.

2. Some general notation and observations. This section is devoted to the notation and preliminaries to be used in the rest of the paper. The necessary notation and definitions include the following:

- The *spectrum* of a matrix $X \in \mathbb{C}^{n\times n}$ is defined and denoted by $\Lambda(X) = \{\lambda \in \mathbb{C} : \det(X - \lambda I) = 0\}$, where $I \in \mathbb{C}^{n\times n}$ denotes the identity matrix.
- For $X, Y \in \mathbb{C}^{n\times n}$, the *spectrum* of the matrix pencil $P(z) = X - zY$ is defined and denoted by $\Lambda(X, Y) = \{\lambda \in \mathbb{C} : \det(X - \lambda Y) = 0\}$.
- The *spectrum* of the perturbed matrix pencil $P(z, t)$ in (3) (for any fixed t), or equivalently, in (4) (for any fixed z), is denoted by $\Lambda(P(z, t))$.
- The *negative* (resp., *positive*) definite property of a Hermitian matrix X is denoted by $X \prec 0$ (resp., $X \succ 0$).
- The real (resp., imaginary) part of a complex number z is denoted by $\mathrm{Re}(z)$ (resp., $\mathrm{Im}(z)$) .

One interesting question that arises from linear perturbation theory concerns conditions which guarantee that all the eigenvalues of the perturbed pencils $P(z, t)$, for $t \in \mathbb{R}$, remain within a particular open subset of $\mathbb{C}$. For any $z \in \mathbb{C}$, we use the following notation which was introduced by Bora and Mehrmann [9]:

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$$\text{sep}_{\mathbb{R}}(z, \Delta A, \Delta B) := \min\{|t| : \det(P(z,t)) = 0,\, t \in \mathbb{R}\}. \tag{5}$$

A necessary and sufficient condition for the existence of a nonzero real t attaining the minimum in (5) is given by the next lemma.

Lemma 1. ([9]) *For any $z \in \mathbb{C}$, $sep_{\mathbb{R}}(z, \Delta A, \Delta B) < \infty$ if and only if $F_z = -\Delta P(z)P^{-1}(z)$ has a nonzero real eigenvalue for $P(z)$ and $\Delta P(z)$ of the perturbed pencil $P(z,t)$ in (4). Moreover, if $sep_{\mathbb{R}}(z, \Delta A, \Delta B) < \infty$, then*

$$sep_{\mathbb{R}}(z, \Delta A, \Delta B) = \frac{1}{\max\limits_{\mu \in \mathbb{R}}\{|\mu| : \mu \in \Lambda(-\Delta P(z)P^{-1}(z))\}}.$$

A central problem arising from Lemma 1 is to find the conditions under which the matrix $F_z = -\Delta P(z)P^{-1}(z)$ has a nonzero real eigenvalue. Under the assumption that F_z has spectral symmetry, that is, $\Lambda(F_z) = \Lambda(F_z^H)$, Bora and Mehrmann have identified some sufficient conditions for this central question [9]. In what follows, we need the next definition.

Definition 2. ([9]) Hermitian Frobenius factors of a matrix $X \in \mathbb{C}^{n\times n}$ are two Hermitian matrices S and T, where S is nonsingular and $X = TS^{-1}$.

The following theorem from [9, Theorem 3.5] is necessary for the remainder.

Theorem 3. ([9]) *Suppose $P(z)$ and $\Delta P(z)$ are the matrix pencils in (4). Then all the eigenvalues of $F_z := -\Delta P(z)P^{-1}(z)$, $z \in \mathbb{C}\backslash\sigma(P(z))$, are real if its Hermitian Frobenius factors $T(z)$ and $S(z)$ exist and satisfy any of the following conditions:*

(i) $T(z)$ and $S(z)^{-1}$ commute.
(ii) $T(z)$ is positive or negative semidefinite.
(iii) $S(z)$ is positive or negative definite.

It is possible that for some matrix pairs (A, B) and some perturbation matrix pairs $(\Delta A, \Delta B)$, there exists a set $\mathbb{C}_g$ such that $\inf\limits_{z\in\delta\mathbb{C}_g} \text{sep}_{\mathbb{R}}(z, \Delta A, \Delta B) = \infty$, where $\delta\mathbb{C}_g = \mathbb{C}\backslash\mathbb{C}_g$. This means that the eigenvalues of the perturbed pairs $(A + t\Delta A, B + t\Delta B)$ always remain inside $\mathbb{C}_g$ as t varies over the real numbers. This results in two important points:

- For any $z \in \delta\mathbb{C}_g$, there may be no real $t \neq 0$ for which $\det P(z,t) = 0$.
- We should expect some examples with complex t or infinite $|t|$ for which $\det(P(z,t)) = 0$.

The second point suggests at least one research direction for characterizing the set $\delta\mathbb{C}_g$, and one new convention, beyond the classical linear perturbation, for the problems arising in control theory or elsewhere with very large $|t|$, or $|t| \to \infty$ [5].

3. Eigenvalue characterization of linearly perturbed Hermitian/skew-Hermitian matrix pencils. In this section, we study some perturbed matrix pencils $P(z,t)$ with $*$-even (H-even) matrix pencils $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$, where A is Hermitian, ΔA is Hermitian positive or negative definite, and both B and ΔB are skew-Hermitian matrices.

We provide the conditions under which, for some purely imaginary numbers z and some nonzero real numbers t, it holds that $\det P(z,t) = 0$. We use the following definition.

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Definition 4. ([19]) The *field of values* (also known as *numerical range*) of a matrix $X \in \mathbb{C}^{n\times n}$ is defined as:

$$\mathcal{F}(X) = \left\{\mathbf{z}^H X\mathbf{z} \in \mathbb{C} : \mathbf{z} \in \mathbb{C}^n, \mathbf{z}^H\mathbf{z} = 1\right\} \subset \mathbb{C},$$

where $\mathbf{z}^H$ denotes the conjugate transpose of $\mathbf{z}$.

Some applications of the field of values in numerical analysis are presented in [8]. We need the following notation:

- We denote

$$f^X_{min} = \min\{\mathrm{Im}(f) : f \in \mathcal{F}(X)\} \quad \text{and} \quad f^X_{max} = \max\{\mathrm{Im}(f) : f \in \mathcal{F}(X)\},$$

  for $X \in \mathbb{C}^{n\times n}$.
- When $X \in \mathbb{C}^{n\times n}$ is Hermitian, then for any $f \in \mathcal{F}(X)$, $\lambda^X_{min} \le f \le \lambda^X_{max}$, where

$$\lambda^X_{min} = \min\{\lambda : \lambda \in \Lambda(X)\} \quad \text{and} \quad \lambda^X_{max} = \max\{\lambda : \lambda \in \Lambda(X)\}.$$

- For any Hermitian $X \in \mathbb{C}^{n\times n}$ and for any $Y \in \mathbb{C}^{n\times n}$, we denote

$$\beta^{X,Y}_{11} = -\frac{\lambda^X_{min}}{f^Y_{min}}, \quad \beta^{X,Y}_{12} = -\frac{\lambda^X_{min}}{f^Y_{max}}, \quad \beta^{X,Y}_{21} = -\frac{\lambda^X_{max}}{f^Y_{min}}, \quad \text{and} \quad \beta^{X,Y}_{22} = -\frac{\lambda^X_{max}}{f^Y_{max}}.$$

- The open upper half-plane and the open lower half-plane [19, p. 9] of $\mathbb{C}$ are denoted by:

$$UHP = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \quad \text{and} \quad LHP = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}.$$

Proposition 5. *Suppose that we are given two $*$-even (H-even) matrix pencils $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ for Hermitian matrices $A, \Delta A \in \mathbb{C}^{n\times n}$ and skew-Hermitian matrices $B, \Delta B \in \mathbb{C}^{n\times n}$. Then, for any purely imaginary number $z = i\gamma$ ($\gamma \in \mathbb{R}$), there exist $r_z = rank(A - zB)$ nonzero real numbers t such that* $\det(P(z,t)) = 0$ *if any of the following conditions holds, where we denote $\beta_{ij} = \beta^{\Delta A,\Delta B}_{ij}$ ($1 \le i,j \le 2$):*

(a) $\Delta A \succ 0$, $\mathcal{F}(\Delta B) \subset UHP$, and either $\gamma > \beta_{12}$ or $\gamma < \beta_{21}$ when $\beta_{12} > \beta_{21}$.
(b) $\Delta A \succ 0$, $\mathcal{F}(\Delta B) \subset LHP$, and either $\gamma < \beta_{11}$ or $\gamma > \beta_{22}$ when $\beta_{22} > \beta_{11}$.
(c) $\Delta A \prec 0$, $\mathcal{F}(\Delta B) \subset UHP$, and either $\gamma > \beta_{11}$ or $\gamma < \beta_{22}$ when $\beta_{11} > \beta_{22}$.
(d) $\Delta A \prec 0$, $\mathcal{F}(\Delta B) \subset LHP$, and either $\gamma < \beta_{12}$ or $\gamma > \beta_{21}$ when $\beta_{21} > \beta_{12}$.

Proof. We prove the case (a). The proof for each of the cases (b), (c), and (d) is the same as that for the case (a) with some appropriate modifications.

We note that when Y is a skew-Hermitian matrix, then iY is Hermitian. This fact simply shows that for any Hermitian matrix X and any skew-Hermitian matrix Y, both $X + iY$ and $X - iY$ are Hermitian. Hence, for $z = i\gamma$ ($\gamma \in \mathbb{R}$), Hermitian matrices A and ΔA and skew-Hermitian matrices B and ΔB, both $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ in (4) are Hermitian.

Clearly, for any nonzero vector $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^H\Delta B\mathbf{x} = ib_{\mathbf{x}}$ is a purely imaginary number. So, for any purely imaginary number $z = i\gamma$ ($\gamma \in \mathbb{R}$) and any nonzero $\mathbf{x} \in \mathbb{C}^n$, it holds that

$$\mathbf{x}^H\Delta P(z)\mathbf{x} = \mathbf{x}^H\Delta A\mathbf{x} - i\gamma\mathbf{x}^H\Delta B\mathbf{x} = \mathbf{x}^H\Delta A\mathbf{x} + \gamma b_{\mathbf{x}}.$$

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When $\Delta A \succ 0$, we have $\mathbf{x}^H \Delta P(z)\mathbf{x} = \mathbf{x}^H \Delta A\mathbf{x} + \gamma b_{\mathbf{x}} \geq \lambda_{min}^{\Delta A} + \gamma b_{\mathbf{x}}$ with $\lambda_{min}^{\Delta A} > 0$. In this case, for $\mathcal{F}(\Delta B) \subset UHP$ and $\gamma > -\dfrac{\lambda_{min}^{\Delta A}}{b_{\mathbf{x}}}$, we get $\mathbf{x}^H \Delta P(z)\mathbf{x} > 0$. Thus, by introducing a fixed lower bound for γ, which is valid for all $b_{\mathbf{x}}$ $(0 \neq \mathbf{x} \in \mathbb{C}^n)$, we can state that when $\Delta A \succ 0$, $\mathcal{F}(\Delta B) \subset UHP$, and $\gamma > -\dfrac{\lambda_{min}^{\Delta A}}{f_{max}^{\Delta B}} = \beta_{12}$, then the Hermitian matrix $\Delta P(z)$ is positive definite. This proves the first part of (a).

We now provide another condition on γ, where for positive definite matrix ΔA and when $\mathcal{F}(\Delta B) \subset UHP$, the Hermitian matrix $\Delta P(z)$ becomes negative definite. To this end, we see that

$$\mathbf{x}^H \Delta P(z)\mathbf{x} = \mathbf{x}^H \Delta A\mathbf{x} + \gamma b_{\mathbf{x}} \leq \lambda_{max}^{\Delta A} + \gamma b_{\mathbf{x}}.$$

This can be negative if $\gamma < -\dfrac{\lambda_{max}^{\Delta A}}{b_{\mathbf{x}}}$. Therefore, it follows that the Hermitian matrix $\Delta P(z)$ becomes negative definite if $\gamma < -\dfrac{\lambda_{max}^{\Delta A}}{f_{min}^{\Delta B}} = \beta_{21}$.

