The Envelope of Tridiagonal Toeplitz Matrices and Block-Shift Matrices

Aik. Aretaki[*], P. Psarrakos[*], and M. Tsatsomeros[†]

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Abstract

The envelope of a square complex matrix is a spectrum encompassing region in the complex plane. It is contained in and is akin to the numerical range in the sense that the envelope is obtained as an infinite intersection of unbounded regions contiguous to cubic curves, rather than half-planes. In this article, the geometry and properties of the envelopes of special matrices are examined. In particular, symmetries of the envelope of a tridiagonal Toeplitz matrix are obtained, and the envelopes of block-shift matrices, Jordan blocks and 2×2 matrices are explicitly characterized.

Key words: eigenvalue, envelope, cubic curve, numerical range, tridiagonal Toeplitz matrix, block-shift matrix, Jordan block.

AMS Subject Classifications: 15A18, 15A60.

1 Introduction

Let $\mathcal{M}_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices. The classical *numerical range* (also known as *field of values*) of a matrix $A \in \mathcal{M}_n(\mathbb{C})$ is defined as the *compact* and *convex* set

$$F(A) = \{x^* A x \in \mathbb{C} : \, x \in \mathbb{C}^n, \; x^* x = 1\},$$

whose basic properties are presented in [7, Chapter 1]. Among them is the well-known spectral containment property $\sigma(A) \subseteq F(A)$, where $\sigma(A)$ denotes the *spectrum* of A. Apparently, the numerical range contains the convex hull of the spectrum, $\mathrm{co}\{\sigma(A)\}$, which reduces to equality in the case of a normal matrix A.

[*]Department of Mathematics, National Technical University of Athens, Greece (kathy@mail.ntua.gr, ppsarr@math.ntua.gr).

[†]Department of Mathematics, Washington State University, Pullman, USA (tsat@wsu.edu).

For a matrix $A \in \mathcal{M}_n(\mathbb{C})$, denote by $H(A) = \frac{A+A^*}{2}$ the *hermitian part* of A and by $S(A) = \frac{A-A^*}{2}$ the *skew-hermitian part* of A. Then $A = H(A) + S(A)$, and the matrices $H(A)$ and $\mathrm{i}\, S(A)$ are hermitian. Let also

$$\delta_1(A) \geq \delta_2(A) \geq \cdots \geq \delta_n(A)$$

be the eigenvalues of $H(A)$ in a nonincreasing order, and $y_1 \in \mathbb{C}^n$ be a unit eigenvector associated with the largest eigenvalue $\delta_1(A)$ of $H(A)$. The eigenvalues of A that are vertices of $\mathrm{co}\{\sigma(A)\}$ are called *extremal* [11].

By [8] (see also [7, Chapter 1]), for any $\theta \in [0, 2\pi)$, the line $\mathcal{L}_\theta = \left\{z \in \mathbb{C} : \mathrm{Re}\, z = \delta_1(e^{\mathrm{i}\theta} A)\right\}$ is a right vertical supporting line of the convex set $F(e^{\mathrm{i}\theta}A) = e^{\mathrm{i}\theta}F(A)$. Moreover, if $y_1(\theta)$ is a unit eigenvector of $H(e^{\mathrm{i}\theta}A)$ associated with its largest eigenvalue $\delta_1(e^{\mathrm{i}\theta}A)$, then $\mathcal{L}_\theta$ is tangential to $F(e^{\mathrm{i}\theta}A)$ at the point $y_1^*(\theta)(e^{\mathrm{i}\theta}A)y_1(\theta)$, which lies on the boundary, denoted by $\partial F(e^{\mathrm{i}\theta}A)$. By the convexity of the numerical range, $y_1^*(\theta)(e^{\mathrm{i}\theta}A)y_1(\theta) \in \partial F(e^{\mathrm{i}\theta}A)$ is a right-most point of $F(e^{\mathrm{i}\theta}A)$, and $F(e^{\mathrm{i}\theta}A)$ lies in the closed half-plane $\mathcal{H}_\theta = \left\{z \in \mathbb{C} : \mathrm{Re}\, z \leq \delta_1(e^{\mathrm{i}\theta}A)\right\}$. As a consequence,

$$F(A) = \bigcap_{\theta \in [0,2\pi)} e^{-\mathrm{i}\theta}\mathcal{H}_\theta.$$

In other words, $F(A)$ is an infinite intersection of half-planes, providing the most commonly used method to draw the numerical range; see [7, 8].

Consider now the real quantities

$$\mathrm{u}(A) = \mathrm{Im}(y_1^* S(A) y_1) \quad \text{and} \quad \mathrm{v}(A) = \|S(A)y_1\|_2^2\,,$$

where $\|\cdot\|_2$ denotes the 2-norm and $|\mathrm{u}(A)| \leq |y_1^* S(A) y_1| \leq \sqrt{\mathrm{v}(A)}$. In [1], Adam and Tsatsomeros introduced and studied the cubic curve

$$\Gamma(A) = \big\{z \in \mathbb{C} : \, [(\delta_1(A) - \mathrm{Re}\, z)^2 + (\mathrm{u}(A) - \mathrm{Im}\, z)^2](\delta_2(A) - \mathrm{Re}\, z) + (\delta_1(A) - \mathrm{Re}\, z)(\mathrm{v}(A) - \mathrm{u}^2(A)) \;=\; 0\big\}\,, \quad (1.1)$$

showing that all the eigenvalues of A lie to its left; namely, $\sigma(A)$ lies in the unbounded closed region

$$\Gamma_{in}(A) = \big\{z \in \mathbb{C} : \, [(\delta_1(A) - \mathrm{Re}\, z)^2 + (\mathrm{u}(A) - \mathrm{Im}\, z)^2](\delta_2(A) - \mathrm{Re}\, z) + (\delta_1(A) - \mathrm{Re}\, z)(\mathrm{v}(A) - \mathrm{u}^2(A)) \;\geq\; 0\big\}\,,$$

which is a subset of the half-plane $\mathcal{H}_0 = \{z \in \mathbb{C} : \mathrm{Re}\, z \leq \delta_1(A)\}$. A description of the cubic curve $\Gamma(A)$ is given in the appendix at the end of the paper.

Motivated by the above, a finer spectrum localization area that is contained in the numerical range is introduced and studied in [10, 11], called the *envelope* of $A \in \mathcal{M}_n(\mathbb{C})$ and defined as

$$\mathcal{E}(A) = \bigcap_{\theta \in [0,2\pi)} e^{-\mathrm{i}\theta}\Gamma_{in}(e^{\mathrm{i}\theta}A).$$

One may immediately observe that $\mathcal{E}(A)$ is generated analogously to the numerical range $F(A)$, by replacing the closed half-planes $\mathcal{H}_\theta$ with the regions $\Gamma_{in}(e^{\mathrm{i}\theta}A)$, $\theta \in [0, 2\pi)$. Since, for any $\theta \in [0, 2\pi)$, $e^{\mathrm{i}\theta}\sigma(A) = \sigma(e^{\mathrm{i}\theta}A) \subseteq \Gamma_{in}(e^{\mathrm{i}\theta}A) \subseteq \mathcal{H}_\theta$, it follows that

$$\sigma(A) \subseteq \mathcal{E}(A) = \bigcap_{\theta \in [0,2\pi)} e^{-\mathrm{i}\theta}\Gamma_{in}(e^{\mathrm{i}\theta}A) \subseteq \bigcap_{\theta \in [0,2\pi)} e^{-\mathrm{i}\theta}\mathcal{H}_\theta = F(A).$$

The envelope $\mathcal{E}(A)$ is a compact subset of the complex plane (since it is a closed subset of the compact numerical range $F(A)$), but it is not necessarily convex or connected. It has, however, a rich structure and it satisfies some of the basic properties of $F(A)$ and $\sigma(A)$ listed next (see [10, 11]):

(P$_1$) $\Gamma(A^T) = \Gamma(A)$, $\Gamma(A^*) = \Gamma(\overline{A}) = \overline{\Gamma(A)}$, $\mathcal{E}(A^T) = \mathcal{E}(A)$ and $\mathcal{E}(A^*) = \mathcal{E}(\overline{A}) = \overline{\mathcal{E}(A)}$. In particular, if A is real, then the curve $\Gamma(A)$ and the envelope $\mathcal{E}(A)$ are symmetric with respect to the real axis.

(P$_2$) For any unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$, $\Gamma(U^*AU) = \Gamma(A)$ and $\mathcal{E}(U^*AU) = \mathcal{E}(A)$.

(P$_3$) For any $b \in \mathbb{C}$, $\Gamma(A + bI_n) = \Gamma(A) + b$ and $\mathcal{E}(A + bI_n) = \mathcal{E}(A) + b$, where I_n denotes the $n \times n$ identity matrix and adding a scalar to a set means adding this scalar to every element of the set.

(P$_4$) For any $r > 0$ and any $a \in \mathbb{C}$, $\Gamma(rA) = r\,\Gamma(A)$ and $\mathcal{E}(aA) = a\,\mathcal{E}(A)$.

(P$_5$) If A is normal and $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_k$ are its simple extremal eigenvalues, then

$$\mathcal{E}(A) = \left(\bigcap_{\theta \in [0,2\pi)} e^{-\mathrm{i}\theta} \left\{ z \in \mathbb{C} :\, \mathrm{Re}\, z \le \delta_2(e^{\mathrm{i}\theta}A) \right\} \right) \cup \left\{ \hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_k \right\}.$$

(P$_6$) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then

$$\bigcap \left\{ \Gamma_{in}(R^{-1}AR) :\, R \in \mathcal{M}_n(\mathbb{C}),\, \det(R) \neq 0 \right\} \subseteq \Gamma_{in}(\mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\})$$

and

$$\bigcap \left\{ \mathcal{E}(R^{-1}AR) :\, R \in \mathcal{M}_n(\mathbb{C}),\, \det(R) \neq 0 \right\} \subseteq \mathcal{E}(\mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}).$$

In this article, we study additional features of the envelope by turning our attention to special types of matrices that have also been studied in the context of the numerical range. In Section 2, we obtain the symmetries of the envelope of a tridiagonal Toeplitz matrix. In Section 3, we construct explicitly the envelopes of block-shift matrices and Jordan blocks. In Section 4, we prove that the envelope of any 2×2 matrix coincides with the spectrum of the matrix. Finally, in Appendix A, we provide an alternative analysis of the cubic curve $\Gamma(A)$ that complements the one provided in [1] and assists in the developments of some of the new results herein.

2 The Envelope of a Tridiagonal Toeplitz Matrix

In this section, we investigate the envelopes of tridiagonal Toeplitz matrices that arise e.g., in the numerical solution of differential equations; they have constant entries along the diagonal, the superdiagonal and the subdiagonal, that is,

$$T_n(c, a, b) = \begin{bmatrix} a & b & \cdots & 0 \\ c & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ 0 & \cdots & c & a \end{bmatrix} \in \mathcal{M}_n(\mathbb{C}), \quad bc \neq 0.$$

As shown in [5, Corollary 4], the numerical range of $T_n(c, a, b)$ coincides with the elliptical disc

$$\{bz + c\overline{z} : z \in F(J_n(0))\} + \{a\},$$

where $J_n(0)$ is the $n \times n$ Jordan block with zero eigenvalue and its numerical range, $F(J_n(0))$, coincides with the circular disc $\mathcal{D}\left(0, \cos\left(\frac{\pi}{n+1}\right)\right)$ centered at the origin and having radius $\cos\left(\frac{\pi}{n+1}\right)$; see [14, Theorem 1]. Moreover, the eigenvalues of $T_n(c, a, b)$ (see [3, Theorem 2.4] and also [6, 9]) are

$$\lambda_j = \lambda_j(T_n(c, a, b)) = a + 2(bc)^{1/2} \cos\left(\frac{j\pi}{n+1}\right), \quad j = 1, 2, \ldots, n, \tag{2.1}$$

and the corresponding eigenvectors $x_j = [\, x_{j,1} \; x_{j,2} \; \cdots \; x_{j,n} \,]^T$ can be chosen to have entries

$$x_{j,k} = \left(\frac{c}{b}\right)^{k/2} \sin\left(\frac{kj\pi}{n+1}\right), \quad k = 1, 2, \ldots, n. \tag{2.2}$$

Clearly, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are simple eigenvalues of $T_n(c, a, b)$ lying on the (complex) line segment

$$\left\{ a + \gamma e^{\mathrm{i}\frac{\arg(b)+\arg(c)}{2}} : -2\sqrt{|bc|}\cos\left(\frac{\pi}{n+1}\right) \le \gamma \le 2\sqrt{|bc|}\cos\left(\frac{\pi}{n+1}\right) \right\},$$

and they are located symmetrically with respect to point a.

