ON THE DISTANCE FROM A WEAKLY NORMAL MATRIX POLYNOMIAL TO MATRIX POLYNOMIALS WITH A PRESCRIBED MULTIPLE EIGENVALUE

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Abstract. Consider an $n \times n$ matrix polynomial $P(\lambda)$. An upper bound for a spectral norm distance from $P(\lambda)$ to the set of $n \times n$ matrix polynomials that have a given scalar $\mu \in \mathbb{C}$ as a multiple eigenvalue was obtained by Papathanasiou and Psarrakos (2008). This paper concerns a refinement of this result for the case of weakly normal matrix polynomials. A modified method is developed and its efficiency is verified by two illustrative examples. The proposed methodology can also be applied to general matrix polynomials.

Keywords: Matrix polynomial, Eigenvalue, Normality, Perturbation, Singular value.

AMS Classification: 15A18, 65F35.

1. Introduction. Let A be an $n \times n$ complex matrix and let μ be a complex number, and denote by $\mathcal{M}_\mu$ the set of $n \times n$ complex matrices that have $\mu \in \mathbb{C}$ as a multiple eigenvalue. Malyshev [10] obtained the following formula for the spectral norm distance from A to $\mathcal{M}_\mu$:

$$\min_{B \in \mathcal{M}_\mu} \|A - B\|_2 = \max_{\gamma \geq 0} s_{2n-1}\left(\begin{bmatrix} A - \mu I & \gamma I_n \\ 0 & A - \mu I \end{bmatrix}\right),$$

where $\|\cdot\|_2$ denotes the spectral matrix norm (i.e., that norm subordinate to the euclidean vector norm) and $s_1(\cdot) \geq s_2(\cdot) \geq s_3(\cdot) \geq \cdots$ are the singular values of the corresponding matrix in a nonincreasing order. Malyshev's work can be considered as a theoretical solution to Wilkinson's problem, that is, the calculation of the distance from a matrix $A \in \mathbb{C}^{n\times n}$ that has all its eigenvalues simple to the $n \times n$ matrices with multiple eigenvalues. Wilkinson introduced this distance in [17], and some bounds for it were computed by Ruhe [15], Wilkinson [18, 19, 20, 21] and Demmel [1].

However, in the non-generic case where A is a normal matrix, it is not clear how one can construct the optimal perturbation based on Malyshev's derivation. In 2004, Ikramov and Nazari [4] showed this point and obtained a modification of Malyshev's method for normal matrices. Moreover, Malyshev's results were extended by Lippert [9] and Gracia [3]; in particular, they computed a spectral norm distance from A to

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the set of matrices that have two prescribed eigenvalues and studied a nearest matrix with the two desired eigenvalues. Nazari and Rajabi [12] refined the method obtained by Lippert and Gracia for the case of normal matrices.

In 2008, Papathanasiou and Psarrakos [14] introduced and studied a spectral norm distance from an $n \times n$ matrix polynomial $P(\lambda)$ to the set of $n \times n$ matrix polynomials that have a given scalar $\mu \in \mathbb{C}$ as a multiple eigenvalue. In particular, generalizing Malyshev's methodology, they computed lower and upper bounds for this distance, constructing an associated perturbation of $P(\lambda)$ to derive the upper bound. Motivated by the above, in this note, we study the case of weakly normal matrix polynomials. In the next section, we give some definitions, and briefly present some of the results of [13, 14]. We also give an example of a normal matrix polynomial where the method described in [14] for the computation of the upper bound is not directly applicable. In Section 3, we prove that the methodology of [14] for the computation of the upper bound is indeed not directly applicable to weakly normal matrix polynomials (see Theorem 3.1), and in Section 4, we obtain a modified procedure to improve the method. The same numerical example is considered to illustrate the validity of the proposed technique. It is remarkable that the proposed technique can be applied to *general* matrix polynomials and not only to weakly normal matrix polynomials; see the discussion in Section 4 and the example of Subsection 4.2.

2. Preliminaries. For given $A_0, A_1, \ldots, A_m \in \mathbb{C}^{n\times n}$, with $\det(A_m) \neq 0$, and a complex variable λ, we define the *matrix polynomial*

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0. \tag{2.1}$$

The study of matrix polynomials, especially with regard to their spectral analysis, has received a great deal of attention and has been used in several applications [2, 6, 7, 11, 16]. Standard references for the theory of matrix polynomials are [2, 11]. Here, some definitions of matrix polynomials are briefly reviewed.

If $P(\lambda_0)x_0 = 0$ for a scalar $\lambda_0 \in \mathbb{C}$ and some nonzero vector $x_0 \in \mathbb{C}^n$, then the scalar λ_0 is called an *eigenvalue* of $P(\lambda)$ and the vector x_0 is known as a *(right) eigenvector* of $P(\lambda)$ corresponding to λ_0. The *spectrum* of $P(\lambda)$, denoted by $\sigma(P)$, is the set of all eigenvalues of $P(\lambda)$. Since the leading matrix-coefficient A_m is nonsingular, the spectrum $\sigma(P)$ contains at most mn distinct finite elements. The multiplicity of an eigenvalue $\lambda_0 \in \sigma(P)$ as a root of the scalar polynomial $\det P(\lambda)$ is the *algebraic multiplicity* of λ_0, and the dimension of the null space of the (constant) matrix $P(\lambda_0)$ is the *geometric multiplicity* of λ_0. The algebraic multiplicity of an eigenvalue is always greater than or equal to its geometric multiplicity. An eigenvalue is called *semisimple* if its algebraic and geometric multiplicities are equal; otherwise, it is called *defective*.

DEFINITION 2.1. Let $P(\lambda)$ be a matrix polynomial as in (2.1). If there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that $U^*P(\lambda)U$ is a diagonal matrix polynomial, then $P(\lambda)$ is *weakly normal.* If, in addition, all the eigenvalues of $P(\lambda)$ are semisimple, then $P(\lambda)$ is *normal.*

The suggested references on weakly normal and normal matrix polynomials, and their properties are [8, 13]. Some of the results of [13] are summarized in the next proposition.

