Birkhoff-James $\varepsilon$-orthogonality sets of vectors and vector-valued polynomials

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Abstract

Consider a complex normed linear space $(X, \| \cdot \|)$, and let $\chi, \psi \in X$ with $\psi \neq 0$. Motivated by recent works on rectangular matrices and on normed linear spaces, we study the Birkhoff-James $\varepsilon$-orthogonality set of $\chi$ with respect to $\psi$, give an alternative definition for this set, and explore its rich structure. We also introduce the Birkhoff-James $\varepsilon$-orthogonality set of polynomials in one complex variable whose coefficients are members of $X$, and survey and record extensions of results on matrix polynomials to these vector-valued polynomials.

Key words: norm, vector-valued polynomial, Birkhoff-James orthogonality, Birkhoff-James $\varepsilon$-orthogonality, numerical range.

AMS Subject Classifications: 46B99, 47A12.

1 Introduction

Let $(A, \| \cdot \|)$ (for simplicity, $A$) be a unital normed algebra over $\mathbb{C}$, and let $A^*$ be the dual space of $A$, i.e., the Banach space of all continuous linear functionals of $A$ (using the induced operator norm). The numerical range (also known as the field of values) of an element $\alpha \in A$ is defined as

$$F(\alpha) = \{ f(\alpha) : f \in A^*, f(1) = 1, \| f \| = 1 \}.$$  (1)

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This set has been studied extensively, and is useful in understanding matrices and operators; see [3, 4, 15, 30] and the references therein. Stampfli and Williams [30, Theorem 4], and later Bonsall and Duncan [4, Lemma 6.22.1], observed that the numerical range $F(\alpha)$ can be written in the form

$$F(\alpha) = \{ \mu \in \mathbb{C} : \|\alpha - \lambda \mathbf{1}\| \geq |\mu - \lambda|, \forall \lambda \in \mathbb{C} \}.$$ 

This means that $F(\alpha)$ is an infinite intersection of closed (circular) disks

$$D(\lambda, \|\alpha - \lambda \mathbf{1}\|) = \{ \mu \in \mathbb{C} : \|\mu - \lambda\| \leq \|\alpha - \lambda \mathbf{1}\|, \lambda \in \mathbb{C} \},$$

namely,

$$F(\alpha) = \bigcap_{\lambda \in \mathbb{C}} \{ \mu \in \mathbb{C} : \|\mu - \lambda\| \leq \|\alpha - \lambda \mathbf{1}\| \} = \bigcap_{\lambda \in \mathbb{C}} D(\lambda, \|\alpha - \lambda \mathbf{1}\|). \quad (2)$$

For two elements $\chi$ and $\psi$ of a complex normed linear space $(X, \|\cdot\|)$, $\chi$ is said to be Birkhoff-James orthogonal to $\psi$, denoted by $\chi \perp_{BJ} \psi$, if $\|\chi + \lambda \psi\| \geq \|\chi\|$ for all $\lambda \in \mathbb{C}$ [2, 18]. This orthogonality is homogeneous, but it is neither symmetric nor additive [18]. Moreover, for any $\varepsilon \in [0,1)$, $\chi$ is called Birkhoff-James $\varepsilon$-orthogonal to $\psi$, denoted by $\chi \perp_{J} \psi$, if $\|\chi + \lambda \psi\| \geq \sqrt{1 - \varepsilon \|\chi\|}$ for all $\lambda \in \mathbb{C}$ [5, 8]. It is worth mentioning that this relation is also homogeneous. In an inner product space $(X, \langle \cdot, \cdot \rangle)$, with the standard orthogonality relation $\perp$, a $\chi \in X$ is called $\varepsilon$-orthogonal to a $\psi \in X$, denoted by $\chi \perp_{\varepsilon} \psi$, if $|\langle \chi, \psi \rangle| \leq \varepsilon \|\chi\| \|\psi\|$. Furthermore, for any $\varepsilon \in [0,1)$, $\chi \perp_{\varepsilon} \psi$ if and only if $\chi \perp_{BJ} \psi$ [5, 8].