The above fundamental results motivate us to consider in what follows the envelope of a tridiagonal Toeplitz matrix.

Theorem 2.1. *The envelope of a tridiagonal Toeplitz matrix $T_n(c, a, b) \in \mathcal{M}_n(\mathbb{C})$, $bc \neq 0$, is symmetric with respect to point a.*

Proof. Due to the translation property (P_3) of the envelope, we have $\mathcal{E}(T_n(c, a, b)) = \mathcal{E}(T_n(c, 0, b)) + \{a\}$. Hence, without loss of generality, we may consider the matrix $T_n(c, 0, b)$ with $bc \neq 0$, which we denote by T_n for brevity. Then it suffices to prove that $\mathcal{E}(T_n)$ is symmetric with respect to the origin, which is true when $\Gamma(e^{\mathrm{i}\theta}T_n) = \Gamma(-e^{\mathrm{i}\theta}T_n)$ for every $\theta \in [0, 2\pi)$.

Keeping in mind equation (1.1), it is enough to prove that for any $\theta \in [0, 2\pi)$,

$$\delta_1(-e^{\mathrm{i}\theta}T_n) = \delta_1(e^{\mathrm{i}\theta}T_n), \;\; \delta_2(-e^{\mathrm{i}\theta}T_n) = \delta_2(e^{\mathrm{i}\theta}T_n),$$
$$\mathrm{v}(-e^{\mathrm{i}\theta}T_n) = \mathrm{v}(e^{\mathrm{i}\theta}T_n), \;\text{ and }\; \mathrm{u}(-e^{\mathrm{i}\theta}T_n) = \mathrm{u}(e^{\mathrm{i}\theta}T_n).$$

Denote $\beta(\theta) = be^{\mathrm{i}\theta} + \overline{c}e^{-\mathrm{i}\theta}$, $\theta \in [0, 2\pi)$. By (2.1), the eigenvalues of the hermitian tridiagonal Toeplitz matrix $H(e^{\mathrm{i}\theta}T_n) = \frac{1}{2}T_n(\overline{\beta(\theta)}, 0, \beta(\theta))$ are

$$\delta_j(e^{\mathrm{i}\theta}T_n) = |\beta(\theta)| \cos\left(\frac{j\pi}{n+1}\right), \quad j = 1, 2, \ldots, n. \tag{2.3}$$

Since $\beta(\theta + \pi) = -\beta(\theta)$ for all $\theta \in [0, 2\pi)$, (2.3) yields the first pair of desired equalities for the eigenvalues.

By (2.2) and the formula $\sum_{j=1}^{n} \sin^2\left(\frac{j\pi}{n+1}\right) = \frac{n+1}{2}$, a unit eigenvector of $H(e^{\mathrm{i}\theta}T_n)$ associated with the largest eigenvalue $\delta_1(e^{\mathrm{i}\theta}T_n)$ is

$$y_1(\theta) = \sqrt{\frac{2}{n+1}}\; D(\theta) \begin{bmatrix} \sin\left(\frac{\pi}{n+1}\right) \\ \sin\left(\frac{2\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{n\pi}{n+1}\right) \end{bmatrix}, \tag{2.4}$$

and a unit eigenvector of $H(e^{\mathrm{i}\theta}T_n)$ associated with the smallest eigenvalue $\delta_n(e^{\mathrm{i}\theta}T_n)$ is

$$\begin{aligned} y_n(\theta) &= \sqrt{\frac{2}{n+1}}\, D(\theta) \begin{bmatrix} \sin\left(\frac{n\pi}{n+1}\right) \\ \sin\left(\frac{2n\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{n^2\pi}{n+1}\right) \end{bmatrix} \\ &= \sqrt{\frac{2}{n+1}}\, D(\theta) \begin{bmatrix} \sin\left(\frac{\pi}{n+1}\right) \\ -\sin\left(\frac{2\pi}{n+1}\right) \\ \vdots \\ (-1)^{n+1}\sin\left(\frac{n\pi}{n+1}\right) \end{bmatrix}, \end{aligned}$$

where

$$D(\theta) = \operatorname{diag}\left\{ \left(\frac{\overline{\beta(\theta)}}{\beta(\theta)}\right)^{1/2}, \left(\frac{\overline{\beta(\theta)}}{\beta(\theta)}\right)^{2/2}, \ldots, \left(\frac{\overline{\beta(\theta)}}{\beta(\theta)}\right)^{n/2} \right\}.$$

Observe now that $y_1(\theta+\pi) = y_n(\theta)$ is a unit eigenvector of

$$H(e^{\mathrm{i}(\theta+\pi)}T_n) = H(-e^{\mathrm{i}\theta}T_n) = -H(e^{\mathrm{i}\theta}T_n) = -\frac{1}{2}T_n(\overline{\beta(\theta)}, 0, \beta(\theta))$$

associated with its largest eigenvalue

$$\delta_1(e^{\mathrm{i}(\theta+\pi)}T_n) = \delta_1(-e^{\mathrm{i}\theta}T_n) = -\delta_n(e^{\mathrm{i}\theta}T_n).$$

One can also see that each entry of the vector $S(e^{\mathrm{i}\theta}T_n)y_1(\theta+\pi) = S(e^{\mathrm{i}\theta}T_n)y_n(\theta)$ is either equal to the corresponding entry of $S(e^{\mathrm{i}\theta}T_n)y_1(\theta)$ (at the even positions) or equal to the corresponding entry of $S(e^{\mathrm{i}\theta}T_n)y_1(\theta)$ negated (at odd positions). As a consequence,

$$\begin{aligned} \mathrm{v}(-e^{\mathrm{i}\theta}T_n) &= \left\|S(-e^{\mathrm{i}\theta}T_n)y_1(\theta+\pi)\right\|_2^2 = \left\|-S(e^{\mathrm{i}\theta}T_n)y_n(\theta)\right\|_2^2 \\ &= (S(e^{\mathrm{i}\theta}T_n)y_n(\theta))^*(S(e^{\mathrm{i}\theta}T_n)y_n(\theta)) = (S(e^{\mathrm{i}\theta}T_n)y_1(\theta))^*(S(e^{\mathrm{i}\theta}T_n)y_1(\theta)) \\ &= \mathrm{v}(e^{\mathrm{i}\theta}T_n). \end{aligned}$$

Moreover, it is straightforward to verify that

$$y_1^*(\theta+\pi)S(e^{\mathrm{i}\theta}T_n)y_1(\theta+\pi) = y_n^*(\theta)S(e^{\mathrm{i}\theta}T_n)y_n(\theta) = -y_1^*(\theta)S(e^{\mathrm{i}\theta}T_n)y_1(\theta),$$

which yields

$$\mathrm{u}(-e^{\mathrm{i}\theta}T_n) = \mathrm{Im}(y_1^*(\theta+\pi)S(-e^{\mathrm{i}\theta}T_n)y_1(\theta+\pi)) = \mathrm{Im}[-(-y_1^*(\theta)S(e^{\mathrm{i}\theta}T_n)y_1(\theta))] = \mathrm{u}(e^{\mathrm{i}\theta}T_n),$$

completing the proof. □

Theorem 2.2. *The envelope of a tridiagonal Toeplitz matrix $T_n(c,a,b) \in \mathcal{M}_n(\mathbb{C})$, $bc \neq 0$, is symmetric with respect to the line*

$$\mathcal{L}(T_n(c,a,b)) = \left\{ a + \gamma e^{\mathrm{i}\frac{\arg(b)+\arg(c)}{2}} : \gamma \in \mathbb{R} \right\}.$$

Proof. Without loss of generality, and for the sake of simplicity, we consider again the matrix $T_n = T_n(c, 0, b)$. We also denote $\theta_0 = \frac{\arg(b)+\arg(c)}{2}$. Then, the envelope $\mathcal{E}(T_n)$ is symmetric with respect to the line $\mathcal{L}(T_n(c, 0, b))$ if and only if $\mathcal{E}(e^{-\mathrm{i}\theta_0}T_n)$ is symmetric with respect to the real axis. By properties (P_1) and (P_4) of the envelope, we observe that

$$\overline{\mathcal{E}(e^{-\mathrm{i}\theta_0}T_n)} = \mathcal{E}(e^{-\mathrm{i}\theta_0}T_n) \;\Leftrightarrow\; \mathcal{E}(e^{\mathrm{i}\theta_0}T_n^*) = \mathcal{E}(e^{-\mathrm{i}\theta_0}T_n) \;\Leftrightarrow\; \mathcal{E}(e^{\mathrm{i}2\theta_0}T_n^*) = \mathcal{E}(T_n),$$

which equivalences hold when

$$\Gamma(e^{\mathrm{i}(\theta+2\theta_0)}T_n^*) = \Gamma(e^{\mathrm{i}\theta}T_n) \;\Leftrightarrow\; \Gamma(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) = \overline{\Gamma(e^{\mathrm{i}\theta}T_n)}, \quad \forall\,\theta \in [0, 2\pi).$$

In order to verify the later equality on the cubic curves, it is sufficient to obtain that

$$\delta_1(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) = \delta_1(e^{\mathrm{i}\theta}T_n), \;\; \delta_2(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) = \delta_2(e^{\mathrm{i}\theta}T_n),$$

$$\mathrm{v}(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) = \mathrm{v}(e^{\mathrm{i}\theta}T_n), \;\text{ and }\; \mathrm{u}(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) = -\mathrm{u}(e^{\mathrm{i}\theta}T_n)$$

for any $\theta \in [0, 2\pi)$. As in the proof of Theorem 2.1, consider the ellipse $\beta(\theta) = be^{\mathrm{i}\theta} + \overline{c}e^{-\mathrm{i}\theta}$, $\theta \in [0, 2\pi)$. Then, according to expression (2.3), the two largest eigenvalues of the hermitian tridiagonal Toeplitz matrix $H(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) = \frac{1}{2}T_n(\overline{\beta(-\theta-2\theta_0)}, 0, \beta(-\theta-2\theta_0))$ are given by

$$\begin{aligned}
\delta_j(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) &= |\beta(-\theta-2\theta_0)| \cos\left(\frac{j\pi}{n+1}\right)\\
&= (|b|^2 + |c|^2 + 2\,\mathrm{Re}((bc)e^{\mathrm{i}2(-\theta-2\theta_0)}))^{1/2} \cos\left(\frac{j\pi}{n+1}\right)\\
&= (|b|^2 + |c|^2 + 2\,\mathrm{Re}(|bc|e^{\mathrm{i}2(-\theta-\theta_0)}))^{1/2} \cos\left(\frac{j\pi}{n+1}\right)\\
&= (|b|^2 + |c|^2 + 2\,\mathrm{Re}(|bc|e^{\mathrm{i}2(\theta+\theta_0)}))^{1/2} \cos\left(\frac{j\pi}{n+1}\right)\\
&= (|b|^2 + |c|^2 + 2\,\mathrm{Re}((bc)e^{\mathrm{i}2\theta}))^{1/2} \cos\left(\frac{j\pi}{n+1}\right)\\
&= |\beta(\theta)| \cos\left(\frac{j\pi}{n+1}\right)\\
&= \delta_j(e^{\mathrm{i}\theta}T_n), \qquad j = 1, 2.
\end{aligned}$$