As the intersection of the two set of γ's obtained above is the empty set, it is true that in this case, $\beta_{12} > \beta_{21}$.

We have shown that for the considered matrices and any $z = i\gamma$ $(\gamma \in \mathbb{R})$ satisfying the first (resp., the second) inequality in case (a), $P(z)$ is Hermitian and $\Delta P(z)$ is Hermitian positive (resp., negative) definite. Hence, for any $z = i\gamma$ $(\gamma \in \mathbb{R})$ with γ satisfying either the first or the second inequality in case (a), there exist $r_z = \operatorname{rank}(A - zB)$ real and nonzero numbers t such that $\det(P(z,t)) = 0$. ∎

Under the conditions of Proposition 5, we expect n nonzero real numbers t in the spectrum $\Lambda(P(z), -\Delta P(z)) = \{t : \ \det(P(z) + t\Delta P(z)) = 0\}$ when $z \notin \Lambda(A,B)$.

Corollary 6. *Suppose that we are given two $*$-even (H-even) matrix pencils $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ for $A, \Delta A, B, \Delta B \in \mathbb{C}^{n\times n}$. Then, for ΔA, ΔB, and $\gamma \in \mathbb{R}$ of $z = i\gamma \notin \Lambda(A,B)$ satisfying any of the conditions (a), (b), (c), or (d) in Proposition 5, all the eigenvalues of $F_z = -\Delta P(z)P^{-1}(z)$ are real and nonzero, and $\Lambda(F_z) = \Lambda(F_z^H)$.*

Proof. We have seen that for the considered matrices and any $z = i\gamma \notin \Lambda(A,B)$ $(\gamma \in \mathbb{R})$ satisfying any of the cases (a), (b), (c), or (d) in Proposition 5, all the eigenvalues, t, of the matrix pair $(P(z), -\Delta P(z))$ are real and nonzero. From Lemma 1, we know that, for $0 \neq t \in \Lambda(P(z), -\Delta P(z))$, we have $\dfrac{1}{t} \in \Lambda(F_z) = \Lambda(-\Delta P(z)P^{-1}(z))$. As $z = i\gamma \notin \Lambda(A,B)$, $P(z)$ is Hermitian and nonsingular. This implies that all the eigenvalues of F_z are real and nonzero, and F_z has Hermitian Frobenius factors $T = -\Delta P(z)$ and $S = P(z)$. The spectral symmetry property, $\Lambda(F_z) = \Lambda(F_z^H)$, of F_z is straightforward via Corollary 2.3 of [9], where it is shown that the existence of Hermitian Frobenius factors for a matrix is necessary and sufficient for its spectral symmetry. ∎

4. Linearly perturbed Hermitian indefinite/skew-Hermitian matrix pencils. In this section, we start by discussing more general problems than in Section 3, and we gradually move on to the closest type of problems related to continuous-time control problems. For MG matrix pencil problems, we characterize the purely imaginary $z = i\gamma$ $(\gamma \in \mathbb{R})$ which may belong to the spectrum of $P(z,t)$ for some nonzero real t. For any 3×3 block Hermitian/skew-Hermitian matrix pencil in continuous-time control problems, we explain that it cannot satisfy conditions under which its eigenvalues for nonzero real perturbation parameter t, go onto the imaginary axis. Moreover, by explaining the difference and the distance between the structure of continuous-time control problems and the MG matrix pencil problems, we provide some numerical approaches for finding

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the nearest problem which, for some nonzero real perturbation parameter t, has at least one eigenvalue z on the imaginary axis.

The main difference between the problems discussed in Section 3 and the problems of this section is that, here, none of the involved matrices is necessarily definite.

Let us start with the next proposition which considers the case where $\mathcal{F}(\Delta B) \subset UHP$ or $\mathcal{F}(\Delta B) \subset LHP$. In the following, we denote $\beta_{ij} = \beta_{ij}^{\Delta A,\Delta B}$ $(1 \leq i,j \leq 2)$ for $\beta_{ij}^{\Delta A,\Delta B}$ defined in Section 3.

Proposition 7. *Suppose that we are given two $*$-even (H-even) matrix pencils $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ for $A, \Delta A, B, \Delta B \in \mathbb{C}^{n\times n}$. Let $z = i\gamma$ $(\gamma \in \mathbb{R})$ be any purely imaginary number. Then there exist $r_z = rank(A - zB)$ nonzero real numbers t such that $\det(P(z,t)) = 0$ if any of the following six conditions holds:*

(a) $\mathcal{F}(\Delta B) \subset UHP$, $\lambda_{min}^{\Delta A} < 0$, and $\gamma > \beta_{11}$.
(b) $\mathcal{F}(\Delta B) \subset UHP$, $\lambda_{max}^{\Delta A} > 0$, and $\gamma < \beta_{21}$.
(c) $\mathcal{F}(\Delta B) \subset UHP$, $\gamma > 0$ if $\lambda_{min}^{\Delta A} = 0$, and $\gamma < 0$ if $\lambda_{max}^{\Delta A} = 0$.
(d) $\mathcal{F}(\Delta B) \subset LHP$, $\gamma < 0$ if $\lambda_{min}^{\Delta A} = 0$, and $\gamma > 0$ if $\lambda_{max}^{\Delta A} = 0$.
(e) $\mathcal{F}(\Delta B) \subset LHP$, $\lambda_{max}^{\Delta A} > 0$, and $\gamma > \beta_{22}$.
(f) $\mathcal{F}(\Delta B) \subset LHP$, $\lambda_{min}^{\Delta A} < 0$, and $\gamma < \beta_{12}$.

Proof. We prove the cases (a), (b), and (c). The cases (d), (e), and (f) can be proven analogously with some appropriate modifications.

For Hermitian matrices A and ΔA, skew-Hermitian matrices B and ΔB, and $z = i\gamma$ $(\gamma \in \mathbb{R})$, we have seen in Proposition 5 that both $P(z)$ and $\Delta P(z)$ are Hermitian. We show that in case (a), $\Delta P(z) = \Delta A - z\Delta B$, for $z = i\gamma$ $(\gamma \in \mathbb{R})$, is positive definite. To do so, observe that

$$\mathbf{x}^H \Delta P(z)\mathbf{x} = \mathbf{x}^H \Delta A\mathbf{x} + \gamma b_{\mathbf{x}} \geq \lambda_{min}^{\Delta A} + \gamma b_{\mathbf{x}},$$

where $ib_{\mathbf{x}} = \mathbf{x}^H \Delta B\mathbf{x}$. As $b_{\mathbf{x}}$ is supposed to be positive and $\lambda_{min}^{\Delta A} < 0$, so $\mathbf{x}^H \Delta P(z)\mathbf{x}$ is positive for any nonzero vector $\mathbf{x} \in \mathbb{C}^n$ when $\gamma > -\dfrac{\lambda_{min}^{\Delta A}}{b_{\mathbf{x}}}$. For giving a fixed lower bound on γ, which is valid for any $b_{\mathbf{x}}$ $(0 \neq \mathbf{x} \in \mathbb{C}^n)$, we use $\gamma > -\dfrac{\lambda_{min}^{\Delta A}}{f_{min}^{\Delta B}} = \beta_{11}$.

Under the conditions of the case (b), we have

$$\mathbf{x}^H \Delta P(z)\mathbf{x} = \mathbf{x}^H \Delta A\mathbf{x} + \gamma b_{\mathbf{x}} \leq \lambda_{max}^{\Delta A} + \gamma b_{\mathbf{x}},$$

where $b_{\mathbf{x}} > 0$ for every nonzero $\mathbf{x} \in \mathbb{C}^n$. When $\lambda_{max}^{\Delta A} > 0$, we have $\mathbf{x}^H \Delta P(z)\mathbf{x} < 0$ for any nonzero $\mathbf{x} \in \mathbb{C}^n$ if $\gamma < -\dfrac{\lambda_{max}^{\Delta A}}{b_x}$. A fixed upper bound for γ is $\gamma < -\dfrac{\lambda_{max}^{\Delta A}}{f_{min}^{\Delta B}} = \beta_{21}$.

For the case (c), we have $\mathbf{x}^H \Delta P(z)\mathbf{x} \geq \lambda_{min}^{\Delta A} + \gamma b_{\mathbf{x}}$. When $\lambda_{min}^{\Delta A} = 0$ and $b_{\mathbf{x}} > 0$, then $\Delta P(z)$ is positive definite if $\gamma > 0$. On the other hand, we have $\mathbf{x}^H \Delta P(z)\mathbf{x} \leq \lambda_{max}^{\Delta A} + \gamma b_{\mathbf{x}}$. Therefore, when $\lambda_{max}^{\Delta A} = 0$, $\Delta P(z)$ is negative definite if $\gamma < 0$. ∎

The next proposition does not involve any inclusion of the field of values $\mathcal{F}(\Delta B)$.

Proposition 8. *Suppose that we are given two $*$-even (H-even) matrix pencils $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ for $A, \Delta A, B, \Delta B \in \mathbb{C}^{n\times n}$. Let $z = i\gamma$ $(\gamma \in \mathbb{R})$ be any purely imaginary number. Then*

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there exist $r_z = rank(A - zB)$ *nonzero real numbers* t *such that* $\det(P(z,t)) = 0$ *if, for every nonzero vector* $\mathbf{x} \in \mathbb{C}^n$ *and* $ib_{\mathbf{x}} = \mathbf{x}^H \Delta B \mathbf{x}$*, either of the following conditions holds:*

(a) $\lambda_{min}^{\Delta A} < 0$ *and* $b_{\mathbf{x}}\gamma > -\lambda_{min}^{\Delta A}$.
(b) $\lambda_{max}^{\Delta A} > 0$ *and* $b_{\mathbf{x}}\gamma < -\lambda_{max}^{\Delta A}$.

Proof. For the given matrices and $z = i\gamma$ ($\gamma \in \mathbb{R}$), it is obtained in the proof of Proposition 5 that both $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ in (4) are Hermitian. When $\lambda_{min}^{\Delta A} < 0$, we see that for any nonzero $\mathbf{x} \in \mathbb{C}^n$,

$$\mathbf{x}^H \Delta P(z)\mathbf{x} = \mathbf{x}^H \Delta A \mathbf{x} + b_{\mathbf{x}}\gamma \geq \lambda_{min}^{\Delta A} + b_{\mathbf{x}}\gamma.$$

In this case, we look for any possible $b_{\mathbf{x}}$ and γ for which $\mathbf{x}^H \Delta P(z)\mathbf{x} > 0$, that is,

$$\lambda_{min}^{\Delta A} + b_{\mathbf{x}}\gamma > 0. \tag{6}$$

As we have no restriction on the set $\mathcal{F}(\Delta B)$, we should expect the cases where, at the same time, $f_{min}^{\Delta B} < 0$ and $f_{max}^{\Delta B} > 0$. For such problems, the possible ordered pairs $(b_{\mathbf{x}}, \gamma)$ satisfying (6) are the ones which localize some region above the right branch or below the left branch of the hyperbola $xy = c$ for $c = -\lambda_{min}^{\Delta A} > 0$, when $b_{\mathbf{x}}$ belongs to the closed interval $[f_{min}^{\Delta B}, f_{max}^{\Delta B}]$.

A sufficient condition for γ to satisfy (6) is $b_{\mathbf{x}}\gamma > -\lambda_{min}^{\Delta A}$, and this happens when for any fixed $\epsilon > 0$, we take $\gamma = \gamma_{\mathbf{x}} = \dfrac{-\lambda_{min}^{\Delta A} + \epsilon}{b_{\mathbf{x}}}$.

