Gershgorin type inclusion-exclusion sets for matrix polynomials

Christina Michailidou*, Vasiliki Panagakou* and Panayiotis Psarrakos*

October 21, 2025

Abstract

In this paper, improvements of the spectrum estimation for matrix polynomials given by the Gershgorin set, the Brauer set, and the Dashnic-Zusmanovich set are derived by substracting regions of the complex plane which do not contain eigenvalues. Geometrical and topological properties of the exclusion sets are obtained, and illustrative examples are presented.

Key-words: Matrix polynomial, eigenvalue, spectrum, inclusion-exclusion set, the Gershgorin set, the Brauer set, the Dashnic-Zusmanovich set.

AMS Subject Classifications: 15A18, 15A22.

1 Introduction

Consider a square complex matrix $A \in \mathbb{C}^{n\times n}$, and let $\sigma(A)$ be its standard spectrum. The celebrated Gershgorin (Geršgorin) circle theorem [8, 22] yields n easily computable disks centered at the diagonal entries of A, whose union (known as the Gershgorin set of A) contains $\sigma(A)$. The excessive simplicity and the applications of the Gershgorin circle theorem have motivated further research on Gershgorin disks and relative sets such as the Brauer set, the Dashnic-Zusmanovich set, the $\mathcal{A}$-Ostrowski set, the Householder set, and others (see [2, 3, 6, 19, 22] and the references therein), which are widely used for estimating the location of the eigenvalues of a matrix. The spectrum estimations given by the Gershgorin set, the Brauer set, and the Dashnic-Zusmanovich set can be refined by subtracting parts (exclusion sets) which do not

*Department of Mathematics, School of Applied Mathematical and Physical Sciences, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece (xristina.michailidou@gmail.com, vaspanagakou@gmail.com, ppsarr@math.ntua.gr).

contain eigenvalues; see [16, 21] for the Gershgorin set, [14] for the Brauer set, and [23] for the Dashnic-Zusmanovich set.

In this paper, we consider $n \times n$ matrix polynomials of the form

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0, \tag{1}$$

where λ is a complex variable, $A_0, A_1, \ldots, A_m \in \mathbb{C}^{n\times n}$ with $A_m \neq 0$, and the determinant $\det P(\lambda)$ *is not identically zero.* The study of matrix polynomials, and particularly their spectrum analysis, has a long history and significant applications in various fields such as control theory, vibrating analysis, queuing theory and differential equations(see [7, 9, 11, 12, 13, 15, 18] and their references).

A scalar $\mu \in \mathbb{C}$ is said to be an *eigenvalue* of $P(\lambda)$ if the homogeneous linear system $P(\mu)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. Such a solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to the eigenvalue μ. The set of all finite eigenvalues of $P(\lambda)$,

$$\sigma(P(\lambda)) = \{\mu \in \mathbb{C} :\ \det P(\mu) = 0\} = \{\mu \in \mathbb{C} :\ 0 \in \sigma(P(\mu))\}$$

(recalling that $\sigma(P(\mu))$ denotes the standard spectrum of the constant matrix $P(\mu)$), is the *finite spectrum* of $P(\lambda)$. The *algebraic multiplicity* of an eigenvalue $\mu \in \sigma(P(\lambda))$ is the multiplicity of μ as a root of the polynomial $\det P(\lambda)$, and it is always greater than or equal to the *geometric multiplicity* of μ, that is, the dimension of the null space of matrix $P(\mu)$. Moreover, it is said that $\mu = \infty$ is an eigenvalue of $P(\lambda)$ if and only if 0 is an eigenvalue of the *reverse* matrix polynomial $\hat{P}(\lambda) = \lambda^m P(1/\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m$, or equivalently, if and only if the leading coefficient matrix A_m is singular. In this case, the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\mu = \infty$ of $P(\lambda)$ are defined as the algebraic multiplicity and the geometric multiplicity of the eigenvalue 0 of $\hat{P}(\lambda)$, respectively.

In the next three sections, the Gershgorin inclusion-exclusion set, the Brauer inclusion-exclusion set, and the Dashnic-Zusmanovich inclusion-exclusion set of constant matrices, introduced respectively in [16, 21], [14], and [23], are extended to matrix polynomials. In particular, the spectrum estimations given by the Gershgorin set, the Brauer set, and the Dashnic-Zusmanovich set of matrix polynomials, which were studied in [17], are clearly improved by subtracting parts of the original inclusion sets (exclusion sets) that do not contain eigenvalues. Geometrical and topological properties of the exclusion sets are investigated, and numerical examples are provided to illustrate the theoretical results and demonstrate the effectiveness of the approach. Of special interest is the case where the original inclusion set is unbounded and the inclusion-exclusion set becomes bounded. The numerical examples were performed in Mathematica 12.1.

2 The Gershgorin inclusion-exclusion set

2.1 The Gershgorin inclusion-exclusion set of a matrix

Consider a square complex matrix $A \in \mathbb{C}^{n\times n}$, define the set $\mathcal{N} = \{1, 2, \ldots, n\}$, and let $(A)_{i,j}$ denote the (i, j)-th entry of A, $i, j \in \mathcal{N}$. In [8], defining the nonnegative quantities $r_i(A) = \sum_{j\in\mathcal{N}\setminus\{i\}} |(A)_{i,j}|$ $(i \in \mathcal{N})$, the *Gershgorin disks* $\mathcal{G}_i(A) = \{\mu \in \mathbb{C} : |\mu - (A)_{i,i}| \leq r_i(A)\}$ $(i \in \mathcal{N})$ and the *Gershgorin set*

$$\mathcal{G}(A) = \bigcup_{i\in\mathcal{N}} \mathcal{G}_i(A)$$

of A were introduced. The latter set contains all the eigenvalues of A, i.e., $\sigma(A) \subseteq \mathcal{G}(A)$.

In [16, 21], the disks $\Delta_{i,j}(A) = \{\mu \in \mathbb{C} : |\mu - (A)_{j,j}| < 2|(A)_{j,i}| - r_j(A)\}$, $i, j \in \mathcal{N}$ with $i \neq j$, and the exclusion set (the set that will be subtracted from the i-th Gershgorin disk)

$$\Delta_i(A) = \bigcup_{j\in\mathcal{N}\setminus\{i\}} \Delta_{i,j}(A)$$

were defined.

Remark 1. If for any $i, j \in \mathcal{N}$ with $i \neq j$, the inequality $2\,|(A)_{j,i}| - r_j(A) \leq 0$ holds, then it is obvious that $\Delta_{i,j}(A) = \emptyset$. As remarked in [16], for any given $j \in \mathcal{N}$, the quantity $2\,|(A)_{j,i}| - r_j(A)$ may be positive for at most one $i \in \mathcal{N} \setminus \{j\}$. This implies that, for any given $j \in \mathcal{N}$, out of all the sets $\Delta_{i,j}(A)$ $(i \in \mathcal{N}\setminus\{j\})$, at most one can be non-empty.

Defining the Gershgorin inclusion-exclusion set for the i-th row of a matrix A, $\Omega_i(A) = \mathcal{G}_i(A) \setminus \Delta_i(A)$, and the Gershgorin inclusion-exclusion set of A

$$\Omega(A) = \bigcup_{i\in\mathcal{N}} \Omega_i(A),$$

from Theorem 2 in [16], it follows that $\sigma(A) \subseteq \Omega(A) \subseteq \mathcal{G}(A)$.

Remark 2. For any $i \in \mathcal{N}$, $\mathcal{G}_i(A)$ is a closed disk and $\Delta_i(A)$ is an open set. Therefore, every $\Omega_i(A) = \mathcal{G}_i(A) \setminus \Delta_i(A) = \mathcal{G}_i(A) \cap (\mathbb{C} \setminus \Delta_i(A))$ is a closed set. We conclude that the set $\Omega(A)$ is non-empty (since it contains the eigenvalues of A) and closed (as a finite union of the closed sets $\Omega_i(A)$, $i \in \mathcal{N}$).

Example 3. For the 4×4 matrix

$$A = \begin{bmatrix} 5 & 4 & -1 & 0 \\ 5 & 2 & 0 & 1 \\ 1 & -1 & -4 & 1 \\ 1 & 1 & 0 & 0.2 \end{bmatrix},$$

the Gershgorin set is given in Figure 1 and the Gershgorin inclusion-exclusion set is given in Figure 2. Here, and in all the figures of the paper, the eigenvalues are marked with dots.

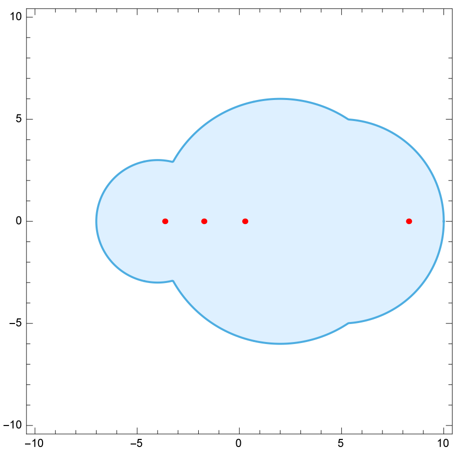


Figure 1: The Gershgorin set of A.

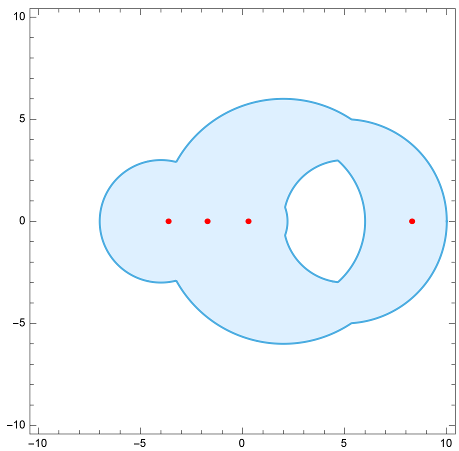


Figure 2: The Gershgorin inclusion-exclusion set of A.

2.2 The Gershgorin inclusion-exclusion set of a matrix polynomial

Consider the matrix polynomial $P(\lambda)$ defined in (1) and the nonnegative functions

$$r_i(P(\lambda)) = \sum_{j\in\mathcal{N}\setminus\{i\}} |(P(\lambda))_{i,j}|, \quad i \in \mathcal{N}. \tag{2}$$

In [17], the *Gershgorin set for the i-th row* of $P(\lambda)$ ($i \in \mathcal{N}$)

$$\mathcal{G}_i(P(\lambda)) = \{\mu \in \mathbb{C} :\, 0 \in \mathcal{G}_i(P(\mu))\} = \{\mu \in \mathbb{C} :\, |(P(\mu))_{i,i}| \le r_i(P(\mu))\}$$

and the (inclusion) *Gershgorin set* of $P(\lambda)$

$$\mathcal{G}(P(\lambda)) = \{\mu \in \mathbb{C} :\, 0 \in \mathcal{G}(P(\mu))\} = \bigcup_{i\in\mathcal{N}} \mathcal{G}_i(P(\lambda))$$

were studied. It is said that $\mu = \infty$ lies in $\mathcal{G}(P(\lambda))$ (or $\mathcal{G}_i(P(\lambda))$) if and only if 0 lies in $\mathcal{G}(\hat{P}(\lambda))$ (resp., $\mathcal{G}_i(\hat{P}(\lambda))$). The Gershgorin set $\mathcal{G}(P(\lambda))$ is a closed set that contains all the eigenvalues of $P(\lambda)$. Furthermore, if $(P(\lambda))_{i,i} = 0$ for some $i \in \mathcal{N}$, then $\mathcal{G}_i(P(\lambda)) = \mathbb{C}$.

In the following proposition, for clarity and reader's convenience, some of the basic properties of the Gershgorin set $\mathcal{G}(P(\lambda))$ are summarized [17].

Proposition 4. Let $i \in \mathcal{N}$.

(i) For any scalar $b \in \mathbb{C}\setminus\{0\}$, it holds that $\mathcal{G}_i(P(b\lambda)) = b^{-1}\mathcal{G}_i(P(\lambda))$, $\mathcal{G}_i(bP(\lambda)) = \mathcal{G}_i(P(\lambda))$, and $\mathcal{G}_i(P(\lambda + b)) = \mathcal{G}_i(P(\lambda)) - b$.

(ii) If all the coefficient matrices $A_0, A_1, \ldots, A_m$ of $P(\lambda)$ have their i-th row real, then $\mathcal{G}_i(P(\lambda))$ is symmetric with respect to the real axis.

(iii) The set $\{\mu \in \mathbb{C} : |(P(\mu))_{i,i}| < r_i(P(\mu))\}$ lies in the interior of $\mathcal{G}_i(P(\lambda))$. Consequently, $\partial\mathcal{G}_i(P(\lambda)) \subseteq \{\mu \in \mathbb{C} : |(P(\mu))_{i,i}| = r_i(P(\mu))\} = \{\mu \in \mathbb{C} : 0 \in \partial\mathcal{G}_i(P(\mu))\}$.

(iv) If a scalar $\mu_0 \in \mathbb{C}$ is an isolated point of $\mathcal{G}_i(P(\lambda))$, then μ_0 is a common root of all polynomials $(P(\lambda))_{i,j}$, $j \in \mathcal{N}$, and thus, it is also an eigenvalue of $P(\lambda)$.

(v) Suppose that the i-th row of A_m is non-zero. If $(A_m)_{i,i} \neq 0$, then $\mathcal{G}_i(P(\lambda))$ is unbounded if and only if $0 \in \mathcal{G}_i(A_m)$. If $(A_m)_{i,i} = 0$, then $\mathcal{G}_i(P(\lambda))$ is unbounded and $0 \in \mathcal{G}_i(A_m)$.

(vi) If A_m does not have a zero row and $\mathcal{G}(P(\lambda))$ is bounded, then the number of connected components of $\mathcal{G}(P(\lambda))$ is less than or equal to nm.

Next, we extend the concepts of the Gershgorin exclusion set and the Gershgorin inclusion-exclusion set to matrix polynomials.