Inspired by (2) and the above definition of Birkhoff-James $\varepsilon$-orthogonality, Chorianopoulos and Psarrakos [7] (for rectangular matrices), and Karamanlis and Psarrakos [20] (for elements of a normed linear space) introduced and studied the following set: For any $\chi, \psi \in X$, with $\psi \neq 0$, and any $\varepsilon \in [0,1)$, the Birkhoff-James $\varepsilon$-orthogonality set of $\chi$ with respect to $\psi$ is defined and denoted by

$$F_{\varepsilon, \|\cdot\|}(\chi; \psi) = \{ \mu \in \mathbb{C} : \psi \perp_{BJ} (\chi - \mu \psi) \}.$$ 

The Birkhoff-James $\varepsilon$-orthogonality set is a direct generalization of the numerical range, and appears to have a rich structure and interesting geometrical properties [7, 20]. In this paper, motivated by (1), we introduce a new (equivalent) definition for the Birkhoff-James $\varepsilon$-orthogonality set, using continuous linear functionals. Based on this definition, in the next section, we obtain some basic properties of the set $F_{\varepsilon, \|\cdot\|}(\chi; \psi)$ such as subadditivity in $\chi$. In Section 3, we introduce the Birkhoff-James $\varepsilon$-orthogonality set of vector-valued polynomials in one complex variable, and investigate its localization in the complex plane. In Sections 4, 5 and 6, we study the connected components of the Birkhoff-James $\varepsilon$-orthogonality set of vector-valued polynomials, the boundary of this set, and the local dimension of its points, respectively. The proof techniques are analogous to existing proofs [22, 24, 25, 27, 28], albeit modified and adapted to the new setting. The main contribution of this effort is a concise generalization to a new concept. Furthermore, the results indicate that the information on Birkhoff-James $\varepsilon$-orthogonality set is useful in understanding vector-valued polynomials.
2 Definition and basic properties

Consider a complex normed linear space \((\mathcal{X}, \|\cdot\|)\) (for simplicity, \(\mathcal{X}\)), and let \(\chi, \psi \in \mathcal{X}\) with \(\psi \neq 0\). For any \(\varepsilon \in [0, 1)\), it is straightforward to see that

\[
F^\varepsilon_{\|\cdot\|}(\chi; \psi) = \left\{ \mu \in \mathbb{C} : \psi \perp_{B.J} (\chi - \mu \psi) \right\}
= \left\{ \mu \in \mathbb{C} : \|\psi - \lambda(\chi - \mu \psi)\| \geq \sqrt{1 - \varepsilon^2} ||\psi||, \forall \lambda \in \mathbb{C} \right\}
= \left\{ \mu \in \mathbb{C} : \left\| \psi - \frac{1}{\lambda}(\chi - \mu \psi) \right\| \geq \sqrt{1 - \varepsilon^2} ||\psi||, \forall \lambda \in \mathbb{C} \setminus \{0\} \right\}
= \left\{ \mu \in \mathbb{C} : \lambda \|\chi - (\mu - \lambda)\psi\| \geq \sqrt{1 - \varepsilon^2} ||\psi|| \|\mu - \lambda\|, \forall \lambda \in \mathbb{C} \right\}
= \left\{ \mu \in \mathbb{C} : ||\chi - (\mu - \lambda)\psi\| \geq \sqrt{1 - \varepsilon^2} ||\psi|| |\mu - \lambda|, \forall \lambda \in \mathbb{C} \right\}
= \bigcap_{\lambda \in \mathbb{C}} D \left( \lambda, \frac{\|\chi - \lambda \psi\|}{\sqrt{1 - \varepsilon^2} ||\psi||} \right).
\]

Corollary 2.2 of [18] implies that \(F^\varepsilon_{\|\cdot\|}(\chi; \psi)\) is always non-empty (see also Proposition 2.1 of [20]), and the defining formula (5) ensures that \(F^\varepsilon_{\|\cdot\|}(\chi; \psi)\) is a compact and convex subset of \(\mathbb{C}\) that lies in the closed disk \(D \left( \frac{\|\chi\|}{\sqrt{1 - \varepsilon^2} ||\psi||} \right)\). Moreover, it is apparent that for any \(0 \leq \varepsilon_1 < \varepsilon_2 < 1\), \(F^\varepsilon_{\|\cdot\|}(\chi; \psi) \subseteq F^\varepsilon_0(\chi; \psi)\). The Birkhoff-James \(\varepsilon\)-orthogonality set is a direct generalization of the standard numerical range. In particular, for \(\mathcal{X} = \mathcal{A}\), \(\chi = \alpha\), \(\psi = 1\) and \(\varepsilon = 0\), we have \(F^0_{\|\cdot\|}(\alpha; 1) = F(\alpha)\); see (2) and (5).