For the case (b) where $\lambda_{max}^{\Delta A} > 0$, we have

$$\mathbf{x}^H \Delta P(z)\mathbf{x} = \mathbf{x}^H \Delta A \mathbf{x} + b_{\mathbf{x}}\gamma \leq \lambda_{max}^{\Delta A} + b_{\mathbf{x}}\gamma.$$

We should look for the possible $b_{\mathbf{x}}$ and γ which yield $\lambda_{max}^{\Delta A} + b_{\mathbf{x}}\gamma < 0$ or, equivalently,

$$b_{\mathbf{x}}\gamma < -\lambda_{max}^{\Delta A}. \tag{7}$$

The possible ordered pairs $(b_{\mathbf{x}}, \gamma)$ which satisfy (7) localize some region above the left branch or below the right branch of the hyperbola $xy = c$ for $c = -\lambda_{max}^{\Delta A} < 0$, when $b_{\mathbf{x}}$ belongs to the closed interval $[f_{min}^{\Delta B}, f_{max}^{\Delta B}]$. If we take $\gamma = \gamma_{\mathbf{x}} = \dfrac{-\lambda_{max}^{\Delta A} - \epsilon}{b_{\mathbf{x}}}$, for any fixed $\epsilon > 0$, then (7) holds. $\square$

As an advantage of applying Proposition 8, we observe that the borders derived by the branches of the hyperbolas $xy = c$ for $c = -\lambda_{min}^{\Delta A}$ and $c = -\lambda_{max}^{\Delta A}$ allow us to visualize information which is helpful in finding and describing the smallest possible value of nonzero real t which results in $\det(P(z,t)) = 0$ for the specified values of $z = i\gamma$ ($\gamma \in \mathbb{R}$).

Example 4.1. The matrices A, B, ΔA, and ΔB are 6×6 symmetric matrices randomly generated by using MATLAB's command `X=rand(n,n)` and then symmetrized using MATLAB's command `X=(X+X')/2`. In this example, we verify the sufficient conditions (a) and (b) of Proposition 8 for the related linearly perturbed matrix pencil $P(z,t)$.

Figure 1(a) displays the hyperbola $xy = c$, where $c = -\lambda_{min}^{\Delta A}$ for $\lambda_{min}^{\Delta A} = -1.97 < 0$. We have $\mathcal{F}(\Delta B) = [-i, i]$, and therefore, $f_{min}^{\Delta B} = -1$ and $f_{max}^{\Delta B} = 1$. Based on Proposition 8 (a), we should look for $\gamma \in \mathbb{R}$ that satisfies $b_{\mathbf{x}}\gamma > -\lambda_{min}^{\Delta A}$ for any $b_{\mathbf{x}}$ ($0 \neq \mathbf{x} \in \mathbb{C}^n$). The y-coordinates of the down triangle symbols under the left branch of hyperbola $xy = c$ yields some values of γ such that $b_{\mathbf{x}}\gamma = (-1)\gamma > -\lambda_{min}^{\Delta A}$, i.e., it is valid only for negative values of $b_{\mathbf{x}}$. Also, the y-coordinates of the upper triangle symbols above the right branch

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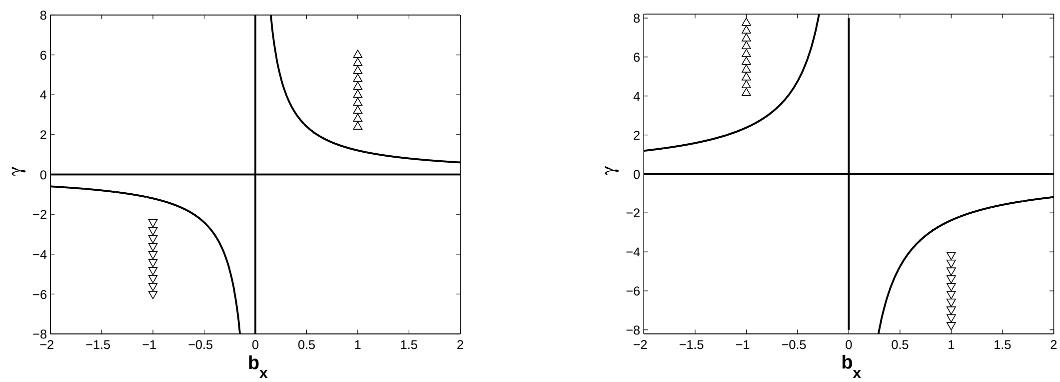
of the same hyperbola show some values of γ such that $b_{\mathbf{x}}\gamma = (+1)\gamma > -\lambda^{\Delta A}_{min}$, that is, it is valid only for positive values of $b_{\mathbf{x}}$. Altogether mean that, for $b_{\mathbf{x}} = -1$ and $z_1 = i\gamma_1$ with $\gamma_1 < -1.97$, there exist $r_{z_1} = \text{rank}(A - z_1B)$ nonzero real numbers t_1 such that $\det(P(z_1, t_1)) = 0$. Also for $b_{\mathbf{x}} = +1$ and $z_2 = i\gamma_2$ with $\gamma_2 > 1.97$, there exist $r_{z_2} = \text{rank}(A - z_2B)$ nonzero real numbers t_2 such that $\det(P(z_2, t_2)) = 0$.

Figure 1(b) displays the hyperbola $xy = c$, where $c = -\lambda^{\Delta A}_{max}$ for $\lambda^{\Delta A}_{max} = 3.99 > 0$. The y-coordinates of the upper triangle symbols above the left branch of hyperbola $xy = c$ yield some values of γ such that for the negative value $b_{\mathbf{x}} = -1$, $b_{\mathbf{x}}\gamma = (-1)\gamma < -\lambda^{\Delta A}_{max}$. Moreover, the y-coordinates of the down triangle symbols under the right branch of that hyperbola show some values of γ such that for the positive value $b_{\mathbf{x}} = +1$, $b_{\mathbf{x}}\gamma = \gamma < -\lambda^{\Delta A}_{max}$. The conclusion is that, for $b_{\mathbf{x}} = -1$ and $z_1 = i\gamma_1$ with $\gamma_1 > 3.99$, there exist $r_{z_1} = \text{rank}(A - z_1B)$ nonzero real numbers t_1 such that $\det(P(z_1, t_1)) = 0$. Also for $b_{\mathbf{x}} = +1$ and $z_2 = i\gamma_2$ with $\gamma_2 < -3.99$, there exist $r_{z_2} = \text{rank}(A - z_2B)$ nonzero real numbers t_2 such that $\det(P(z_2, t_2)) = 0$.



(a) $xy = -\lambda^{\Delta A}_{min}$ and γ's which satisfy (6).

(b) $xy = -\lambda^{\Delta A}_{max}$ and γ's which satisfy (7).

Fig. 1: The values of γ which satisfy (6) or (7).

We continue with some more general results which do not need conditions like the inclusions $\mathcal{F}(\Delta B) \subset UHP$ or $\mathcal{F}(\Delta B) \subset LHP$. Then we revisit the problems with the same conditions as those in Proposition 8 (i.e., $\lambda^{\Delta A}_{min} < 0$ or $\lambda^{\Delta A}_{max} > 0$) and provide some fixed bounds for γ. The next definition will be used in this regard.

Definition 9. ([19]) The *numerical radius* of a matrix $X \in \mathbb{C}^{n\times n}$ is defined and denoted by:

$$r(X) = \sup\{|z| : z \in \mathcal{F}(X)\} = \sup\{|\mathbf{z}^H X\mathbf{z}| : \mathbf{z} \in \mathbb{C}^n, \mathbf{z}^H\mathbf{z} = 1\}.$$

For the matrix 2-norm of $X \in \mathbb{C}^{n\times n}$, it is known (see [19]) that $\dfrac{\|X\|_2}{2} \leq r(X) \leq \|X\|_2$. For more information on the ways of computing numerical radius of matrices, we refer to [16, 27, 31].

The next proposition is a partner of Proposition 7 with the difference that here we use numerical radius of matrix ΔB.

Proposition 10. *Suppose we are given two $*$-even (H-even) matrix pencils $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ for $A, \Delta A, B, \Delta B \in \mathbb{C}^{n\times n}$. Let $z = i\gamma$ ($\gamma \in \mathbb{R}$) be any purely imaginary number. Then there*

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exist $r_z = rank(A - zB)$ *nonzero real numbers* t *such that* $\det(P(z,t)) = 0$ *if any of the following conditions holds:*

(a) $\lambda_{min}^{\Delta A} > 0$ *and* $0 < \gamma < \frac{\lambda_{min}^{\Delta A}}{r(\Delta B)}$.
(b) $\lambda_{min}^{\Delta A} > 0$ *and* $\frac{-\lambda_{min}^{\Delta A}}{r(DB)} < \gamma < 0$.
(c) $\lambda_{max}^{\Delta A} < 0$ *and* $0 < \gamma < \frac{-\lambda_{max}^{\Delta A}}{r(\Delta B)}$.
(d) $\lambda_{max}^{\Delta A} < 0$ *and* $\frac{\lambda_{max}^{\Delta A}}{r(DB)} < \gamma < 0$.

Proof. We only prove the cases (a) and (c). The proofs for the cases (b) and (d) are almost the same.

For every $n \times n$ matrix ΔB and every nonzero vector $\mathbf{x} \in \mathbb{C}^n$, we have

$$|ib_{\mathbf{x}}| = |\mathbf{x}^H \Delta B \mathbf{x}| \leq r(\Delta B).$$

This means that $-r(\Delta B) \leq b_{\mathbf{x}} \leq r(\Delta B)$, which in turn yields

$$-\gamma r(\Delta B) \leq \gamma b_{\mathbf{x}} \leq \gamma r(\Delta B) \quad \text{when} \quad \gamma > 0,$$

and

$$\gamma r(\Delta B) \leq \gamma b_{\mathbf{x}} \leq -\gamma r(\Delta B) \quad \text{when} \quad \gamma < 0.$$

Now the proof of the case (a) is clear because for every nonzero $\mathbf{x} \in \mathbb{C}^n$ and $\gamma > 0$, it holds that

$$\mathbf{x}^H \Delta P(z) \mathbf{x} = \mathbf{x}^H \Delta A \mathbf{x} + b_{\mathbf{x}} \gamma \geq \lambda_{min}^{\Delta A} - \gamma r(\Delta B).$$

So, $\Delta P(z)$ is positive definite if $0 < \gamma < \frac{\lambda_{min}^{\Delta A}}{r(\Delta B)}$.

The proof of the case (c) follows from the fact that $\Delta P(z)$ is negative definite if for $\gamma > 0$ and every nonzero $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^H \Delta P(z) \mathbf{x} = \mathbf{x}^H \Delta A \mathbf{x} + b_{\mathbf{x}} \gamma \leq \lambda_{max}^{\Delta A} + \gamma r(\Delta B)$. ∎

Example 4.2. In this example, we consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \Delta A = \begin{bmatrix} 15 & 1 \\ 1 & 26 \end{bmatrix}, \quad \text{and} \quad \Delta B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

to illustrate the values of γ which satisfy the cases (a) and (b) of Proposition 10. Since $\Delta A \succ 0$, we have $\lambda_{min}^{\Delta A} > 0$. Based on the cases (a) and (b) of Proposition 10, for

$$\gamma \in (-\lambda_{min}^{\Delta A}/r(\Delta B),\ 0) \cup (0,\ \lambda_{min}^{\Delta A}/r(\Delta B)) = (-14.91,\ 0) \cup (0,\ 14.91),$$

we also have $\Delta P(i\gamma) \succ 0$, that is, both eigenvalues of $\Delta P(i\gamma)$ are positive for any γ in this set. Figure 2 shows that both eigenvalues of $\Delta P(i\gamma)$ (blue and red) are positive for any γ in $(-20.75,\ 18.75)$. This confirms the validity of the interval found by the cases (a) and (b) of Proposition 10.