Definition 5. For $i, j \in \mathcal{N}$ with $i \neq j$, define the sets

$$\begin{aligned} \Delta_{i,j}(P(\lambda)) &= \{\mu \in \mathbb{C} : 0 \in \Delta_{i,j}(P(\mu))\} \\ &= \{\mu \in \mathbb{C} : |(P(\mu))_{j,j}| < 2|(P(\mu))_{j,i}| - r_j(P(\mu))\}, \end{aligned}$$

and the *Gershgorin exclusion set for the i-th row* of the matrix polynomial $P(\lambda)$

$$\Delta_i(P(\lambda)) = \bigcup_{j \in \mathcal{N}\backslash\{i\}} \Delta_{i,j}(P(\lambda)).$$

From this definition, it follows immediately that for any $i \in \mathcal{N}$,

$$\begin{aligned} \Delta_i(\hat{P}(\lambda))\backslash\{0\} &= \left\{\mu \in \mathbb{C}\backslash\{0\} : 0 \in \Delta_i(\hat{P}(\lambda))\right\} = \left\{\mu \in \mathbb{C}\backslash\{0\} : 0 \in \Delta_i(\mu^m P(\mu^{-1}))\right\} \\ &= \left\{\mu \in \mathbb{C}\backslash\{0\} : 0 \in \Delta_i(P(\mu^{-1}))\right\} = \left\{\mu \in \mathbb{C}\backslash\{0\} : \mu^{-1} \in \Delta_i(P(\lambda))\right\}. \end{aligned}$$

Moreover, we say that $\mu = \infty$ lies in $\Delta_i(P(\lambda))$ (or $\Delta_{i,j}(P(\lambda))$) if and only if 0 lies in $\Delta_i(\hat{P}(\lambda))$ (resp., $\Delta_{i,j}(\hat{P}(\lambda))$).

Remark 6. The set $\Delta_{i,j}(P(\lambda))$ is not necessarily non-empty. For example, it is empty (in a trivial way) if $2|(P(\lambda))_{j,i}| \leq r_j(P(\lambda))$ for every $\lambda \in \mathbb{C}$.

Remark 7. Unlike the case of constant matrices studied in Remark 1, for an $n \times n$ matrix polynomial $P(\lambda)$ and a given $j \in \mathcal{N}$, we may have more than one non-empty exclusion sets $\Delta_{i,j}(P(\lambda))$ ($i \in \mathcal{N}\backslash\{j\}$). By Remark 1, it follows that these non-empty exclusion sets have no common points. For example, if we consider the matrix polynomial

$$P(\lambda) = \begin{bmatrix} 8 & \lambda^3 & \lambda^3 + 11 \\ \lambda^3 & 0 & 27 \\ \lambda^3 - 1 & 9 & -3 \end{bmatrix},$$

then $\Delta_{1,2}(P(\lambda)) = \{\mu \in \mathbb{C} : |\mu| > 3\}$ and $\Delta_{3,2}(P(\lambda)) = \{\mu \in \mathbb{C} : |\mu| < 3\}$ are non-empty and have no common points.

Remark 8. It is clear that if $|(P(\lambda))_{j,j}| < 2|(P(\lambda))_{j,i}| - r_j(P(\lambda))$ for every $\lambda \in \mathbb{C}$, then $\Delta_{i,j}(P(\lambda)) = \mathbb{C}$. Thus, in the special case where $(P(\lambda))_{j,j} = 0$ and $2|(P(\lambda))_{j,i}| > r_j(P(\lambda))$ for every $\lambda \in \mathbb{C}$, $\Delta_{i,j}(P(\lambda))$ coincides with the complex plane.

Definition 9. The *Gershgorin inclusion-exclusion set* of the matrix polynomial $P(\lambda)$ is defined as

$$\Omega(P(\lambda)) = \bigcup_{i \in \mathcal{N}} \Omega_i(P(\lambda)),$$

where

$$\Omega_i(P(\lambda)) = \mathcal{G}_i(P(\lambda)) \setminus \Delta_i(P(\lambda)), \quad i \in \mathcal{N}.$$

Theorem 10. All the eigenvalues of the matrix polynomial $P(\lambda)$ lie in the Gershgorin inclusion-exclusion set $\Omega(P(\lambda))$.

Proof. For every finite eigenvalue $\mu \in \sigma(P(\lambda))$, we have that $0 \in \sigma(P(\mu))$. We know from [17] that there exists some i such that $0 \in \mathcal{G}_i(P(\mu))$. Moreover, by [16], it holds that $0 \notin \Delta_i(P(\mu))$, or equivalently, $\mu \notin \Delta_i(P(\lambda))$. Therefore,

$$\mu \in \Omega_i(P(\lambda)) \subseteq \Omega(P(\lambda)).$$

If $\mu = \infty$ is an eigenvalue of $P(\lambda)$, then $0 \in \sigma(\hat{P}(\mu)) \subseteq \Omega(\hat{P}(\mu))$, and hence, $\mu \in \Omega(P(\lambda))$. $\square$

We continue with the study of geometrical and topological properties of the exclusion set $\Delta_{i,j}(P(\lambda))$.

Proposition 11. Let $i, j \in \mathcal{N}$ with $i \neq j$. Then the following hold:

(i) The set $\Delta_{i,j}(P(\lambda))$ is open (consequently, the set $\Delta_i(P(\lambda))$ is also open).

(ii) For any scalar $b \in \mathbb{C}\backslash\{0\}$, it holds that $\Delta_{i,j}(P(b\lambda)) = b^{-1}\Delta_{i,j}(P(\lambda))$, $\Delta_{i,j}(bP(\lambda)) = \Delta_{i,j}(P(\lambda))$, and $\Delta_{i,j}(P(\lambda + b)) = \Delta_{i,j}(P(\lambda)) - b$ (consequently, $\Delta_i(P(\lambda))$ satisfies these properties as well).

(iii) If all the coefficient matrices $A_0, A_1, \ldots, A_m$ of $P(\lambda)$ have their j-th row real, then $\Delta_{i,j}(P(\lambda))$ is symmetric with respect to the real axis (consequently, if $A_0, A_1, \ldots, A_m$ are real, then $\Omega(P(\lambda))$ is symmetric with respect to the real axis).

Proof. (i) For any point $\mu \in \Delta_{i,j}(P(\lambda))$, it holds that $|(P(\mu))_{j,j}| < 2|(P(\mu))_{j,i}| - r_j(P(\mu))$. By continuity, it is clear that, for any $\hat{\mu} \in \mathbb{C}$ close enough to μ, $|(P(\hat{\mu}))_{j,j}| < 2|(P(\hat{\mu}))_{j,i}| - r_j(P(\hat{\mu}))$, which means that $\hat{\mu} \in \Delta_{i,j}(P(\lambda))$. Therefore, the set $\Delta_{i,j}(P(\lambda))$ is open.

(ii) We observe that

$$\mu \in \Delta_{i,j}(P(b\lambda)) \Leftrightarrow |(P(b\mu))_{j,j}| < 2|(P(b\mu))_{j,i}| - r_j(P(b\mu)) \Leftrightarrow \frac{\mu}{b} \in \Delta_{i,j}(P(\lambda)),$$

$$\mu \in \Delta_{i,j}(bP(\lambda)) \Leftrightarrow |b(P(\mu))_{j,j}| < 2|b(P(\mu))_{j,i}| - r_j(bP(\mu)) \Leftrightarrow \mu \in \Delta_{i,j}(P(\lambda))$$

and

$$\mu \in \Delta_{i,j}(P(\lambda + b)) \Leftrightarrow |(P(\mu + b))_{j,j}| < 2|(P(\mu + b))_{j,i}| - r_j(P(\mu + b)) \Leftrightarrow \mu - b \in \Delta_{i,j}(P(\lambda)).$$

(iii) Suppose that all the coefficient matrices $A_0, A_1, \ldots, A_m$ have their j-th row real. If $\mu \in \Delta_{i,j}(P(\lambda))$, then

$$\left|\sum_{k=0}^{m}(A_k)_{j,j}\mu^k\right| \leq 2\left|\sum_{k=0}^{m}(A_k)_{j,i}\mu^k\right| - \sum_{p\in\mathcal{N}\setminus\{j\}}\left|\sum_{k=0}^{m}(A_k)_{j,p}\mu^k\right|,$$

or

$$\left|\overline{\sum_{k=0}^{m}(A_k)_{j,j}\mu^k}\right| \leq 2\left|\overline{\sum_{k=0}^{m}(A_k)_{j,i}\mu^k}\right| - \sum_{p\in\mathcal{N}\setminus\{j\}}\left|\overline{\sum_{k=0}^{m}(A_k)_{j,p}\mu^k}\right|,$$

or

$$\left|\sum_{k=0}^{m}(A_k)_{j,j}\overline{\mu}^k\right| \leq 2\left|\sum_{k=0}^{m}(A_k)_{j,i}\overline{\mu}^k\right| - \sum_{p\in\mathcal{N}\setminus\{j\}}\left|\sum_{k=0}^{m}(A_k)_{j,p}\overline{\mu}^k\right|.$$

This means that $\overline{\mu} \in \Delta_{i,j}(P(\lambda))$. □

Remark 12. The Gershgorin set $\mathcal{G}_i(P(\lambda))$ for the i-th row of a matrix polynomial $P(\lambda)$ $(i \in \mathcal{N})$ is closed and the corresponding exclusion set $\Delta_i(P(\lambda))$ is open. Therefore, the set $\Omega_i(P(\lambda)) = \mathcal{G}_i(P(\lambda)) \setminus \Delta_i(P(\lambda)) = \mathcal{G}_i(P(\lambda)) \cap (\mathbb{C} \setminus \Delta_i(P(\lambda)))$ is closed. Clearly, the Gershgorin inclusion-exclusion set $\Omega(P(\lambda))$ is closed as a finite union of closed sets.

In the next three propositions, we study the unboundedness of the Gershgorin exclusion sets. In Proposition 13, we consider the case of zero diagonal entries of the leading coefficient matrix, while in Propositions 14 and 15, we consider the case of non-zero diagonal entries of the leading coefficient matrix.

Proposition 13. Suppose that for some $j \in \mathcal{N}$, it holds that $(A_m)_{j,j} = 0$. If there exists an $i \in \mathcal{N}\setminus\{j\}$ such that $\Delta_{i,j}(A_m)$ is non-empty (or equivalently, $2\,|(A_m)_{j,i}| > r_j(A_m)$), then this i is unique, $0 \in \Delta_{i,j}(A_m)$ $(\subseteq \Delta_i(A_m))$ and the sets $\Delta_{i,j}(P(\lambda))$ and $\Delta_i(P(\lambda))$ are unbounded.

Proof. Let $j \in \mathcal{N}$ with $(A_m)_{j,j} = 0$. Then, for any $i \in \mathcal{N}\setminus\{j\}$,

$$\Delta_{i,j}(A_m) = \{\mu \in \mathbb{C} : \, |\mu| < 2\,|(A_m)_{j,i}| - r_j(A_m)\}\,.$$

If for some $i \in \mathcal{N}\backslash\{j\}$, the relation $2\,|(A_m)_{j,i}| - r_j(A_m) > 0$ holds, then by Remark 1, this i is unique (always referring to this particular j). Moreover, $0 \in \Delta_{i,j}(A_m)$ (by definition), $2\,|(A_m)_{j,i}| > r_j(A_m) \geq 0$ (i.e., the j-th row of the matrix A_m is not zero), and

$$\begin{aligned}
\Delta_{i,j}(P(\lambda))\backslash\{0\} \;&=\; \{\mu \in \mathbb{C}\backslash\{0\} : |(P(\mu))_{j,j}| < 2\,|(P(\mu))_{j,i}| - r_j(P(\mu))\} \\
&\overset{(A_m)_{j,j}=0}{=} \left\{ \mu \in \mathbb{C}\backslash\{0\} : \left|\sum_{k=0}^{m-1}(A_k)_{j,j}\mu^k\right| < 2\left|\sum_{k=0}^{m}(A_k)_{j,i}\mu^k\right| - \sum_{p\in\mathcal{N}\backslash\{j\}}\left|\sum_{k=0}^{m}(A_k)_{j,p}\mu^k\right| \right\} \\
&= \left\{ \mu \in \mathbb{C}\backslash\{0\} : \left|\sum_{k=0}^{m-1}(A_k)_{j,j}\frac{\mu^k}{\mu^m}\right| < 2\left|\sum_{k=0}^{m}(A_k)_{j,i}\frac{\mu^k}{\mu^m}\right| - \sum_{p\in\mathcal{N}\backslash\{j\}}\left|\sum_{k=0}^{m}(A_k)_{j,p}\frac{\mu^k}{\mu^m}\right| \right\}.
\end{aligned}$$

By the assumption $0 < 2\,|(A_m)_{j,i}| - r_j(A_m)$, it follows that for sufficiently large $|\mu|$, $\mu \in \Delta_{i,j}(P(\lambda))$. Furthermore, there exists a real number $M > 0$ such that for any $\mu \in \mathbb{C}$ with $|\mu| \geq M$, $\mu \in \Delta_{i,j}(P(\lambda))$. Thus, $\{\mu \in \mathbb{C} : |\mu| \geq M\} \subseteq \Delta_{i,j}(P(\lambda)) \subseteq \Delta_i(P(\lambda))$. □

Proposition 14. Suppose that for some $j \in \mathcal{N}$, it holds that $(A_m)_{j,j} \neq 0$. If there exists an $i \in \mathcal{N}\backslash\{j\}$ such that $0 \in \Delta_{i,j}(A_m)$, then the sets $\Delta_{i,j}(P(\lambda))$ and $\Delta_i(P(\lambda))$ are unbounded.