Remark 2.1. Let \(\chi, \psi \in \mathcal{X}\) be nonzero, with \(\psi\) not a scalar multiple of \(\chi\), and consider the distance from \(\psi\) to \(\text{span}\{\chi\}\), \(\text{dist}(\psi, \text{span}\{\chi\}) = \inf_{\lambda \in \mathbb{C}} \|\psi - \lambda \chi\|\). Then, for any \(\varepsilon \in [0, 1)\), it follows

\[
0 \in F^\varepsilon_{\|\cdot\|}(\chi; \psi) \iff \psi \perp_{B.J} \chi \iff \|\psi - \lambda \chi\| \geq \sqrt{1 - \varepsilon^2} ||\psi||, \forall \lambda \in \mathbb{C} \\
\iff \inf_{\lambda \in \mathbb{C}} \|\psi - \lambda \chi\| \geq \sqrt{1 - \varepsilon^2} ||\psi|| \quad (\psi \notin \text{span}\{\chi\}) \\
\iff \text{dist}(\psi, \text{span}\{\chi\}) \geq \sqrt{1 - \varepsilon^2} ||\psi||.
\]

Clearly, for \(\varepsilon = 0\), \(0 \notin F^0_{\|\cdot\|}(\chi; \psi)\) if and only if \(\text{dist}(\psi, \text{span}\{\chi\}) = ||\psi||\). Moreover, if \(0 \notin F^0_{\|\cdot\|}(\chi; \psi)\) (or equivalently, if \(\text{dist}(\psi, \text{span}\{\chi\}) < ||\psi||\)), then by Theorems 3.1 and 3.5 of [20] (see also Properties (P6) and (P8) below), there is a unique number \(\varepsilon_0 \in [0, 1)\) such that the origin lies on the boundary \(\partial F^{\varepsilon_0}_{\|\cdot\|}(\chi; \psi)\) and \(\text{dist}(\psi, \text{span}\{\chi\}) = \sqrt{1 - \varepsilon_0^2} ||\psi||\). This number \(\varepsilon_0\) is the smallest value of the parameter \(\varepsilon \in [0, 1)\) with \(0 \in F^\varepsilon_{\|\cdot\|}(\chi; \psi)\).

We remark that in the remainder of the paper, the zero vector is always considered as a scalar multiple of \(\psi\).

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Let $\chi, \psi \in X$ with $\psi \neq 0$. Next, for convenience, we summarize the results of [20] (see also [6, 7] for rectangular matrices), describing basic properties of the Birkhoff-James $\varepsilon$-orthogonality set.

(P1) For any $a, b \in \mathbb{C}$ and any $\varepsilon \in [0, 1)$, $F\varepsilon_{||\cdot||}(a \chi + b \psi; \psi) = a F\varepsilon_{||\cdot||}(\chi; \psi) + b$.

(P2) For any nonzero $b \in \mathbb{C}$ and any $\varepsilon \in [0, 1)$, $F\varepsilon_{||\cdot||}(\chi; b \psi) = \frac{1}{b} F\varepsilon_{||\cdot||}(\chi; \psi)$.

(P3) If $\chi$ is a nonzero element of $\mathcal{X}$, then for any $\varepsilon \in [0, 1)$,

$$\left\{ \mu^{-1} \in \mathbb{C} : \mu \in F\varepsilon_{||\cdot||}(\chi; \psi), |\mu| \geq \frac{||\chi||_{||\psi||}}{||\psi||} \right\} \subset F\varepsilon_{||\cdot||}(\psi; \chi).$$

(P4) Let $||\cdot||_\alpha$ and $||\cdot||_\beta$ be two equivalent norms acting in $X$, i.e., there exist two real numbers $C, c > 0$ such that $c ||\xi||_\alpha \leq ||\xi||_\beta \leq C ||\xi||_\alpha$ for all $\xi \in X$. Then for any $\varepsilon \in [0, 1)$, it holds that $F\varepsilon_{||\cdot||_\alpha}(\chi; \psi) \subset F\varepsilon_{||\cdot||_\beta}(\chi; \psi)$, where

$$\varepsilon' = \sqrt{1 - \frac{e^2(1 - \varepsilon^2)}{e^2}}.$$

(P5) $\chi = a \psi$ for some $a \in \mathbb{C}$ if and only if $F\varepsilon_{||\cdot||}(\chi; \psi) = \{a\}$ for every $\varepsilon \in [0, 1)$.