Moreover, by (2.4), a unit eigenvector of $H(e^{-\mathrm{i}(\theta+2\theta_0)}T_n)$ corresponding to the largest eigenvalue $\delta_1(e^{-\mathrm{i}(\theta+2\theta_0)}T_n)$ is given by

$$y_1(-\theta-2\theta_0) = \sqrt{\frac{2}{n+1}}\; D(-\theta-2\theta_0) \begin{bmatrix} \sin\left(\frac{\pi}{n+1}\right) \\ \sin\left(\frac{2\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{n\pi}{n+1}\right) \end{bmatrix},$$

where $D(\theta)$ is as in the proof of Theorem 2.1. To further simplify the exposition of our calculations, we denote

$$\gamma(\theta) = be^{\mathrm{i}\theta} - \overline{c}e^{-\mathrm{i}\theta}, \quad \theta \in [0, 2\pi)$$

and

$$w_1 = \sqrt{\frac{2}{n+1}} \begin{bmatrix} \sin\left(\frac{\pi}{n+1}\right) & \sin\left(\frac{2\pi}{n+1}\right) & \cdots & \sin\left(\frac{n\pi}{n+1}\right) \end{bmatrix}^T \in \mathbb{R}^n.$$

Then, the following computations ensue:

$$\begin{aligned}
|\gamma(-\theta-2\theta_0)| &= [(be^{-\mathrm{i}(\theta+2\theta_0)} - \overline{c}e^{\mathrm{i}(\theta+2\theta_0)})(\overline{b}e^{\mathrm{i}(\theta+2\theta_0)} - ce^{-\mathrm{i}(\theta+2\theta_0)})]^{1/2} \\
&= (|b|^2 - (bc)e^{-\mathrm{i}2(\theta+2\theta_0)} - (\overline{bc})e^{\mathrm{i}2(\theta+2\theta_0)} + |c|^2)^{1/2} \\
&= (|b|^2 - |bc|e^{\mathrm{i}2\theta_0}e^{-\mathrm{i}2(\theta+2\theta_0)} - |bc|e^{-\mathrm{i}2\theta_0}e^{\mathrm{i}2(\theta+2\theta_0)} + |c|^2)^{1/2} \\
&= (|b|^2 - 2\,\mathrm{Re}(|bc|e^{\mathrm{i}2(\theta+\theta_0)}) + |c|^2)^{1/2},
\end{aligned}$$

$$\begin{aligned}
|\gamma(\theta)| &= [(be^{\mathrm{i}\theta} - \overline{c}e^{-\mathrm{i}\theta})(\overline{b}e^{-\mathrm{i}\theta} - ce^{\mathrm{i}\theta})]^{1/2} \\
&= (|b|^2 - (bc)e^{\mathrm{i}2\theta} - (\overline{bc})e^{-\mathrm{i}2\theta} + |c|^2)^{1/2} \\
&= (|b|^2 - |bc|e^{\mathrm{i}2\theta_0}e^{\mathrm{i}2\theta} - |bc|e^{-\mathrm{i}2\theta_0}e^{-\mathrm{i}2\theta} + |c|^2)^{1/2} \\
&= (|b|^2 - 2\,\mathrm{Re}(|bc|e^{\mathrm{i}2(\theta+\theta_0)}) + |c|^2)^{1/2},
\end{aligned}$$

$$\begin{aligned}
\gamma(-\theta-2\theta_0)\overline{\beta(-\theta-2\theta_0)} &= (be^{-\mathrm{i}(\theta+2\theta_0)} - \overline{c}e^{\mathrm{i}(\theta+2\theta_0)})(\overline{b}e^{\mathrm{i}(\theta+2\theta_0)} + ce^{-\mathrm{i}(\theta+2\theta_0)}) \\
&= |b|^2 + (bc)e^{-\mathrm{i}2(\theta+2\theta_0)} - (\overline{bc})e^{\mathrm{i}2(\theta+2\theta_0)} - |c|^2 \\
&= |b|^2 + |bc|e^{\mathrm{i}2\theta_0}e^{-\mathrm{i}2(\theta+2\theta_0)} - |bc|e^{-\mathrm{i}2\theta_0}e^{\mathrm{i}2(\theta+2\theta_0)} - |c|^2 \\
&= |b|^2 - \mathrm{i}2\,\mathrm{Im}(|bc|e^{\mathrm{i}2(\theta+\theta_0)}) - |c|^2,
\end{aligned}$$

$$\begin{aligned}
\overline{\gamma(\theta)}\beta(\theta) &= (\overline{b}e^{-\mathrm{i}\theta} - ce^{\mathrm{i}\theta})(be^{\mathrm{i}\theta} + \overline{c}e^{-\mathrm{i}\theta}) \\
&= |b|^2 + (\overline{bc})e^{-\mathrm{i}2\theta} - (bc)e^{\mathrm{i}2\theta} - |c|^2 \\
&= |b|^2 + |bc|e^{-\mathrm{i}2\theta_0}e^{-\mathrm{i}2\theta} - |bc|e^{\mathrm{i}2\theta_0}e^{\mathrm{i}2\theta} - |c|^2 \\
&= |b|^2 - \mathrm{i}2\,\mathrm{Im}(|bc|e^{\mathrm{i}2(\theta+\theta_0)}) - |c|^2,
\end{aligned}$$

$$\begin{aligned}
\gamma(-\theta-2\theta_0)\overline{\beta(\theta)} &= (be^{-\mathrm{i}(\theta+2\theta_0)} - \overline{c}e^{\mathrm{i}(\theta+2\theta_0)})(\overline{b}e^{-\mathrm{i}\theta} + ce^{\mathrm{i}\theta}) \\
&= |b|^2e^{-\mathrm{i}2(\theta+\theta_0)} + (bc)e^{-\mathrm{i}2\theta_0} - (\overline{bc})e^{\mathrm{i}2\theta_0} - |c|^2e^{\mathrm{i}2(\theta+\theta_0)} \\
&= |b|^2e^{-\mathrm{i}2(\theta+\theta_0)} - |c|^2e^{\mathrm{i}2(\theta+\theta_0)},
\end{aligned}$$

and

$$\begin{aligned}
\overline{\gamma(\theta)}\beta(-\theta-2\theta_0) &= (\overline{b}e^{-\mathrm{i}\theta} - ce^{\mathrm{i}\theta})(be^{-\mathrm{i}(\theta+2\theta_0)} + \overline{c}e^{\mathrm{i}(\theta+2\theta_0)}) \\
&= |b|^2e^{-\mathrm{i}2(\theta+\theta_0)} + (\overline{bc})e^{\mathrm{i}2\theta_0} - (bc)e^{-\mathrm{i}2\theta_0} - |c|^2e^{\mathrm{i}2(\theta+\theta_0)} \\
&= |b|^2e^{-\mathrm{i}2(\theta+\theta_0)} - |c|^2e^{\mathrm{i}2(\theta+\theta_0)}.
\end{aligned}$$

Thus,

$$|\gamma(-\theta-2\theta_0)| = |\gamma(\theta)|, \quad \gamma(-\theta-2\theta_0)\overline{\beta(-\theta-2\theta_0)} = \overline{\gamma(\theta)}\beta(\theta),$$

$$\gamma(-\theta-2\theta_0)\overline{\beta(\theta)} = \overline{\gamma(\theta)}\beta(-\theta-2\theta_0), \text{ and } \frac{\gamma(-\theta-2\theta_0)^2\overline{\beta(-\theta-2\theta_0)}}{\beta(-\theta-2\theta_0)} = \frac{\overline{\gamma(\theta)}^2\beta(\theta)}{\overline{\beta(\theta)}}.$$

The above relations yield

$$
\begin{aligned}
\mathrm{v}(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) &= \left\| S(e^{-\mathrm{i}(\theta+2\theta_0)}T_n)y_1(-\theta-2\theta_0) \right\|_2^2 \\
&= -w_1^T D(-\theta-2\theta_0)^{-1} S(e^{-\mathrm{i}(\theta+2\theta_0)}T_n)^2 D(-\theta-2\theta_0)w_1 \\
&= w_1^T \begin{bmatrix}
\frac{|\gamma(\theta)|^2}{4} & 0 & \frac{-\overline{\gamma(\theta)}^2\beta(\theta)}{4\overline{\beta(\theta)}} & 0 & & 0 \\
0 & \frac{|\gamma(\theta)|^2}{2} & 0 & \ddots & & \\
\frac{-\gamma(\theta)^2\overline{\beta(\theta)}}{4\beta(\theta)} & 0 & \ddots & \ddots & \ddots & \\
0 & \ddots & \ddots & \ddots & \ddots & \frac{-\overline{\gamma(\theta)}^2\beta(\theta)}{4\overline{\beta(\theta)}} \\
 & & \ddots & & \frac{|\gamma(\theta)|^2}{2} & 0 \\
0 & & & \frac{-\gamma(\theta)^2\overline{\beta(\theta)}}{4\beta(\theta)} & 0 & \frac{|\gamma(\theta)|^2}{4}
\end{bmatrix} w_1 \\
&= -w_1^T\, \overline{D(\theta)^{-1}S(e^{\mathrm{i}\theta}T_n)^2 D(\theta)}\, w_1 \\
&= \left\| \overline{S(e^{\mathrm{i}\theta}T_n)D(\theta)w_1} \right\|_2^2 \\
&= \left\| S(e^{\mathrm{i}\theta}T_n)y_1(\theta) \right\|_2^2 \\
&= \mathrm{v}(e^{\mathrm{i}\theta}T_n)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{u}(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) &= \mathrm{Im}(y_1^*(-\theta-2\theta_0)S(e^{-\mathrm{i}(\theta+2\theta_0)}T_n)y_1(-\theta-2\theta_0)) \\
&= \mathrm{Im}\left(w_1^T D(-\theta-2\theta_0)^{-1} S(e^{-\mathrm{i}(\theta+2\theta_0)}T_n) D(-\theta-2\theta_0)w_1 \right) \\
&= \mathrm{Im}\left(\frac{1}{2}\, w_1^T\, T_n\left(-\gamma(\theta)\left(\frac{\beta(\theta)}{\overline{\beta(\theta)}}\right)^{-1/2}, 0, \overline{\gamma(\theta)}\left(\frac{\beta(\theta)}{\overline{\beta(\theta)}}\right)^{1/2} \right) w_1 \right) \\
&= \mathrm{Im}\left(w_1^T\, \overline{D(\theta)^{-1}S(e^{\mathrm{i}\theta}T_n)D(\theta)}\, w_1 \right) \\
&= -\mathrm{Im}\left(w_1^T D(\theta)^{-1} S(e^{\mathrm{i}\theta}T_n) D(\theta) w_1 \right) \\
&= -\mathrm{u}(e^{\mathrm{i}\theta}T_n),
\end{aligned}
$$

completing the proof. □

Example 2.3. Consider the 5×5 tridiagonal Toeplitz matrices $T_5(2+3\mathrm{i}, 0, -1-\mathrm{i})$ and $T_5(2+3\mathrm{i}, 0, 0.8-\mathrm{i})$. Their envelopes are illustrated by the unshaded areas in the left and right parts of Figure 1, respectively. The envelope of $T_5(2{+}3\mathrm{i}, 0, -1{-}\mathrm{i})$ consists of three connected components, the envelope of $T_5(2+3\mathrm{i}, 0, 0.8-\mathrm{i})$ is connected, and the eigenvalues of the matrices are marked with $+$'s. The symmetry results in Theorems 2.1 and 2.2 (with respect to the origin and the straight line determined by the eigenvalues) are confirmed. It is worth noting that the numerical range appears, as a by-product of our drawing technique, in all of our plots of an envelope; indeed, the numerical range is depicted as the outer outlined elliptical region.