Proof. Suppose that $(A_m)_{j,j} \neq 0$ and $0 \in \Delta_{i,j}(A_m)$ for some $i \in \mathcal{N}\backslash\{j\}$. Then, we have

$$|(A_m)_{j,j}| < 2\,|(A_m)_{j,i}| - r_j(A_m).$$

However, $A_m = \hat{P}(0)$ (for the reverse matrix polynomial $\hat{P}(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_m$), and consequently,

$$\left|\hat{P}(0)_{j,j}\right| < 2\left|\hat{P}(0)_{j,i}\right| - r_j(\hat{P}(0)).$$

By continuity, there is a real number $r > 0$ such that

$$\left|\hat{P}(\mu)_{j,j}\right| < 2\left|\hat{P}(\mu)_{j,i}\right| - r_j(\hat{P}(\mu))$$

for every $\mu \in \mathbb{C}$ with $|\mu| \leq r$ (i.e., 0 cannot be an isolated point of the open set $\Delta_{i,j}(\hat{P}(\lambda))$). This in turn implies

$$|P(\mu)_{j,j}| < 2\,|P(\mu)_{j,i}| - r_j(P(\mu))$$

for every $\mu \in \mathbb{C}$ with $|\mu| \geq r^{-1}$. Consequently, $\{\mu \in \mathbb{C} : |\mu| \geq r^{-1}\} \subseteq \Delta_{i,j}(P(\lambda)) \subseteq \Delta_i(P(\lambda))$, and the sets $\Delta_{i,j}(P(\lambda))$ and $\Delta_i(P(\lambda))$ are unbounded. □

Proposition 15. Suppose that for some $j \in \mathcal{N}$, it holds that $(A_m)_{j,j} \neq 0$. If there exists an $i \in \mathcal{N}\backslash\{j\}$ such that the set $\Delta_{i,j}(P(\lambda))$ is unbounded, then 0 lies in the closure of $\Delta_{i,j}(A_m)$ and $(A_m)_{j,i} \neq 0$.

Proof. Suppose that $(A_m)_{j,j} \neq 0$ and $\Delta_{i,j}(P(\lambda))$ is unbounded for some $i \in \mathcal{N}\backslash\{j\}$. Clearly, 0 lies in $\Delta_{i,j}(\hat{P}(\lambda))$. Thus, since 0 cannot be an isolated point of the open set $\Delta_{i,j}(\hat{P}(\lambda))$ (see the proof of Proposition 14), there is a sequence $\{\mu_\ell\}_{\ell=1,2,\dots}$ in $\Delta_{i,j}(P(\lambda))\setminus\{0\}$ such that $|\mu_\ell|\to\infty$. Then, for every positive integer ℓ, we have

$$|(P(\mu_\ell))_{j,j}| < 2\,|(P(\mu_\ell))_{j,i}| - r_j(P(\mu_\ell)),$$

or

$$\left|\sum_{k=0}^{m}(A_k)_{j,j}\mu_\ell^k\right| < 2\left|\sum_{k=0}^{m}(A_k)_{j,i}\mu_\ell^k\right| - \sum_{p\in\mathcal{N}\setminus\{j\}}\left|\sum_{k=0}^{m}(A_k)_{j,p}\mu_\ell^k\right|,$$

or

$$\left|\sum_{k=0}^{m}(A_k)_{j,j}\frac{1}{\mu_\ell^{m-k}}\right| < 2\left|\sum_{k=0}^{m}(A_k)_{j,i}\frac{1}{\mu_\ell^{m-k}}\right| - \sum_{p\in\mathcal{N}\setminus\{j\}}\left|\sum_{k=0}^{m}(A_k)_{j,p}\frac{1}{\mu_\ell^{m-k}}\right|.$$

As a consequence, for $\ell\to\infty$, it holds that

$$|(A_m)_{j,j}| \le 2\,|(A_m)_{j,i}| - r_j(A_m),$$

implying that 0 lies in the closure of $\Delta_{i,j}(A_m)$ and $(A_m)_{j,i}\neq 0$. $\square$

It is worth noting that if $(A_m)_{j,j}=0$, then $\Delta_{i,j}(A_m)$ is non-empty if and only if $0\in\Delta_{i,j}(A_m)$. By Proposition 4 (v) and the proofs of Propositions 13 and 14, the next result follows readily.

Corollary 16. Suppose that for some $i,j\in\mathcal{N}$ with $i\neq j$, the i-th row of A_m is non-zero and one of the following holds:

(i) $(A_m)_{i,i}=0$ and $0\in\Delta_{i,j}(A_m)$.

(ii) $(A_m)_{i,i}\neq 0$, $0\in\mathcal{G}_i(A_m)$, and $0\in\Delta_{i,j}(A_m)$.

Then $\mathcal{G}_i(P(\lambda))$ is unbounded and $\Omega_i(P(\lambda))$ is bounded.

Theorem 17. If $\mu_0\in\mathbb{C}$ is an isolated point of $\mathbb{C}\setminus\Delta_{i,j}(P(\lambda))$ for some $i,j\in\mathcal{N}$ with $i\neq j$, then μ_0 is a common root of all polynomials $(P(\lambda))_{j,s}$, $s\in\mathcal{N}$, and consequently, μ_0 is an eigenvalue of $P(\lambda)$.

Proof. Suppose that $(P(\mu_0))_{j,s}\neq 0$ for some $s\in\mathcal{N}$. Since μ_0 is an isolated point of $\mathbb{C}\setminus\Delta_{i,j}(P(\lambda))$, it follows that μ_0 lies on the boundary of $\mathbb{C}\setminus\Delta_{i,j}(P(\lambda))$, and there is an $\varepsilon>0$ such that the closed disk $D(\mu_0,\varepsilon)=\{\lambda\in\mathbb{C}:\ |\lambda-\mu_0|\le\varepsilon\}$ does not contain any other point of $\mathbb{C}\setminus\Delta_{i,j}(P(\lambda))$. The set $\mathbb{C}\setminus\Delta_{i,j}(P(\lambda))$ can be described as follows:

$$\begin{aligned}
\mathbb{C}\setminus\Delta_{i,j}(P(\lambda)) &= \{\mu\in\mathbb{C}:\ |P(\mu)_{j,j}|\ge 2|P(\mu)_{j,i}|-r_j(P(\mu))\}\\
&= \left\{\mu\in\mathbb{C}:\ \frac{r_j(P(\mu))+|P(\mu)_{j,j}|-|P(\mu)_{j,i}|}{|(P(\mu))_{j,i}|}\ge 1\right\}\\
&= \left\{\mu\in\mathbb{C}:\ \log\frac{r_j(P(\mu))+|P(\mu)_{j,j}|-|P(\mu)_{j,i}|}{|(P(\mu))_{j,i}|}\ge 0\right\}.
\end{aligned}$$

Consider the function

$$\begin{aligned}\varphi(\lambda) &= \log \frac{r_j(P(\lambda)) + |P(\lambda)_{j,j}| - |P(\lambda)_{j,i}|}{|(P(\lambda))_{j,i}|} \\ &= \log \left\| \left[\frac{(P(\lambda))_{j,1}}{(P(\lambda))_{j,i}}, \ldots, \frac{(P(\lambda))_{j,i-1}}{(P(\lambda))_{j,i}}, \frac{(P(\lambda))_{j,i+1}}{(P(\lambda))_{j,i}}, \ldots, \frac{(P(\lambda))_{j,n}}{(P(\lambda))_{j,i}} \right] \right\|_1,\end{aligned}$$

with $\lambda \in D(\mu_0, \varepsilon)$. This function is subharmonic and satisfies the Maximum Principle [1, 4]. By definition, $\varphi(\lambda)$ is zero on the boundary of $\mathbb{C} \setminus \Delta_{i,j}(P(\lambda))$, nonnegative in the interior of $\mathbb{C} \setminus \Delta_{i,j}(P(\lambda))$, and negative in the set $\Delta_{i,j}(P(\lambda))$. Since $\mu_0 \in \partial(\mathbb{C} \setminus \Delta_{i,j}(P(\lambda)))$, we have

$$|(P(\mu_0))_{j,i}| = r_j(P(\mu_0)) + |(P(\mu_0))_{j,j}| - |(P(\mu_0))_{j,i}| = \sum_{p \in \mathcal{N} \setminus \{i\}} |(P(\mu_0))_{j,p}|.$$

Thus, the function $\varphi(\lambda)$ is zero at the center μ_0 of $D(\mu_0, \varepsilon)$ and negative in the rest of the disk. Since $\varphi(\lambda)$ satisfies the Maximum Principle, it attains its maximum value on the boundary of the disk, which is a contradiction. Therefore,

$$0 = (P(\mu_0))_{j,i} = \sum_{p \in \mathcal{N} \setminus \{i\}} |(P(\mu_0))_{j,p}|,$$

and μ_0 is a common root of all polynomials $(P(\lambda))_{j,s}$, $s \in \mathcal{N}$. $\square$

Example 18. **(i)** Consider the 3×3 matrix polynomial

$$P(\lambda) = \begin{bmatrix} \lambda^2 & 4 & 1 \\ 3 & \lambda^2 + 2\lambda - 2 & 0 \\ 0.5\mathrm{i} & 0 & \lambda^2 + 2 \end{bmatrix}.$$

The Gershgorin set $\mathcal{G}(P(\lambda))$ in Figure 3 and the Gershgorin inclusion-exclusion set $\Omega(P(\lambda))$ in Figure 4 are both bounded and contain the eigenvalues of $P(\lambda)$. The improvement of the original estimation of the spectrum is clear. In Figure 3, the curves shown in blue, orange and purple correspond to the boundaries of the sets $\mathcal{G}_1(P(\lambda))$, $\mathcal{G}_2(P(\lambda))$ and $\mathcal{G}_3(P(\lambda))$, respectively. Analogous boundary curves for the corresponding inclusion sets are presented in Figures 5, 7, and 9.

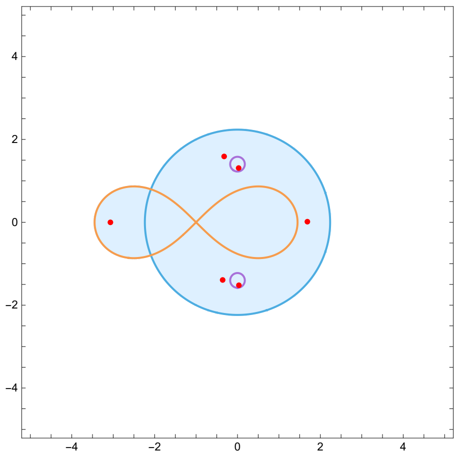


Figure 3: The Gershgorin set of $P(\lambda)$.

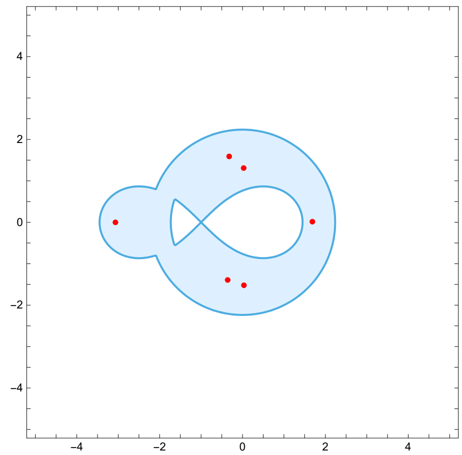


Figure 4: The Gershgorin inclusion-exclusion set of $P(\lambda)$.

(ii) Consider the 3×3 matrix polynomial

$$Q(\lambda) = \begin{bmatrix} -2\mathrm{i}\,\lambda + 2 & 4\mathrm{i}\,\lambda^2 + \mathrm{i}\,\lambda + 1 + 2\mathrm{i} & 2\mathrm{i}\,\lambda^2 + \lambda + 2 \\ -12\mathrm{i}\,\lambda^2 + 5\mathrm{i}\,\lambda + 1 + \mathrm{i} & -6\mathrm{i}\,\lambda^2 + 3\mathrm{i}\,\lambda + 4\mathrm{i} & (2 - 2\mathrm{i})\lambda^2 - 4\lambda - 5\mathrm{i} \\ 9\mathrm{i}\,\lambda^2 + \mathrm{i}\,\lambda + 1 - \mathrm{i} & 2\mathrm{i}\,\lambda^2 - \mathrm{i}\,\lambda & 12\mathrm{i}\,\lambda^2 + 2\mathrm{i} \end{bmatrix}.$$

The Gershgorin set $\mathcal{G}(Q(\lambda))$ in Figure 5 is unbounded (in particular, $\mathcal{G}(Q(\lambda)) = \mathcal{G}_2(Q(\lambda)) = \mathbb{C}$) and the Gershgorin inclusion-exclusion set $\Omega(Q(\lambda))$ in Figure 6 is bounded, i.e., the original estimation of the spectrum is significantly improved. It is worth mentioning that if Q_2 is the leading coefficient matrix of $Q(\lambda)$, then $(Q_2)_{2,2} = -6\mathrm{i} \neq 0$, $0 \in \mathcal{G}_2(Q_2)$, and $\Delta_{2,1}(Q_2)$ is non-empty and contains the origin; this means that the (sufficient) conditions of Corollary 16 (ii) hold for $i = 2$ and $j = 1$.

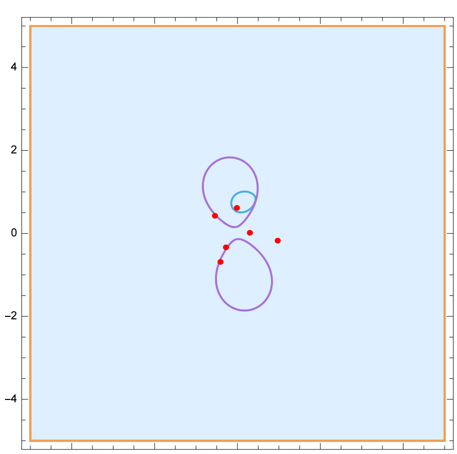


Figure 5: The Gershgorin set of $Q(\lambda)$.

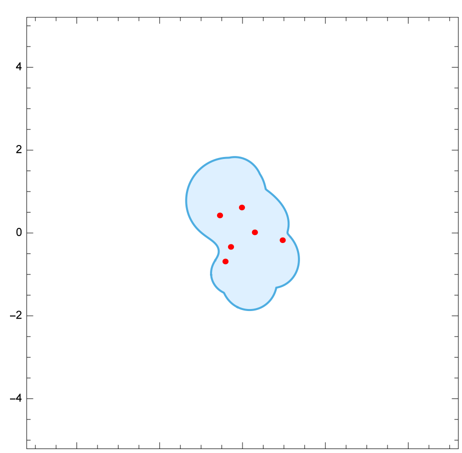


Figure 6: The Gershgorin inclusion-exclusion set of $Q(\lambda)$.