It is worth noting that $\Lambda(A,B) = \{-1.32 + 0.50i, 1.32 + 0.50i\}$, which means that the spectrum is symmetric with respect to the imaginary axis. A close verification shows that, in finite precision, for $z = i\gamma$ and $t = t(z) \in \Lambda(P(z), -\Delta P(z))$, $\min\{|t(z)|\}$ is a continuous descending (resp., ascending) function of γ for $\frac{-\lambda_{min}^{\Delta A}}{r(DB)} < \gamma < 0.55$ (resp., for $0.55 < \gamma < \frac{\lambda_{min}^{\Delta A}}{r(\Delta B)}$). This means that the minimum positive value of t which brings at least one (in this example both) of the finite eigenvalues z of $P(z,t)$ on the imaginary axis is $t = 0.0265$. For this value of t, we have $\Lambda(A + t\Delta A, B + t\Delta B) = \{0.55i, 0.63i\}$.

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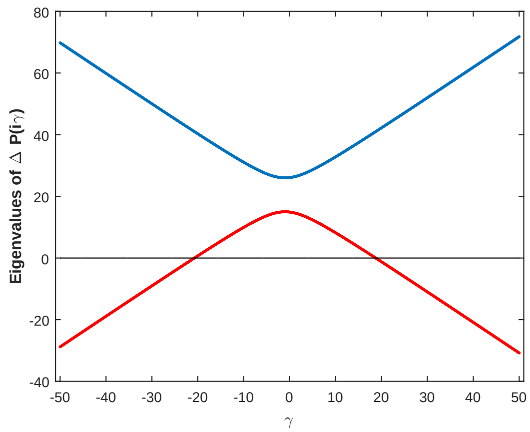

Fig. 2: *Eigenvalues of* $\Delta P(i\gamma)$ *versus* γ *in Example 4.2.*

The next proposition provides some fixed bounds for γ when $\lambda_{min}^{\Delta A} < 0$ or $\lambda_{max}^{\Delta A} > 0$.

PROPOSITION 11. *Suppose we are given two* $*$*-even (*H*-even) matrix pencils* $P(z) = A - zB$ *and* $\Delta P(z) = \Delta A - z\Delta B$ *for* $A, \Delta A, B, \Delta B \in \mathbb{C}^{n\times n}$. *Let* $z = i\gamma$ $(\gamma \in \mathbb{R})$ *be any purely imaginary number. Then there exist* $r_z = rank(A - zB)$ *nonzero real numbers* t *such that* $\det(P(z,t)) = 0$ *if any of the following conditions holds:*

(a) $\lambda_{min}^{\Delta A} < 0$ *and* $-\dfrac{\lambda_{min}^{\Delta A}}{r(\Delta B)} < \gamma$.

(b) $\lambda_{max}^{\Delta A} > 0$ *and* $\gamma < -\dfrac{\lambda_{max}^{\Delta A}}{r(\Delta B)}$.

Proof. The details of the proof are the same as those of the proof of Proposition 10, so we omit them. ∎

4.1. Special 3 × 3 block Hermitian/skew-Hermitian matrix pencils. We are now in a position to consider structured linear perturbation for a special family of 3×3 block $*$-even (H-even) matrix pencils $P(z) = A - zB$ for which the continuous-time control is a particular case of it. The matrix pencils and their perturbation matrix pencils we consider here have the following structure:

$$A = \begin{bmatrix} M_0 & M_1 & M_2 \\ M_1^H & M_4 & M_5 \\ M_2^H & M_5^H & M_6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & M_3 & 0 \\ -M_3^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(8)

$$\Delta A = \begin{bmatrix} \Delta M_0 & \Delta M_1 & \Delta M_2 \\ \Delta M_1^H & \Delta M_4 & \Delta M_5 \\ \Delta M_2^H & \Delta M_5^H & \Delta M_6 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0 & \Delta M_3 & 0 \\ -\Delta M_3^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $M_0, M_1, M_3, M_4, \Delta M_0, \Delta M_1, \Delta M_3, \Delta M_4 \in \mathbb{C}^{n_1\times n_1}$, $M_2, M_5, \Delta M_2, \Delta M_5 \in \mathbb{C}^{n_1\times n_2}$, and $M_6, \Delta M_6 \in \mathbb{C}^{n_2\times n_2}$, with $n = 2n_1 + n_2$. In addition, it is assumed that M_4, M_6, ΔM_4, and ΔM_6 are Hermitian, M_3 is invertible, and $M_3 + \Delta M_3$ remains invertible.

We provide the conditions under which, for this special family of matrix pencils, there exist some sets of purely imaginary $z = i\gamma$ $(\gamma \in \mathbb{R})$ which imply the existence of some nonzero real eigenvalues, t, for the matrix pair $(P(z), -\Delta P(z))$, where $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$.

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For any matrix $X \in \mathbb{C}^{m\times n}$, the set

$$\mathcal{W}(X) = \left\{\mathbf{y}^H X\mathbf{z} : \mathbf{y} \in \mathbb{C}^m,\, \mathbf{z} \in \mathbb{C}^n,\, \mathbf{y}^H\mathbf{y} = \mathbf{z}^H\mathbf{z} = 1\right\} = \{z \in \mathbb{C} : |z| \leq \|X\|_2\}, \tag{9}$$

was studied in [6], and it will be used in what follows. It is apparent that for any $z = \mathrm{Re}(z)+i\mathrm{Im}(z) \in \mathcal{W}(X)$, the real part $\mathrm{Re}(z)$ and the imaginary part $\mathrm{Im}(z)$ satisfy

$$\mathrm{Re}(z), \mathrm{Im}(z) \in [-\|X\|_2, \|X\|_2] = [-\sigma_{max}(X), \sigma_{max}(X)], \tag{10}$$

where $\sigma_{max}(X)$ denotes the largest singular value of X.

Theorem 12. *Suppose we are given two $*$-even (H-even) matrix pencils $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ where each of $A, \Delta A, B, \Delta B \in \mathbb{C}^{n\times n}$ have their corresponding block structure presented in (8). Let $z = i\gamma$ ($\gamma \in \mathbb{R}$) be any purely imaginary number. Denote*

$$\delta m_k = \mathbf{x}_i^H \Delta M_k \mathbf{x}_j, \quad k = 1, 2, 3, 5,$$

and

$$s_1 = \sum_{k=0,4,6} \lambda_{min}^{\Delta M_k}, \quad s_2 = 2\left(\min_{\mathbf{x}_1^H\mathbf{x}_1=1}(\mathrm{Re}(\delta m_1)) - \sum_{k=2,5}\sigma_{max}(\Delta M_k)\right),$$

$$s_3 = \sum_{k=0,4,6} \lambda_{max}^{\Delta M_k}, \quad \textit{and} \quad s_4 = 2\left(\max_{\mathbf{x}_1^H\mathbf{x}_1=1}(\mathrm{Re}(\delta m_1)) + \sum_{k=2,5}\sigma_{max}(\Delta M_k)\right).$$

Then there exist $r_z = rank(A - zB)$ nonzero real numbers t such that $\det(P(z,t)) = 0$ if any of the following conditions holds:

(a) $s_1 + s_2 > 0$ and $\gamma \in (-c, c)$ for $c = \dfrac{s_1+s_2}{2\sigma_{max}(\Delta M_3)}$.

(b) $s_3 + s_4 < 0$ and $\gamma \in (-d, d)$ for $d = -\dfrac{s_3+s_4}{2\sigma_{max}(\Delta M_3)}$.

Proof. For the given matrices and any purely imaginary number $z = i\gamma$ ($\gamma \in \mathbb{R}$), it is already known that both $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ are Hermitian. We show that, for any $z = i\gamma$ satisfying the condition (a), $\Delta P(z)$ is positive definite. This proves that there exist $r_z = \mathrm{rank}(A - zB)$ nonzero real numbers t such that $\det(P(z,t)) = 0$. We do not give the details of the proof of the case (b), since it can be done in the same way.

To proceed, we partition an arbitrary vector $\mathbf{x} \in \mathbb{C}^n$ as $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}$ where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^{n_1}$ and $\mathbf{x}_3 \in \mathbb{C}^{n_2}$ for n_1 and n_2 defined as in (8). Without loss of generality, we assume that $\mathbf{x}_i^H\mathbf{x}_i = 1$, $i = 1, 2, 3$. Then $\mathbf{x}^H\Delta P(i\gamma)\mathbf{x}$ takes the form:

$$\begin{bmatrix} \mathbf{x}_1^H & \mathbf{x}_2^H & \mathbf{x}_3^H \end{bmatrix} \begin{bmatrix} \Delta M_0 & \Delta M_1 - i\gamma\Delta M_3 & \Delta M_2 \\ (\Delta M_1 - i\gamma\Delta M_3)^H & \Delta M_4 & \Delta M_5 \\ \Delta M_2^H & \Delta M_5^H & \Delta M_6 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}$$

$$= \mathbf{x}_1^H\Delta M_0\mathbf{x}_1 + \overbrace{\mathbf{x}_1^H\Delta M_1\mathbf{x}_2}^{\delta m_1} - \overbrace{i\gamma\mathbf{x}_1^H\Delta M_3\mathbf{x}_2}^{i\gamma\delta m_3} + \overbrace{\mathbf{x}_1^H\Delta M_2\mathbf{x}_3}^{\delta m_2} + \overbrace{\mathbf{x}_2^H\Delta M_1^H\mathbf{x}_1}^{\overline{\delta m_1}} - \overbrace{\overline{i\gamma}\mathbf{x}_2^H\Delta M_3^H\mathbf{x}_1}^{\overline{i\gamma\delta m_3}}$$
$$+ \mathbf{x}_2^H\Delta M_4\mathbf{x}_2 + \overbrace{\mathbf{x}_2^H\Delta M_5\mathbf{x}_3}^{\delta m_5} + \overbrace{\mathbf{x}_3^H\Delta M_2^H\mathbf{x}_1}^{\overline{\delta m_2}} + \overbrace{\mathbf{x}_3^H\Delta M_5^H\mathbf{x}_2}^{\overline{\delta m_5}} + \mathbf{x}_3^H\Delta M_6\mathbf{x}_3.$$

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We use $\delta m_k + \overline{\delta m}_k = 2\mathrm{Re}(\delta m_k)$, $k = 1, 2, 5$, and $-i\gamma\delta m_3 - \overline{i\gamma\delta m}_3 = 2\mathrm{Im}(\delta m_3)\gamma$ to summarize:

$$\mathbf{x}^H \Delta P(i\gamma)\mathbf{x} = 2 \sum_{k=1,2,5} \mathrm{Re}(\delta m_k) + 2\mathrm{Im}(\delta m_3)\gamma + \mathbf{x}_1^H \Delta M_0 \mathbf{x}_1 + \mathbf{x}_2^H \Delta M_4 \mathbf{x}_2 + \mathbf{x}_3^H \Delta M_6 \mathbf{x}_3. \tag{11}$$

Here, ΔM_0, ΔM_4 and ΔM_6 are Hermitian, so $\delta m_k \geq \lambda_{min}^{\Delta M_k}$ for $k = 0, 4, 6$. We know that both $\mathrm{Re}(\delta m_k)$ and $\mathrm{Im}(\delta m_k) \geq -\sigma_{max}(\Delta M_k)$ for any $k = 2, 5$. Hence, for the assumed s_1 and s_2 and positive γ, it follows

$$\mathbf{x}^H \Delta P(i\gamma)\mathbf{x} \geq s_1 + s_2 - 2\sigma_{max}(\Delta M_3)\gamma.$$

This shows that when $s_1 + s_2 > 0$ and $0 < \gamma < c = \dfrac{s_1 + s_2}{2\sigma_{max}(\Delta M_3)}$, we have $\Delta P(i\gamma) \succ 0$.