PROPOSITION 2.2. [13] *Let $P(\lambda)$ be a matrix polynomial as in (2.1). Then $P(\lambda)$ is weakly normal if and only if one of the following (equivalent) conditions holds.*

(i) *For every $\mu \in \mathbb{C}$, the matrix $P(\mu)$ is normal.*
(ii) *$A_0, \ldots, A_m$ are normal and mutually commuting (i.e., $A_iA_j = A_jA_i$; $i, j = 0, \ldots, m$).*
(iii) *Each linear combination of $A_0, A_1, \ldots, A_m$ is a normal matrix.*
(iv) *There exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that U^*A_jU is diagonal for every $j = 0, 1, \ldots, m$.*

As mentioned, Papathanasiou and Psarrakos [14] introduced a spectral norm distance from a matrix polynomial $P(\lambda)$ to the matrix polynomials that have μ as a multiple eigenvalue, and computed lower and upper bounds for this distance. Consider (additive) perturbations of $P(\lambda)$ of the form

$$Q(\lambda) = P(\lambda) + \Delta(\lambda) = (A_m + \Delta_m)\lambda^m + \cdots + (A_1 + \Delta_1)\lambda + A_0 + \Delta_0, \tag{2.2}$$

where the matrices $\Delta_0, \Delta_1, \ldots, \Delta_m \in \mathbb{C}^{n\times n}$ are arbitrary. For a given parameter $\epsilon \geq 0$ and a given set of nonnegative weights $\mathrm{w} = \{w_0, w_1, \ldots, w_m\}$ with $w_0 > 0$, define the class of admissible perturbed matrix polynomials

$$\mathcal{B}(P, \epsilon, \mathrm{w}) = \{Q(\lambda) \text{ as in (2.2)} : \|\Delta_j\|_2 \leq \epsilon\, w_j,\ j = 0, 1, \ldots, m\},$$

and the scalar polynomial $w(\lambda) = w_m\lambda^m + w_{m-1}\lambda^{m-1} + \cdots + w_1\lambda + w_0$. Note that the weights $w_0, w_1, \ldots, w_m$ allow freedom in how perturbations are measured.

For any real number $\gamma \in [0, +\infty)$, we define the $2n \times 2n$ matrix polynomial

$$F\,[P(\lambda); \gamma] = \begin{bmatrix} P(\lambda) & 0 \\ \gamma P'(\lambda) & P(\lambda) \end{bmatrix},$$

where $P'(\lambda)$ denotes the derivative of $P(\lambda)$ with respect to λ.

LEMMA 2.3. [14, Lemma 17] *Let $\mu \in \mathbb{C}$ and $\gamma_* > 0$ be a point where the singular value $s_{2n-1}(F[P(\mu); \gamma])$ attains its maximum value, and denote $s_* = s_{2n-1}(F[P(\mu); \gamma_*]) > 0$. Then there exists a pair $\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} \in \mathbb{C}^{2n}$*

$(u_k(\gamma_*), v_k(\gamma_*) \in \mathbb{C}^n,\ k=1,2)$ *of left and right singular vectors of* $F[P(\mu);\gamma_*]$ *corresponding to* s_**, respectively, such that*

(1) $u_2^*(\gamma_*)P'(\mu)v_1(\gamma_*) = 0$, *and*
(2) *the* $n \times 2$ *matrices* $U(\gamma_*) = [u_1(\gamma_*)\ u_2(\gamma_*)]$ *and* $V(\gamma_*) = [v_1(\gamma_*)\ v_2(\gamma_*)]$ *satisfy* $U^*(\gamma_*)U(\gamma_*) = V^*(\gamma_*)V(\gamma_*)$.

Moreover, it is remarkable that (1) implies (2) (see the proof of Lemma 17 in [14]).

Consider the quantity $\phi = \frac{w'(|\mu|)}{w(|\mu|)}\frac{\bar{\mu}}{|\mu|}$, where, by convention, we set $\frac{\bar{\mu}}{|\mu|} = 0$ whenever $\mu = 0$. Also let $V(\gamma_*)^\dagger$ be the *Moore-Penrose pseudoinverse* of $V(\gamma_*)$. For the pair of singular vectors $\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} \in \mathbb{C}^{2n}$ of Lemma 2.3, define the $n \times n$ matrix

$$\Delta_{\gamma_*} = -s_* U(\gamma_*) \begin{bmatrix} 1 & -\gamma_*\phi \\ 0 & 1 \end{bmatrix} V(\gamma_*)^\dagger.$$

Theorem 2.4. [14, Theorem 19] *Let* $P(\lambda)$ *be a matrix polynomial as in (2.1), and let* $\mathrm{w} = \{w_0, w_1, \ldots, w_m\}$*, with* $w_0 > 0$*, be a set of nonnegative weights. Suppose that* $\mu \in \mathbb{C}\backslash\sigma(P')$*,* $\gamma_* > 0$ *is a point where the singular value* $s_{2n-1}(F[P(\mu);\gamma])$ *attains its maximum value, and* $s_* = s_{2n-1}(F[P(\mu);\gamma_*]) > 0$*. Then, for the pair of singular vectors* $\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} \in \mathbb{C}^{2n}$ *of Lemma 2.3, we have*

$$\begin{aligned} &\min\{\epsilon \geq 0 : \exists\, Q(\lambda) \in \mathcal{B}(P,\epsilon,\mathrm{w}) \text{ with } \mu \text{ as a multiple eigenvalue}\} \\ &\leq \frac{s_*}{w(|\mu|)} \left\| V(\gamma_*) \begin{bmatrix} 1 & -\gamma_*\,\phi \\ 0 & 1 \end{bmatrix} V(\gamma_*)^\dagger \right\|_2 . \end{aligned}$$

Moreover, the perturbed matrix polynomial

$$Q_{\gamma_*}(\lambda) = P(\lambda) + \Delta_{\gamma_*}(\lambda) = P(\lambda) + \sum_{j=0}^{m} \frac{w_j}{w(|\mu|)} \left(\frac{\overline{\mu}}{|\mu|}\right)^j \Delta_{\gamma_*}\,\lambda^j, \tag{2.3}$$

lies on the boundary of the set $\mathcal{B}\left(P, \frac{s_*}{w(|\mu|)}\left\|V(\gamma_*)\begin{bmatrix} 1 & -\gamma_*\,\phi \\ 0 & 1 \end{bmatrix} V(\gamma_*)^\dagger\right\|_2, \mathrm{w}\right)$ *and has* μ *as a (multiple) defective eigenvalue.*

Some numerical examples in Section 8 of [14] illustrate the effectiveness of the upper bound of Theorem 2.4. In all these examples, s_* is a simple singular value, and consequently, the singular vectors $\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} \in \mathbb{C}^{2n}$ of Lemma 2.3 are directly computable (due to their essential uniqueness). When s_* is not simple, the vector spaces formed by the left and right singular vectors corresponding to s_*

are at least two dimensional, and (1) and (2) of Lemma 2.3 hold only for particular consistent pairs of left and right singular vectors.