(iii) For the 3×3 matrix polynomial

$$R(\lambda) = \begin{bmatrix} -2\lambda & \lambda^3 + 4\mathrm{i}\,\lambda^2 + \lambda & 2\mathrm{i}\,\lambda^3 + 2\mathrm{i}\,\lambda^2 + \lambda \\ -\lambda^2 + \mathrm{i} & \lambda^3 + \lambda + 2 & 2\mathrm{i}\,\lambda^3 + 3\mathrm{i}\,\lambda^2 - 2\lambda \\ \mathrm{i}\,\lambda^3 + \mathrm{i}\,\lambda^2 + \mathrm{i}\,\lambda + 1 - \mathrm{i} & -\lambda & 0.6\mathrm{i}\,\lambda^2 + 2 \end{bmatrix},$$

both the Gershgorin set $\mathcal{G}(R(\lambda))$ in Figure 7 and the Gershgorin inclusion-exclusion set $\Omega(R(\lambda))$ in Figure 8 are unbounded. In the Gershgorin inclusion-exclusion set, we observe two subtracted regions. It is worth mentioning that if R_3 is the leading coefficient matrix of $R(\lambda)$, then 0 lies in $\mathcal{G}_1(R_3)$, $\mathcal{G}_2(R_3)$ and $\mathcal{G}_3(R_3)$, and the sets $\Delta_{1,2}(R_3)$, $\Delta_{2,1}(R_3)$ and $\Delta_{2,3}(R_3)$ are empty; i.e., the conditions of Corollary 16 do not hold for $i = 1$ and $j = 2$, and for $i = 2$ and $j = 1$ or $j = 3$.

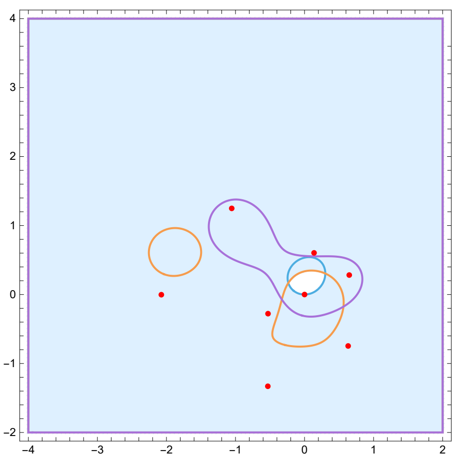


Figure 7: Gershgorin set of $R(\lambda)$.

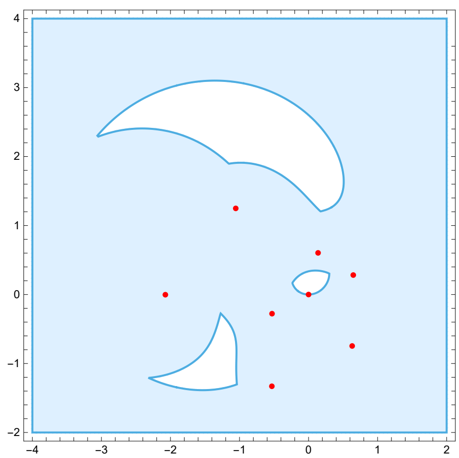


Figure 8: Gershgorin inclusion-exclusion set of $R(\lambda)$.

Example 19. To verify Theorem 17, consider the 3×3 matrix polynomial

$$P(\lambda) = \begin{bmatrix} 0 & \lambda + 1 & 0 \\ \mathrm{i} & \lambda & -\mathrm{i} \\ \lambda & \lambda - 5 & \lambda + 10 \end{bmatrix}.$$

The eigenvalues of $P(\lambda)$ are -5 and -1. The exclusion set $\Delta_{2,1}(P(\lambda))$ coincides with $\mathbb{C} \setminus \{-1\}$, so the set $\mathbb{C} \setminus \Delta_{2,1}(P(\lambda))$ is the singleton $\{-1\}$, which is a common root of the entries of the first row of $P(\lambda)$ and an eigenvalue of $P(\lambda)$. Therefore, the set $\Omega_2(P(\lambda))$ is the singleton $\{-1\}$.

2.3 The weighted Gershgorin inclusion-exclusion set of a matrix polynomial

Gershgorin himself recognized in [8] that by using similarity transformations $X^{-1}AX$, where $X = \mathrm{diag}\,\{x_1, x_2, \ldots, x_n\}$, with $x_i > 0$ ($i \in \mathcal{N}$), the disks $\mathcal{G}_i(A)$ can be improved. Cameron and Psarrakos in [3] generalized the notion of weighted Gershgorin sets for matrix polynomials through a suitable matrix norm. Here, we introduce the weighted Gershgorin inclusion-exclusion sets for a matrix polynomial $P(\lambda)$ as in (1).

Definition 20. If $X \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive entries, then we define the *weighted Gershgorin set of* $P(\lambda)$ *with respect to* X as $\mathcal{G}^X(P(\lambda)) = \mathcal{G}(X^{-1}\,P(\lambda)\,X)$.

By Definition 2.1 in [17] and Definition 20, the weighted Gershgorin set of the matrix polynomial $P(\lambda)$ can be written as

$$\mathcal{G}^X(P(\lambda)) = \bigcup_{i\in\mathcal{N}} \mathcal{G}_i^X(P(\lambda)),$$

where

$$\mathcal{G}_i^X(P(\lambda)) = \mathcal{G}_i(X^{-1}\,P(\lambda)\,X) = \left\{ \mu \in \mathbb{C} : \, |P(\mu)_{i,i}| \le \sum_{j\in\mathcal{N}\setminus\{i\}} |P(\mu)_{i,j}| x_j x_i^{-1} \right\}, \quad i \in \mathcal{N}.$$

As noted in [3], all the eigenvalues of $P(\lambda)$ lie in the Gershgorin set $\mathcal{G}^X(P(\lambda))$.

Definition 21. The *weighted Gershgorin inclusion-exclusion set* of $P(\lambda)$ is defined as

$$\Omega^X(P(\lambda)) = \bigcup_{i\in\mathcal{N}} \Omega_i^X(P(\lambda)),$$

where

$$\Omega_i^X(P(\lambda)) = \mathcal{G}_i^X(P(\lambda)) \setminus \Delta_i(X^{-1}P(\lambda)\,X), \quad i \in \mathcal{N}.$$

It is apparent that the matrix polynomials $X^{-1}\,P(\lambda)\,X$ and $P(\lambda)$ have the same eigenvalues with the same algebraic and geometric multiplicities. Moreover, since $X^{-1}\,P(\lambda)\,X$ is generated by multiplying each entry $(P(\lambda))_{i,j}$ of $P(\lambda)$ with $x_j x_i^{-1}$, the diagonal entries of $X^{-1}\,P(\lambda)\,X$ are the same as the diagonal entries of $P(\lambda)$, while the non-diagonal entries of $X^{-1}\,P(\lambda)\,X$ are scalar (positive) multiples of the corresponding non-diagonal entries of $P(\lambda)$. As a consequence, it is easy to verify that the weighted Gershgorin inclusion-exclusion set $\Omega^X(P(\lambda))$ $(\subseteq \mathcal{G}^X(P(\lambda)))$ contains all the eigenvalues of the matrix polynomial $P(\lambda)$ and satisfies the properties obtained in the previous subsection. For example, by Theorem 17, if $\mu_0 \in \mathbb{C}$ is an isolated point of $\mathbb{C}\setminus\Delta_{i,j}(X^{-1}\,P(\lambda)\,X)$ for some $i,j \in \mathcal{N}$ with $i \neq j$, then μ_0 is a common root of all polynomials $x_s x_j^{-1}(P(\lambda))_{j,s}$ $(s \in \mathcal{N})$, or equivalently, μ_0 is a common root of all polynomials $(P(\lambda))_{j,s}$ $(s \in \mathcal{N})$ (and hence, μ_0 is an eigenvalue of $P(\lambda)$).

Furthermore, it is clear that constructing the intersection $\bigcap_X \Omega^X(P(\lambda))$ for an appropriate selection of diagonal matrices X with positive diagonal entries may yield an improved estimation of the spectrum $\sigma(P(\lambda))$.

Example 22. Consider the 3×3 matrix polynomial

$$P(\lambda) = \begin{bmatrix} 2\mathrm{i}\,\lambda^2 - 2\mathrm{i}\,\lambda + 2 & 4\mathrm{i}\,\lambda^2 + \mathrm{i}\,\lambda + 1 + 2\mathrm{i} & \lambda + 2 \\ -0.6\lambda^2 + 5\mathrm{i}\,\lambda + 1 + \mathrm{i} & (2 - 2\mathrm{i})\lambda^2 + 3\mathrm{i}\,\lambda + 4\mathrm{i} & 0.1\lambda^2 - 4\lambda - 5\mathrm{i} \\ \mathrm{i}\,\lambda^2 + \mathrm{i}\,\lambda + 1 - \mathrm{i} & 2\mathrm{i}\,\lambda^2 - \mathrm{i}\,\lambda & 6\mathrm{i}\,\lambda^2 + 2\mathrm{i} \end{bmatrix},$$

and the diagonal matrices $X_1 = I$, $X_2 = \mathrm{diag}\,\{4, 1, 1\}$, $X_3 = \mathrm{diag}\,\{4, 1, 2\}$, $X_4 = \mathrm{diag}\,\{4, 1.5, 3\}$, $X_5 = \mathrm{diag}\,\{4, 2, 3\}$, and $X_6 = \mathrm{diag}\,\{4, 1.5, 1.4\}$. In Figures 9–12, we illustrate the Gershgorin set $\mathcal{G}(P(\lambda))$, the intersection of weighted Gershgorin sets $\bigcap\limits_{k=1,2,\ldots,6} \mathcal{G}^{X_k}(P(\lambda))$, the Gershgorin inclusion-exclusion set $\Omega(P(\lambda))$, and the intersection of weighted Gershgorin inclusion-exclusion sets $\bigcap\limits_{k=1,2,\ldots,6} \Omega^{X_k}(P(\lambda))$, respectively. It is worth noting that $\mathcal{G}(P(\lambda))$ and $\Omega(P(\lambda))$ are unbounded, while $\bigcap\limits_{k=1,2,\ldots,6} \mathcal{G}^{X_k}(P(\lambda))$ and $\bigcap\limits_{k=1,2,\ldots,6} \Omega^{X_k}(P(\lambda))$ are bounded.

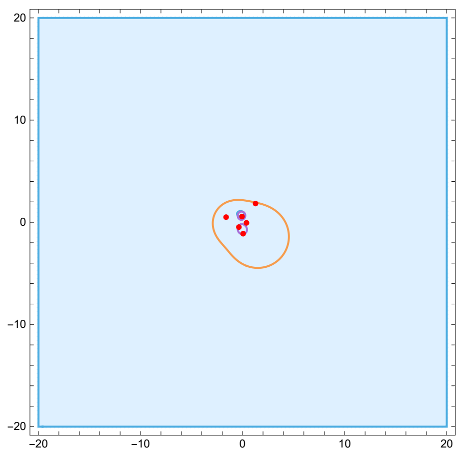


Figure 9: The Gershgorin set of $P(\lambda)$.

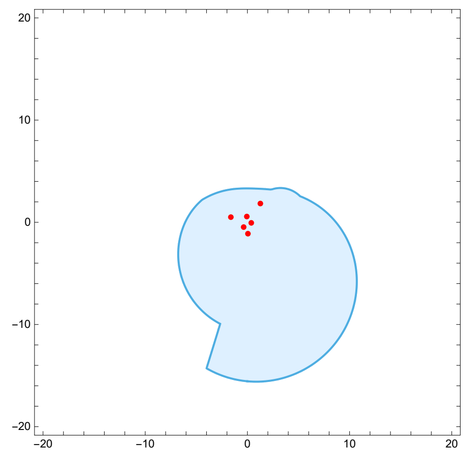


Figure 10: The intersection of weighted Gershgorin sets of $P(\lambda)$.

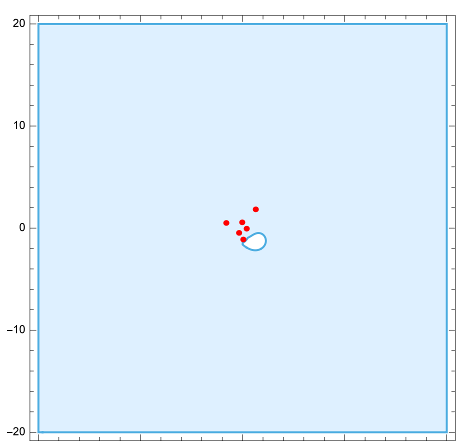


Figure 11: The Gershgorin inclusion-exclusion set of $P(\lambda)$.

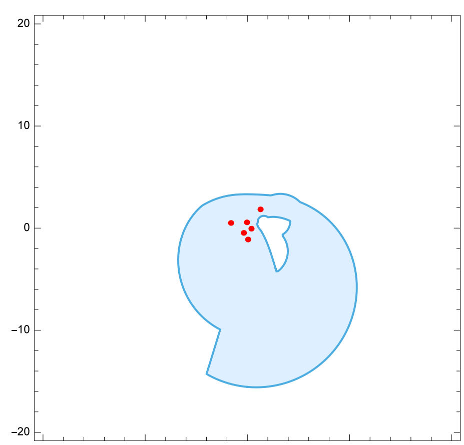


Figure 12: The intersection of weighted Gershgorin inclusion-exclusion sets of $P(\lambda)$.