(P6) If $\chi$ is not a scalar multiple of $\psi$, then for any $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, $F\varepsilon_{||\cdot||}(\chi; \psi)$ lies in the interior of $F\varepsilon_{||\cdot||}(\chi; \psi)$.

(P7) If $\chi$ is not a scalar multiple of $\psi$, then for any $\varepsilon \in (0, 1)$, $F\varepsilon_{||\cdot||}(\chi; \psi)$ has a non-empty interior.

(P8) If $\chi$ is not a scalar multiple of $\psi$, then for any bounded region $\Omega \subset \mathbb{C}$, there is an $\varepsilon_\Omega \in [0, 1)$ such that $\Omega \subset F\varepsilon_{||\cdot||}(\chi; \psi)$. (This means that if $\chi$ is not a scalar multiple of $\psi$, then $F\varepsilon_{||\cdot||}(\chi; \psi)$ can be arbitrarily large for $\varepsilon$ sufficiently close to 1.)

(P9) Let $\mu_0 \in F\varepsilon_{||\cdot||}(\chi; \psi)$ for some $\varepsilon \in [0, 1)$.

(i) The scalar $\mu_0$ lies on the boundary $\partial F\varepsilon_{||\cdot||}(\chi; \psi)$ if and only if

$$\inf_{\lambda \in \mathbb{C}} \left\{ ||\chi - \lambda \psi|| - \sqrt{1 - \varepsilon^2} ||\psi|| |\mu_0 - \lambda| \right\} = 0.$$

(ii) If $\varepsilon > 0$, then $\mu_0 \in \partial F\varepsilon_{||\cdot||}(\chi; \psi)$ if and only if

$$\min_{\lambda \in \mathbb{C}} \left\{ ||\chi - \lambda \psi|| - \sqrt{1 - \varepsilon^2} ||\psi|| |\mu_0 - \lambda| \right\} = 0,$$

or equivalently, if and only if $||\chi - \lambda_0 \psi|| = \sqrt{1 - \varepsilon^2} ||\psi|| |\mu_0 - \lambda_0|$ for some $\lambda_0 \in \mathbb{C}$.

(P10) For any $\varepsilon \in (0, 1)$,

$$\text{Int} \left[ F\varepsilon_{||\cdot||}(\chi; \psi) \right] = \left\{ \mu \in \mathbb{C} : ||\chi - \lambda \psi|| > \sqrt{1 - \varepsilon^2} ||\psi|| |\mu - \lambda|, \forall \lambda \in \mathbb{C} \right\}.$$

(P11) If the norm $||\cdot||$ is induced by an inner product $\langle \cdot, \cdot \rangle$, then for any $\varepsilon \in [0, 1)$,

$$F\varepsilon_{||\cdot||}(\chi; \psi) = \mathcal{D} \left( \frac{\langle \chi, \psi \rangle}{||\psi||^2}, \frac{\varepsilon}{\sqrt{1 - \varepsilon^2} ||\psi||} \right).$$
Figure 1: The sets $F^{0.5}_{∥·∥_1}(χ; ψ)$ (left), $F^{0.65}_{∥·∥_1}(χ; ψ)$ (middle), and $F^{0.5}_{∥·∥_1}(χ − 3ψ; 2ψ)$ (right).

Example 2.1. Consider the $2 \times 4$ complex matrices $χ = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 + i & 0 & −1 & −11 i \end{bmatrix}$ and $ψ = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & −1 & −1 & 1 \end{bmatrix}$. The Birkhoff-James $\varepsilon$-orthogonality sets $F^{0.5}_{∥·∥_1}(χ; ψ)$, $F^{0.65}_{∥·∥_1}(χ; ψ)$ and $F^{0.5}_{∥·∥_1}(χ − 3ψ; 2ψ) = \frac{1}{2} F^{0.5}_{∥·∥_1}(χ; ψ) − \frac{3}{2}$ are estimated by the unshaded regions in the left, middle and right parts of Figure 2, respectively. Each estimation results from having drawn 1000 circles of the form $∂D \left( λ, \frac{∥χ − λψ∥}{\sqrt{1 − \varepsilon^2 ∥ψ∥}} \right)$; see the defining formula (5). The compactness and the convexity of the sets are apparent, and Properties $(P_1)$, $(P_2)$, $(P_6)$ and $(P_7)$ are verified.