Now let us consider the normal (in particular, diagonal) matrix polynomial

$$P(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \lambda + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{2.4}$$

that is borrowed from [13, Section 3]. Also let the set of weights $w = \{1, 1, 1\}$ and the scalar $\mu = -4$. The singular value $s_5(F[P(-4); \gamma])$ attains its maximum value at $\gamma_* = 2.0180$, and at this point, we have $s_* = s_5(F[P(-4); 2.0180]) = s_4(F[P(-4); 2.0180]) = 12.8841$; i.e., s_* is a multiple singular value of matrix $F[P(-4); 2.0180]$. A left and a right singular vectors of $F[P(-4); 2.0180]$ corresponding to s_* are

$$\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8407 \\ 0 \\ 0 \\ 0.5416 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5416 \\ 0 \\ 0 \\ 0.8407 \\ 0 \end{bmatrix},$$

respectively, and they yield the perturbed matrix polynomial (see (2.3))

$$Q_{\gamma_*}(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.0664 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -0.0664 & 0 \\ 0 & 0 & 3 \end{bmatrix} \lambda + \begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.9336 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

One can see that $\mu = -4$ is not a multiple eigenvalue of $Q_{\gamma_*}(\lambda)$. Moreover, properties (1) and (2) of Lemma 2.3 do not hold since $u_2^*(\gamma_*)P'(\mu)v_1(\gamma_*) = -2.6396 \neq 0$ and $\|U^*(\gamma_*)U(\gamma_*) - V^*(\gamma_*)V(\gamma_*)\|_2 = 0.4134 \neq 0$.

Clearly, this example verifies that the computation of appropriate singular vectors which satisfy (1) and (2) of Lemma 2.3 is still an open problem when s_* is a multiple singular value. In the next section, we obtain that for weakly normal matrix polynomials, s_* is always a multiple singular value, and in Section 4, we solve the problem of calculation of the desired singular vectors of Lemma 2.3.

3. Weakly normal matrix polynomials. In this section, by extending the analysis performed in [5], we prove that s_* is always a multiple singular value of $F[P(\mu); \gamma_*]$ when $P(\lambda)$ is a weakly normal matrix polynomial.

Let $P(\lambda)$ be a weakly normal matrix polynomial, and let $\mu \in \mathbb{C} \backslash \sigma(P')$. By Proposition 2.2 (iv), it follows that there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that all matrices $U^*A_0U, U^*A_1U, \ldots, U^*A_mU$ are diagonal. Hence, $U^*P(\mu)U$ and $U^*P'(\mu)U$ are also diagonal matrices; in particular,

$$U^*P(\mu)U = \operatorname{diag}\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \quad \text{and} \quad U^*P'(\mu)U = \operatorname{diag}\{\xi_1, \xi_2, \ldots, \xi_n\},$$

where all scalars $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}$ are *nonzero* (recall that $P'(\mu)$ is nonsingular) and, without loss of generality, we assume that

$$|\zeta_1| \geq |\zeta_2| \geq \cdots \geq |\zeta_n| \, .$$

As a consequence,

$$\begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix} F[P(\mu);\gamma] \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} U^* & 0 \\ 0 & U^* \end{bmatrix} \begin{bmatrix} P(\mu) & 0 \\ \gamma P'(\mu) & P(\mu) \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$$
$$= \begin{bmatrix} \operatorname{diag}\{\zeta_1, \zeta_2, \ldots, \zeta_n\} & 0 \\ \gamma \operatorname{diag}\{\xi_1, \xi_2, \ldots, \xi_n\} & \operatorname{diag}\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \end{bmatrix}.$$

It is straightforward to verify that there is a $2n \times 2n$ permutation matrix R such that

$$R \begin{bmatrix} \operatorname{diag}\{\zeta_1, \zeta_2, \ldots, \zeta_n\} & 0 \\ \gamma \operatorname{diag}\{\xi_1, \xi_2, \ldots, \xi_n\} & \operatorname{diag}\{\zeta_1, \zeta_2, \ldots, \zeta_n\} \end{bmatrix} R^T$$
$$= \begin{bmatrix} \zeta_1 & 0 \\ \gamma\,\xi_1 & \zeta_1 \end{bmatrix} \oplus \begin{bmatrix} \zeta_2 & 0 \\ \gamma\,\xi_2 & \zeta_2 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \zeta_n & 0 \\ \gamma\,\xi_n & \zeta_n \end{bmatrix}.$$

The fact that the singular values of a matrix are invariant under unitary similarity transformations implies that the $2n \times 2n$ matrices

$$F[P(\mu);\gamma] \quad \text{and} \quad \begin{bmatrix} \zeta_1 & 0 \\ \gamma\,\xi_1 & \zeta_1 \end{bmatrix} \oplus \begin{bmatrix} \zeta_2 & 0 \\ \gamma\,\xi_2 & \zeta_2 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \zeta_n & 0 \\ \gamma\,\xi_n & \zeta_n \end{bmatrix},$$

have the same singular values. Therefore, in what follows, we are focused on the singular values of the matrix $\begin{bmatrix} \zeta_1 & 0 \\ \gamma\,\xi_1 & \zeta_1 \end{bmatrix} \oplus \begin{bmatrix} \zeta_2 & 0 \\ \gamma\,\xi_2 & \zeta_2 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \zeta_n & 0 \\ \gamma\,\xi_n & \zeta_n \end{bmatrix}$, which are the union of the singular values of $\begin{bmatrix} \zeta_i & 0 \\ \gamma\,\xi_i & \zeta_i \end{bmatrix}$, $i = 1, 2, \ldots, n$.

For any $i = 1, 2, \ldots, n$, let $s_{i,1}(\gamma) \geq s_{i,2}(\gamma)$ be the singular values of $\begin{bmatrix} \zeta_i & 0 \\ \gamma\,\xi_i & \zeta_i \end{bmatrix}$, and consider the characteristic polynomial of the matrix

$$\begin{bmatrix} \zeta_i & 0 \\ \gamma\,\xi_i & \zeta_i \end{bmatrix}^* \begin{bmatrix} \zeta_i & 0 \\ \gamma\,\xi_i & \zeta_i \end{bmatrix} = \begin{bmatrix} |\zeta_i|^2 + \gamma^2 |\xi_i|^2 & \gamma\,\overline{\xi}_i\,\zeta_i \\ \gamma\,\xi_i\,\overline{\zeta}_i & |\zeta_i|^2 \end{bmatrix},$$

that is,

$$\det\left(tI - \begin{bmatrix} |\zeta_i|^2 + \gamma^2\,|\xi_i|^2 & \gamma\,\overline{\xi}_i\,\zeta_i \\ \gamma\,\xi_i\,\overline{\zeta}_i & |\zeta_i|^2 \end{bmatrix}\right) = t^2 - \left(2\,|\zeta_i|^2 + \gamma^2\,|\xi_i|^2\right)t + |\zeta_i|^4 .$$