Example 23. Consider the 3×3 matrix polynomial

$$Q(\lambda) = \begin{bmatrix} \lambda + 6 & -\lambda^2 + \lambda + 1 & 0 \\ -\lambda^2 + 5 + \mathrm{i} & 2\lambda + 3 + \mathrm{i} & 0 \\ 2\lambda & 7 + \mathrm{i} & 6\lambda \end{bmatrix},$$

and the diagonal matrices $X_1 = I$, $X_2 = \mathrm{diag}\{1, 0.5, 1\}$, and $X_3 = \mathrm{diag}\{0.2, 0.5, 5\}$. In Figures 13–16, we illustrate the Gershgorin set $\mathcal{G}(Q(\lambda))$, the intersection of weighted Gershgorin sets $\bigcap\limits_{k=1,2,3} \mathcal{G}^{X_k}(Q(\lambda))$, the Gershgorin inclusion-exclusion set $\Omega(Q(\lambda))$, and the intersection of weighted Gershgorin inclusion-exclusion sets $\bigcap\limits_{k=1,2,3} \Omega^{X_k}(Q(\lambda))$, respectively. It is worth mentioning that $\mathcal{G}(Q(\lambda))$ and $\bigcap\limits_{k=1,2,3} \mathcal{G}^{X_k}(Q(\lambda))$ are unbounded, while $\Omega(Q(\lambda))$ and $\bigcap\limits_{k=1,2,3} \Omega^{X_k}(Q(\lambda))$ are bounded.

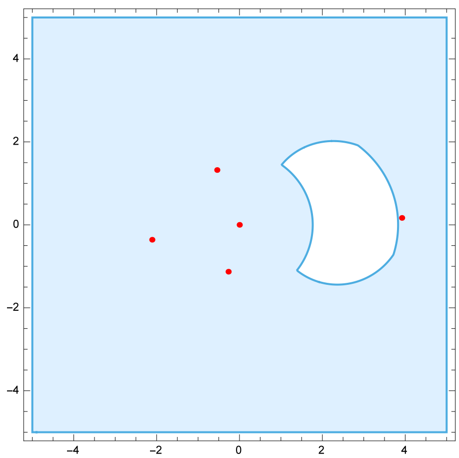


Figure 13: The Gershgorin set of $Q(\lambda)$.

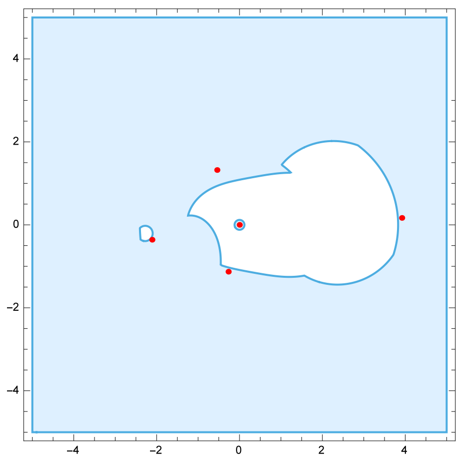


Figure 14: The intersection of weighted Gershgorin sets of $Q(\lambda)$.

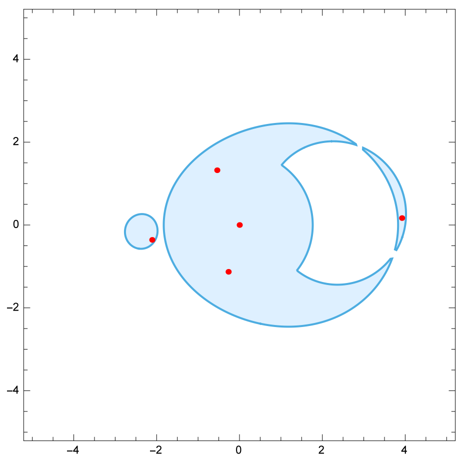


Figure 15: The Gershgorin inclusion-exclusion set of $Q(\lambda)$.

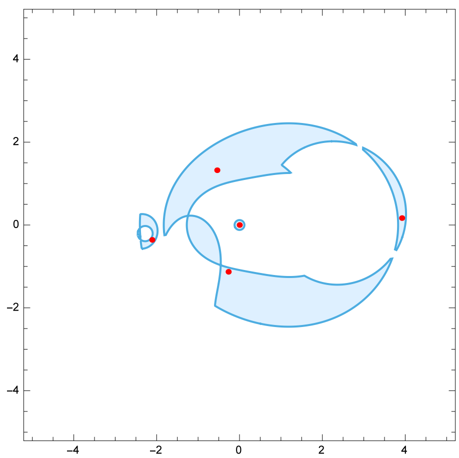


Figure 16: The intersection of weighted Gershgorin inclusion-exclusion sets of $Q(\lambda)$.

3 The Brauer inclusion-exclusion set

3.1 The Brauer inclusion-exclusion set of a matrix

The Brauer set of a square complex matrix $A \in \mathbb{C}^{n\times n}$ is defined as [2, 22]

$$\mathcal{B}(A) = \bigcup_{\substack{i,j\in\mathcal{N}\\ i\neq j}} \mathcal{B}_{i,j}(A) = \bigcup_{\substack{i,j\in\mathcal{N}\\ i>j}} \mathcal{B}_{i,j}(A),$$

where

$$\mathcal{B}_{i,j}(A) = \mathcal{B}_{j,i}(A) = \{\mu \in \mathbb{C} : \, |\mu - (A)_{i,i}|\, |\mu - (A)_{j,j}| \leq r_i(A) r_j(A)\}, \quad i,j \in \mathcal{N}, \;\; i \neq j.$$

The Brauer set is the union of $(n-1)+(n-2)+\cdots+2+1 = \dfrac{n(n-1)}{2}$ Cassini ovals $\mathcal{B}_{i,j}(A)$, $i > j$. Moreover, it contains all the eigenvalues of the matrix A and it is a subset of the Gershgorin set $\mathcal{G}(A)$, i.e. $\sigma(A) \subseteq \mathcal{B}(A) \subseteq \mathcal{G}(A)$.

In [14], the set

$$L_{s,i}(A) = \left\{\mu \in \mathbb{C} : \, |\mu - (A)_{s,s}|\, (|\mu - (A)_{i,i}| + r_i^s(A)) < \left(|(A)_{s,i}| - r_s^i(A)\right) |(A)_{i,s}|\right\}, \qquad (3)$$

where $r_t^k(A) = r_t(A) - |(A)_{t,k}|$ with $k \neq t$, and the exclusion set

$$L_i(A) = \bigcup_{s\in\mathcal{N}\setminus\{i\}} L_{s,i}(A)$$

were introduced. It is worth noting that the set $L_i(A)$ may be empty. Defining the *Brauer inclusion-exclusion set for the i-th row and the j-th row* of A

$$\Phi_{i,j}(A) = \mathcal{B}_{i,j}(A) \setminus L_i(A), \quad i,j \in \mathcal{N}, \;\; i \neq j,$$

and the *Brauer inclusion-exclusion set* of A

$$\Phi(A) = \bigcup_{\substack{i,j\in\mathcal{N}\\ i\neq j}} \Phi_{i,j}(A), \qquad (4)$$

it is known that $\sigma(A) \subseteq \Phi(A) \subseteq \mathcal{B}(A)$ [14, Theorem 4].

Remark 24. For any $i, s \in \mathcal{N}$ with $i \neq s$, it follows that $L_{i,s}(A)$ and $L_{s,i}(A)$ are not necessarily equal. Thus, $\Phi_{i,j}(A)$ and $\Phi_{j,i}(A)$ are not necessarily equal, unlike the Brauer sets $\mathcal{B}_{i,j}(A)$ and $\mathcal{B}_{j,i}(A)$ (which coincide). For this reason, we consider the formula (4).

Remark 25. For any $i, j \in \mathcal{N}$ with $i \neq j$, the Brauer set $\mathcal{B}_{i,j}(A)$ is closed and the Brauer exclusion set $L_i(A)$ is open. Therefore, any $\Phi_{i,j}(A) = \mathcal{B}_{i,j}(A) \setminus L_i(A) = \mathcal{B}_{i,j}(A) \cap (\mathbb{C} \setminus L_i(A))$ $(i \neq j)$ is a closed set. We conclude that the set $\Phi(A)$ is non-empty (since it contains the eigenvalues of the matrix A) and closed as a finite union of the closed sets $\Phi_{i,j}(A)$.

Example 26. For the 4×4 matrix A of Example 3, the Brauer set and the Brauer inclusion-exclusion set of A are illustrated in Figures 17 and 18, respectively.

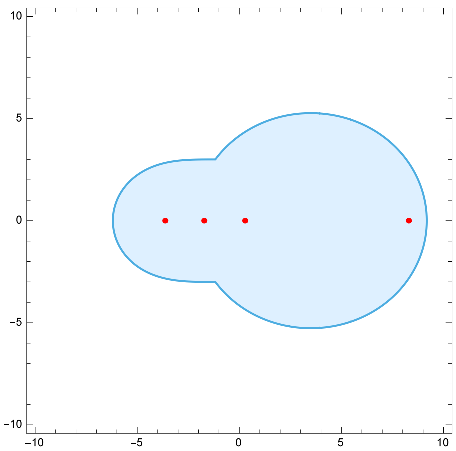


Figure 17: The Brauer set of A.

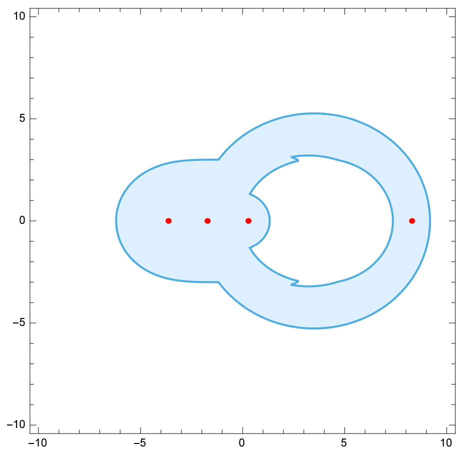


Figure 18: The Brauer inclusion-exclusion set of A.

3.2 The Brauer inclusion-exclusion set of a matrix polynomial

Consider a matrix polynomial $P(\lambda)$ as in (1) and the nonnegative functions $r_i(P(\lambda))$ $(i \in \mathcal{N})$ defined in (2). In [17], the sets

$$\mathcal{B}_{i,j}(P(\lambda)) = \{\mu \in \mathbb{C} :\ 0 \in \mathcal{B}_{i,j}(P(\mu))\} = \{\mu \in \mathbb{C} :\ |(P(\mu))_{i,i}|\,|(P(\mu))_{j,j}| \leq r_i(P(\mu))r_j(P(\mu))\}$$

$(i, j \in \mathcal{N},\ i \neq j)$ and the (inclusion) *Brauer set* of $P(\lambda)$

$$\mathcal{B}(P(\lambda)) = \{\mu \in \mathbb{C} :\ 0 \in \mathcal{B}(P(\mu))\} = \bigcup_{\substack{i,j \in \mathcal{N} \\ i \neq j}} \mathcal{B}_{i,j}(P(\lambda)) = \bigcup_{\substack{i,j \in \mathcal{N} \\ i > j}} \mathcal{B}_{i,j}(P(\lambda))$$

were studied. It is said that $\mu = \infty$ lies in $\mathcal{B}(P(\lambda))$ (or $\mathcal{B}_{i,j}(P(\lambda))$) if and only if 0 lies in $\mathcal{B}(\hat{P}(\lambda))$ (resp., $\mathcal{B}_{i,j}(\hat{P}(\lambda))$). As noted in [17], $\mathcal{B}(P(\lambda))$ is a closed subset of $\mathcal{G}(P(\lambda))$ that contains all the eigenvalues of $P(\lambda)$. Furthermore, if $(P(\lambda))_{i,i} = 0$ or $(P(\lambda))_{j,j} = 0$, then $\mathcal{B}_{i,j}(P(\lambda)) = \mathbb{C}$.

For clarity and reader's convenience, the following proposition summarizes some of the basic properties of the Brauer set $\mathcal{B}(P(\lambda))$ [17].

Proposition 27. Let $i, j \in \mathcal{N}$ with $i \neq j$.

(i) For any scalar $b \in \mathbb{C}\backslash\{0\}$, it holds that $\mathcal{B}_{i,j}(P(b\lambda)) = b^{-1}\mathcal{B}_{i,j}(P(\lambda))$, $\mathcal{B}_{i,j}(bP(\lambda)) = \mathcal{B}_{i,j}(P(\lambda))$, and $\mathcal{B}_{i,j}(P(\lambda + b)) = \mathcal{B}_{i,j}(P(\lambda)) - b$.

(ii) If all the coefficient matrices $A_0, A_1, \ldots, A_m$ of $P(\lambda)$ have their i-th row and j-th row real, then $\mathcal{B}_{i,j}(P(\lambda))$ is symmetric with respect to the real axis.

(iii) The set $\{\mu \in \mathbb{C} : \, |(P(\mu))_{i,i}| \, |(P(\mu))_{j,j}| < r_i(P(\mu))r_j(P(\mu))\}$ lies in the interior of $\mathcal{B}_{i,j}(P(\lambda))$. Consequently, $\partial\mathcal{B}_{i,j}(P(\lambda)) \subseteq \{\mu \in \mathbb{C} : \, |(P(\mu))_{i,i}| \, |(P(\mu))_{j,j}| = r_i(P(\mu))r_j(P(\mu))\} = \{\mu \in \mathbb{C} : \, 0 \in \partial\mathcal{B}_{i,j}(P(\mu))\}$.

(iv) If a scalar $\mu_0 \in \mathbb{C}$ is an isolated point of $\mathcal{B}_{i,j}(P(\lambda))$, then μ_0 is a root of either $(P(\lambda))_{i,i}$ or $(P(\lambda))_{j,j}$, and μ_0 is a common root of all polynomials $(P(\lambda))_{i,p}$, $p \in \mathcal{N}\backslash\{i\}$, or a common root of all polynomials $(P(\lambda))_{j,p}$, $p \in \mathcal{N}\backslash\{j\}$.

(v) Suppose that the i-th row and the j-th row of A_m are non-zero. If $(A_m)_{i,i} \neq 0$ and $(A_m)_{j,j} \neq 0$, then $\mathcal{B}_{i,j}(P(\lambda))$ is unbounded if and only if $0 \in \mathcal{B}_{i,j}(A_m)$. If $(A_m)_{i,i} = (A_m)_{j,j} = 0$, then $\mathcal{B}_{i,j}(P(\lambda))$ is unbounded and $0 \in \mathcal{B}_{i,j}(A_m)$.