Now, recalling (11), we can assert that when $\gamma \leq 0$,

$$\mathbf{x}^H \Delta P(i\gamma)\mathbf{x} \geq s_1 + s_2 + 2\mathrm{Im}(\delta m_3)\gamma \geq s_1 + s_2 + 2\sigma_{max}(\Delta M_3)\gamma.$$

Therefore, when $\gamma \in (-c, 0]$, we have $\Delta P(i\gamma) \succ 0$. □

Example 4.3. This is an example with block matrices $A, B, \Delta A, \Delta B \in \mathbb{C}^{5\times 5}$, as in (8), with M_1, M_3, $M_4, \Delta M_1, \Delta M_3, \Delta M_4 \in \mathbb{C}^{2\times 2}$, $M_2, M_5, \Delta M_2, \Delta M_5 \in \mathbb{C}^{2\times 1}$, and $M_6, \Delta M_6 \in \mathbb{C}^{1\times 1}$. Here, M_1, M_2, M_3, M_5, $\Delta M_1, \Delta M_2, \Delta M_3, \Delta M_5$ are randomly generated by MATLAB's command randn. The diagonal blocks in ΔA are the diagonal matrices $\Delta M_0 = \mathrm{diag}(6, 5.5)$, $\Delta M_4 = \mathrm{diag}(4.2, 7.1)$ and $\Delta M_6 = 4.2$. M_0 is the 2×2 zero matrix, and M_4 and M_6 are symmetric positive definite matrices randomly generated by MATLAB's command gallery('randsvd',k,-1e1) for $k = 2, 1$, respectively. Here, we report a representative example with the specification as follows. In this example, we have $s_1 + s_2 = 8.075 > 0$, $\sigma_{max}(\Delta M_3) = 1.36$, and $c = 2.96$. So, according to the sufficient conditions (a) of Theorem 12, there exist $r_z = \mathrm{rank}(A - zB)$ nonzero real numbers t such that $\det(P(z, t)) = 0$ if $\gamma \in (-c, c) = (-2.96, 2.96)$. Figure 3(a) shows the values of the eigenvalues of $\Delta P(i\gamma)$ when γ varies in the open interval $[-20, 20]$. Figure 3(b) shows the zoomed case where we can see that, for $\gamma \in (-3.99, 3.99)$, all the five eigenvalues of $\Delta P(i\gamma)$ are positive. Our investigation by MATLAB's function isspd [29] for checking the symmetric positive definiteness of $\Delta P(i\gamma)$, when γ varies in the open interval $[-20, 20]$, supports the same interval $(-3.99, 3.99)$.

The finite eigenvalues of $P(z)$ are $\Lambda(A, B) = \{-1.67 + 0.65i, -1.67 - 0.65i, 1.67 + 0.65i, 1.67 - 0.65i\}$ which means that two eigenvalues are located on the open left half-plane and the two other eigenvalues are located on the open right half-plane, that is, the spectrum is symmetric with respect to origin. Verifying, in finite precision, $\min\{|t(z)|\}$ for $z = i\gamma$, $t = t(z) \in \Lambda(P(z), -\Delta P(z))$, and $-2.96 = -c < \gamma < c = 2.96$, we found that the minimum positive value of t which brings at least one (in this example all) of the finite eigenvalues z of $P(z, t)$ on the imaginary axis is $t = 0.0136$. For this value of t, $\Lambda(A + t\Delta A, B + t\Delta B) = \{-0.93i, 0.93i, -0.54i, 0.54i\}$.

Let us denote

$$\begin{aligned} c_1 &= -\frac{\lambda_{min}^{\Delta M_4} + \lambda_{min}^{\Delta M_6}}{2} - \min_{\mathbf{x}_1^H\mathbf{x}_1=1} (\mathrm{Re}(\delta m_1)) + \sum_{k=2,5} \sigma_{max}(\Delta M_k), \\ c_2 &= -\frac{\lambda_{max}^{\Delta M_4} + \lambda_{max}^{\Delta M_6}}{2} - \min_{\mathbf{x}_1^H\mathbf{x}_1=1} (\mathrm{Re}(\delta m_1)) - \sum_{k=2,5} \sigma_{max}(\Delta M_k). \end{aligned} \tag{12}$$

The next corollary considers the case where $\mathcal{F}(\Delta M_3) \subset \mathbb{R}$, that is, $\mathcal{F}(\Delta M_3)$ includes only real numbers and $\mathrm{Im}(\delta m_3)$ in Theorem 12 is zero for all nonzero vectors $\mathbf{x} \in \mathbb{C}^n$.

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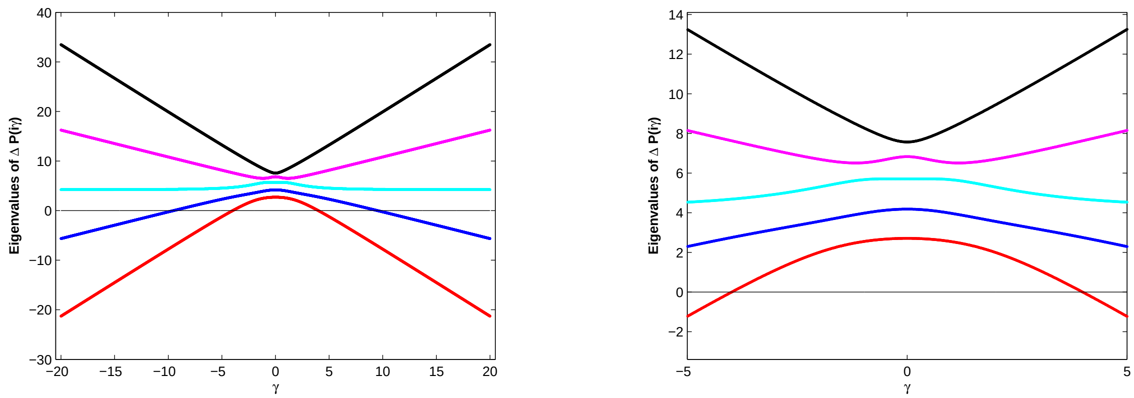

(a) Global illustration (b) Zoomed illustration

Fig. 3: Eigenvalues of $\Delta P(i\gamma)$ as γ varies in $[-20, 20]$.

Corollary 13. *Suppose we are given two $*$-even (H-even) matrix pencils $P(z) = A - zB$ and $\Delta P(z) = \Delta A - z\Delta B$ where each of $A, \Delta A, B, \Delta B \in \mathbb{C}^{n\times n}$ have their corresponding block structure presented in (8). Let $z = i\gamma$ ($\gamma \in \mathbb{R}$) be any purely imaginary number, and let c_1, c_2 be the constants defined in (12). Then there exist $r_z = \text{rank}(A - zB)$ nonzero real numbers t such that $\det(P(z,t)) = 0$ if any of the following conditions holds:*

(a) $c_1 < 0$, $\mathcal{F}(\Delta M_3) \subset \mathbb{R}$, and γ is an arbitrary real number.
(b) $c_2 > 0$, $\mathcal{F}(\Delta M_3) \subset \mathbb{R}$, and γ is an arbitrary real number.

4.2. Problems arising in continuous-time control. For the continuous-time control problems studied in [9, Subsection 4.1], using our notation, the matrices A, ΔA, B, and ΔB have the same structure and properties as those in (8) with a special restriction that for the continuous-time control problems the diagonal blocks M_0 and ΔM_0 are zero matrices. More precisely, we have the following matrix pencils and their perturbation matrix pencils

$$A = \begin{bmatrix} 0 & M_1 & M_2 \\ M_1^H & M_4 & M_5 \\ M_2^H & M_5^H & M_6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & M_3 & 0 \\ -M_3^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\Delta A = \begin{bmatrix} 0 & \Delta M_1 & \Delta M_2 \\ \Delta M_1^H & \Delta M_4 & \Delta M_5 \\ \Delta M_2^H & \Delta M_5^H & \Delta M_6 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0 & \Delta M_3 & 0 \\ -\Delta M_3^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

where all the assumptions are identical with the problem in (8) except that here $M_0 = 0$ and $\Delta M_0 = 0$. Evidently, this difference makes it impossible (for this kind of problems) to find some conditions under which $\Delta P(z) = \Delta A - z\Delta B$ becomes positive or negative definite for some $z = i\gamma$ (with $\gamma \in \mathbb{R}$), since at least one diagonal entry of $\Delta P(z)$ is always zero; see [10, Theorem 6.23]. One may also apply a generalization of [14, Proposition 16.1] to complex matrices on 2×2 block representation of $\Delta P(z)$, that is, on

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$$\Delta P(z) = \left[\begin{array}{c|cc} 0 & \Delta M_1 - z\Delta M_3 & \Delta M_2 \\ \hline \Delta M_1^H + z\Delta M_3^H & \Delta M_4 & \Delta M_5 \\ \Delta M_2^H & \Delta M_5^H & \Delta M_6 \end{array}\right].$$

to show that $\Delta P(z)$ cannot be positive definite.

Nevertheless, for any purely imaginary number $z = i\gamma$ ($\gamma \in \mathbb{R}$), both $P(z)$ and $\Delta P(z)$ are still Hermitian. Therefore, the most natural aim is to look for the nearest linear perturbed block structured matrix pencil (i.e., both modified A and ΔA remain Hermitian and both modified B and ΔB remain skew-Hermitian) that can satisfy necessary and/or sufficient conditions for the existence of nonzero real t such that $\det(P(z,t)) = 0$ becomes available.

A way to get around this issue can be applying on ΔA a complex extension of the method provided in [17], where it has been proved that for an arbitrary real matrix X, the matrix $\bar{X} = \dfrac{Y+H}{2}$, with H the symmetric polar factor of $Y = \dfrac{X+X^T}{2}$, is the nearest symmetric positive semidefinite matrix to X, with respect to the Frobenius norm. Another alternative can be an adaptation of [11] where for a Hermitian matrix pair (X, Y), one wishes to find the distance to the nearest definite matrix pair. This means that we should find, for instance, the distance to the nearest definite matrix pair $(\overline{\Delta A}, -\overline{\Delta B}) = (\Delta A + \Delta\tilde{A}, -(\Delta B + \Delta\tilde{B}))$ such that there exists some real γ with $\overline{\Delta A} - (i\gamma)\overline{\Delta B} \succ 0$.

Because of the special block structure of the continuous-time control problems, we may suggest either of the following numerical approaches:

(1) Looking for the minimum change in the diagonal block ΔM_0 of ΔA such that $\Delta P(i\gamma)$ can satisfy conditions which ensure that $\det(P(i\gamma, t)) = 0$ for some nonzero real t.
(2) Looking for the minimum change in the diagonal blocks ΔM_0, ΔM_4 and ΔM_6 of ΔA such that the existence of the conditions mentioned in the numerical approach 1 above is guaranteed.

It should be noted that by using any of the above methods, we use a linear perturbation that does not preserve the block structure of the existing matrices (at least in terms of keeping ΔM_0 as a zero matrix). However, the matrices A and ΔA remain Hermitian, and the matrices B and ΔB remain skew-Hermitian.

In the following example, we use the approach (2) to impose changes on ΔM_0, ΔM_4, and ΔM_6.