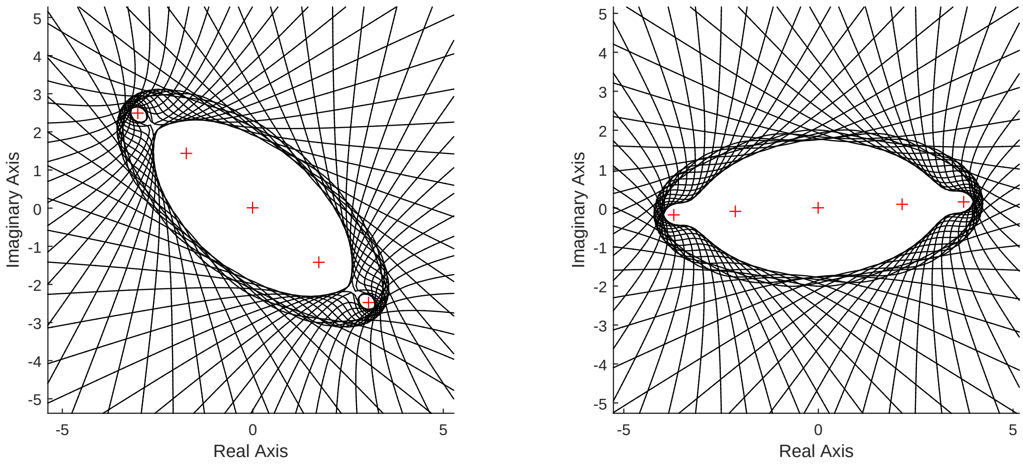


Figure 1: The envelopes $\mathcal{E}(T_5(2+3\mathrm{i}, 0, -1-\mathrm{i}))$ (left) and $\mathcal{E}(T_5(2+3\mathrm{i}, 0, 0.8-\mathrm{i}))$ (right).

3 The Envelope of a Block-Shift Matrix

A square matrix of the block form

$$A = \begin{bmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_m \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{3.1}$$

with $m > 1$ and the zero blocks along the main diagonal being square, is called a *block-shift matrix*. The next lemma can be found in [13] (see Theorem 1 and Remarks 2 and 4), and it is stated here for the sake of completeness.

Lemma 3.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following conditions are equivalent:*

(i) A is permutationally similar to a block-shift matrix.

(ii) For every nonzero $a \in \mathbb{C}$, A is diagonally similar to aA.

(iii) There is a nonzero $a \in \mathbb{C}$, which is not a root of unity, such that A is diagonally similar to aA,

(iv) For every $\theta \in [0, 2\pi)$, A is unitarily diagonally similar to $e^{i\theta} A$.

Lemma 3.2. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A is permutationally similar to a block-shift matrix if and only if for every $\theta \in [0, 2\pi)$, there exists a unitary diagonal matrix $U_\theta \in \mathcal{M}_n(\mathbb{C})$ such that $H(A) = U_\theta^* H(e^{i\theta} A) U_\theta$ and $S(A) = U_\theta^* S(e^{i\theta} A) U_\theta$.*

Proof. By Lemma 3.1, A is permutationally similar to a block-shift matrix if and only if for every $\theta \in [0, 2\pi)$, there is a unitary diagonal matrix U_θ such that $A = U_\theta^* e^{i\theta} A U_\theta$, or equivalently, $H(A) = U_\theta^* H(e^{i\theta} A) U_\theta$ and $S(A) = U_\theta^* S(e^{i\theta} A) U_\theta$. □

Using the above lemmas, we are now able to show that the envelope of a block-shift matrix is a circular disc centered at the origin.

Theorem 3.3. *Let $A \in \mathcal{M}_n(\mathbb{C})$ $(n \geq 3)$ be a block-shift matrix. Then $\mathcal{E}(A)$ coincides with the circular disc $\mathcal{D}(0, R)$ centered at the origin, with radius*

$$R = \left(\delta_1^2(A) - \left(\sqrt{2\delta_1(A)(\delta_1(A) - \delta_2(A))} - \sqrt{\mathrm{v}(A)} \right)^2 \right)^{1/2}.$$

Proof. Suppose $A \in \mathcal{M}_n(\mathbb{C})$ is a block-shift matrix as in (3.1). Lemma 3.2 asserts that for any $\theta \in [0, 2\pi)$, $H(e^{\mathrm{i}\theta}A) = U_\theta H(A) U_\theta^*$ and $S(e^{\mathrm{i}\theta}A) = U_\theta S(A) U_\theta^*$ for some unitary $U_\theta \in \mathcal{M}_n(\mathbb{C})$. Thus, the eigenvalues of $H(e^{\mathrm{i}\theta}A)$ remain constant (independently of the angle $\theta \in [0, 2\pi)$) and equal to $\delta_1(A) \geq \delta_2(A) \geq \cdots \geq \delta_n(A)$.

Consider a unit eigenvector y_1 of $H(A)$ associated with the largest eigenvalue $\delta_1(A)$. Then $U_\theta\, y_1$ is a unit eigenvector of $H(e^{\mathrm{i}\theta}A)$ associated with $\delta_1(A)$, and hence,

$$\mathrm{v}(e^{\mathrm{i}\theta}A) = \left\| S(e^{\mathrm{i}\theta}A) U_\theta y_1 \right\|_2^2 = \left\| U_\theta^* S(e^{\mathrm{i}\theta}A) U_\theta y_1 \right\|_2^2 = \| S(A) y_1 \|_2^2 = \mathrm{v}(A)$$

and

$$\mathrm{u}(e^{\mathrm{i}\theta}A) = \mathrm{Im}(y_1^* U_\theta^* S(e^{\mathrm{i}\theta}A) U_\theta y_1) = \mathrm{Im}(y_1^* S(A) y_1) = \mathrm{u}(A).$$

It follows that all rotations of A have the same cubic curve; that is, $\Gamma(e^{\mathrm{i}\theta}A) = \Gamma(A)$ for all $\theta \in [0, 2\pi)$.

We are now interested in the type of the cubic curve $\Gamma(A)$. It is known by [12, Theorem 1] that $F(A) = \mathcal{D}(0, r(A))$, where $r(A)$ is the numerical radius of A. As a consequence, $\delta_1(A) + \mathrm{i}\mathrm{u}(A) = r(A) > 0$ (the right-most point of $F(A)$), $\mathrm{u}(A) = 0$, and (1.1) takes the form

$$\Gamma(A) = \left\{ z \in \mathbb{C} :\, (\delta_2(A) - \mathrm{Re}\, z)[(\delta_1(A) - \mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2] + (\delta_1(A) - \mathrm{Re}\, z)\mathrm{v}(A) = 0 \right\}. \quad (3.2)$$

Recall that $\Gamma(A)$ is symmetric with respect to the real axis (see [10] or the appendix below), and every eigenvalue of A lies to the left of the curve. Since A has only the zero eigenvalue of multiplicity $n \geq 3$, Theorem 3.2 in [10] ensures that $\Gamma(A)$ is connected with no closed branch. These observations lead us to the conclusion that, carrying out the rotation of $\Gamma(A)$ about the origin, the envelope $\mathcal{E}(A)$ coincides with a circular disc centered at the origin.

The radius of the disc can be determined by calculating the shortest distance from the origin to the curve $\Gamma(A)$. To achieve this, we have to minimize $d = \sqrt{s^2 + t^2}$, or equivalently, d^2, subject to $s + \mathrm{i}\,t \in \Gamma(A)$ $(s, t \in \mathbb{R})$. Since the curve $\Gamma(A)$ lies in the vertical zone $\{s + \mathrm{i}\,t \in \mathbb{C} :\, s, t \in \mathbb{R},\, \delta_2(A) < s \leq \delta_1(A)\}$, (3.2) can be written as

$$\Gamma(A) = \{s + \mathrm{i}\,t \in \mathbb{C} :\, s, t \in \mathbb{R},\, f(s, t) = 0\},$$

where

$$f(s, t) = (\delta_1(A) - s)^2 + t^2 + \frac{(\delta_1(A) - s)\mathrm{v}(A)}{\delta_2(A) - s}.$$

Solving $f(s, t) = 0$ for $d^2 = s^2 + t^2$, we obtain

$$d^2 = d^2(s) = \delta_1(A)(2s - \delta_1(A)) - \frac{(\delta_1(A) - s)\mathrm{v}(A)}{\delta_2(A) - s}.$$

Hence, minimizing d^2 subject to $f(s,t) = 0$ is equivalent to minimizing d^2 with respect to s, with $\delta_2(A) < s \leq \delta_1(A)$. Therefore,

$$(d^2(s))' = 2\delta_1(A) - \frac{\mathrm{v}(A)(\delta_1(A) - \delta_2(A))}{(\delta_2(A) - s)^2} = 0$$

results into

$$(\delta_2(A) - s)^2 = \frac{\mathrm{v}(A)(\delta_1(A) - \delta_2(A))}{2\delta_1(A)} \; (> 0),$$

and due to $\delta_2(A) < s \leq \delta_1(A)$, we have

$$s = \delta_2(A) + \sqrt{\frac{\mathrm{v}(A)(\delta_1(A) - \delta_2(A))}{2\delta_1(A)}}.$$

Hence, the minimum distance is

$$\begin{aligned} R &= \left(2\delta_1(A)\delta_2(A) - \delta_1(A)^2 - \mathrm{v}(A) + 2\sqrt{2\mathrm{v}(A)\delta_1(A)(\delta_1(A) - \delta_2(A))}\right)^{1/2} \\ &= \left(\delta_1^2(A) - \left(\sqrt{2\delta_1(A)(\delta_1(A) - \delta_2(A))} - \sqrt{\mathrm{v}(A)}\right)^2\right)^{1/2}, \end{aligned}$$

and the proof is complete. □

Remark 3.4. As mentioned in the above proof, the numerical range of a block-shift matrix is also a circular disc centered at the origin [4, 12, 13]. The numerical radius of a block-shift matrix A is $r(A) = \delta_1(A)$. Hence, it is straightforward to verify that

$$r(A)^2 - R^2 = \left(\sqrt{2r(A)(r(A) - \delta_2(A))} - \sqrt{\mathrm{v}(A)}\right)^2.$$

Example 3.5. Consider a 6×6 block-shift matrix as in (3.1) with $m = 3$:

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 3\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The envelope $\mathcal{E}(A)$ is illustrated in Figure 2 by the unshaded area, exhibiting the circular shape proven in Theorem 3.3. Its radius is $R = 2.7416$ since $\delta_1 = r(A) = 3.1495$, $\delta_2 = 0.9522$ and $\mathrm{v}(A) = 4.7094$. The zero eigenvalue of A is marked with a $+$ and coincides with the center of the circle.

We next apply Theorem 3.3 to determine the envelope of a Jordan block with zero eigenvalue.