The positive square roots of the eigenvalues of matrix $\begin{bmatrix} |\zeta_i|^2 + \gamma\,|\xi_i|^2 & \gamma\,\overline{\xi}_i\,\zeta_i \\ \gamma\,\xi_i\,\overline{\zeta}_i & |\zeta_i|^2 \end{bmatrix}$ are the singular values of the matrix $\begin{bmatrix} \zeta_i & 0 \\ \gamma\,\xi_i & \zeta_i \end{bmatrix}$, namely,

$$s_{i,1}(\gamma) = \sqrt{|\zeta_i|^2 + \frac{\gamma^2\,|\xi_i|^2}{2} + \gamma\,|\xi_i|\sqrt{|\zeta_i|^2 + \frac{\gamma^2\,|\xi_i|^2}{4}}},$$

and

$$s_{i,2}(\gamma) = \sqrt{|\zeta_i|^2 + \frac{\gamma^2\,|\xi_i|^2}{2} - \gamma\,|\xi_i|\sqrt{|\zeta_i|^2 + \frac{\gamma^2\,|\xi_i|^2}{4}}}.$$

As $\gamma \geq 0$ increases, $s_{i,1}(\gamma)$ increases and $\lim_{\gamma\to+\infty} s_{i,1}(\gamma) = +\infty$, while $s_{i,2}(\gamma)$ decreases and $\lim_{\gamma\to+\infty} s_{i,2}(\gamma) = 0$ (recall that $|\xi_i| > 0$, $i = 1, 2, \ldots, n$). Also, it is apparent that

$$s_{i,2}(\gamma) \leq |\zeta_i| \leq s_{i,1}(\gamma) \quad \text{and} \quad s_{i,1}(0) = s_{i,2}(0) = |\zeta_i| .$$

Next we consider two cases with respect to $|\zeta_{n-1}|$ and $|\zeta_n|$.

Case 1. Suppose $|\zeta_n| < |\zeta_{n-1}|$. At $\gamma = 0$, it holds that $s_{n,1}(0) = |\zeta_n| < |\zeta_{n-1}| = s_{n-1,2}(0)$. According to the above discussion, as the nonnegative variable γ increases from zero, the functions

$$s_{1,1}(\gamma),\ s_{2,1}(\gamma),\ \ldots,\ s_{n-1,1}(\gamma),\ s_{n,1}(\gamma),$$

increase to $+\infty$, whereas the functions

$$s_{1,2}(\gamma),\ s_{2,2}(\gamma),\ \ldots,\ s_{n-1,2}(\gamma),\ s_{n,2}(\gamma),$$

decrease to 0. Let (γ_0, s_0) be the first point in $\mathbb{R}^2$ where the graph of the increasing function $s_{n,1}(\gamma)$ intersects the graph of one of the $n-1$ decreasing functions $s_{1,2}(\gamma), s_{2,2}(\gamma), \ldots, s_{n-1,2}(\gamma)$, say $s_{\kappa,2}(\gamma)$ (for some $\kappa \in \{1, 2, \ldots, n-1\}$). Note that by the definitions of $s_{i,1}(\gamma)$ and $s_{i,2}(\gamma)$ ($i = 1, 2, \ldots, n$), s_0 lies in the open interval $(0, |\zeta_{n-1}|)$ and the graph of $s_{n,1}(\gamma)$ cannot intersect the graph of one of the increasing functions $s_{1,1}(\gamma), s_{2,1}(\gamma), \ldots, s_{n-1,1}(\gamma)$ for $\gamma \leq \gamma_0$.

Since $s_{n,2}(\gamma)$ and $s_{\kappa,2}(\gamma)$ are both decreasing functions in $\gamma \geq 0$, it follows that (see Figure 4.1 below, where $\kappa = n - 1 = 2$)

$$\gamma_* = \gamma_0 \text{ and } s_* = s_0 = s_{2n-1}(F[P(\mu);\gamma_*]) = s_{n,1}(\gamma_*) = s_{\kappa,2}(\gamma_*) = s_{2n-2}(F[P(\mu);\gamma_*]).$$

Hence, when $|\zeta_n| < |\zeta_{n-1}|$, γ_* is the minimum positive root of one of the equations

$$s_{n,1}(\gamma) = s_{n-1,2}(\gamma), \quad s_{n,1}(\gamma) = s_{n-2,2}(\gamma), \quad \ldots, \quad s_{n,1}(\gamma) = s_{1,2}(\gamma),$$

and s_* is a multiple singular value of $F[P(\mu);\gamma_*]$.

Case 2. Suppose $|\zeta_n| = |\zeta_{n-1}|$. Then, it follows that $s_{n,1}(\gamma) = s_{n-1,1}(\gamma)$ and $s_{n,2}(\gamma) = s_{n-1,2}(\gamma)$. Moreover, one can see that at $\gamma = 0$,

$$s_{n,1}(0) = s_{n,2}(0) = s_{n-1,1}(0) = s_{n-1,2}(0) = |\zeta_n| = |\zeta_{n-1}|\,,$$

i.e.,

$$\begin{aligned} s_{2n}(F[P(\mu);0]) &= s_{2n-1}(F[P(\mu);0]) = s_{2n-2}(F[P(\mu);0]) \\ &= s_{2n-3}(F[P(\mu);0]) = |\zeta_n| = |\zeta_{n-1}|\,. \end{aligned}$$

Since $s_{n,2}(\gamma)$ and $s_{n-1,2}(\gamma))$ are decreasing functions in $\gamma \geq 0$, $s_{2n-1}(F[P(\mu);\gamma])$ attains its maximum value s_* at $\gamma = 0 = \gamma_*$, and s_* is a multiple singular value of $F[P(\mu);0]$. In this non-generic case, an upper bound and an associate perturbed matrix polynomial can be computed by the method described in Section 6 of [14].

Hence, we have the following result.