(vi) If all the diagonal entries of A_m are non-zero and $\mathcal{B}(P(\lambda))$ is bounded, then the number of connected components of $\mathcal{B}(P(\lambda))$ is less than or equal to nm.

Remark 28. By Theorem 4.6 in [17], its proof and the relative discussion therein, it follows that the condition "$(A_m)_{i,i} = (A_m)_{j,j} = 0$" in Proposition 27 (v) can be replaced by the condition "$(A_m)_{i,i} = 0$ and a non-diagonal entry of the j-th row of A_m is non-zero". These two conditions make sure that the origin is not an isolated point of $\mathcal{B}_{i,j}(\hat{P}(\lambda))$.

We now extend the concepts of the Brauer exclusion set and the Brauer inclusion-exclusion set to matrix polynomials.

Definition 29. For $i, s \in \mathcal{N}$ with $i \neq s$, define the sets

$$\begin{aligned} L_{s,i}(P(\lambda)) &= \{\mu \in \mathbb{C} : \, 0 \in L_{s,i}(P(\mu))\} \\ &= \left\{\mu \in \mathbb{C} : \, |(P(\mu))_{s,s}| \left(|(P(\mu))_{i,i}| + r_i^s\left(P(\mu)\right)\right) < \left(|(P(\mu))_{s,i}| - r_s^i(P(\mu))\right) |(P(\mu))_{i,s}|\right\}, \end{aligned}$$

and the *i-th Brauer exclusion set*

$$L_i(P(\lambda)) = \bigcup_{s \in \mathcal{N}\backslash\{i\}} L_{s,i}(P(\lambda)).$$

From the above definition, it follows immediately that for any $i \in \mathcal{N}$,

$$\begin{aligned} L_i(\hat{P}(\lambda))\backslash\{0\} &= \left\{\mu \in \mathbb{C}\backslash\{0\} : \, 0 \in L_i(\hat{P}(\lambda))\right\} = \left\{\mu \in \mathbb{C}\backslash\{0\} : \, 0 \in L_i(\mu^m P(\mu^{-1}))\right\} \\ &= \left\{\mu \in \mathbb{C}\backslash\{0\} : \, 0 \in L_i(P(\mu^{-1}))\right\} = \left\{\mu \in \mathbb{C}\backslash\{0\} : \, \mu^{-1} \in L_i(P(\lambda))\right\}. \end{aligned}$$

Moreover, $\mu = \infty$ lies in $L_i(P(\lambda))$ (or $L_{s,i}(P(\lambda))$) if and only if 0 lies in $L_i(\hat{P}(\lambda))$ (resp., $L_{s,i}(\hat{P}(\lambda))$).

Remark 30. As in the case of the Gershgorin exclusion sets, $L_{s,i}(P(\lambda))$ is not necessarily non-empty. For example, it is empty (in a trivial way) if for every $\lambda \in \mathbb{C}$, $|(P(\lambda))_{s,i}| \leq r_s^i(P(\lambda))$ or $(P(\lambda))_{i,s} = 0$.

Definition 31. The *Brauer inclusion-exclusion set* of the matrix polynomial $P(\lambda)$ is defined as

$$\Phi(P(\lambda)) = \bigcup_{\substack{i,j\in\mathcal{N}\\ i\neq j}} \Phi_{i,j}(P(\lambda)),$$

where

$$\Phi_{i,j}(P(\lambda)) = \mathcal{B}_{i,j}(P(\lambda)) \setminus L_i(P(\lambda)), \quad i,j \in \mathcal{N}, \ \ i \neq j.$$

Remark 32. Just as with constant matrices, while it holds that $\mathcal{B}_{i,j}(P(\lambda)) = \mathcal{B}_{j,i}(P(\lambda))$ for each $i \neq j$, we have that $\Phi_{i,j}(P(\lambda)) \neq \Phi_{j,i}(P(\lambda))$.

Theorem 33. All the eigenvalues of the matrix polynomial $P(\lambda)$ lie in the Brauer inclusion-exclusion set $\Phi(P(\lambda))$.

Proof. For any finite eigenvalue $\mu \in \sigma(P(\lambda))$, we have that $0 \in \sigma(P(\mu))$. By [17], there exist $i,j \in \mathcal{N}$ with $i \neq j$ such that $0 \in \mathcal{B}_{i,j}(P(\mu))$. Additionally, by Theorem 4 of [14], we have that $0 \notin L_i(P(\mu))$, and therefore, $\mu \notin L_i(P(\lambda))$. Thus,

$$\mu \in \Phi_{i,j}(P(\lambda)) \subseteq \Phi(P(\lambda)).$$

If $\mu = \infty$, then $0 \in \sigma(\hat{P}(\mu)) \subseteq \Phi(\hat{P}(\lambda))$, and consequently, $\mu \in \Phi(P(\lambda))$. $\square$

We continue with the study of geometrical and topological properties of the Brauer exclusion set. The next proposition can be obtained similarly to Proposition 11.

Proposition 34. Let $i,j \in \mathcal{N}$ with $i \neq j$. Then, the following hold:

(i) The set $L_{i,j}(P(\lambda))$ is open (consequently, the set $L_i(P(\lambda))$ is also open).

(ii) For any scalar $b \in \mathbb{C} \setminus \{0\}$, it holds that $L_{i,j}(P(b\lambda)) = b^{-1}L_{i,j}(P(\lambda))$, $L_{i,j}(bP(\lambda)) = L_{i,j}(P(\lambda))$, and $L_{i,j}(P(\lambda + b)) = L_{i,j}(P(\lambda)) - b$ (consequently, $L_i(P(\lambda))$ satisfies these properties as well).

(iii) If all the coefficient matrices $A_0, A_1, \ldots, A_m$ of $P(\lambda)$ have their i-th row and j-th row real, then $L_{i,j}(P(\lambda))$ is symmetric with respect to the real axis (consequently, if $A_0, A_1, \ldots, A_m$ are real, then $\Phi(P(\lambda))$ is symmetric with respect to the real axis).

Remark 35. For any $i,j \in \mathcal{N}$ with $i \neq j$, the set $\mathcal{B}_{i,j}(P(\lambda))$ is closed [17, Proposition 4.2] and the set $L_i(P(\lambda))$ is open. Therefore, the set $\Phi_{i,j}(P(\lambda)) = \mathcal{B}_{i,j}(P(\lambda)) \setminus L_i(P(\lambda)) = \mathcal{B}_{i,j}(P(\lambda)) \cap (\mathbb{C} \setminus L_i(P(\lambda)))$ is closed. Hence, the Brauer inclusion-exclusion set $\Phi(P(\lambda))$ is also closed as a finite union of closed sets.

In the next three propositions, we study the unboundedness of the Brauer exclusion sets. In Proposition 36, we consider the case of zero diagonal entries of the leading coefficient matrix, while in Propositions 37 and 38, we consider the case of non-zero diagonal entries of the leading coefficient matrix.

Proposition 36. Suppose that for some $s \in \mathcal{N}$, it holds that $(A_m)_{s,s} = 0$. If there exists an $i \in \mathcal{N} \setminus \{s\}$ such that $2\,|(A_m)_{s,i}| - r_s(A_m) > 0$ and $(A_m)_{i,s} \neq 0$, then i is unique, $0 \in L_{s,i}(A_m)$ ($\subseteq L_i(P(\lambda))$) and the sets $L_{s,i}(P(\lambda))$ and $L_i(P(\lambda))$ are unbounded.

Proof. Let $s \in \mathcal{N}$ with $(A_m)_{s,s} = 0$. Then, for any $i \in \mathcal{N} \setminus \{s\}$, we have

$$\begin{aligned} L_{s,i}(A_m) &= \left\{\mu \in \mathbb{C} :\, |\mu| \left(|\mu - (A_m)_{i,i}| + r_i^s(A_m)\right) < \left(|(A_m)_{s,i}| - r_s^i(A_m)\right) |(A_m)_{i,s}|\right\} \\ &= \left\{\mu \in \mathbb{C} :\, |\mu| \left(|\mu - (A_m)_{i,i}| + r_i^s(A_m)\right) < \left(2|(A_m)_{s,i}| - r_s(A_m)\right) |(A_m)_{i,s}|\right\}. \end{aligned}$$

As in the proof of Proposition 13, if the quantity $2|(A_m)_{s,i}| - r_s(A_m)$ is positive for some $i \in \mathcal{N} \setminus \{s\}$, then this i is unique (referring to this particular s).

Suppose that there exists an $i \in \mathcal{N} \setminus \{s\}$ such that $2|(A_m)_{s,i}| - r_i(A_m) > 0$ and $(A_m)_{i,s} \neq 0$, or equivalently, $L_{s,i}(A_m)$ is non-empty. Then, it follows that $0 \in L_{s,i}(A_m)$ and (by definition) $2\,|(A_m)_{s,i}| > r_s(A_m) > 0$, i.e., the s-th row of the matrix A_m is nonzero. Moreover,

$$\begin{aligned} L_{s,i}(P(\lambda)) \setminus \{0\} &= \left\{\mu \in \mathbb{C} \setminus \{0\} :\, |(P(\mu))_{s,s}| \left(|(P(\mu))_{i,i}| + r_i^s\left(P(\mu)\right)\right) < \left(|(P(\mu))_{s,i}| - r_s^i(P(\mu))\right) |(P(\mu))_{i,s}|\right\} \\ &\overset{(A_m)_{s,s}=0}{=} \Bigg\{\mu \in \mathbb{C} \setminus \{0\} :\, \left|\sum_{k=0}^{m-1} (A_k)_{s,s} \mu^k\right| \left(\left|\sum_{k=0}^{m} (A_k)_{i,i} \mu^k\right| + \sum_{p \in \mathcal{N} \setminus \{i,s\}} \left|\sum_{k=0}^{m} (A_k)_{i,p} \mu^k\right|\right) \\ &\qquad < \left(2\left|\sum_{k=0}^{m} (A_k)_{s,i} \mu^k\right| - \sum_{p \in \mathcal{N} \setminus \{s\}} \left|\sum_{k=0}^{m} (A_k)_{s,p} \mu^k\right|\right) \left|\sum_{k=0}^{m} (A_k)_{i,s} \mu^k\right|\Bigg\} \\ &= \Bigg\{\mu \in \mathbb{C} \setminus \{0\} :\, \left|\sum_{k=0}^{m-1} (A_k)_{s,s} \frac{\mu^k}{\mu^m}\right| \left(\left|\sum_{k=0}^{m} (A_k)_{i,i} \frac{\mu^k}{\mu^m}\right| + \sum_{p \in \mathcal{N} \setminus \{i,s\}} \left|\sum_{k=0}^{m} (A_k)_{i,p} \frac{\mu^k}{\mu^m}\right|\right) \\ &\qquad < \left(2\left|\sum_{k=0}^{m} (A_k)_{s,i} \frac{\mu^k}{\mu^m}\right| - \sum_{p \in \mathcal{N} \setminus \{s\}} \left|\sum_{k=0}^{m} (A_k)_{s,p} \frac{\mu^k}{\mu^m}\right|\right) \left|\sum_{k=0}^{m} (A_k)_{i,s} \frac{\mu^k}{\mu^m}\right|\Bigg\}. \end{aligned}$$

From the assumption $0 < \left(2|(A_m)_{s,i}| - r_s(A_m)\right) |(A_m)_{i,s}|$, it follows that for sufficiently large $|\mu|$, μ lies in $L_{s,i}(P(\lambda))$. Furthermore, there exists a positive real number $M > 0$ such that for any $\mu \in \mathbb{C}$ with $|\mu| \geq M$, it holds that $\mu \in L_{s,i}(P(\lambda))$. Hence, $\{\mu \in \mathbb{C} :\, |\mu| \geq M\} \subseteq L_{s,i}(P(\lambda)) \subseteq L_i(P(\lambda))$. □

Proposition 37. Suppose that for some $s \in \mathcal{N}$, it holds that $(A_m)_{s,s} \neq 0$. If there exists an $i \in \mathcal{N} \setminus \{s\}$ such that $0 \in L_{s,i}(A_m)$, then the sets $L_{s,i}(P(\lambda))$ and $L_i(P(\lambda))$ are unbounded.

Proof. Let $(A_m)_{s,s} \neq 0$, and suppose that there exists an $i \in \mathcal{N} \setminus \{s\}$ such that $0 \in L_{s,i}(A_m)$. Recalling that $A_m = \hat{P}(0)$ and following the steps in the proof of Proposition 14, we can verify that the sets $L_{s,i}(P(\lambda))$ and $L_i(P(\lambda))$ are unbounded. □

Proposition 38. Suppose that for some $s \in \mathcal{N}$, it holds that $(A_m)_{s,s} \neq 0$. If there exists an $i \in \mathcal{N} \setminus \{s\}$ such that the set $L_{s,i}\left(P(\lambda)\right)$ is unbounded, then 0 lies in the closure of $L_{s,i}(A_m)$.