Let $Χ^*$ denote the dual space of $Χ$, i.e., the complex normed linear space of all continuous linear functionals of $Χ$ (using the induced operator norm).

Definition 2.1. Let $χ, ψ ∈ X$ with $ψ ≠ 0$. For any $ε ∈ [0, 1)$, define the sets $L_ε(ψ) = \{ f ∈ X^*: f(ψ) = \sqrt{1 − \varepsilon^2 ∥ψ∥}$ and $∥f∥ ≤ 1 \}$ and $Ω_ε(χ; ψ) = \{ \frac{f(χ)}{\sqrt{1 − \varepsilon^2 ∥ψ∥}} : f ∈ L_ε(ψ) \}$.

Lemma 2.1. For any nonzero vector $ψ ∈ X$ and any $ε ∈ [0, 1)$, the set $L_ε(ψ)$ is non-empty, closed and convex.

Proof. Consider an element $χ ∈ X$ which is not a scalar multiple of $ψ$. From Corollary 2.1 in [20], the Birkhoff-James $ε$-orthogonality set $F^{ε}_{∥·∥_1}(χ; ψ)$ is not empty. So, there exists at least one complex number $µ$ in the set $F^{ε}_{∥·∥_1}(χ; ψ)$. In the 2-dimensional vector subspace $Y = \text{span}\{χ, ψ\}$, we define the linear functional $f_0 ∈ Y^*$ such that 

$$f_0(z_1χ + z_2ψ) = z_1µ\sqrt{1 − \varepsilon^2 ∥ψ∥} + z_2\sqrt{1 − \varepsilon^2 ∥ψ∥}, \quad z_1, z_2 ∈ C.$$
Then \( f_0(\chi) = \mu \sqrt{1 - \varepsilon^2} \|\psi\| \) and \( f_0(\psi) = \sqrt{1 - \varepsilon^2} \|\psi\| \). Since \( \mu \in F_{\|\cdot\|}(\chi; \psi) \), we have that for every \( \lambda \in \mathbb{C} \),
\[
\|\chi - \lambda \psi\| \geq \sqrt{1 - \varepsilon^2} \|\psi\| |\mu - \lambda| = |f_0(\chi) - \lambda f_0(\psi)| = |f_0(\chi - \lambda \psi)|,
\]
and \( \|f_0\| \leq 1 \) (as a continuous linear functional defined in the 2-dimensional subspace \( Y \)). Applying the Hahn-Banach extension theorem, there is an extension of \( f_0 \), say \( f \in X^* \), such that
\[
f(\chi) = \mu \sqrt{1 - \varepsilon^2} \|\psi\|, \quad f(\psi) = \sqrt{1 - \varepsilon^2} \|\psi\| \quad \text{and} \quad \|f\| = \|f_0\| \leq 1.
\]
Then, \( f \in L_\varepsilon(\psi) \), and the set \( L_\varepsilon(\psi) \) is non-empty.

For the closedness of the set \( L_\varepsilon(\psi) \), it is enough to see that the set \( X^* \setminus L_\varepsilon(\psi) \) is open. Indeed, if a linear functional \( f \in X^* \) does not belong to \( L_\varepsilon(\psi) \), then
\[
f(\psi) \neq \sqrt{1 - \varepsilon^2} \|\psi\| \quad \text{or} \quad \|f\| > 1.
\]
Consequently, by the continuity of the norm, there is a neighborhood \( G_f \subset X^* \) of \( f \) such that for any \( g \in G_f \),
\[
g(\psi) \neq \sqrt{1 - \varepsilon^2} \|\psi\| \quad \text{or} \quad \|g\| > 1,
\]
and so \( G_f \subset X^* \setminus L_\varepsilon(\psi) \).