Example 4.4. We use the same fixed matrices and the randomly generated matrices as those in Example 4.3. The difference between these two examples is that here, in addition to $M_0 = 0$, we have $\Delta M_0 = 0$. To follow the approach (2) above, we may look for the minimum nonnegative real number α such that, for $\Delta M_0 = 0$ replaced by $\alpha I_{2\times 2}$, for ΔM_4 replaced by $\Delta M_4 + \alpha I_{2\times 2}$, and for ΔM_6 replaced by $\Delta M_6 + \alpha$, there exists a real number γ such that the modified $\Delta P(i\gamma)$ is Hermitian positive definite. Based on our numerical experiments in MATLAB, where we let α varying in $[0, 2)$, we estimate the minimum possible value $\alpha = 0.95$, for which $\Delta P(i\gamma) \succ 0$. Indeed, for this value of α, we have $(-c, c) = (-0.32, 0.32)$ by the formula given in the case (a) of Theorem 12. This is a subset of what we found in practice, that is, $(-0.7, 0.7)$. Figure 4 illustrates the applicable values of γ versus α. To check the positive definiteness of $\Delta P(i\gamma)$, we used MATLAB's function isspd [29]. We can use the same idea as in Example 4.3 to compute and use $\min\{|t(z)|\}$ of the nearest matrix pair $(P_\alpha(z), -\Delta P_\alpha(z))$ with $\alpha = 0.95$, $z = i\gamma$, and $\gamma \in (-c, c) = (-0.32, 0.32)$.

5. Special 3 × 3 block non-Hermitian perturbed matrix pairs. In this section, we consider the matrix pencils arising in discrete-time control (item III in Section 1) and address the question of finding

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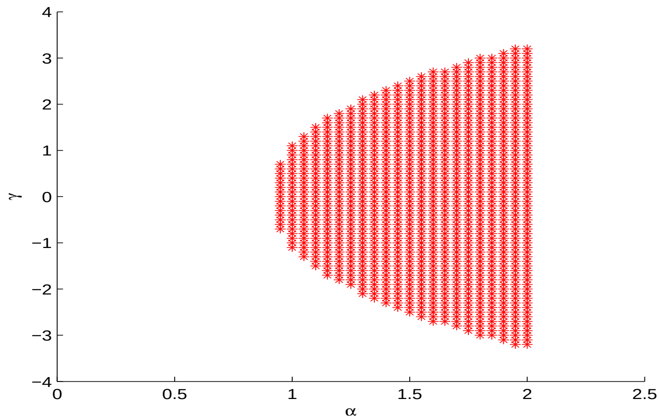

Fig. 4: *Values of γ versus α.*

the smallest nonzero real value t such that the perturbed matrix pencil has one or more eigenvalues on the unit circle. For such problems, the $n \times n$ matrices A and B of the matrix pencil related to the discrete-time control and the $n \times n$ matrices ΔA and ΔB of the perturbation matrix pencil are as follows:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & M_1 & M_2 \\ -M_3^H & M_4 & M_5 \\ 0 & M_5^H & M_6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & M_3 & 0 \\ -M_1^H & 0 & 0 \\ -M_2^H & 0 & 0 \end{bmatrix}, \\
\Delta A &= \begin{bmatrix} 0 & \Delta M_1 & \Delta M_2 \\ -\Delta M_3^H & \Delta M_4 & \Delta M_5 \\ 0 & \Delta M_5^H & \Delta M_6 \end{bmatrix}, \quad \text{and} \quad \Delta B = \begin{bmatrix} 0 & \Delta M_3 & 0 \\ -\Delta M_1^H & 0 & 0 \\ -\Delta M_2^H & 0 & 0 \end{bmatrix},
\end{aligned} \tag{14}
$$

where $M_1, M_3, M_4, \Delta M_1, \Delta M_3, \Delta M_4 \in \mathbb{C}^{n_1 \times n_1}$, $M_6, \Delta M_6 \in \mathbb{C}^{n_2 \times n_2}$, and $M_2, M_5, \Delta M_2, \Delta M_5 \in \mathbb{C}^{n_1 \times n_2}$, with $n = 2n_1 + n_2$. In addition, it is assumed that M_4, M_6, ΔM_4, and ΔM_6 are Hermitian.

As in Section 4.2, here, it is not possible for $\Delta P(z) = \Delta A - z\Delta B$ to be positive or negative definite, because at least one diagonal entry of $\Delta P(z)$ is always zero; see [10, Theorem 6.23]. Therefore, we use the following three phases to apply the results and the treatment suggested in Section 4:

- (P_1) We use the approach introduced and analyzed in [35] to transform the matrix pairs (A, B) and $(\Delta A, \Delta B)$ in (14) to the matrix pairs in the form (13).
- (P_2) Then, using the same arguments as those in Section 4.2, we look for the nearest matrix pencil related to the obtained continuous-time matrix pencil for which we can find the smallest nonzero real number t that brings some eigenvalues of the perturbed matrix pencil on the imaginary axis.
- (P_3) Since the transformation we use in phase (P_1) is one-to-one and invertible [35], we can use the results obtained in phase (P_2) to find the smallest nonzero real number t which brings some eigenvalues of the perturbed (nearest) matrix pencil related to the discrete-time problem on the unit circle.

In the continuation of this section, we first review the Cayley transformation and its generalization to matrix pairs. Then, we will discuss in detail the (safer) way of implementing the above three phases.

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5.1. The Cayley transformation and its generalization to matrix pairs. Let's start with the Cayley transformation and its generalization to matrix pairs. The Cayley transformation $\mathbf{c} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, is defined by $\mu = \mathbf{c}(\lambda) = (\lambda - 1)(\lambda + 1)^{-1}$. This transformation can be generalized to the space of matrix pairs by $\mathbf{C}(A, B) = (A - B, A + B)$ [20, 25] (where we use $\mathbf{C}$ instead of $\mathbf{c}$). If we apply it on the discrete-time matrix pair $(A_d, B_d) = (A, B)$ in (14), then we get $(\widetilde{A}, \widetilde{B})$, where each eigenvalue pair $(\lambda, \bar{\lambda}^{-1})$ of $A_d - \lambda B_d$ is transformed to the eigenvalue pair $(\mu, -\bar{\mu})$ of $\widetilde{A} - \mu\widetilde{B}$, with $\mu = \mathbf{c}(\lambda)$ and $-\bar{\mu} = \mathbf{c}(\bar{\lambda}^{-1})$. That is, the eigenvalues of $(\widetilde{A}, \widetilde{B})$ have the same symmetric pattern as the eigenvalues of (A_c, B_c) in (1). As $\widetilde{A} - \mu\widetilde{B}$ does not have the same block structure as $A_c - \lambda B_c$, it cannot be put into the continuous-time control setting. There exist some suggestions ([20, 22, 25, 26]) to remedy this lackness, but still their proper applicability requires the nonsingularity of certain matrices associated with the blocks in A_d and B_d, which may not always hold [25]. Even if the nonsingularity of the submatrices in the block matrices A_d and B_d holds, with the presence of matrix inversions, the resulting Hamiltonian matrix may be still hard to interpret. Therefore, we use an one-to-one transformation, reviewed in next section, to connect the discrete-time matrix pair and the continuous-time matrix pair directly.

5.2. One-to-one transformation to continuous-time matrix pencil. What we actually do here is to apply the same idea as in the one-to-one transformation in [35] on the matrix pairs of (14) to get the following:

$$\begin{aligned}(A_c, B_c) &= \mathbf{T}(A_d, B_d),\\ (\Delta A_c, \Delta B_c) &= \mathbf{T}(\Delta A_d, \Delta B_d).\end{aligned} \tag{15}$$

More precisely, for the matrices M_i, $i = 1, 2, \ldots, 6$, in (14), and for

$$F = \begin{bmatrix} M_3 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} M_4 & M_5 \\ M_5^H & M_6 \end{bmatrix},$$

the transformation can be described by the following two steps (recalling that $\mu = \mathbf{c}(\lambda)$):

Step 1:

$$\begin{bmatrix} 0 & G \\ -F^H & D \end{bmatrix} - \lambda \begin{bmatrix} 0 & F \\ -G^H & 0 \end{bmatrix} \quad \longrightarrow \quad \widetilde{A}_d - \lambda\widetilde{B}_d,$$

where $\widetilde{A}_d - \lambda\widetilde{B}_d = \begin{bmatrix} 0 & G - F \\ (G - F)^H & D \end{bmatrix} - \mu \begin{bmatrix} 0 & G + F \\ -(G + F)^H & D \end{bmatrix}$.

Step 2:

$$\widetilde{A}_d - \lambda\widetilde{B}_d \quad \longrightarrow \quad \begin{bmatrix} 0 & G - F \\ (G - F)^H & D \end{bmatrix} - \mu \begin{bmatrix} 0 & G + F \\ -(G + F)^H & 0 \end{bmatrix}.$$

Similarly, for the matrices ΔM_i, $i = 1, 2, \ldots, 6$, in (14), and for

$$\Delta F = \begin{bmatrix} \Delta M_3 & 0 \end{bmatrix}, \quad \Delta G = \begin{bmatrix} \Delta M_1 & \Delta M_2 \end{bmatrix} \quad \text{and} \quad \Delta D = \begin{bmatrix} \Delta M_4 & \Delta M_5 \\ \Delta M_5^H & \Delta M_6 \end{bmatrix},$$

the transformation can be described by the following two steps (recalling that $\mu = \mathbf{c}(\lambda)$):

Step 1:

$$\begin{bmatrix} 0 & \Delta G \\ -\Delta F^H & \Delta D \end{bmatrix} - \lambda \begin{bmatrix} 0 & \Delta F \\ -\Delta G^H & 0 \end{bmatrix} \quad \longrightarrow \quad \widetilde{\Delta A}_d - \lambda\widetilde{\Delta B}_d,$$

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where $\widetilde{\Delta A}_d - \lambda\widetilde{\Delta B}_d = \begin{bmatrix} 0 & \Delta G - \Delta F \\ (\Delta G - \Delta F)^H & \Delta D \end{bmatrix} - \mu \begin{bmatrix} 0 & \Delta G + \Delta F \\ -(\Delta G + \Delta F)^H & \Delta D \end{bmatrix}$.

Step 2:

$$\widetilde{\Delta A}_d - \lambda\widetilde{\Delta B}_d \quad \longrightarrow \quad \begin{bmatrix} 0 & \Delta G - \Delta F \\ (\Delta G - \Delta F)^H & \Delta D \end{bmatrix} - \mu \begin{bmatrix} 0 & \Delta G + \Delta F \\ -(\Delta G + \Delta F)^H & 0 \end{bmatrix}.$$

This means that the transformation $\mathbf{T}$ is just the Cayley transformation $\mathbf{C}$ followed by a droping transformation which drops matrix D (resp., ΔD) from the second block matrix of the matrix pair $(\widetilde{A}_d, \widetilde{B}_d)$ (resp., $(\widetilde{\Delta A}_d, \widetilde{\Delta B}_d)$) to get the matrix pair (A_c, B_c) (resp., $(\Delta A_c, \Delta B_c)$). Also its inverse is an adding transformation, which adds matrix D (resp., ΔD) to the second block matrix of the matrix pair (A_c, B_c) (resp., $(\Delta A_c, \Delta B_c)$), followed by the inverse of the Cayley transformation to get (A_d, B_d) (resp., $(\Delta A_d, \Delta B_d)$). This transformation establishes an equivalence relation between the eigenstructures of the matrix pencils $A_d - \lambda B_d$ (resp., $\Delta A_d - \lambda\Delta B_d$) and $A_c - \mu B_c$ (resp., $\Delta A_c - \mu\Delta B_c$); in particular, $\lambda \in \Lambda(A_d, B_d)\backslash\{-1, \infty\}$ if and only if $\mu \in \Lambda(A_c, B_c)\backslash\{\infty, 1\}$ [35, Theorem 16]. It is interesting that λ and μ have the same partial, algebraic and geometric multiplicities. More properties in this regard can be found in [35, Section 4].