THEOREM 3.1. *Let $P(\lambda)$ in (2.1) be a weakly normal matrix polynomial, and let $\mu \in \mathbb{C}\backslash\sigma(P')$. If $\gamma_* > 0$ is a point where the singular value $s_{2n-1}(F[P(\mu);\gamma])$ attains its maximum value, then $s_* = s_{2n-1}(F[P(\mu);\gamma_*]) > 0$ is a multiple singular value of $F[P(\mu);\gamma_*]$.*

4. Computing the desired singular vectors. In this section, we apply a technique described in [4] (see also the proof of Lemma 5 in [10]) to compute a consistent pair of left and right singular vectors of $F[P(\mu);\gamma_*]$ corresponding to the singular value s_*, so that (1) and (2) of Lemma 2.3 hold. We remark that the proposed methodology can be applied to general matrix polynomials when the singular value s_* is not simple (since Lemma 4.1 below and the relative analysis are valid for any $n \times n$ matrix polynomial), and not only to weakly normal matrix polynomials; see also the example of Subsection 4.2.

4.1. The case of multiplicity 2. First we consider the case where $\gamma_* > 0$ and the multiplicity of the singular value $s_* > 0$ is equal to 2, and we work on the example of Section 2.

Recall that for the normal matrix polynomial $P(\lambda)$ in (2.4) and for $\mu = -4$, the singular value $s_{2n-1}(F[P(\mu);\gamma]) = s_5(F[P(-4);\gamma])$ attains its maximum value at $\gamma_* = 2.0180$ and $s_* = s_5(F[P(-4);2.0180]) = s_4(F[P(-4);2.0180]) = 12.8841$ (i.e., s_*

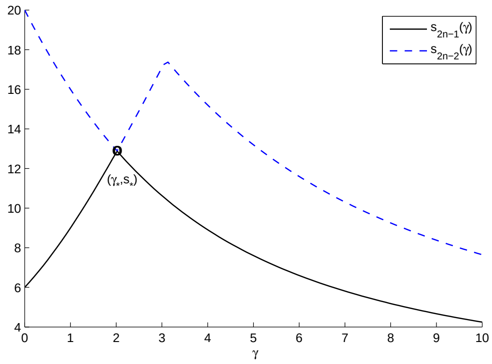

Fig. 4.1. *The singular values* $s_{2n-1}(F[P(\mu);\gamma])$ *(solid line) and* $s_{2n-2}(F[P(\mu);\gamma])$ *(dashed line).*

is a double singular value of $F[P(-4);2.0180])$. Two pairs of left and a right singular vectors of $F[P(-4);2.0180]$ corresponding to s_*, which do not satisfy properties (1) and (2) of Lemma 2.3 are

$$\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8407 \\ 0 \\ 0 \\ 0.5416 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5416 \\ 0 \\ 0 \\ 0.8407 \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \hat{u}_1(\gamma_*) \\ \hat{u}_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.4222 \\ 0 \\ 0 \\ 0.9065 \end{bmatrix}, \quad \begin{bmatrix} \hat{v}_1(\gamma_*) \\ \hat{v}_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.9065 \\ 0 \\ 0 \\ 0.4222 \end{bmatrix}.$$

In particular, we have

$$u_2(\gamma_*)^* P'(-4) v_1(\gamma_*) = -2.6396 \neq 0 \quad \text{and} \quad \hat{u}_2(\gamma_*)^* P'(-4) \hat{v}_1(\gamma_*) = 4.1089 \neq 0.$$

In Figure 4.1, the graphs of

$$s_{2n-1}(F[P(\mu);\gamma]) = s_5(F[P(-4);\gamma]) \quad \text{and} \quad s_{2n-2}(F[P(\mu);\gamma]) = s_4(F[P(-4);\gamma]),$$

are plotted for $\gamma \in [0,10]$, and their common point $(\gamma_*, s_*) = (2.0180, 12.8841)$ is marked with "$\circ$". With respect to the discussion in the previous section, it is worth noting that in this example, the graph of $s_{2,2}(\gamma)$ (that is, $s_{n-1,2}(\gamma)$) is the graph of the decreasing functions $s_{1,2}(\gamma)$ and $s_{2,2}(\gamma)$ that intersects first the graph of the increasing function $s_{3,1}(\gamma)$ (that is, $s_{n,1}(\gamma)$). Moreover, it is apparent that $s_{2n-1}(F[P(\mu);\gamma])$ and $s_{2n-2}(F[P(\mu);\gamma])$ are non-differentiable functions at γ_*.

Since $s_* = s_5(F[P(-4); 2.0180]) = s_4(F[P(-4); 2.0180]) = 12.8841$ is a double singular value, the pairs of unit vectors

$$\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} \hat{u}_1(\gamma_*) \\ \hat{u}_2(\gamma_*) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} \hat{v}_1(\gamma_*) \\ \hat{v}_2(\gamma_*) \end{bmatrix}$$

form orthonormal bases of the left and right singular subspaces corresponding to s_*, respectively. So, recalling that in Lemma 2.3, assertion (1) yields assertion (2), henceforth we are looking for a pair of unit vectors

$$(4.1) \quad \begin{bmatrix} \tilde{u}_1(\gamma_*) \\ \tilde{u}_2(\gamma_*) \end{bmatrix} = \alpha \begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix} + \beta \begin{bmatrix} \hat{u}_1(\gamma_*) \\ \hat{u}_2(\gamma_*) \end{bmatrix}, \quad \begin{bmatrix} \tilde{v}_1(\gamma_*) \\ \tilde{v}_2(\gamma_*) \end{bmatrix} = \alpha \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} + \beta \begin{bmatrix} \hat{v}_1(\gamma_*) \\ \hat{v}_2(\gamma_*) \end{bmatrix}$$

such that

$$(4.2) \qquad \tilde{u}_2(\gamma_*)^* P'(\mu) \tilde{v}_1(\gamma_*) = 0,$$

where the scalars $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha|^2 + |\beta|^2 = 1$. By substituting the unknown singular vectors of (4.1) into (4.2), we obtain

$$(4.3) \qquad \begin{bmatrix} \overline{\alpha} & \overline{\beta} \end{bmatrix} M \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0,$$

where

$$(4.4) \qquad M = \begin{bmatrix} u_2(\gamma_*)^* P'(\mu) v_1(\gamma_*) & u_2(\gamma_*)^* P'(\mu) \hat{v}_1(\gamma_*) \\ \hat{u}_2(\gamma_*)^* P'(\mu) v_1(\gamma_*) & \hat{u}_2(\gamma_*)^* P'(\mu) \hat{v}_1(\gamma_*) \end{bmatrix}.$$

LEMMA 4.1. *The matrix M in (4.4) is always hermitian.*

Proof. Recall that γ_* and s_* are positive. By the proof of Lemma 17 in [14], it follows that the diagonal entries of matrix M are real.