Finally, for the convexity, we consider two linear functionals \( f, g \in L_\varepsilon(\psi) \). It is easy to see that for any \( t \in [0, 1] \),
\[
[(1-t)f + tg](\psi) = (1-t)f(\psi) + t g(\psi) = \sqrt{1 - \varepsilon^2} \|\psi\|
\]
and
\[
\|(1-t)f + tg\| \leq (1-t)\|f\| + t \|g\| \leq 1,
\]
and hence, \( (1-t)f + tg \) lies in \( L_\varepsilon(\psi) \).

We have proved that for \( \psi \neq 0 \), the set \( L_\varepsilon(\psi) \) is non-empty. As a consequence, the region \( \Omega_\varepsilon(\chi; \psi) \) is non-empty. Moreover, the set \( \Omega_\varepsilon(\chi; \psi) \) coincides with the Birkhoff-James \( \varepsilon \)-orthogonality set \( F_{\|\cdot\|}^\varepsilon(\chi; \psi) \).

**Theorem 2.2.** Let \( \chi, \psi \in X \), with \( \psi \neq 0 \). For every \( \varepsilon \in [0, 1) \), it holds that
\[
\Omega_\varepsilon(\chi; \psi) = F_{\|\cdot\|}^\varepsilon(\chi; \psi).
\]
Proposition 2.3. Let $\mu \in \Omega_\varepsilon(\chi; \psi)$. Then, $\mu = \frac{f_\mu(\chi)}{\sqrt{1 - \varepsilon^2 \|\psi\|}}$ for some linear functional $f_\mu \in L_\varepsilon(\psi)$. For every $\lambda \in \mathbb{C}$, we have

$$\sqrt{1 - \varepsilon^2 \|\psi\|} |\mu - \lambda| = \left| \sqrt{1 - \varepsilon^2 \|\psi\|} \frac{f_\mu(\chi) - \sqrt{1 - \varepsilon^2 \|\psi\|} \lambda}{\sqrt{1 - \varepsilon^2 \|\psi\|}} \right|$$

$$\leq |f_\mu(\chi - \lambda\psi)|$$

$$\leq \|f_\mu\| |\chi - \lambda\psi|$$

Thus, $\mu \in F_{\|\|}^\varepsilon(\chi; \psi)$, and clearly, $\Omega_\varepsilon(\chi; \psi) \subseteq F_{\|\|}^\varepsilon(\chi; \psi)$.

For the converse, we consider two cases:

(i) Suppose that $\chi = c\psi$ for a constant $c \in \mathbb{C}$. Then, by Property $(P_5)$, $F_{\|\|}^\varepsilon(\chi; \psi) = F_{\|\|}^\varepsilon(c\psi; \psi) = \{c\}$. Also,

$$\frac{f(\chi)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} = \frac{f(c\psi)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} = \frac{cf(\psi)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} = c, \quad \forall f \in L_\varepsilon(\psi),$$

and hence, $\Omega_\varepsilon(c\psi; \psi) = \{c\}$.

(ii) Suppose that $\chi, \psi \in \mathcal{X}$ are nonzero and linearly independent, and consider a scalar $\mu \in F_{\|\|}^\varepsilon(\chi; \psi)$. By the proof of Lemma 2.1, there exists a continuous linear functional $f_\mu \in L_\varepsilon(\psi)$ such that $f_\mu(\chi) = \mu \sqrt{1 - \varepsilon^2 \|\psi\|}$. Thus, $\mu \in \Omega_\varepsilon(\chi; \psi)$, and the proof is complete.

The above alternative definition of the Birkhoff-James $\varepsilon$-orthogonality set yields readily the subadditivity of $F_{\|\|}^\varepsilon(\chi; \psi)$ in $\chi$, which is necessary for the proofs of our results in Sections 5 and 6.

Proposition 2.3. Let $\chi_1, \chi_2, \psi \in \mathcal{X}$, with $\psi \neq 0$. Then, it holds that

$$F_{\|\|}^\varepsilon(\chi_1 + \chi_2; \psi) \subseteq F_{\|\|}^\varepsilon(\chi_1; \psi) + F_{\|\|}^\varepsilon(\chi_2; \psi).$$