In the following example, we use the one-to-one transformation $\mathbf{T}$ in (15) to transform the discrete-time matrix pencil to the continuous-time matrix pencil. Then, we look for the minimum change in the diagonal blocks ΔM_0, ΔM_4 and ΔM_6 of ΔA_c such that $\Delta P_c(i\gamma) = \Delta A_c - i\gamma\Delta B_c$ has conditions which ensure that for some nonzero real t, and some $\gamma \in \mathbb{R}$, $\det(P(i\gamma, t)) = 0$. For the estimated minimum change, α, we let γ vary in $(-c, c)$ (for c computed via Theorem 12) and check the positive definiteness of $\Delta P_c(i\gamma)$. Finally, using the equivalence relation between the eigenvalues of both discrete-time matrix pencil and the continuous-time matrix pencil, we recover and illustrate the complex numbers z on the unit circle which ensure that the matrix pencil $\Delta P_d(z) = \Delta A_d - z\Delta B_d$ become positive definite. We remark that here the matrix pencil $\Delta P_d(z)$ is the one that has one-to-one equivalence relation with the nearest matrix pencil obtained for the continuous-time problem.

Example 5.1. For the discrete-time matrix pencil examined here, we use the same fixed matrices and the randomly generated matrices as those in Example 4.3. The difference is that, in addition to $M_0 = 0$ in Example 4.3, here we have $\Delta M_0 = 0$. We look for the minimum nonnegative real number α such that for $\Delta M_0 = 0$ replaced by $\alpha I_{2\times 2}$, for ΔM_4 replaced by $\Delta M_4 + \alpha I_{2\times 2}$, and for ΔM_6 replaced by $\Delta M_6 + \alpha$, there exists a real interval $(-c, c)$ such that for any $\gamma \in (-c, c)$, the matrix $\Delta P_c(i\gamma)$ associated with the obtained equivalence continuous-time matrix pencil becomes Hermitian positive definite. Then, we use $\lambda = \frac{1+\mu}{1-\mu}$ to recover the corresponding values $z = \lambda$ on the unit circle for which $\Delta P_d(z)$ (associated with the nearest matrix pencil obtained for the continuous-time problem) becomes positive definite. One of our numerical experiments in MATLAB shows that the minimum possible value of α for which $\Delta P_c(i\gamma) \succ 0$ is $\alpha = 0.9$. For the modified problem obtained with this value of α, Theorem 12 (a) provides us with $(-c, c) = (-1.23, 1.23)$. In practice, all the values of $\gamma \in (-c, c)$ satisfy $\Delta P_c(i\gamma) \succ 0$. Figure 5(a) shows these values of γ's versus $\alpha = 0.9$ for the nearest matrix pencil obtained for the continuous-time problem. The small red circles in Figure 5(b) show the values of $z = \lambda$ on the unit circle such that $\Delta P_d(z) \succ 0$. We checked the positive definiteness of the mentioned matrices, using MATLAB's function `isspd` [29].

6. Conclusions. We have characterized the eigenvalues of the perturbed matrix pencil $P(z, t)$ when the perturbation parameter t is real and nonzero. In our study, the first two families of matrix pencils and their perturbation matrix pencils have general or block Hermitian/skew-Hermitian matrix pairs. A specific block structure of the second family of the considered matrix pencils appears in continuous-time control. The

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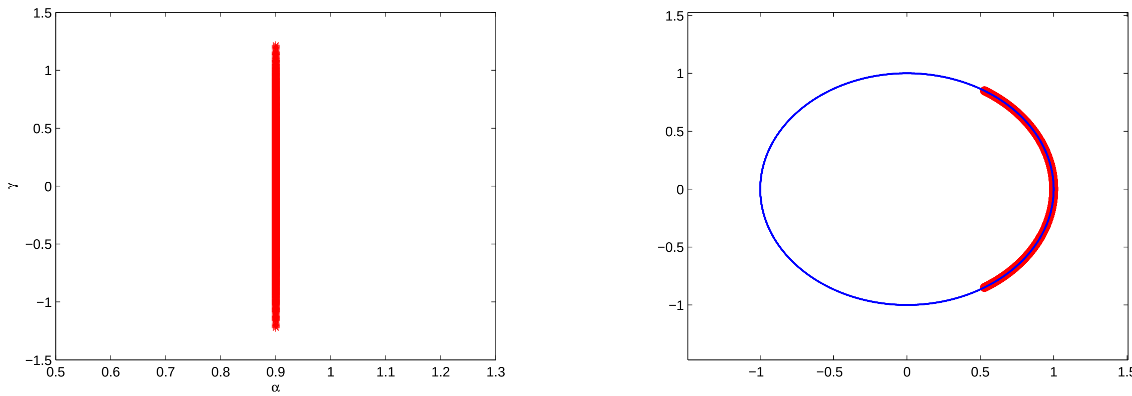

(a) γ values (b) z values

Fig. 5: The values of γ (resp., z with $|z| = 1$) which make $\Delta P_c(i\gamma) \succ 0$ (resp., $\Delta P_d(z) \succ 0$) for $\alpha = 0.9$.

third family of matrix pencils and their perturbation matrix pencils is a special block non-Hermitian/non-Hermitian matrix pairs which arise in discrete-time control.

Our approach of dealing with the above problems is different than those in the literature, in the sense that it provides some practical techniques which result in some bounds on the set of specified z, such that for some nonzero real t, $\det(P(z,t)) = 0$. What we have achieved for more general matrix pairs, in all the three families, is to apply an idea of inverse eigenvalue problem for introducing some subsets of z on the imaginary axis or on the unit circle (depend on the family of the problem) for which the matrix pair $(P(z), -\Delta P(z))$ has at least one nonzero real eigenvalue. We arrived to distinguish that for continuous-time control problems (resp., discrete-time control problems), with z on the imaginary axis (resp., on the unit circle), it is not guaranteed that the matrix pair $(P(z), -\Delta P(z))$ has at least one nonzero real eigenvalue. Therefore, for each of the control problems, we have suggested to look for the nearest block problem with the same structure (except for the matrix ΔM_0) which at least the existence of one nonzero real eigenvalue of the matrix pair $(P(z), -\Delta P(z))$ can be guaranteed.

There are several interesting questions which should be considered in future works. One of them is to strengthen the proposed upper and lower bounds for the set of z on the imaginary axis (resp., the unit circle) when we study the two first families (resp., the third family) of the problems. Another research direction is the study of problems arising in control theory or elsewhere with very large $|t|$, or $|t| \to \infty$.

Appendix A. Direct eigenvalue characterization of discrete-time matrix pencil.

For the block matrices in (14), each one of the conditions (i), (ii), and (iii) of Theorem 3 proved in [9, Theorem 3.5] guarantees that all the eigenvalues of $(\Delta A - z\Delta B)(A - zB)^{-1}$ are real. These conditions can also be used as sufficient conditions for $(A + t\Delta A, B + t\Delta B)$, $t \in \mathbb{R}$, to have a complex eigenvalue z with $|z| = 1$. However, Bora and Mehrmann preferred to introduce some other necessary and sufficient conditions for this problem (see Theorem 5.2 in [9]).

Due to the existence of the zero diagonal blocks M_0 and ΔM_0 in the matrices A and ΔA, respectively, there is no guarantee that the pair $(P(z), -\Delta P(z))$ has a nonzero real eigenvalue t; see Sections 4.2 and

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5. This deficiency also prevents the matrix $S(z)$ from being positive or negative definite. For the same reason, the matrix $T(z)$ cannot be positive or negative semidefinite. One reasonable and reliable suggestion is applying similar numerical approaches as those suggested in Section 5. Nevertheless, let us use Theorem 3 and derive some sufficient conditions under which, for a subset of z on the unit circle, all the eigenvalues of $(P(z), -\Delta P(z))$ are real.

Under the assumption that $\Lambda(F_z) = \Lambda(F_z^H)$, for $F_z = -\Delta P(z)P^{-1}(z)$, it has been proved in [9, Theorem 5.1] that the matrices:

(16)
$$T(z) = \begin{bmatrix} 0 & \Delta M_1 - z\Delta M_3 & \Delta M_2 \\ (\Delta M_1 - z\Delta M_3)^H & \Delta M_4 & \Delta M_5 \\ \Delta M_2^H & \Delta M_5^H & \Delta M_6 \end{bmatrix} \quad \text{and} \quad S(z) = \begin{bmatrix} 0 & M_1 - zM_3 & M_2 \\ (M_1 - zM_3)^H & M_4 & M_5 \\ M_2^H & M_5^H & M_6 \end{bmatrix}$$

are Hermitian Frobenius factors (see Definition 2) of F_z when $z \in \mathbb{C}$ and $|z| = 1$. In what follows, we provide some new sufficient conditions for which (see (4)) all the eigenvalues of the pair $(P(z), -\Delta P(z))$ are real and nonzero when $|z| = 1$. The sufficient conditions we obtain are equivalent to the sufficient conditions (ii) and (iii) of Theorem 3.

To proceed, let us rewrite the matrix $S(z)$ in (16) as:

$$S(z) = S_1 - zS_2 - \bar{z}S_3, \tag{17}$$

where

$$S_1 = \begin{bmatrix} 0 & M_1 & M_2 \\ M_1^H & M_4 & M_5 \\ M_2^H & M_5^H & M_6 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & M_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad S_3 = \begin{bmatrix} 0 & 0 & 0 \\ M_3^H & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $S_3 = S_2^H$, and thus, using the notation $s_{2\mathbf{x}} = \mathbf{x}^H S_2 \mathbf{x}$ and $s_{3\mathbf{x}} = \mathbf{x}^H S_3 \mathbf{x}$, the quadratic form of $S(z)$ (for any $\mathbf{x} \in \mathbb{C}^n$) becomes

$$\mathbf{x}^H S(z)\mathbf{x} = \mathbf{x}^H S_1 \mathbf{x} - z\mathbf{x}^H S_2 \mathbf{x} - \bar{z}\mathbf{x}^H S_2^H \mathbf{x} = \mathbf{x}^H S_1 \mathbf{x} - zs_{2\mathbf{x}} - \bar{z}\bar{s}_{2\mathbf{x}}. \tag{18}$$

We use the notation $\overrightarrow{Op}$ as geometric representation of the complex number p, that is, the vector from the origin to p. We also use $\langle \overrightarrow{Op_1}, \overrightarrow{Op_2} \rangle$ for the standard inner product of two (geometric) vectors $\overrightarrow{Op_1}$ and $\overrightarrow{Op_2}$ ($p_1, p_2 \in \mathbb{C}$).

Since $T(z)$ and $S(z)$ have the same structure, the details given for $S(z)$ imply that

$$T(z) = T_1 - zT_2 - \bar{z}T_3, \tag{19}$$

where T_1, T_2, and T_3 have the same structure as S_1, S_2, and S_3, respectively, but all the block matrices X_i are to be replaced by ΔX_i. Therefore, for any nonzero vector $\mathbf{x} \in \mathbb{C}^n$, we have

$$\mathbf{x}^H T(z)\mathbf{x} = \mathbf{x}^H T_1 \mathbf{x} - z\mathbf{x}^H T_2 \mathbf{x} - \bar{z}\mathbf{x}^H T_2^H \mathbf{x} = \mathbf{x}^H T_1 \mathbf{x} - zt_{2\mathbf{x}} - \bar{z}\bar{t}_{2\mathbf{x}}.$$

Our first list of sufficient conditions which ensure the existence of nonzero real eigenvalues for $(P(z), -\Delta P(z))$ is as follows. Here, we use the notion of numerical radius $r(X)$ defined in Definition 9 for any $X \in \mathbb{C}^{n\times n}$.