Theorem 3.6. *Let $n \geq 3$ and consider the $n \times n$ Jordan block*

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \quad \lambda \in \mathbb{C}.$$

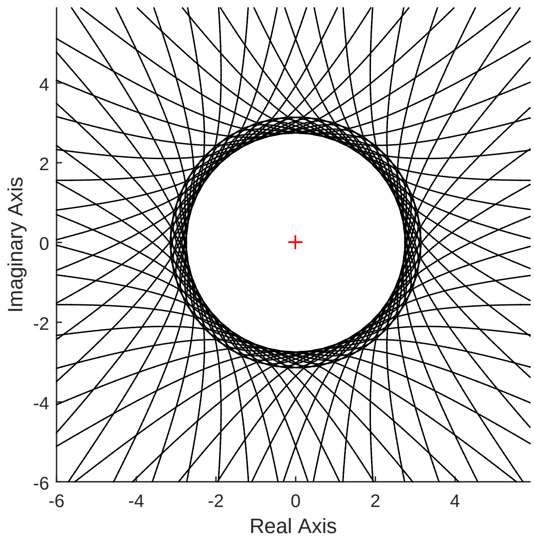


Figure 2: The envelope of a block-shift matrix.

The envelope $\mathcal{E}(J_n(\lambda))$ *is the circular disc* $\mathcal{D}(\lambda, R)$ *with radius*

$$R = \left(\frac{-4\delta_1^4 + 2\delta_1^2 + 2(n+1)\delta_1\delta_2 + \delta_2^2 - n}{n+1} + 2\sqrt{2\delta_1(\delta_1 - \delta_2)\frac{4\delta_1^4 - (n+3)\delta_1^2 - \delta_2^2 + n}{n+1}} \right)^{\frac{1}{2}},$$

where $\delta_j = \cos\left(\frac{j\pi}{n+1}\right)$, $j = 1, 2$. *Moreover, for* $n = 2$, $\mathcal{E}(J_2(\lambda)) = \{\lambda\}$.

Proof. Denote by J_n the basic $n \times n$ Jordan block $J_n(0)$. By the translation property (P_3) of the envelope, we have

$$\mathcal{E}(J_n(\lambda)) = \mathcal{E}(J_n + \lambda I_n) = \mathcal{E}(J_n) + \lambda.$$

By Theorem 3.3, $\mathcal{E}(J_n) = \mathcal{D}(0, R_n)$, where

$$R_n = \left(2\delta_1(J_n)\delta_2(J_n) - \delta_1^2(J_n) - \mathrm{v}(J_n) + 2\sqrt{2\mathrm{v}(J_n)\delta_1(J_n)(\delta_1(J_n) - \delta_2(J_n))}\right)^{1/2}. \quad (3.3)$$

In the sequel, we will compute the quantities $\delta_1(J_n)$, $\delta_2(J_n)$ and $\mathrm{v}(J_n) = \|S(J_n)y_1\|_2^2$, with y_1 a unit eigenvector of $H(J_n)$ corresponding to $\delta_1(J_n)$.

Following the notation given in the previous section for tridiagonal Toeplitz matrices, we notice that $H(J_n) = T_n\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ and its eigenvalues are given explicitly by (2.1); that is, $\delta_j(J_n) = \cos\left(\frac{j\pi}{n+1}\right)$, $j = 1, 2, \ldots, n$. Moreover, a unit eigenvector y_1 of $H(J_n)$ associated to $\delta_1(J_n)$ can be readily calculated by (2.2), that is,

$$y_1 = \sqrt{\frac{2}{n+1}} \begin{bmatrix} \sin\left(\frac{\pi}{n+1}\right) & \sin\left(\frac{2\pi}{n+1}\right) & \cdots & \sin\left(\frac{n\pi}{n+1}\right) \end{bmatrix}^T.$$

As a consequence,

$$
\begin{aligned}
\mathrm{v}(J_n) = \|S(J_n)y_1\|_2^2 &= \left\| T_n\left(-\frac{1}{2},0,\frac{1}{2}\right) y_1 \right\|_2^2 \\
&= \frac{2}{n+1}\left[\sin\left(\tfrac{\pi}{n+1}\right) \quad \cdots \quad \sin\left(\tfrac{n\pi}{n+1}\right)\right]
\begin{bmatrix}
\frac14 & 0 & -\frac14 & 0 & \cdots & 0 \\
0 & \frac12 & 0 & -\frac14 & & \vdots \\
-\frac14 & 0 & \frac12 & 0 & -\frac14 & \\
& \ddots & & \ddots & & \ddots \\
& -\frac14 & 0 & \frac12 & 0 & -\frac14 \\
\vdots & & -\frac14 & 0 & \frac12 & 0 \\
0 & \cdots & 0 & -\frac14 & 0 & \frac14
\end{bmatrix}
\begin{bmatrix}
\sin\left(\frac{\pi}{n+1}\right) \\ \vdots \\ \sin\left(\frac{n\pi}{n+1}\right)
\end{bmatrix} \\
&= \frac{2}{n+1}\left[\sin\left(\tfrac{\pi}{n+1}\right) \quad \cdots \quad \sin\left(\tfrac{n\pi}{n+1}\right)\right]
\begin{bmatrix}
\frac14\sin\left(\frac{\pi}{n+1}\right) - \frac14\sin\left(\frac{3\pi}{n+1}\right) \\
\frac12\sin\left(\frac{2\pi}{n+1}\right) - \frac14\sin\left(\frac{4\pi}{n+1}\right) \\
-\frac14\sin\left(\frac{\pi}{n+1}\right) + \frac12\sin\left(\frac{3\pi}{n+1}\right) - \frac14\sin\left(\frac{5\pi}{n+1}\right) \\
\vdots \\
-\frac14\sin\left(\frac{(n-4)\pi}{n+1}\right) + \frac12\sin\left(\frac{(n-2)\pi}{n+1}\right) - \frac14\sin\left(\frac{n\pi}{n+1}\right) \\
-\frac14\sin\left(\frac{(n-3)\pi}{n+1}\right) + \frac12\sin\left(\frac{(n-1)\pi}{n+1}\right) \\
-\frac14\sin\left(\frac{(n-2)\pi}{n+1}\right) + \frac14\sin\left(\frac{n\pi}{n+1}\right)
\end{bmatrix} \\
&= \frac{2}{n+1}\Bigg[\sin\left(\frac{\pi}{n+1}\right)\left(\frac14\sin\left(\frac{\pi}{n+1}\right) - \frac14\sin\left(\frac{3\pi}{n+1}\right)\right) \\
&\qquad + \sin\left(\frac{2\pi}{n+1}\right)\left(\frac12\sin\left(\frac{2\pi}{n+1}\right) - \frac14\sin\left(\frac{4\pi}{n+1}\right)\right) \\
&\qquad + \sin\left(\frac{3\pi}{n+1}\right)\left(-\frac14\sin\left(\frac{\pi}{n+1}\right) + \frac12\sin\left(\frac{3\pi}{n+1}\right) - \frac14\sin\left(\frac{5\pi}{n+1}\right)\right) \\
&\qquad + \cdots \\
&\qquad + \sin\left(\frac{(n-2)\pi}{n+1}\right)\left(-\frac14\sin\left(\frac{(n-4)\pi}{n+1}\right) + \frac12\sin\left(\frac{(n-2)\pi}{n+1}\right) - \frac14\sin\left(\frac{n\pi}{n+1}\right)\right) \\
&\qquad + \sin\left(\frac{(n-1)\pi}{n+1}\right)\left(-\frac14\sin\left(\frac{(n-3)\pi}{n+1}\right) + \frac12\sin\left(\frac{(n-1)\pi}{n+1}\right)\right) \\
&\qquad + \sin\left(\frac{n\pi}{n+1}\right)\left(-\frac14\sin\left(\frac{(n-2)\pi}{n+1}\right) + \frac14\sin\left(\frac{n\pi}{n+1}\right)\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{n+1}\left[\frac{1}{4}\sin^2\left(\frac{\pi}{n+1}\right) - \frac{1}{4}\sin\left(\frac{\pi}{n+1}\right)\sin\left(\frac{3\pi}{n+1}\right)\right. \\
&\quad +\frac{1}{2}\sin^2\left(\frac{2\pi}{n+1}\right) - \frac{1}{4}\sin\left(\frac{2\pi}{n+1}\right)\sin\left(\frac{4\pi}{n+1}\right) \\
&\quad -\frac{1}{4}\sin\left(\frac{3\pi}{n+1}\right)\sin\left(\frac{\pi}{n+1}\right) + \frac{1}{2}\sin^2\left(\frac{3\pi}{n+1}\right) - \frac{1}{4}\sin\left(\frac{3\pi}{n+1}\right)\sin\left(\frac{5\pi}{n+1}\right) \\
&\quad - \cdots \\
&\quad -\frac{1}{4}\sin\left(\frac{(n-2)\pi}{n+1}\right)\sin\left(\frac{(n-4)\pi}{n+1}\right) + \frac{1}{2}\sin^2\left(\frac{(n-2)\pi}{n+1}\right) - \frac{1}{4}\sin\left(\frac{(n-2)\pi}{n+1}\right)\sin\left(\frac{n\pi}{n+1}\right) \\
&\quad -\frac{1}{4}\sin\left(\frac{(n-1)\pi}{n+1}\right)\sin\left(\frac{(n-3)\pi}{n+1}\right) + \frac{1}{2}\sin^2\left(\frac{(n-1)\pi}{n+1}\right) \\
&\quad \left. -\frac{1}{4}\sin\left(\frac{n\pi}{n+1}\right)\sin\left(\frac{(n-2)\pi}{n+1}\right) + \frac{1}{4}\sin^2\left(\frac{n\pi}{n+1}\right)\right] \\
&= \frac{1}{2(n+1)}\left[\sin^2\left(\frac{\pi}{n+1}\right) + 2\sum_{j=2}^{n-1}\sin^2\left(\frac{j\pi}{n+1}\right) + \sin^2\left(\frac{n\pi}{n+1}\right) - 2\sum_{j=1}^{n-2}\sin\left(\frac{j\pi}{n+1}\right)\sin\left(\frac{(j+2)\pi}{n+1}\right)\right] \\
&= \frac{1}{2(n+1)}\left[\sum_{j=1}^{n-2}\left(\sin\left(\frac{j\pi}{n+1}\right) - \sin\left(\frac{(j+2)\pi}{n+1}\right)\right)^2 + \sin^2\left(\frac{2\pi}{n+1}\right) + \sin^2\left(\frac{(n-1)\pi}{n+1}\right)\right].
\end{aligned}
$$