Proof. It is easy to see that

$$F_{\|\|}^\varepsilon(\chi_1 + \chi_2; \psi) = \Omega_\varepsilon(\chi_1 + \chi_2; \psi)$$

$$= \left\{ \frac{f(\chi_1 + \chi_2)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} : f \in L_\varepsilon(\psi) \right\}$$

$$= \left\{ \frac{f(\chi_1)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} + \frac{f(\chi_2)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} : f \in L_\varepsilon(\psi) \right\}$$

$$\subseteq \left\{ \frac{f(\chi_1)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} : f \in L_\varepsilon(\psi) \right\} + \left\{ \frac{g(\chi_2)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} : g \in L_\varepsilon(\psi) \right\}$$

$$= \Omega_\varepsilon(\chi_1; \psi) + \Omega_\varepsilon(\chi_2; \psi)$$

$$= F_{\|\|}^\varepsilon(\chi_1; \psi) + F_{\|\|}^\varepsilon(\chi_2; \psi).$$
Example 2.2. Consider the sequences \( \chi_1 = \{1, \frac{1}{2 - i}, \frac{1}{2 - i^2}, \frac{1}{2 - i^3}, \ldots \} \), \( \chi_2 = \{1, \frac{1}{2 - 2i}, \frac{1}{2 - 2i^2}, \frac{1}{2 - 2i^3}, \ldots \} \) and \( \psi = \{1, \frac{1}{1 + i}, \frac{1}{1 + i^2}, \frac{1}{1 + i^3}, \ldots \} \) of the complex normed linear space \( \ell^1 \). The Birkhoff-James \( \varepsilon \)-orthogonality sets \( F_{\|\cdot\|_1}^{0,4} (\chi_1; \psi) \), \( F_{\|\cdot\|_1}^{0,4} (\chi_2; \psi) \) and \( F_{\|\cdot\|_1}^{0,4} (\chi_1 + \chi_2; \psi) \) are estimated by the unshaded regions in the left, middle and right parts of Figure 2, respectively. Each estimation results from having drawn 500 circles; see the defining formula (5). The compactness and the convexity of the sets, Property (P7), and the subadditivity of Proposition 2.3 are verified.

Proposition 2.4. Let \( \chi, \psi \in X \), with \( \psi \neq 0 \), \( \chi \) not a scalar multiple of \( \psi \), and \( \varepsilon \in [0, 1) \). If \( \mu \in \partial F_{\|\cdot\|_1}^{\varepsilon,4} (\chi; \psi) \), then for every continuous linear functional \( f_\mu \in L_\varepsilon (\psi) \) such that \( \mu = \frac{f_\mu (\chi)}{\sqrt{1 - \varepsilon^2 \|\psi\|^2}} \), it holds that \( \|f_\mu\| = 1 \).

Proof. Let \( \mu \in \partial F_{\|\cdot\|_1}^{\varepsilon,4} (\chi; \psi) \). Then, by Property (Pb),

\[
\inf_{\lambda \in \mathbb{C}} \left\{ \|\chi - \lambda \psi\| - \sqrt{1 - \varepsilon^2 \|\psi\|^2} \|\mu - \lambda\| \right\} = 0.
\]

For every \( f_\mu \in L_\varepsilon (\psi) \) with \( \mu = \frac{f_\mu (\chi)}{\sqrt{1 - \varepsilon^2 \|\psi\|^2}} \), we have

\[
0 = \inf_{\lambda \in \mathbb{C}} \left\{ \|\chi - \lambda \psi\| - \sqrt{1 - \varepsilon^2 \|\psi\|^2} \left| \frac{f_\mu (\chi) - \sqrt{1 - \varepsilon^2 \|\psi\|^2} \lambda}{\sqrt{1 - \varepsilon^2 \|\psi\|^2}} \right| \right\}
\]

\[
= \inf_{\lambda \in \mathbb{C}} \{ \|\chi - \lambda \psi\| - |f_\mu (\chi) - \lambda f_\mu (\psi)| \}
\]

\[
= \inf_{\lambda \in \mathbb{C}} \{ \|\chi - \lambda \psi\| - |f_\mu (\chi - \lambda \psi)| \}
\]

\[
= - \sup_{\lambda \in \mathbb{C}} \left\{ \frac{|f_\mu (\chi - \lambda \psi)|}{\|\chi - \lambda \psi\|} - 1 \right\},
\]

and we conclude that \( \|f_\mu\| = 1 \).
Proposition 2.5. Let $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$, $\chi$ not a scalar multiple of $\psi$, and $\varepsilon \in [0, 1)$. Then, it holds that

$$\max \left\{ \Re \mu : \mu \in F^\varepsilon_{\| \cdot \|} (\chi; \psi) \right\} \leq \inf_{a > 0} \frac{1}{a} \left\{ \frac{\| \psi + a\chi \|}{\sqrt{1 - \varepsilon^2 \| \psi \|}} - 1 \right\}.$$ 