PROPOSITION 14. *Suppose that we are given a matrix pencil (A, B) and an associated perturbation matrix pencil $(\Delta A, \Delta B)$ with A, B, ΔA, and ΔB as in (14). Let z be a complex number such that $|z| = 1$. Suppose also that $S(z)$, S_1, and S_2 are the matrices defined in (17), and $T(z)$, T_1, and T_2 are those of (19). Then all the eigenvalues of $(P(z), -\Delta P(z))$ are nonzero and real under any of the following conditions:*

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(a) $S_1 \succ 0$ *and* $\lambda_{min}^{S_1} > 2r(S_2)$.
(b) $S_1 \prec 0$ *and* $\lambda_{max}^{S_1} < -2r(S_2)$.
(c) $T_1 \succ 0$ *and* $\lambda_{min}^{T_1} \geq 2r(T_2)$.
(d) $T_1 \prec 0$ *and* $\lambda_{max}^{T_1} \leq -2r(T_2)$.

Proof. We prove the cases (a) and (b). The proof for the cases (c) and (d) are simple modifications of the proofs of (a) and (b), respectively.

For any such z, the matrix $S(z)$ is Hermitian. So, based on the sufficient condition (iii) in Theorem 3, we only need to find the conditions that for any complex number z on the unit circle, the matrix $S(z)$ becomes positive or negative definite. We represent the complex number z on the unit circle and $s_{2\mathbf{x}}$ in (18) as $z = z_1 + iz_2$ and $s_{2\mathbf{x}} = s_{21\mathbf{x}} + is_{22\mathbf{x}}$.

For the case (a), we notice that the quadratic form (18) of $S(z)$ satisfies

$$\mathbf{x}^H S(z)\mathbf{x} = \mathbf{x}^H S_1\mathbf{x} - zs_{2\mathbf{x}} - \overline{z}\overline{s}_{2\mathbf{x}} = \mathbf{x}^H S_1\mathbf{x} - 2(z_1 s_{21\mathbf{x}} - z_2 s_{22\mathbf{x}}) = \mathbf{x}^H S_1\mathbf{x} - 2\langle\overrightarrow{Oz}, \overrightarrow{O\overline{s}_{2\mathbf{x}}}\rangle \geq \lambda_{min}^{S_1} - 2\langle\overrightarrow{Oz}, \overrightarrow{O\overline{s}_{2\mathbf{x}}}\rangle$$

for $\langle\overrightarrow{Oz}, \overrightarrow{O\overline{s}_{2\mathbf{x}}}\rangle = |\overrightarrow{Oz}|\,|\overrightarrow{O\overline{s}_{2\mathbf{x}}}|\cos(\theta)$, where θ is the angle between $\overrightarrow{Oz}$ and $\overrightarrow{O\overline{s}_{2\mathbf{x}}}$. As $|\overrightarrow{Oz}| = 1$, and for any nonzero $\mathbf{x} \in \mathbb{C}^n$, it holds that $|\overrightarrow{O\overline{s}_{2\mathbf{x}}}| \leq r(\overline{S_2}) = r(S_2)$, and hence, $S(z)$ is positive definite if $\lambda_{min}^{S_1} > 2r(S_2)$.

For the case (b), with the same arguments as in the case (a), we have

$$\mathbf{x}^H S(z)\mathbf{x} = \mathbf{x}^H S_1\mathbf{x} - 2\langle\overrightarrow{Oz}, \overrightarrow{O\overline{s}_{2\mathbf{x}}}\rangle \leq \lambda_{max}^{S_1} + 2\langle\overrightarrow{Oz}, \overrightarrow{O\overline{s}_{2\mathbf{x}}}\rangle.$$

So, $S(z)$ is negative definite if $\lambda_{max}^{S_1} < -2|\overrightarrow{Oz}|\,|\overrightarrow{O\overline{s}_{2\mathbf{x}}}|$ or $\lambda_{max}^{S_1} < -2r(S_2)$. ∎

The following notes concerning Proposition 14 are useful:

- Neither of S_1 and T_1 can be positive or negative definite, because their first diagonal block matrices are zero matrices.
- For applying (a) and (b) of Proposition 14, we need to assess positive or negative definiteness of S_1 and depending on that property of S_1, we should compute either $\lambda_{min}^{S_1}$ or $\lambda_{max}^{S_1}$. We also need to compute $r(S_2)$. The same assessments on T_1 and the same computations for $\lambda_{min}^{T_1}$ or $\lambda_{max}^{T_1}$, and for $r(T_2)$, should be done when we want to apply (c) and (d) of Proposition 14.
- The relation $r(X) \leq \|X\|_2$ (for $X \in \mathbb{C}^{n\times n}$) implies that any of the cases (a), (b), (c), and (d) of Proposition 14 holds if we replace $r(\cdot)$ by $\|\cdot\|_2$. This yields bounds on $\lambda_{min}^{S_1}$, $\lambda_{max}^{S_1}$, $\lambda_{min}^{T_1}$, or $\lambda_{max}^{T_1}$ that may not necessarily be sharp [31]. Anyway, in cases where computational performance is the main preference, we should use the norm $\|\cdot\|_2$ instead of $r(\cdot)$.

The next proposition uses the block structure of $S(z)$ and $T(z)$ to provide some sufficient conditions for which all the eigenvalues of the pair $(P(z), -\Delta P(z))$, for A, B, ΔA, and ΔB as in (14), are nonzero and real. The proposition does not involve positive or negative definiteness of either of S_1 or T_1, but instead it provides the sufficient conditions under which $S(z)$ (resp., $T(z)$) is either positive or negative definite (resp., semidefinite). Here, the set $\mathcal{W}(X)$ defined by (9) is required.

Proposition 15. *Suppose that we are given a matrix pencil (A, B) and an associated perturbation matrix pencil $(\Delta A, \Delta B)$ with A, B, ΔA, and ΔB as in (14). Let z be a complex number such that $|z| = 1$. Suppose also that $S(z)$ and $T(z)$ are the matrices defined in (16). Then all the eigenvalues of the pair $(P(z), -\Delta P(z))$ are nonzero and real under any of the following conditions:*

(a) $\lambda_{min}^{M_4} + \lambda_{min}^{M_6} > 2 \sum_{j=1,2,3,5} \sigma_{max}(M_j)$.

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(b) $\lambda_{max}^{M_4} + \lambda_{max}^{M_6} + 2\sum_{j=1,2,5} \sigma_{max}(M_j) < -2\sigma_{max}(M_3)$.

(c) $\lambda_{min}^{\Delta M_4} + \lambda_{min}^{\Delta M_6} \geq 2\sum_{j=1,2,3,5} \sigma_{max}(\Delta M_j)$.

(d) $\lambda_{max}^{\Delta M_4} + \lambda_{max}^{\Delta M_6} + 2\sum_{j=1,2,5} \sigma_{max}(\Delta M_j) \leq -2\sigma_{max}(\Delta M_3)$.

Proof. To prove the case (a) (resp., the case (b)), we use Theorem 3 and obtain the conditions under which $S(z)$ becomes positive (resp., negative) definite. The cases (c) and (d) can be proved in the same way, using the block structure of $T(z)$, but the conditions for positive or negative semidefiniteness of $T(z)$ should be sought for.

For an arbitrary nonzero vector $\mathbf{x} \in \mathbb{C}^n$, let $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}$, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^{n_1}$ and $\mathbf{x}_3 \in \mathbb{C}^{n_2}$ for n_1 and n_2 defined in (14). Then for

$$m_k = \mathbf{x}_i^H M_k \mathbf{x}_j, \quad k = 1, 2, 3, 5,$$

we have

$$\begin{aligned} \mathbf{x}^H S(z)\mathbf{x} &= \begin{bmatrix} \mathbf{x}_1^H & \mathbf{x}_2^H & \mathbf{x}_3^H \end{bmatrix} \begin{bmatrix} 0 & M_1 - zM_3 & M_2 \\ (M_1 - zM_3)^H & M_4 & M_5 \\ M_2^H & M_5^H & M_6 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \\ &= \overbrace{\mathbf{x}_1^H M_1 \mathbf{x}_2}^{m_1} - \overbrace{z\mathbf{x}_1^H M_3 \mathbf{x}_2}^{zm_3} + \overbrace{\mathbf{x}_1^H M_2 \mathbf{x}_3}^{m_2} + \overbrace{\mathbf{x}_2^H M_1^H \mathbf{x}_1}^{\overline{m}_1} - \overbrace{\bar{z}\mathbf{x}_2^H M_3^H \mathbf{x}_1}^{\bar{z}\overline{m}_3} \\ &\quad + \mathbf{x}_2^H M_4 \mathbf{x}_2 + \overbrace{\mathbf{x}_2^H M_5 \mathbf{x}_3}^{m_5} + \overbrace{\mathbf{x}_3^H M_2^H \mathbf{x}_1}^{\overline{m}_2} + \overbrace{\mathbf{x}_3^H M_5^H \mathbf{x}_2}^{\overline{m}_5} + \mathbf{x}_3^H M_6 \mathbf{x}_3. \end{aligned}$$

Using $m_k + \overline{m}_k = 2\mathrm{Re}(m_k)$ for $k = 1, 2, 5$, and $zm_3 + \bar{z}\overline{m}_3 = 2|\overrightarrow{Oz}|\,|\overrightarrow{Om_3}|\cos(\theta)$, where θ is the angle between $\overrightarrow{Oz}$ and $\overrightarrow{O\overline{m}_3}$, it follows

$$\mathbf{x}^H S(z)\mathbf{x} = 2\sum_{j=1,2,5} \mathrm{Re}(m_j) - 2|\overrightarrow{Oz}|\,|\overrightarrow{O\overline{m}_3}|\cos(\theta) + \mathbf{x}_2^H M_4 \mathbf{x}_2 + \mathbf{x}_3^H M_6 \mathbf{x}_3. \tag{20}$$

Now the inclusion (10), the relation (20), the Hermitian property of M_4 and M_6, and the assumption $|\overrightarrow{Oz}| = 1$, together with the definition of $m_3 = \mathbf{x}_1^H M_3 \mathbf{x}_2$ can be used to show that $\mathbf{x}^H S(z)\mathbf{x} > 0$ for any nonzero $\mathbf{x} \in \mathbb{C}^n$ provided that

$$\sum_{j=4,6} \lambda_{min}^{M_j} > 2\sigma_{max}(M_1) + 2\sigma_{max}(M_2) + 2\sigma_{max}(M_5) + 2\sigma_{max}(M_3).$$

This proves the case (a).

For the case (b), the inclusion (10) and the expression (20) can be used to show that, for any nonzero $\mathbf{x} \in \mathbb{C}^n$,

$$\mathbf{x}^H S(z)\mathbf{x} \leq 2\sigma_{max}(M_1) + 2\sigma_{max}(M_2) + 2\sigma_{max}(M_5) - 2|\overrightarrow{Oz}|\,|\overrightarrow{O\overline{m}_3}|\cos(\theta) + \lambda_{max}^{M_4} + \lambda_{max}^{M_6}.$$

The most challenging case for the right-hand side of the latter relation to be negative is when

$$|\overrightarrow{O\overline{m}_3}| = \max\{|w| : w \in \mathcal{W}(M_3)\} = \sigma_{max}(M_3) = \|M_3\|_2,$$

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and $\cos(\theta) = -1$. This means that, for any $z \in \mathbb{C}$ such that $|z| = 1$, $S(z)$ is negative definite if, for any nonzero $\mathbf{x} \in \mathbb{C}^n$,

$$2\sigma_{max}(M_1) + 2\sigma_{max}(M_2) + 2\sigma_{max}(M_5) + \lambda_{max}^{M_4} + \lambda_{max}^{M_6} < -2\sigma_{max}(M_3).$$

This proves the case (b). □

Although in practice the sufficient conditions provided by Proposition 15 are not possible to be fulfilled, it seems that maybe they can provide a way to estimate the distance of $S(z)$ (resp., $T(z)$) to a positive or negative definite (resp., semidefinite) matrix.

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