Applying now the sum-to-product trigonometric identity, and keeping in mind the relation

$$
\sum_{j=1}^{n}\cos^2\left(\frac{j\pi}{n+1}\right) = n - \sum_{j=1}^{n}\sin^2\left(\frac{j\pi}{n+1}\right) = n - \frac{n+1}{2} = \frac{n-1}{2},
$$

we get

$$
\begin{aligned}
\mathrm{v}(J_n) &= \frac{1}{2(n+1)}\left[\sum_{j=1}^{n-2}4\left(\sin^2\left(\frac{\pi}{n+1}\right)\cos^2\left(\frac{(j+1)\pi}{n+1}\right)\right) + 2 - \cos^2\left(\frac{2\pi}{n+1}\right) - \cos^2\left(\frac{(n-1)\pi}{n+1}\right)\right] \\
&= \frac{1}{2(n+1)}\left[4\sin^2\left(\frac{\pi}{n+1}\right)\sum_{j=1}^{n-2}\cos^2\left(\frac{(j+1)\pi}{n+1}\right) + 2 - 2\cos^2\left(\frac{2\pi}{n+1}\right)\right] \\
&= \frac{1}{2(n+1)}\left[4\left(1-\cos^2\left(\frac{\pi}{n+1}\right)\right)\sum_{j=1}^{n-2}\cos^2\left(\frac{(j+1)\pi}{n+1}\right) + 2 - 2\cos^2\left(\frac{2\pi}{n+1}\right)\right] \\
&= \frac{1}{2(n+1)}\left[4\left(1-\cos^2\left(\frac{\pi}{n+1}\right)\right)\left(\sum_{j=1}^{n}\cos^2\left(\frac{j\pi}{n+1}\right) - \cos^2\left(\frac{\pi}{n+1}\right) - \cos^2\left(\frac{n\pi}{n+1}\right)\right)\right. \\
&\qquad \left. + 2 - 2\cos^2\left(\frac{2\pi}{n+1}\right)\right] \\
&= \frac{1}{2(n+1)}\left[4\left(1-\cos^2\left(\frac{\pi}{n+1}\right)\right)\left(\frac{n-1}{2} - 2\cos^2\left(\frac{\pi}{n+1}\right)\right) + 2 - 2\cos^2\left(\frac{2\pi}{n+1}\right)\right].
\end{aligned}
$$

Substituting $\delta_j = \delta_j(J_n) = \cos\left(\frac{j\pi}{n+1}\right)$, $j = 1, 2$, into the above, we obtain

$$\begin{aligned}
\mathrm{v}(J_n) &= \frac{1}{2(n+1)}\left[4(1-\delta_1^2)\left(\frac{n-1}{2} - 2\delta_1^2\right) + 2 - 2\delta_2^2\right] \\
&= \frac{1}{n+1}\left[2(1-\delta_1^2)\left(\frac{n-1}{2} - 2\delta_1^2\right) + 1 - \delta_2^2\right] \\
&= \frac{1}{n+1}\left[(1-\delta_1^2)(n-1-4\delta_1^2) + 1 - \delta_2^2\right] \\
&= \frac{1}{n+1}\left[4\delta_1^4 - (n+3)\delta_1^2 - \delta_2^2 + n\right].
\end{aligned}$$

In turn, we substitute the above expression for $\mathrm{v}(J_n)$ into (3.3) to derive

$$\begin{aligned}
R_n^2 &= 2\delta_1\delta_2 - \delta_1^2 - \frac{4\delta_1^4 - (n+3)\delta_1^2 - \delta_2^2 + n}{n+1} + 2\sqrt{2\delta_1(\delta_1-\delta_2)\frac{4\delta_1^4 - (n+3)\delta_1^2 - \delta_2^2 + n}{n+1}} \\
&= \frac{-4\delta_1^4 + 2\delta_1^2 + 2(n+1)\delta_1\delta_2 + \delta_2^2 - n}{n+1} + 2\sqrt{2\delta_1(\delta_1-\delta_2)\frac{4\delta_1^4 - (n+3)\delta_1^2 - \delta_2^2 + n}{n+1}}.
\end{aligned}$$

For the 2×2 Jordan block $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we have

$$\delta_1(J_2) = -\delta_2(J_2) = 1/2, \quad \mathrm{u}(J_2) = 0 \ \text{ and } \ \mathrm{v}(J_2) = \left\| \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\|_2^2 = 1/4.$$

According to the discussion in the proof of Theorem 3.3, the curve $\Gamma(e^{\mathrm{i}\theta} J_2)$ remains unchanged during all rotations $\theta \in [0, 2\pi)$, with equation

$$\left(\operatorname{Re} z + \frac{1}{2}\right)\left[\left(\operatorname{Re} z - \frac{1}{2}\right)^2 + (\operatorname{Im} z)^2\right] + \frac{1}{4}\left(\operatorname{Re} z - \frac{1}{2}\right) = 0 \tag{3.4}$$

and discriminant $\Delta = (\delta_1 - \delta_2)^2 - 4(\mathrm{v}(J_2) - \mathrm{u}^2(J_2)) = 0$. Taking into account the case (b) of the classification mentioned in the appendix, the curve is singular with a node at the origin $(0, 0) = \left(\frac{\delta_1+\delta_2}{2}, \mathrm{u}\right)$. Hence, the rotation of the curve about the origin yields $\mathcal{E}(J_2) = \{0\}$. ☐

4 The Envelope of a 2×2 Matrix

In this section, we derive that the envelope of every 2×2 complex matrix coincides with the point set of the spectrum of the matrix.

Theorem 4.1. *Let A be a 2×2 complex matrix. Then $\mathcal{E}(A) = \sigma(A)$.*

Proof. According to Schur's triangularization theorem, $A \in \mathcal{M}_2(\mathbb{C})$ is unitarily similar to an upper triangular matrix $T = \begin{bmatrix} \lambda & \alpha \\ 0 & \mu \end{bmatrix}$, where λ and μ are the eigenvalues of A, and $\alpha \in \mathbb{C}$.

If $\alpha = 0$, then by [11, Corollary 4.2], $\mathcal{E}(A) = \mathcal{E}(T) = \sigma(A)$.

If $\alpha \neq 0$ and $\lambda = \mu$, then properties (P_2), (P_3) and (P_4) of the envelope yield

$$\mathcal{E}(A) = \mathcal{E}(T) = \mathcal{E}(\lambda I_2 + \alpha J_2(0)) = \lambda + \alpha\mathcal{E}(J_2(0)) = \{\lambda\} = \sigma(A).$$

Suppose now that $\alpha \neq 0$ and $\lambda \neq \mu$. By Lemma 1.3.1 in [7], there exists a unitary matrix $U \in \mathcal{M}_2(\mathbb{C})$ such that

$$U^*\left(T - \frac{\mathrm{tr}(T)}{2}I_2\right)U = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix} = \begin{bmatrix} 0 & |c|e^{\mathrm{i}\theta_1} \\ |d|e^{\mathrm{i}\theta_2} & 0 \end{bmatrix},$$

for some $c, d \in \mathbb{C}$ and $\theta_1, \theta_2 \in [0, 2\pi)$. Consider the unitary matrix $V = \begin{bmatrix} 1 & 0 \\ 0 & e^{\mathrm{i}\frac{\theta_2-\theta_1}{2}} \end{bmatrix}$ for which

$$V^*U^*\left(T - \frac{\mathrm{tr}(T)}{2}I_2\right)UV = e^{\mathrm{i}\frac{\theta_1+\theta_2}{2}}\begin{bmatrix} 0 & |c| \\ |d| & 0 \end{bmatrix}.$$

Due to the unitary invariance property (P_2) and the translation property (P_3) of the envelope, it follows that

$$\mathcal{E}(A) = e^{\mathrm{i}\frac{\theta_1+\theta_2}{2}}\mathcal{E}\left(\begin{bmatrix} 0 & |c| \\ |d| & 0 \end{bmatrix}\right) + \frac{\mathrm{tr}(T)}{2}.$$

If $cd = 0$, then we get one of the trivial cases discussed above. Thus, it suffices to describe the envelope of the matrix $B = \begin{bmatrix} 0 & |c| \\ |d| & 0 \end{bmatrix}$, with $cd \neq 0$. Notice that $\pm\sqrt{|cd|}$ are the eigenvalues of B. Moreover, using for brevity the notation $\delta_{j,\theta} = \delta_j(e^{\mathrm{i}\theta}B)$ $(j = 1, 2)$, $\mathrm{u}_\theta = \mathrm{u}(e^{\mathrm{i}\theta}B)$ and $\mathrm{v}_\theta = \mathrm{v}(e^{\mathrm{i}\theta}B)$, we have

$$\mathcal{E}(B) = \bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\theta}\Gamma_{in}(e^{\mathrm{i}\theta}B), \tag{4.1}$$

where

$$\Gamma_{in}(e^{\mathrm{i}\theta}B) = \left\{s + \mathrm{i}\,t \in \mathbb{C} :\ s, t \in \mathbb{R},\ (\delta_{2,\theta} - s)[(\delta_{1,\theta} - s)^2 + (\mathrm{u}_\theta - t)^2] + (\delta_{1,\theta} - s)(\mathrm{v}_\theta - \mathrm{u}_\theta^2) \geq 0\right\}.$$

Next, observe that

$$H(e^{\mathrm{i}\theta}B) = \begin{bmatrix} 0 & \frac{e^{\mathrm{i}\theta}|c|+e^{-\mathrm{i}\theta}|d|}{2} \\ \frac{e^{-\mathrm{i}\theta}|c|+e^{\mathrm{i}\theta}|d|}{2} & 0 \end{bmatrix},\quad S(e^{\mathrm{i}\theta}B) = \begin{bmatrix} 0 & \frac{e^{\mathrm{i}\theta}|c|-e^{-\mathrm{i}\theta}|d|}{2} \\ \frac{-e^{-\mathrm{i}\theta}|c|+e^{\mathrm{i}\theta}|d|}{2} & 0 \end{bmatrix}$$

$$\text{and}\quad \delta_{1,\theta} = -\delta_{2,\theta} = \frac{\left|e^{\mathrm{i}\theta}|c| + e^{-\mathrm{i}\theta}|d|\right|}{2} = \frac{\sqrt{|c|^2 + |d|^2 + 2|cd|\cos(2\theta)}}{2}, \tag{4.2}$$

as well as that a unit eigenvector of $H(e^{\mathrm{i}\theta}B)$ corresponding to the eigenvalue $\delta_{1,\theta}$ is

$$y_{1,\theta} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ \frac{e^{-\mathrm{i}\theta}|c|+e^{\mathrm{i}\theta}|d|}{2\delta_{1,\theta}} \end{bmatrix}.$$

Hence,

$$\mathrm{v}_\theta = \left\|S(e^{\mathrm{i}\theta}B)y_{1,\theta}\right\|_2^2 = \frac{1}{4}\left|e^{\mathrm{i}\theta}|c| - e^{-\mathrm{i}\theta}|d|\right|^2 = \frac{|c|^2 + |d|^2 - 2|cd|\cos(2\theta)}{4} \tag{4.3}$$

and

$$\begin{aligned} \mathrm{i}\,\mathrm{u}_\theta &= y_{1,\theta}^* S(e^{\mathrm{i}\theta}B)y_{1,\theta} = \frac{1}{2}\begin{bmatrix} 1 & \frac{e^{\mathrm{i}\theta}|c|+e^{-\mathrm{i}\theta}|d|}{2\delta_{1,\theta}} \end{bmatrix}\begin{bmatrix} 0 & \frac{e^{\mathrm{i}\theta}|c|-e^{-\mathrm{i}\theta}|d|}{2} \\ \frac{-e^{-\mathrm{i}\theta}|c|+e^{\mathrm{i}\theta}|d|}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \frac{e^{-\mathrm{i}\theta}|c|+e^{\mathrm{i}\theta}|d|}{2\delta_{1,\theta}} \end{bmatrix} \\ &= \frac{1}{8\delta_{1,\theta}}[(e^{\mathrm{i}\theta}|c| - e^{-\mathrm{i}\theta}|d|)(e^{-\mathrm{i}\theta}|c| + e^{\mathrm{i}\theta}|d|) - (e^{-\mathrm{i}\theta}|c| - e^{\mathrm{i}\theta}|d|)(e^{\mathrm{i}\theta}|c| + e^{-\mathrm{i}\theta}|d|)] \\ &= \mathrm{i}\,\frac{|cd|\sin(2\theta)}{2\delta_{1,\theta}}. \end{aligned} \tag{4.4}$$