By the definition of the pairs of singular vectors

$$\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{u}_1(\gamma_*) \\ \hat{u}_2(\gamma_*) \end{bmatrix}, \begin{bmatrix} \hat{v}_1(\gamma_*) \\ \hat{v}_2(\gamma_*) \end{bmatrix}$$

of $F[P(\mu);\gamma_*]$ corresponding to s_*, we have

$$\begin{cases} \begin{bmatrix} P(\mu) & 0 \\ \gamma_* P'(\mu) & P(\mu) \end{bmatrix} \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} = s_* \begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix}, \\ \begin{bmatrix} P(\mu) & 0 \\ \gamma_* P'(\mu) & P(\mu) \end{bmatrix} \begin{bmatrix} \hat{v}_1(\gamma_*) \\ \hat{v}_2(\gamma_*) \end{bmatrix} = s_* \begin{bmatrix} \hat{u}_1(\gamma_*) \\ \hat{u}_2(\gamma_*) \end{bmatrix}, \end{cases}$$

or equivalently,

$$(4.5)\qquad \begin{cases} P(\mu)v_1(\gamma_*) = s_* u_1(\gamma_*), \\ \gamma_* P'(\mu)v_1(\gamma_*) + P(\mu)v_2(\gamma_*) = s_* u_2(\gamma_*), \\ P(\mu)\hat{v}_1(\gamma_*) = s_* \hat{u}_1(\gamma_*), \\ \gamma_* P'(\mu)\hat{v}_1(\gamma_*) + P(\mu)\hat{v}_2(\gamma_*) = s_* \hat{u}_2(\gamma_*), \end{cases}$$

and

$$\begin{cases} \begin{bmatrix} u_1(\gamma_*)^* & u_2(\gamma_*)^* \end{bmatrix} \begin{bmatrix} P(\mu) & 0 \\ \gamma_* P'(\mu) & P(\mu) \end{bmatrix} = s_* \begin{bmatrix} v_1(\gamma_*)^* & v_2(\gamma_*)^* \end{bmatrix}, \\ \begin{bmatrix} \hat{u}_1(\gamma_*)^* & \hat{u}_2(\gamma_*)^* \end{bmatrix} \begin{bmatrix} P(\mu) & 0 \\ \gamma_* P'(\mu) & P(\mu) \end{bmatrix} = s_* \begin{bmatrix} \hat{v}_1(\gamma_*)^* & \hat{v}_2(\gamma_*)^* \end{bmatrix}, \end{cases}$$

or equivalently,

$$(4.6)\qquad \begin{cases} u_1(\gamma_*)^* P(\mu) + \gamma_* u_2(\gamma_*)^* P'(\mu) = s_* v_1(\gamma_*)^*, \\ u_2(\gamma_*)^* P(\mu) = s_* v_2(\gamma_*)^*, \\ \hat{u}_1(\gamma_*)^* P(\mu) + \gamma_* \hat{u}_2(\gamma_*)^* P'(\mu) = s_* \hat{v}_1(\gamma_*)^*, \\ \hat{u}_2(\gamma_*)^* P(\mu) = s_* \hat{v}_2(\gamma_*)^*. \end{cases}$$

By pre-multiplying the fourth equation in (4.5) by $u_2(\gamma_*)^*$, and post-multiplying the second equation of (4.6) by $\hat{v}_2(\gamma_*)$, we obtain

$$\gamma_* u_2(\gamma_*)^* P'(\mu)\hat{v}_1(\gamma_*) + u_2(\gamma_*)^* P(\mu)\hat{v}_2(\gamma_*) = s_* u_2(\gamma_*)^* \hat{u}_2(\gamma_*),$$

and

$$u_2(\gamma_*)^* P(\mu)\hat{v}_2(\gamma_*) = s_* v_2(\gamma_*)^* \hat{v}_2(\gamma_*),$$

respectively. As a consequence,

$$(4.7)\qquad \gamma_* u_2(\gamma_*)^* P'(\mu)\hat{v}_1(\gamma_*) = s_* \left(u_2(\gamma_*)^* \hat{u}_2(\gamma_*) - v_2(\gamma_*)^* \hat{v}_2(\gamma_*) \right).$$

Performing similar calculations, one can verify that

$$(4.8)\qquad \gamma_* \hat{u}_2(\gamma_*)^* P'(\mu) v_1(\gamma_*) = s_* \left(\hat{u}_2(\gamma_*)^* u_2(\gamma_*) - \hat{v}_2(\gamma_*)^* v_2(\gamma_*) \right).$$

Clearly, equations (4.7) and (4.8) imply that the non-diagonal entries of matrix M are complex conjugates of each other. □

By Lemma 2.3 (1), equation (4.3) has always a nontrivial (i.e., nonzero) solution. Hence, the hermitian matrix M in (4.4) cannot be (positive or negative) definite. If M is singular semidefinite, then (4.3) holds for any unit eigenvector of M corresponding to 0. So, we may assume that M is an indefinite hermitian matrix; this is the case in our numerical example, where the matrix M has a negative and a positive diagonal entries, namely, -2.6396 and 4.1089.

To derive an explicit solution of (4.3), suppose that $\eta_1, \eta_2 \in \mathbb{C}$ are the (real) eigenvalues of matrix M, with $\eta_1 > 0 > \eta_2$, and let $\omega_1, \omega_2 \in \mathbb{C}^2$ be unit eigenvectors of M corresponding to η_1 and η_2, respectively. Then, it is straightforward to see (keeping in mind the orthogonality of the eigenvectors) that the unit vector

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \sqrt{\frac{|\eta_2|}{|\eta_1| + |\eta_2|}}\,\omega_1 + \sqrt{\frac{|\eta_1|}{|\eta_1| + |\eta_2|}}\,\omega_2$$

satisfies

$$\begin{bmatrix} \overline{\alpha} & \overline{\beta} \end{bmatrix} M \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{|\eta_1|\,\eta_2}{|\eta_1| + |\eta_2|} + \frac{|\eta_2|\,\eta_1}{|\eta_1| + |\eta_2|} = 0.$$

Finally, in order to verify the validity of this refinement, we return again to the normal matrix polynomial $P(\lambda)$ in (2.4), and by applying the above methodology, we obtain $\alpha = 0.7803$ and $\beta = 0.6254$. Consequently, the desired vectors in (4.1) are (approximately)

$$\begin{bmatrix} \tilde{u}_1(\gamma_*) \\ \tilde{u}_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.6560 \\ -0.2640 \\ 0 \\ 0.4226 \\ 0.5669 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{v}_1(\gamma_*) \\ \tilde{v}_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4226 \\ -0.5669 \\ 0 \\ 0.6560 \\ 0.2640 \end{bmatrix}.$$

In particular, it holds that

$$\tilde{u}_2^*(\gamma_*) P'(-4) \tilde{v}_1(\gamma_*) = -4.4409 \cdot 10^{-16},$$

and for the $n \times 2$ matrices $\tilde{U}(\gamma_*) = [\tilde{u}_1(\gamma_*)\ \tilde{u}_2(\gamma_*)]$ and $\tilde{V}(\gamma_*) = [\tilde{v}_1(\gamma_*)\ \tilde{v}_2(\gamma_*)]$, we have

$$\left\| \tilde{U}^*(\gamma_*)\tilde{U}(\gamma_*) - \tilde{V}^*(\gamma_*)\tilde{V}(\gamma_*) \right\|_2 = 1.1383 \cdot 10^{-6}.$$

Thus, Lemma 2.3 is verified.