Proof. Suppose that $(A_m)_{s,s} \neq 0$ and $L_{s,i}(P(\lambda))$ is unbounded for some $i \in \mathcal{N} \setminus \{s\}$. Clearly, 0 lies in $L_{s,i}(\hat{P}(\lambda))$. Thus, since 0 cannot be an isolated point of $L_{s,i}(\hat{P}(\lambda))$, there is a sequence $\{\mu_\ell\}_{\ell=1,2,\ldots}$ in $L_{s,i}(P(\lambda)) \setminus \{0\}$ such that $|\mu_\ell| \to \infty$. This implies that for every positive integer ℓ,

$$|(P(\mu_\ell))_{s,s}| \left(|(P(\mu_\ell))_{i,i}| + r_i^s(P(\mu_\ell))\right) < \left(2\,|(P(\mu_\ell))_{s,i}| - r_s(P(\mu_\ell))\right) |(P(\mu_\ell))_{i,s}|,$$

or

$$\begin{aligned}&\left|\sum_{k=0}^{m}(A_k)_{s,s}\mu_\ell^k\right| \left(\left|\sum_{k=0}^{m}(A_k)_{i,i}\mu_\ell^k\right| + \sum_{p\in\mathcal{N}\setminus\{i,s\}}\left|\sum_{k=0}^{m}(A_k)_{i,p}\mu_\ell^k\right|\right)\\ &< \left(2\left|\sum_{k=0}^{m}(A_k)_{s,i}\mu_\ell^k\right| - \sum_{p\in\mathcal{N}\setminus\{s\}}\left|\sum_{k=0}^{m}(A_k)_{s,p}\mu_\ell^k\right|\right)\left|\sum_{k=0}^{m}(A_k)_{i,s}\mu_\ell^k\right|,\end{aligned}$$

or

$$\begin{aligned}&\left|\sum_{k=0}^{m}(A_k)_{s,s}\frac{\mu_\ell^k}{\mu_\ell^m}\right| \left(\left|\sum_{k=0}^{m}(A_k)_{i,i}\frac{\mu_\ell^k}{\mu_\ell^m}\right| + \sum_{p\in\mathcal{N}\setminus\{i,s\}}\left|\sum_{k=0}^{m}(A_k)_{i,p}\frac{\mu_\ell^k}{\mu_\ell^m}\right|\right)\\ &< \left(2\left|\sum_{k=0}^{m}(A_k)_{s,i}\frac{\mu_\ell^k}{\mu_\ell^m}\right| - \sum_{p\in\mathcal{N}\setminus\{s\}}\left|\sum_{k=0}^{m}(A_k)_{s,p}\frac{\mu_\ell^k}{\mu_\ell^m}\right|\right)\left|\sum_{k=0}^{m}(A_k)_{i,s}\frac{\mu_\ell^k}{\mu_\ell^m}\right|.\end{aligned}$$

As $\ell \to \infty$,

$$|(A_m)_{s,s}| \left(|(A_m)_{i,i}| + \sum_{p\in\mathcal{N}\setminus\{i,s\}}|(A_m)_{i,p}|\right) \leq \left(2\,|(A_m)_{s,i}| - \sum_{p\in\mathcal{N}\setminus\{s\}}|(A_m)_{s,p}|\right)|(A_m)_{i,s}|.$$

Therefore, 0 lies in the closure of $L_{s,i}(A_m)$. □

It is worth noting that if $(A_m)_{s,s} = 0$, then $L_{s,i}(A_m)$ is non-empty if and only if $0 \in L_{s,i}(A_m)$. By Proposition 27 (v), Remark 28, and the proofs of Propositions 36 and 37, the next result follows readily.

Corollary 39. Suppose that for some $i, j, s \in \mathcal{N}$ with $i \neq j, s$, the i-th row and the j-th row of A_m are non-zero, and one of the following holds:

(i) $(A_m)_{i,i} = 0$, a non-diagonal entry of the j-th row of A_m is non-zero, and $0 \in L_{s,i}(A_m)$.

(ii) $(A_m)_{i,i} \neq 0$, $(A_m)_{j,j} \neq 0$, $0 \in \mathcal{B}_{i,j}(A_m)$, and $0 \in L_{s,i}(A_m)$.

Then $\mathcal{B}_{i,j}(P(\lambda))$ is unbounded and $\Phi_{i,j}(P(\lambda))$ is bounded.

Remark 40. In the case where the matrix polynomial $P(\lambda)$ is 2×2, it follows that

$$\begin{aligned}\mathcal{B}(P(\lambda)) = \mathcal{B}_{1,2}(P(\lambda)) &= \{\mu \in \mathbb{C} : |(P(\mu))_{1,1}|\,|(P(\mu))_{2,2}| \leq r_1(P(\mu))r_2(P(\mu))\}\\ &= \{\mu \in \mathbb{C} : |(P(\mu))_{1,1}|\,|(P(\mu))_{2,2}| \leq |(P(\mu))_{1,2}|\,|(P(\mu))_{2,1}|\},\end{aligned}$$

$$\begin{aligned} L_{1,2}(P(\lambda)) &= \left\{\mu \in \mathbb{C} : |(P(\mu))_{1,1}| \left(|(P(\mu))_{2,2}| + r_2^1(P(\mu))\right) < \left(|(P(\mu))_{1,2}| - r_1^2(P(\mu))\right) |(P(\mu))_{2,1}|\right\} \\ &= \left\{\mu \in \mathbb{C} : |(P(\mu))_{1,1}|\,|(P(\mu))_{2,2}| < |(P(\mu))_{1,2}|\,|(P(\mu))_{2,1}|\right\} = L_{2,1}(P(\lambda)), \end{aligned}$$

and

$$L_2(P(\lambda)) = L_1(P(\lambda)).$$

As a consequence, the Brauer inclusion-exclusion set is

$$\Phi(P(\lambda)) = \{\mu \in \mathbb{C} : |(P(\mu))_{1,1}|\,|(P(\mu))_{2,2}| = |(P(\mu))_{1,2}|\,|(P(\mu))_{2,1}|\}$$

and coincides with the boundary of the Brauer set $\mathcal{B}(P(\lambda))$.

Example 41. **(i)** Consider the 3×3 matrix polynomial $P(\lambda)$ in Example 18 (i). The Brauer set $\mathcal{B}(P(\lambda))$ in Figure 19 and the Brauer inclusion-exclusion set $\Phi(P(\lambda))$ in Figure 20 are bounded, and the improvement of the original estimation of the spectrum is clear.

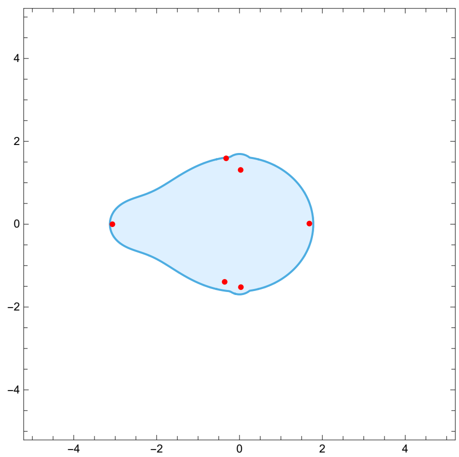


Figure 19: The Brauer set of $P(\lambda)$.

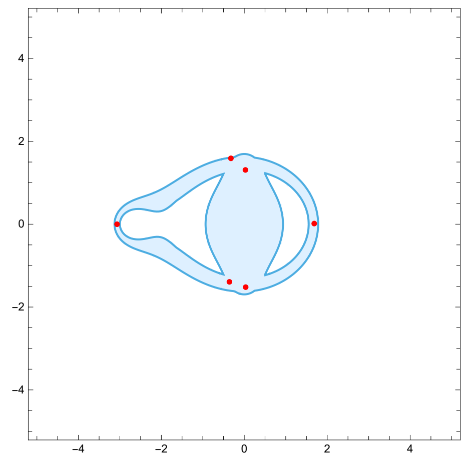


Figure 20: The Brauer inclusion-exclusion set of $P(\lambda)$.

(ii) The 3×3 matrix polynomial

$$Q(\lambda) = \begin{bmatrix} \lambda^2+1 & 5\lambda^2+\lambda & \lambda \\ -5\lambda^2+\lambda & 2\lambda & \lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}$$

has zero as an eigenvalue of algebraic multiplicity 3 and three (finite) simple nonzero eigenvalues. The Brauer set $\mathcal{B}(Q(\lambda))$ in Figure 21 is unbounded (in fact, it coincides with $\mathcal{B}_{1,2}(P(\lambda))$), while the Brauer inclusion-exclusion set $\Phi(Q(\lambda))$ in Figure 22 is bounded. The original estimation of the spectrum is significantly improved, and the symmetry of Proposition 34 (iii) is confirmed. It is worth mentioning that if Q_2 is the leading coefficient matrix of $Q(\lambda)$, then $(Q_2)_{2,2} = 0$, $(Q_2)_{1,2} = 5 \neq 0$, and $L_{1,2}(Q_2) = \{\mu \in \mathbb{C} : |\mu - 1|\,|\mu| < 25\}$ is non-empty; this means that the (sufficient) conditions of Corollary 39 (i) hold for $i = 2$ and $j = s = 1$.

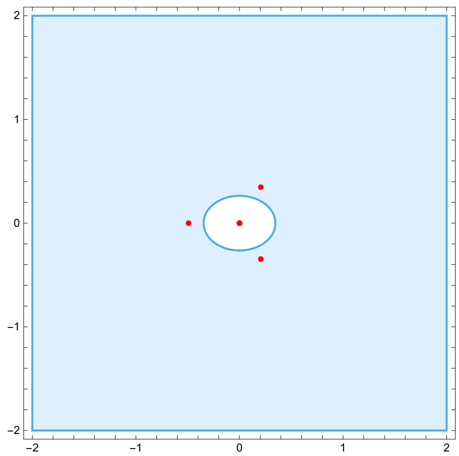


Figure 21: The Brauer set of $Q(\lambda)$.

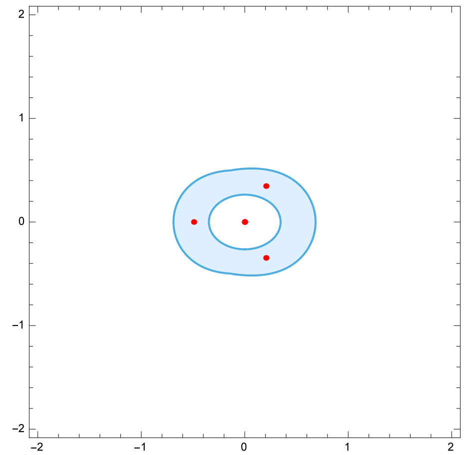


Figure 22: The Brauer inclusion-exclusion set of $Q(\lambda)$.

Remark 42. It is known that the Brauer set provides a more accurate approximation of the spectrum of a matrix polynomial than the Gershgorin set. A question that arises in a natural way is whether the Brauer inclusion-exclusion set is a better approximation compared to the Gershgorin inclusion-exclusion set as well. Our experiments indicate that the Brauer inclusion-exclusion set of a matrix polynomial is not necessarily better than the Gershgorin inclusion-exclusion set.

4 The Dashnic-Zusmanovich set

For a square complex matrix $A \in \mathbb{C}^{n\times n}$, the Dashnic-Zusmanovich set is defined as [6]

$$\mathcal{D}(A) = \bigcap_{i\in\mathcal{N}} \bigcup_{j\in\mathcal{N}\setminus\{i\}} \mathcal{D}_{i,j}(A),$$

where

$$\mathcal{D}_{i,j}(A) = \{\mu \in \mathbb{C} : \, |\mu - (A)_{i,i}| \left(|\mu - (A)_{j,j}| - r_j^i(A)\right) \leq r_i(A)|(A)_{j,i}|\},$$

and it contains all the eigenvalues of A, i.e., $\sigma(A) \subseteq \mathcal{D}(A)$. Additionally, by [5], we know that $\mathcal{D}(A) \subseteq \mathcal{B}(A) \subseteq \mathcal{G}(A)$, i.e., the Dashnic-Zusmanovich set is a better approximation of the spectrum of A compared to the Gershgorin and Brauer sets.

In [23], the Dashnic-Zusmanovich inclusion-exclusion set of A was introduced as

$$\Theta(A) = \bigcap_{i\in\mathcal{N}} \bigcup_{j\in\mathcal{N}\setminus\{i\}} \left(\mathcal{D}_{i,j}(A) \setminus L_{i,j}(A)\right),$$

where $L_{i,j}(A)$ is given by (3). According to Theorem 5 in [23], it holds that $\sigma(A) \subseteq \Theta(A)$.

Consider a matrix polynomial $P(\lambda)$ as in (1) and the nonnegative functions $r_i(P(\lambda))$ ($i \in \mathcal{N}$) defined in (2). The (inclusion) *Dashnic-Zusmanovich set* of $P(\lambda)$ is defined as [17]

$$\mathcal{D}(P(\lambda)) = \{\mu \in \mathbb{C} : \, 0 \in \mathcal{D}(P(\mu))\} = \bigcap_{i\in\mathcal{N}} \bigcup_{j\in\mathcal{N}\setminus\{i\}} \mathcal{D}_{i,j}(P(\lambda)),$$

where

$$\begin{aligned}\mathcal{D}_{i,j}(P(\lambda)) &= \{\mu \in \mathbb{C} : \, 0 \in \mathcal{D}_{i,j}(P(\mu))\} \\ &= \{\mu \in \mathbb{C} : \, |(P(\mu))_{i,i}| \left(|(P(\mu))_{j,j}| - r_j(P(\mu)) + |(P(\mu))_{j,i}|\right) \le r_i(P(\mu))|(P(\mu))_{j,i}|\} \,.\end{aligned}$$

Definition 43. The *Dashnic-Zusmanovich inclusion-exclusion set* of $P(\lambda)$ is defined as

$$\Theta(P(\lambda)) = \bigcap_{i\in\mathcal{N}} \bigcup_{j\in\mathcal{N}\setminus\{i\}} \left(\mathcal{D}_{i,j}(P(\lambda)) \setminus L_{i,j}(P(\lambda))\right),$$

where (see Definition 29)

$$\begin{aligned}L_{i,j}(P(\lambda)) &= \{\mu \in \mathbb{C} : \, 0 \in L_{i,j}(P(\mu))\} \\ &= \left\{\mu \in \mathbb{C} : \, |(P(\mu))_{i,i}| \left(|(P(\mu))_{j,j}| + r_j^i(P(\mu))\right) < \left(|(P(\mu))_{i,j}| - r_i^j(P(\mu))\right) |(P(\mu))_{j,i}|\right\} .\end{aligned}$$

Remark 44. Basic properties of the Dashnic-Zusmanovich set $\mathcal{D}(P(\lambda))$ can be found in Section 5.2 of [17], while geometric and topological properties of the exclusion sets $L_{i,j}(P(\lambda))$ are discussed in Section 3.2.