It is apparent from (4.1) that

$$\mathcal{E}(B) \subseteq \Gamma_{in}(B) \cap e^{-\frac{i\pi}{2}}\Gamma_{in}(e^{\frac{i\pi}{2}}B) \cap e^{\frac{i\pi}{2}}\Gamma_{in}(e^{-\frac{i\pi}{2}}B).$$

To prove that the above intersection coincides with the spectrum of A, we need to calculate the quantities (4.2), (4.3) and (4.4) for the angles $\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$ and $\theta_3 = -\frac{\pi}{2}$, that is,

$$\delta_{1,0} = \frac{|c|+|d|}{2}, \quad \mathrm{v}_0 = \left(\frac{|c|-|d|}{2}\right)^2, \quad \mathrm{u}_0 = 0,$$

and

$$\delta_{1,\frac{\pi}{2}} = \delta_{1,-\frac{\pi}{2}} = \frac{||c|-|d||}{2}, \quad \mathrm{v}_{\frac{\pi}{2}} = \mathrm{v}_{-\frac{\pi}{2}} = \left(\frac{|c|+|d|}{2}\right)^2, \quad \mathrm{u}_{\frac{\pi}{2}} = \mathrm{u}_{-\frac{\pi}{2}} = 0.$$

For $\theta_1 = 0$, the discriminant of the cubic curve $\Gamma(B)$ is $\Delta_0 = (\delta_{1,0}-\delta_{2,0})^2 - 4(\mathrm{v}_0 - \mathrm{u}_0^2) = 4|cd| > 0$. From the case (a) described in the appendix, it follows that the region

$$\Gamma_{in}(B) = \left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s,t \in \mathbb{R}, \ (s+\delta_{1,0})t^2 \le (\delta_{1,0} - s)(s^2 - |cd|)\right\}$$

comprises two branches; a closed bounded branch lying in the vertical zone

$$\left\{s + \mathrm{i}\,t \in \mathbb{C} : \ \sqrt{|cd|} \le s \le \delta_{1,0}, \ t \in \mathbb{R}\right\}$$

and an unbounded branch lying in the closed half-plane $\left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s \le -\sqrt{|cd|}, \ t \in \mathbb{R}\right\}$.

For $\theta_2 = \frac{\pi}{2}$ and $\theta_3 = -\frac{\pi}{2}$, the cubic curves $\Gamma(e^{\frac{i\pi}{2}}B)$ and $\Gamma(e^{-\frac{i\pi}{2}}B)$ are identical, and their common discriminant is $\Delta_{\frac{\pi}{2}} = \Delta_{-\frac{\pi}{2}} = \left(\delta_{1,\frac{\pi}{2}} - \delta_{2,\frac{\pi}{2}}\right)^2 - 4\left(\mathrm{v}_{\frac{\pi}{2}} - \mathrm{u}_{\frac{\pi}{2}}^2\right) = -4|cd| < 0$. The case (e) in the appendix reveals that $\Gamma_{in}(e^{\frac{i\pi}{2}}B) = \Gamma_{in}(e^{-\frac{i\pi}{2}}B)$ is an unbounded region lying in the closed half-plane $\left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s \le \delta_{1,\frac{\pi}{2}}, \ t \in \mathbb{R}\right\}$. As a consequence, we have the rotations

$$\begin{aligned} e^{-\frac{i\pi}{2}}\Gamma_{in}(e^{\frac{i\pi}{2}}B) &= e^{-\frac{i\pi}{2}}\left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s,t \in \mathbb{R}, \ \left(s + \delta_{1,\frac{\pi}{2}}\right)t^2 \le \left(\delta_{1,\frac{\pi}{2}} - s\right)(s^2 + |cd|)\right\} \\ &= \left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s,t \in \mathbb{R}, \ \left(\delta_{1,\frac{\pi}{2}} - t\right)s^2 \le \left(\delta_{1,\frac{\pi}{2}} + t\right)(t^2 + |cd|)\right\} \\ &\subseteq \left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s \in \mathbb{R}, \ t \ge -\delta_{1,\frac{\pi}{2}}\right\} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} e^{\frac{i\pi}{2}}\Gamma_{in}(e^{-\frac{i\pi}{2}}B) &= e^{\frac{i\pi}{2}}\left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s,t \in \mathbb{R}, \ \left(s + \delta_{1,\frac{\pi}{2}}\right)t^2 \le \left(\delta_{1,\frac{\pi}{2}} - s\right)(s^2 + |cd|)\right\} \\ &= \left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s,t \in \mathbb{R}, \ \left(\delta_{1,\frac{\pi}{2}} + t\right)s^2 \le \left(\delta_{1,\frac{\pi}{2}} - t\right)(t^2 + |cd|)\right\} \\ &\subseteq \left\{s + \mathrm{i}\,t \in \mathbb{C} : \ s \in \mathbb{R}, \ t \le \delta_{1,\frac{\pi}{2}}\right\}. \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), it is now clear that both regions $e^{-\frac{i\pi}{2}}\Gamma_{in}(e^{\frac{i\pi}{2}}B)$ and $e^{\frac{i\pi}{2}}\Gamma_{in}(e^{-\frac{i\pi}{2}}B)$ are symmetric with respect to the imaginary axis. Moreover, $e^{-\frac{i\pi}{2}}\Gamma_{in}(e^{\frac{i\pi}{2}}B)$ is a reflection of $e^{\frac{i\pi}{2}}\Gamma_{in}(e^{-\frac{i\pi}{2}}B)$ with about the real axis and the origin.

It is straightforward to identify the points at which the curves $e^{-\frac{i\pi}{2}}\Gamma(e^{\frac{i\pi}{2}}B)$ and $e^{\frac{i\pi}{2}}\Gamma(e^{-\frac{i\pi}{2}}B)$ meet. Indeed, the equations

$$(\delta_{1,\frac{\pi}{2}} - t)s^2 = (\delta_{1,\frac{\pi}{2}} + t)(t^2 + |cd|) \quad \text{and} \quad (\delta_{1,\frac{\pi}{2}} + t)s^2 = (\delta_{1,\frac{\pi}{2}} - t)(t^2 + |cd|) \tag{4.7}$$

yield readily that

$$s^2 = t^2 + |cd|. \tag{4.8}$$

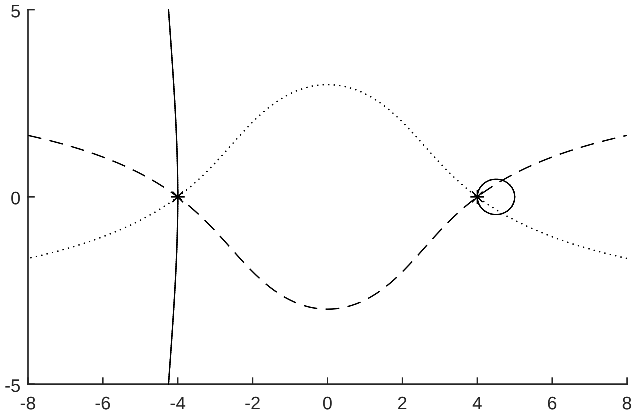


Figure 3: The curves $\Gamma(B)$ (solid curve), $e^{-\frac{\mathrm{i}\pi}{2}}\Gamma(e^{\frac{\mathrm{i}\pi}{2}}B)$ (dashed curve) and $e^{\frac{\mathrm{i}\pi}{2}}\Gamma(e^{-\frac{\mathrm{i}\pi}{2}}B)$ (dotted curve) intersect at the eigenvalues of B.

Substituting (4.8) into any of the two equations in (4.7) implies that $t = 0$. As a consequence, the curves $e^{-\frac{\mathrm{i}\pi}{2}}\Gamma(e^{\frac{\mathrm{i}\pi}{2}}B)$ and $e^{\frac{\mathrm{i}\pi}{2}}\Gamma(e^{-\frac{\mathrm{i}\pi}{2}}B)$ intersect at $\pm\sqrt{|cd|}$, and thus, the intersection $e^{-\frac{\mathrm{i}\pi}{2}}\Gamma_{in}(e^{\frac{\mathrm{i}\pi}{2}}B) \cap e^{\frac{\mathrm{i}\pi}{2}}\Gamma_{in}(e^{-\frac{\mathrm{i}\pi}{2}}B)$ lies in the vertical zone $\left\{s + \mathrm{i}\,t \in \mathbb{C} : -\sqrt{|cd|} \le s \le \sqrt{|cd|},\, t \in \mathbb{R}\right\}$. Hence,

$$\Gamma_{in}(B) \cap e^{-\frac{\mathrm{i}\pi}{2}}\Gamma_{in}(e^{\frac{\mathrm{i}\pi}{2}}B) \cap e^{\frac{\mathrm{i}\pi}{2}}\Gamma_{in}(e^{-\frac{\mathrm{i}\pi}{2}}B) = \{\pm\sqrt{|cd|}\} = \sigma(B),$$

and the proof is complete. □

Our last example illustrates the observations in the above result and the concepts in its proof.

Example 4.2. Consider the 2×2 matrix $B = \begin{bmatrix} 0 & 2 \\ 8 & 0 \end{bmatrix}$, with spectrum $\sigma(B) = \{-4, 4\}$. The curves $\Gamma(B)$, $e^{-\frac{\mathrm{i}\pi}{2}}\Gamma(e^{\frac{\mathrm{i}\pi}{2}}B)$ and $e^{\frac{\mathrm{i}\pi}{2}}\Gamma(e^{-\frac{\mathrm{i}\pi}{2}}B)$ are illustrated in Figure 3 by the solid, dashed and dotted curves, respectively. As one may observe, they all meet at only two points, the eigenvalues of B, which are marked by $*$'s.

Appendix A An Alternative Analysis of $\Gamma(A)$

The cubic curve $\Gamma(A)$ defined in (1.1) is introduced and studied in [1] and subsequently led to the consideration of the envelope in [10, 11]. In this appendix, we present an alternative analysis and classification of $\Gamma(A)$, used in the main part of the paper.

By definition, $\Gamma(A)$ is the locus of the points $z = s + \mathrm{i}\,t$, with coordinates $s \in [\delta_2(A), \delta_1(A)]$

and $t \in \mathbb{R}$, such that $f_A(s,t) = 0$, where

$$f_A(s,t) = [(\delta_1(A) - s)^2 + (\mathrm{u}(A) - t)^2](\delta_2(A) - s) + (\delta_1(A) - s)(\mathrm{v}(A) - \mathrm{u}^2(A)) \qquad \text{(A.1)}$$

is a real polynomial in two variables of total degree 3. Changing variables $s \mapsto x + \delta_2(A)$ and $t \mapsto y + \mathrm{u}(A)$ in (A.1), converts $f_A(s,t) = 0$ into a more amenable equation. In particular, consider

$$\begin{aligned} F_A(x,y) \;\; = \;\; & -x^3 + 2(\delta_1(A) - \delta_2(A))x^2 - [(\delta_1(A) - \delta_2(A))^2 + \mathrm{v}(A) - \mathrm{u}^2(A)]x \\ & + (\delta_1(A) - \delta_2(A))(\mathrm{v}(A) - \mathrm{u}^2(A)) - xy^2, \end{aligned}$$

and let us denote $\alpha(A) = \delta_1(A) - \delta_2(A) \geq 0$ and $\beta(A) = \mathrm{v}(A) - \mathrm{u}^2(A) \geq 0$. Then (A.1) is transformed into its canonical form with respect to the new coordinates $x \in [0, \alpha(A)]$ and $y \in \mathbb{R}$, that is,

$$F_A(x,y) = 0,$$

or equivalently,

$$xy^2 = -x^3 + 2\alpha(A)x^2 - [\alpha^2(A) + \beta(A)]x + \alpha(A)\beta(A),$$

or equivalently,

$$xy^2 = -(x - \alpha(A))(x^2 - \alpha(A)x + \beta(A)). \qquad \text{(A.2)}$$

According to Newton's classification of cubic curves [2], (A.2) belongs to the class of *defective hyperbolas*. The curve has only one real asymptote, the vertical axis $x = 0$ (or $s = \delta_2(A)$) and it is symmetric with respect to the horizontal axis $y = 0$ (or $t = \mathrm{u}(A)$). The cubic polynomial $P(x) = -(x - \alpha(A))(x^2 - \alpha(A)x + \beta(A))$ has at most three real nonnegative roots, the nature of which classifies (A.2) into five different categories. Specifically, we consider the discriminant $\Delta = \alpha(A)^2 - 4\beta(A)$ of the quadratic factor of $P(x)$, and we distinguish the following cases:

(a) Suppose that $P(x)$ has three distinct positive roots

$$x_1 = \alpha(A), \quad x_2 = \frac{\alpha(A) + \sqrt{\Delta}}{2} \quad \text{and} \quad x_3 = \frac{\alpha(A) - \sqrt{\Delta}}{2}.$$

In this case, $\Delta > 0$, $\delta_1(A) > \delta_2(A)$ and $\mathrm{v}(A) > \mathrm{u}^2(A)$, and $\Gamma(A)$ is a conchoidal hyperbola with an oval at its convexity. The oval forms a bounded region lying in the zone oriented by the vertical lines determined by the roots $x_1 > x_2$, while the hyperbola forms an unbounded region lying in the left half-plane determined by the root x_3; see Figure 4(a).