Moreover, using the matrices $\tilde{U}(\gamma_*)$ and $\tilde{V}(\gamma_*)$, Theorem 2.4 yields the upper bound 0.9465 for the distance from $P(\lambda)$ to the set of 3×3 quadratic matrix polynomials that have $\mu = -4$ as a multiple eigenvalue, and the perturbed matrix polynomial

$$\tilde{Q}_{\gamma_*}(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.0680 & 0.0152 \\ 0 & -0.1552 & 0.5986 \end{bmatrix} \lambda^2 + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -0.0680 & -0.0152 \\ 0 & 0.1552 & 3.4014 \end{bmatrix} \lambda + \begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.9320 & 0.0152 \\ 0 & -0.1552 & 1.5986 \end{bmatrix}$$

that lies on the boundary of $\mathcal{B}(P, 0.9465, \mathrm{w})$ and has spectrum

$$\sigma\left(\tilde{Q}_{\gamma_*}(\lambda)\right) = \{1,\ 2,\ 4.1982,\ -0.5140,\ -4.0000 \pm \mathrm{i}\, 0.0031\}.$$

In addition, the lower bound 0.4031 of the distance is given by Theorem 11 in [14]. (All computations were performed in Matlab with 16 significant digits.)

4.2. An example of a general matrix polynomial. In this subsection, we give an example to illustrate that the methodology proposed in the previous subsection can be applied to general matrix polynomials. In particular, we consider the matrix polynomial

$$P(\lambda) = \begin{bmatrix} -3 & -4 & -5 \\ 4 & 5 & 3 \\ -3 & 5 & 5 \end{bmatrix} \lambda^2 + \begin{bmatrix} 3 & -2 & 5 \\ -1 & -1 & -1 \\ 3 & 2 & -1 \end{bmatrix} \lambda + \begin{bmatrix} -2 & 5 & 2 \\ -4 & 0 & -1 \\ 1 & 3 & -3 \end{bmatrix},$$

which is not weakly normal, and the scalar $\mu = 3$. We also choose the spectral norms of the matrix-coefficients as the corresponding weights, i.e., $w = \{6.1031,\ 6.2464,\ 11.2766\}$. The graphs of $s_6(F[P(3);\gamma])$, $s_5(F[P(3);\gamma])$ and $s_4(F[P(3);\gamma])$, for $\gamma \in [0,5]$, are plotted in Figure 4.2.

The function $s_5(F[P(3);\gamma])$ attains its maximum at $\gamma_* = 2.5926$, and the singular value $s_5(F[P(3);2.5926])$ is of multiplicity 2. In particular, Matlab generates the singular value $s_5(F[P(3);2.5926]) = 14.8953$ with corresponding pair of singular vectors

$$\begin{bmatrix} u_1(\gamma_*) \\ u_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} -0.3918 \\ 0.5038 \\ -0.6469 \\ -0.0468 \\ -0.3439 \\ 0.2317 \end{bmatrix}, \quad \begin{bmatrix} v_1(\gamma_*) \\ v_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0.3859 \\ -0.1845 \\ 0.1766 \\ -0.8711 \\ -0.0131 \\ 0.1638 \end{bmatrix},$$

and the singular value $s_4(F[P(3);2.5926]) = 14.8956$ with corresponding pair of singular vectors

$$\begin{bmatrix} \hat{u}_1(\gamma_*) \\ \hat{u}_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} -0.1205 \\ -0.3037 \\ 0.0849 \\ -0.7653 \\ -0.5057 \\ -0.2114 \end{bmatrix}, \quad \begin{bmatrix} \hat{v}_1(\gamma_*) \\ \hat{v}_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} -0.0008 \\ -0.5160 \\ 0.7465 \\ 0.2843 \\ -0.2913 \\ 0.1042 \end{bmatrix}.$$

The method described in the previous subsection yields the scalars $\alpha = -0.5245$ and

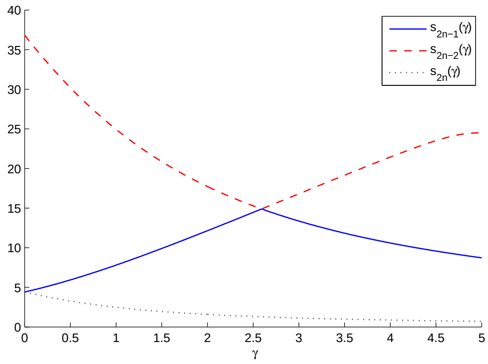

Fig. 4.2. *The graphs of* $s_6(F[P(3);\gamma])$ *(doted line),* $s_5(F[P(3);\gamma])$ *(solid line) and* $s_4(F[P(3);\gamma])$ *(dashed line).*

$\beta = 0.8514$, and the vectors

$$\begin{bmatrix} \tilde{u}_1(\gamma_*) \\ \tilde{u}_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} 0.1029 \\ -0.5228 \\ 0.4116 \\ -0.6270 \\ -0.2502 \\ -0.3015 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{v}_1(\gamma_*) \\ \tilde{v}_2(\gamma_*) \end{bmatrix} = \begin{bmatrix} -0.2031 \\ -0.3425 \\ 0.5429 \\ 0.6989 \\ -0.2411 \\ 0.0028 \end{bmatrix}$$

which satisfy

$$\tilde{u}_2^*(\gamma_*)P'(3)\tilde{v}_1(\gamma_*) = 4.4409 \cdot 10^{-16}.$$

For the $n \times 2$ matrices $\tilde{U}(\gamma_*) = [\tilde{u}_1(\gamma_*)\ \tilde{u}_2(\gamma_*)]$ and $\tilde{V}(\gamma_*) = [\tilde{v}_1(\gamma_*)\ \tilde{v}_2(\gamma_*)]$, we have

$$\left\| \tilde{U}^*(\gamma_*)\tilde{U}(\gamma_*) - \tilde{V}^*(\gamma_*)\tilde{V}(\gamma_*) \right\|_2 = 6.1503 \cdot 10^{-6}.$$