Example 45. **(i)** Consider the 3×3 matrix polynomial

$$P(\lambda) = \begin{bmatrix} \lambda^2 & 9 & 1 \\ 5 & \lambda^2 + 2\lambda & 0 \\ \mathrm{i} & 0 & \lambda^2 \end{bmatrix}.$$

The Dashnic-Zusmanovich set $\mathcal{D}(P(\lambda))$ in Figure 23 and the Dashnic-Zusmanovich inclusion-exclusion set $\Theta(P(\lambda))$ in Figure 24 are bounded, and the improvement of the original estimation of the spectrum is obvious.

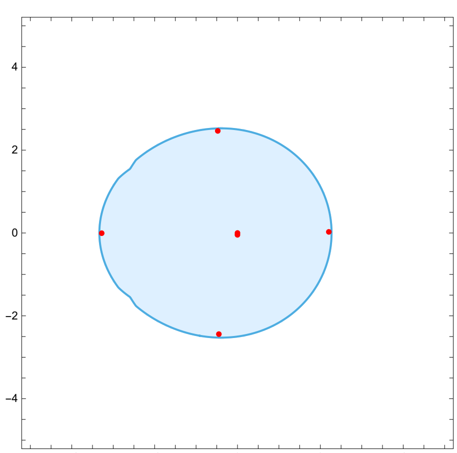


Figure 23: The Dashnic-Zusmanovich set of $P(\lambda)$.

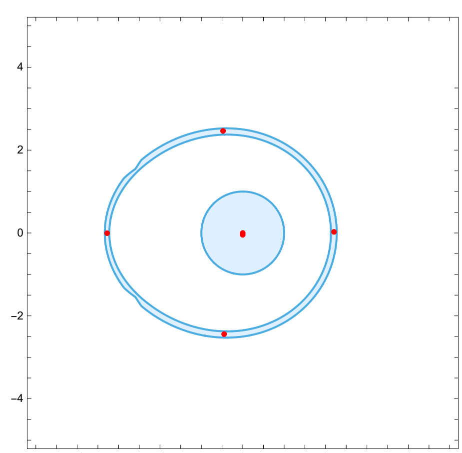


Figure 24: The Dashnic-Zusmanovich inclusion-exclusion set of $P(\lambda)$.

(ii) Consider the 3×3 matrix polynomial

$$Q(\lambda) = \begin{bmatrix} 2\lambda^2 + 13\lambda - 23\mathrm{i} & -8\lambda + 6 - \mathrm{i} & \lambda - 1 \\ -7\lambda^2 + 2\lambda & -4\mathrm{i}\,\lambda + 9 - 2\mathrm{i} & \lambda + 4 \\ \lambda^2 & -\lambda - 0.3 - 2\mathrm{i} & \lambda^2 + 3\lambda + 25 \end{bmatrix},$$

which has five finite eigenvalues and one infinite eigenvalue, all of them of algebraic multiplicity one. The Dashnic-Zusmanovich set $\mathcal{D}(Q(\lambda))$ in Figure 25 is unbounded, while the Dashnic-Zusmanovich inclusion-exclusion set $\Theta(Q(\lambda))$ in Figure 26 is the union of a bounded set (which has two connected components and contains all the finite eigenvalues of $Q(\lambda)$) and ∞. Apparently, the original estimation of the spectrum is significantly improved.

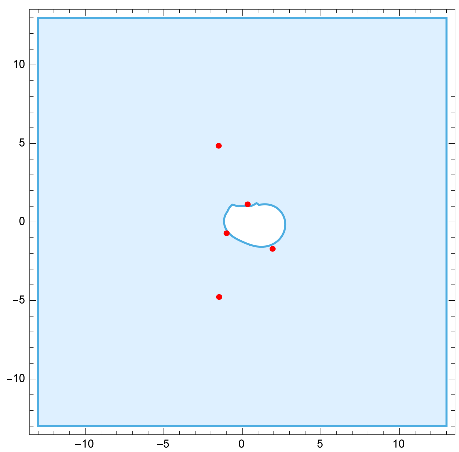


Figure 25: The Dashnic-Zusmanovich set of $Q(\lambda)$.

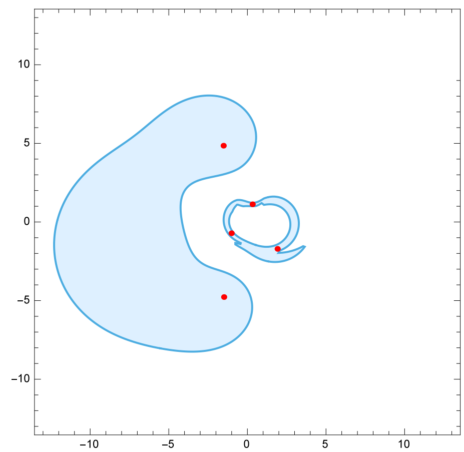


Figure 26: The Dashnic-Zusmanovich inclusion-exclusion set of $Q(\lambda)$.

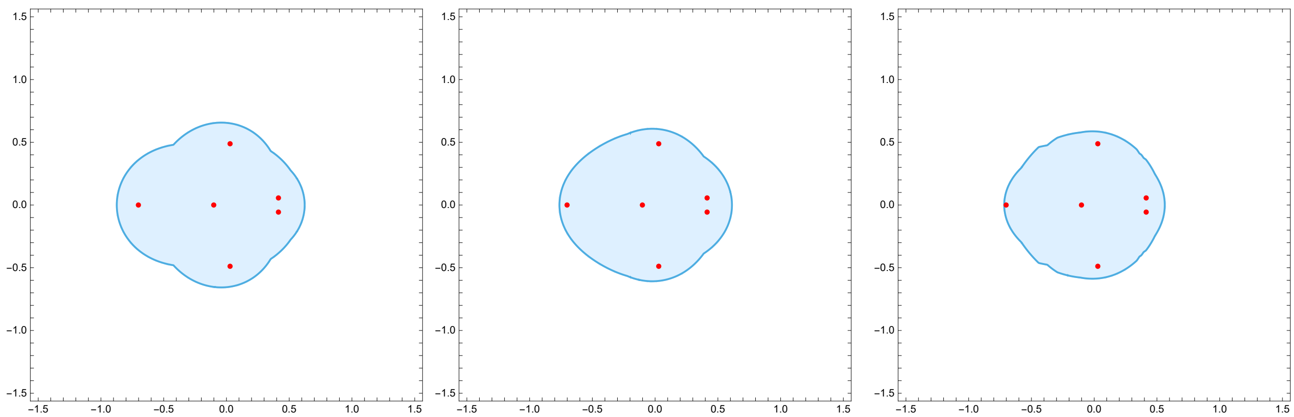
Remark 46. Several spectrum localizations are obtained for matrix polynomials in [10] and for quadratic matrix polynomials in [20], by considering linearizations. For the 3×3 quadratic matrix polynomials

$$P(\lambda) = \begin{bmatrix} \lambda^2 - 0.0532\lambda + 0.0663 & 0.1306\lambda - 0.2776 & 0.0533 - 0.078\lambda \\ 0.0516\lambda - 0.1538 & \lambda^2 - 0.1687\lambda - 0.0166 & 0.0523 - 0.2835\lambda \\ 0.2792 - 0.1381\lambda & -0.1572\lambda - 0.0392 & \lambda^2 + 0.1385\lambda - 0.1444 \end{bmatrix}$$

and

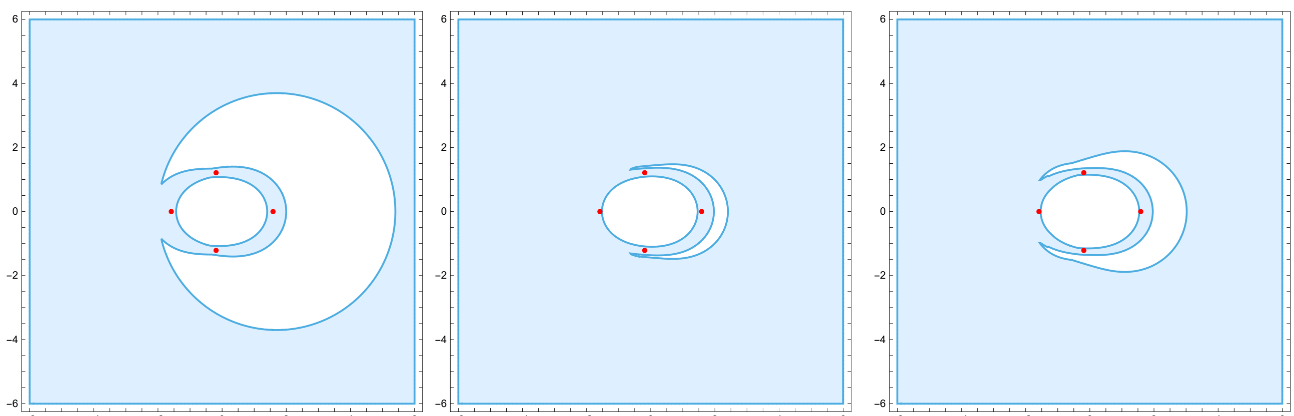
$$Q(\lambda) = \begin{bmatrix} 0.1827\lambda^2 + 0.0533\lambda + 1 & -0.4997\lambda^2 - 0.1386\lambda & -0.1792\lambda \\ -0.4997\lambda^2 - 0.1386\lambda & 0.1506\lambda^2 + 0.1723\lambda + 1 & -0.1637\lambda \\ -0.1792\lambda & -0.1637\lambda & 0.1578\lambda + 1 \end{bmatrix}$$

in Examples 5 and 6 of [20], respectively, the Gershgorin, Brauer and Dashnic-Zusmanovich inclusion-exclusion sets in Figures 27a–27c and 28a–28c imply that our estimations are comparable to, and can be more precise than, the bounds in [10, 20].



(a) The Gershgorin inclusion-exclusion set of $P(\lambda)$.
(b) The Brauer inclusion-exclusion set of $P(\lambda)$.
(c) The Dashnic-Zusmanovich inclusion-exclusion set of $P(\lambda)$.

Figure 27: The Gershgorin, Brauer and Dashnic-Zusmanovich inclusion-exclusion sets of $P(\lambda)$.



(a) The Gershgorin inclusion-exclusion set of $Q(\lambda)$.
(b) The Brauer inclusion-exclusion set of $Q(\lambda)$.
(c) The Dashnic-Zusmanovich inclusion-exclusion set of $Q(\lambda)$.

Figure 28: The Gershgorin, Brauer and Dashnic-Zusmanovich inclusion-exclusion sets of $Q(\lambda)$.

References

[1] S. Boyd and C.A. Desoer, Subharmonic functions and performance bounds on linear time-invariant feedback systems, *IMA Journal of Mathematical Control and Information*, **2** (1985), 153–170.

[2] A. Brauer, Limits for the characteristic roots of a matrix II, *Duke Mathematical Journal*, **14** (1947), 21–26.

[3] T.R. Cameron and P.J. Psarrakos, On Householder sets for matrix polynomials, *Linear Algebra and its Applications*, **585** (2020), 105–126.

[4] J.B. Conway, *Functions of One Complex Variable*, Springer-Verlag, New York, 1973.

[5] L. Cvetkovic, V. Kostic, and R.S. Varga, A new Gershgorin-type eigenvalue inclusion set, *Electronic Transactions on Numerical Analysis*, **18** (2004), 73–80.

[6] L.S. Dashnic and M.S. Zusmanovich, O nekotoryh kriteriyah regulyarnosti matric i lokalizacii ih spectra, *Zh. Vychisl. Matem. i Matem. Fiz.*, **5** (1970), 1092–1097.

[7] B. Friedland, *Control System Design: An Introduction to State-Space Methods*, Dover Publications, Mineola, New York, 2012.

[8] S. Gershgorin, Über die Abgrenzung der Eigenwerte einer Matrix, *Izvestiya Akademii Nauk SSSR, Seriya Fizicheskaya-Matematicheskaya*, **6** (1931), 749–754.

[9] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials* (second edition), Society for Industrial and Applied Mathematics, Philadelphia, 2009.

[10] N.J. Higham and F. Tisseur, Bounds for eigenvalues of matrix polynomials, *Linear Algebra and its Applications*, **258** (2003), 5–22.

[11] S.G. Kelly, *Mechanical Vibrations: Theory and Applications*, Cengage Learning, 2012.

[12] P. Lancaster, *Lambda-Matrices and Vibrating Systems*. Dover Publications, 2002.

[13] P. Lancaster and M. Tismenetsky, *The Theory of Matrices* (second edition), Academic Press, Orlando, 1985.

[14] S. Li, C. Li, and Y. Li, *Exclusion sets for eigenvalues of matrices*, arXiv (2017).

[15] A.S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Translations of Mathematical Monographs, Vol. 71, American Mathematical Society, Providence, 1988.

[16] A. Melman, Gershgorin disk fragments, *Mathematics Magazine*, **83** (2010), 123–129.

[17] C. Michailidou and P. Psarrakos, Gershgorin type sets for eigenvalues of matrix polynomials, *The Electronic Journal of Linear Algebra*, **34** (2018), 652–674.

[18] M.F. Neuts, *Matrix-Geometric Solutions in Stochastic Models*, Springer, Berlin, 1984.

[19] A.M. Ostrowski, Über das Nichtverschwinden einer Klasse von Determinanten und die Lokalisierung der charakteristischen Wurzeln von Matrizen, *Compositio Mathematica*, **9** (1951), 209–226.

[20] N. Roy and S. Bora, Locating eigenvalues of quadratic matrix polynomials, *Linear Algebra and its Applications*, **649** (2022), 452–490.

[21] H. Schneider, Regions of exclusion for the latent roots of a matrix, *Proceedings of the American Mathematical Society*, **5** (1954), 320–322.

[22] R.S. Varga, *Geršgorin and His Circles*, Springer Series in Computational Mathematics, Vol. 36, Springer-Verlag, Berlin, 2004.

[23] J. Zhao and L. She, Exclusion sets in Dashnic Zusmanovich localization sets, *Journal of Inequalities and Applications*, **228** (2019), 1–10.