(b) Suppose that $P(x)$ has two equal positive roots

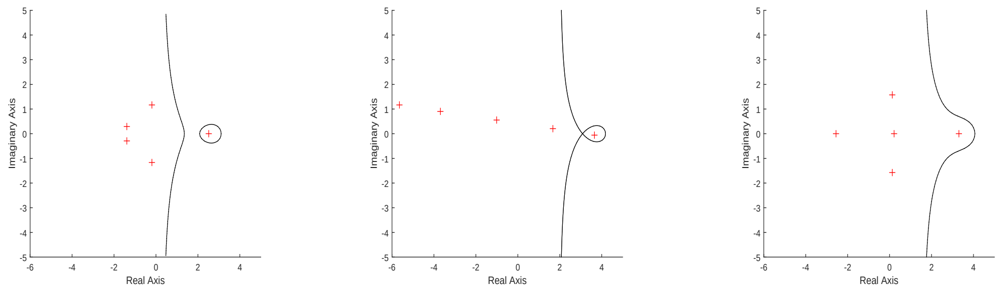
$$x_1 = \alpha(A) > \frac{\alpha(A)}{2} = x_2 = x_3.$$

In this case, $\Delta = 0$, $\delta_1(A) > \delta_2(A)$ and $\mathrm{v}(A) > \mathrm{u}^2(A)$, and $\Gamma(A)$ is a curve where the conchoidal hyperbola and the oval coalesce (folium of Descartes), intersecting each other at the node $(x_2, y(x_2)) = \left(\frac{\alpha(A)}{2}, 0\right)$; see Figure 4(b).

(c) Suppose that $P(x)$ has only the zero root

$$x_1 = x_2 = x_3 = 0.$$

In this case, $\Delta = 0$, $\delta_1(A) = \delta_2(A)$ and $\mathrm{v}(A) = \mathrm{u}^2(A)$, and $\Gamma(A)$ coincides with the vertical axis $x = 0$.



(a) (b) (e)

Figure 4: Different types of cubic curves (A.2), corresponding to the cases (a) $\Delta > 0$, (b) $\Delta = 0$ and (e) $\Delta < 0$.

(d) Suppose that $P(x)$ has two equal positive roots

$$x_1 = x_2 = \alpha(A) > \frac{\alpha(A)}{2} = x_3.$$

In this case, $\Delta > 0$, $\delta_1(A) > \delta_2(A)$ and $\mathrm{v}(A) = \mathrm{u}^2(A)$, and $\Gamma(A)$ degenerates to the vertical axis $x = 0$ with an isolated point-acnode $(x_1, y(x_1)) = (\alpha(A), 0)$.

(e) Suppose that $P(x)$ has only one real root

$$x_1 = \alpha(A).$$

In this case, $\Delta < 0$, $\delta_1(A) \geq \delta_2(A)$ and $\mathrm{v}(A) > \mathrm{u}^2(A)$, and $\Gamma(A)$ is a pure conchoidal curve (degenerating to the line $x = 0$ whether $\delta_1(A) = \delta_2(A)$) with no oval, node or isolated point; see Figure 4(e).

The aforementioned description verifies that $\Gamma(A)$ is a nonsingular curve in cases (a), (c) and (e). An essential attribute of a nonsingular cubic curve is the measure of how much it deviates from a straight line, namely, its curvature. Affine transformations preserve the curvature, and therefore, we shall use the curvature formula for $F_A(x, y) = 0$ in (A.2). The symmetry of the curve with respect to the horizontal axis $y = 0$ permits us to restrict to the positive quadrant and specialize to the curve

$$y = y(x) = \sqrt{\left(\frac{\alpha(A)}{x} - 1\right)(x^2 - \alpha(A)x + \beta(A))} > 0, \tag{A.3}$$

at which we apply the curvature formula

$$\kappa(x) = \frac{|y''(x)|}{((y'(x))^2 + 1)^{3/2}}.$$

Now we want to find out how large $\kappa(x)$ can get. Our search for maxima starts studying the critical points of $\kappa(x)$, which occur at points $x \in [0, \alpha(A)]$ where the first derivative

$$\kappa'(x) = \frac{y'''(x)(1 + (y'(x))^2) - 3y'(x)(y''(x))^2}{((y'(x))^2 + 1)^{5/2}}$$

vanishes. Thus,

$$y'''(x)(1+(y'(x))^2) - 3y'(x)(y''(x))^2 = 0. \tag{A.4}$$

Using ordinary differential calculus in (A.3), we calculate (for $x, y \neq 0$)

$$y'(x) = \frac{-2x^3 + 2\alpha(A)x^2 - \alpha(A)\beta(A)}{2x^2 y},$$

$$y''(x) = \frac{\beta(A)\left[4x^4 - 12\alpha(A)x^3 + 12\alpha^2(A)x^2 - 4\alpha(A)(\alpha^2(A)+\beta(A))x + 3\alpha^2(A)\beta(A)\right]}{4x^4y^3},$$

$$\begin{aligned} y'''(x) &= 3\alpha(A)\beta^2(A) \\ &\quad \frac{-20x^4 + 50\alpha(A)x^3 - (42\alpha^2(A) + 8\beta(A))x^2 + 12\alpha(A)(\alpha^2(A)+\beta(A))x - 5\alpha^2(A)\beta(A)}{8x^6y^5} \\ &\quad + 3\beta(A)\frac{(x-\alpha(A))^5}{x^4y^5}. \end{aligned}$$

If we substitute these derivatives into (A.4), we derive

$$(-2x + \alpha(A))(4x^4 - 8\alpha(A)x^3 + 6\alpha^2(A)x^2 - 2\alpha^3(A)x + \alpha^2(A)\beta(A)) = 0.$$

So the critical points of $\kappa(x)$ are $x = \frac{\alpha(A)}{2}$, or the real roots of the quartic polynomial

$$Q(x) = 4x^4 - 8\alpha(A)x^3 + 6\alpha^2(A)x^2 - 2\alpha^3(A)x + \alpha^2(A)\beta(A).$$

Using Sturm's theorem, we can count the number of distinct real roots of $Q(x)$ in $[0, \alpha(A)]$ in terms of the number of variations in sign of the values of the Sturm's sequence at the endpoints of the interval. Hence, we firstly compute the Sturm sequence for $Q(x)$:

$$\begin{aligned} Q_0(x) &= 4x^4 - 8\alpha(A)x^3 + 6\alpha^2(A)x^2 - 2\alpha^3(A)x + \alpha^2(A)\beta(A), \\ Q_1(x) &= 16x^3 - 24\alpha(A)x^2 + 12\alpha^2(A)x - 2\alpha^3(A), \\ Q_2(x) &= \frac{\alpha^4(A)}{4} - \alpha^2(A)\beta(A) = \frac{\alpha^2(A)\Delta}{4}. \end{aligned}$$

Then we evaluate $\{Q_0(x), Q_1(x), Q_2(x)\}$ at $x = 0$ and $x = \alpha(A)$, and we obtain $\mathcal{S}(0) = \left\{\alpha^2(A)\beta(A), -\alpha^3(A), \frac{\alpha^2(A)\Delta}{4}\right\}$ and $\mathcal{S}(\alpha(A)) = \left\{\alpha^2(A)\beta(A), \alpha^3(A), \frac{\alpha^2(A)\Delta}{4}\right\}$, respectively. The curve $y = y(x)$ in (A.3) is nonsingular whenever $\Delta > 0$ or $\Delta < 0$. Therefore, we have:

1. If $\Delta > 0$, we can see by case (a) that (A.3) is the upper half of a conchoidal hyperbola and an oval with $x \in \left(0, \frac{\alpha(A)-\sqrt{\Delta}}{2}\right] \cup \left[\frac{\alpha(A)+\sqrt{\Delta}}{2}, \alpha(A)\right]$. Evidently, the point at which the curve attains its maximum curvature lies on the oval and occurs at one of the two distinct real roots of the polynomial $Q(x)$. This is due to the fact that $\mathcal{S}(0) = \{+, -, +\}$ contains 2 sign changes whereas $\mathcal{S}(\alpha(A)) = \{+, +, +\}$ has no sign change. Also, the value $x = \frac{\alpha(A)}{2}$ does not verify $y = y(x)$ in (A.3).

   By the fundamental theorem of algebra and the fact that the non-real roots of any polynomial equation come in complex conjugate pairs, we expect exactly two real roots $0 < x_1 < x_2 \leq \alpha(A)$ of $Q(x) = 0$.

2. If $\Delta < 0$, then (A.3) is the upper half of a pure conchoidal curve with no singularities in the interval $[0, \alpha(A)]$ (see case (e)). Apparently, $Q(x)$ has no real roots, since $\mathcal{S}(0) = \{+, -, -\}$ contains 1 sign change, as well as $\mathcal{S}(\alpha(A)) = \{+, +, -\}$. This implies that the maximum curvature of the curve occurs at the point $(x, y) = \left(\frac{\alpha(A)}{2}, \frac{\sqrt{-\Delta}}{2}\right)$.

Likewise, we also consider the symmetric points with respect to the axis $y = 0$ and sum up the aforementioned approach in the next proposition.

Proposition A.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Suppose that the cubic curve $\Gamma(A)$ defined in (1.1) is nonsingular and $\delta_1(A) > \delta_2(A)$. If $\Delta < 0$, then the maximum curvature of $\Gamma(A)$ occurs at the points $\frac{\delta_1(A)+\delta_2(A)}{2} + \mathrm{i}\left(\mathrm{u}(A) \pm \frac{\sqrt{-\Delta}}{2}\right)$. If $\Delta > 0$, then the maximum curvature of $\Gamma(A)$ occurs at the points $z \in \Gamma(A)$ such that $\operatorname{Re} z$ is the largest real root of the polynomial*

$$\begin{aligned} Q(s) &= (s-\delta_1(A))(s-\delta_2(A))[(2s-\delta_1(A)-\delta_2(A))^2+(\delta_1(A)-\delta_2(A))^2] \\ &\quad + \frac{1}{4}(\delta_1(A)-\delta_2(A))^2[(\delta_1(A)-\delta_2(A))^2-\Delta]. \end{aligned}$$

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