Moreover, we obtain the upper bound 0.2256 for the distance from $P(\lambda)$ to the set of 3×3 matrix polynomials that have $\mu = 3$ as a multiple eigenvalue, and the perturbed matrix polynomial

$$\tilde{Q}_{\gamma_*}(\lambda) = \begin{bmatrix} -1.6611 & -4.4373 & -5.0271 \\ 2.8986 & 4.8091 & 3.7479 \\ -1.3058 & 4.8002 & 4.4998 \end{bmatrix} \lambda^2 + \begin{bmatrix} 3.7416 & -2.2422 & 4.9850 \\ -1.6101 & -1.1057 & -0.5857 \\ 3.9385 & 1.8893 & -1.2771 \end{bmatrix} \lambda$$
$$+ \begin{bmatrix} -1.2754 & 4.7633 & 1.9854 \\ -4.5961 & -0.1033 & -0.5952 \\ 1.9170 & 2.8919 & -3.2707 \end{bmatrix},$$

that lies on the boundary of $\mathcal{B}(P, 0.2256, \mathrm{w})$ and has spectrum

$$\sigma\left(\tilde{Q}_{\gamma_*}(\lambda)\right) = \{-0.8468,\ -0.4656,\ 0.8085 \pm \mathrm{i}\,0.6406,\ 3.0000 \pm \mathrm{i}\,0.0048\}.$$

In addition, the lower bound 0.0586 of the distance is given by Theorem 11 of [14].

4.3. The case of multiplicity greater than 2. Suppose that $\gamma_* > 0$, and the multiplicity of the singular value $s_* > 0$ is $r \geq 3$. For weakly normal matrix polynomials, this means that the graph of the increasing function $s_{n,1}(\gamma)$ intersects the graphs of more than one of the $n-1$ decreasing functions $s_{1,2}(\gamma), s_{2,2}(\gamma), \ldots, s_{n-1,2}(\gamma)$, at the point (γ_*, s_*).

Let also

$$\left[\begin{array}{c} u_1^{(1)}(\gamma_*) \\ u_2^{(1)}(\gamma_*) \end{array}\right], \left[\begin{array}{c} u_1^{(2)}(\gamma_*) \\ u_2^{(2)}(\gamma_*) \end{array}\right], \ldots, \left[\begin{array}{c} u_1^{(r)}(\gamma_*) \\ u_2^{(r)}(\gamma_*) \end{array}\right],$$

and

$$\left[\begin{array}{c} v_1^{(1)}(\gamma_*) \\ v_2^{(1)}(\gamma_*) \end{array}\right], \left[\begin{array}{c} v_1^{(2)}(\gamma_*) \\ v_2^{(2)}(\gamma_*) \end{array}\right], \ldots, \left[\begin{array}{c} v_1^{(r)}(\gamma_*) \\ v_2^{(r)}(\gamma_*) \end{array}\right],$$

be orthonormal bases of the left and right singular subspaces of $F[P(\mu);\gamma_*]$ corresponding to s_*, respectively. Then, we are looking for a pair of unit vectors

$$\left[\begin{array}{c} \tilde{u}_1(\gamma_*) \\ \tilde{u}_2(\gamma_*) \end{array}\right] = \sum_{j=1}^{r} \alpha_j \left[\begin{array}{c} u_1^{(j)}(\gamma_*) \\ u_2^{(j)}(\gamma_*) \end{array}\right], \quad \left[\begin{array}{c} \tilde{v}_1(\gamma_*) \\ \tilde{v}_2(\gamma_*) \end{array}\right] = \sum_{j=1}^{r} \alpha_j \left[\begin{array}{c} v_1^{(j)}(\gamma_*) \\ v_2^{(j)}(\gamma_*) \end{array}\right], \tag{4.9}$$

such that

$$\tilde{u}_2(\gamma_*)^* P'(\mu) \tilde{v}_1(\gamma_*) = 0, \tag{4.10}$$

where the scalars $\alpha_1, \alpha_2, \ldots, \alpha_r \in \mathbb{C}$ satisfy $|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_r|^2 = 1$.

Following the arguments of the methodology described in Subsection 4.1, we can compute the desired vectors in (4.9) that satisfy (4.10). In particular, we need to find a solution of the equation

$$\left[\begin{array}{cccc} \overline{\alpha}_1 & \overline{\alpha}_2 & \cdots & \overline{\alpha}_r \end{array}\right] M_r \left[\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{array}\right] = 0, \tag{4.11}$$

where the $r \times r$ matrix

$$M_r = \left[\begin{array}{cccc} u_2^{(1)}(\gamma_*)^* P'(\mu) v_1^{(1)}(\gamma_*) & u_2^{(1)}(\gamma_*)^* P'(\mu) v_1^{(2)}(\gamma_*) & \cdots & u_2^{(1)}(\gamma_*)^* P'(\mu) v_1^{(r)}(\gamma_*) \\ u_2^{(2)}(\gamma_*)^* P'(\mu) v_1^{(1)}(\gamma_*) & u_2^{(2)}(\gamma_*)^* P'(\mu) v_1^{(2)}(\gamma_*) & \cdots & u_2^{(2)}(\gamma_*)^* P'(\mu) v_1^{(r)}(\gamma_*) \\ \vdots & \vdots & \ddots & \vdots \\ u_2^{(r)}(\gamma_*)^* P'(\mu) v_1^{(1)}(\gamma_*) & u_2^{(r)}(\gamma_*)^* P'(\mu) v_1^{(2)}(\gamma_*) & \cdots & u_2^{(r)}(\gamma_*)^* P'(\mu) v_1^{(r)}(\gamma_*) \end{array}\right]$$

is hermitian and not definite. Considering a unit eigenvector $\omega_{\max} \in \mathbb{C}^r$ of M_r corresponding to the maximum eigenvalue $\eta_{\max} \geq 0$ of M_r and an eigenvector $\omega_{\min} \in \mathbb{C}^r$ corresponding to the minimum eigenvalue $\eta_{\min} \leq 0$ of M_r, it is straightforward to verify that the unit vector

$$\sqrt{\frac{|\eta_{\min}|}{|\eta_{\max}| + |\eta_{\min}|}}\,\omega_{\max} + \sqrt{\frac{|\eta_{\max}|}{|\eta_{\max}| + |\eta_{\min}|}}\,\omega_{\min}$$

satisfies (4.11).

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