Proof. Consider a continuous linear functional $f \in L^\varepsilon(\psi)$. Then, for any $a > 0$, we have

$$\frac{f(\chi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} = \frac{1}{a} \left[ \frac{f(\psi + a\chi - \psi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} - \frac{f(\psi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} - 1 \right].$$

Hence,

$$\Re \frac{f(\chi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} = \Re \frac{1}{a} \left[ \frac{f(\psi + a\chi - \psi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} - 1 \right] = \Re \frac{f(\psi + a\chi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} - 1,$$

and consequently,

$$\Re \frac{f(\chi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} + \frac{1}{a} = \frac{1}{a} \Re \frac{f(\psi + a\chi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} \leq \frac{1}{a} \left\{ \frac{|f(\psi + a\chi)|}{\sqrt{1 - \varepsilon^2 \| \psi \|}} \right\}.$$

Thus, for any $a > 0$,

$$\Re \frac{f(\chi)}{\sqrt{1 - \varepsilon^2 \| \psi \|}} \leq \frac{1}{a} \left\{ \frac{|f(\psi + a\chi)|}{\sqrt{1 - \varepsilon^2 \| \psi \|}} - 1 \right\} \leq \frac{1}{a} \left\{ \frac{\| \psi + a\chi \|}{\sqrt{1 - \varepsilon^2 \| \psi \|}} - 1 \right\},$$

and the proof is complete.

3 Vector-valued polynomials

Consider a vector-valued polynomial

$$P(z) = \chi_m z^m + \chi_{m-1} z^{m-1} + \cdots + \chi_1 z + \chi_0,$$

with vector coefficients $\chi_i \in \mathcal{X}$ ($i = 0, 1, \ldots, m$), $\chi_m \neq 0$, and a scalar variable $z \in \mathbb{C}$. Vector-valued polynomials appear in the approximation of vector-valued functions [1, 29]. Moreover, special cases of vector-valued polynomials such as square matrix polynomials [9, 10, 11, 19, 21], rectangular matrix polynomials [9, 19] and operator polynomials [12, 17, 23, 26], appear in many applications like systems of differential-algebraic equations, linear system theory, control theory, vibration analysis of structural systems, and acoustics.

For any $\varepsilon \in [0, 1)$, and any nonzero vector $\psi \in \mathcal{X}$ such that $F^\varepsilon_{\| \cdot \|} (\chi_m; \psi) \neq \{0\}$, we can define the Birkhoff-James $\varepsilon$-orthogonality set of $P(z)$ with respect to $\psi$. 

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Definition 3.1. Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F_\varepsilon^\varepsilon(\chi_m; \psi) \neq \{0\}$. The Birkhoff-James $\varepsilon$-orthogonality set of $P(z)$ with respect to $\psi$ is defined and denoted by

$$W_\varepsilon^{\varepsilon}(P(z); \psi) = \left\{ \mu \in \mathbb{C} : 0 \in F_\varepsilon^\varepsilon(P(\mu); \psi) \right\}$$

$$= \left\{ \mu \in \mathbb{C} : f(P(\mu)) = 0, f \in L_\varepsilon(\psi) \right\}$$

$$= \left\{ \mu \in \mathbb{C} : f(\chi_m)\mu^m + f(\chi_{m-1})\mu^{m-1} + \cdots + f(\chi_1)\mu + f(\chi_0) = 0, f \in L_\varepsilon(\psi) \right\}$$

$$= \left\{ \mu \in \mathbb{C} : \psi \perp_{BJ} P(\mu) \right\}$$

$$= \left\{ \mu \in \mathbb{C} : ||P(\mu) - \lambda \psi|| \geq \sqrt{1 - \varepsilon^2} ||\lambda||, \forall \lambda \in \mathbb{C} \right\}. \tag{7}$$

Note that for $\chi_m \neq 0$ and $\varepsilon \in (0,1)$, the condition $F_\varepsilon^\varepsilon(\chi_m; \psi) \neq \{0\}$ is always satisfied; see Properties $(P_3)$ and $(P_7)$.

Since the set $L_\varepsilon(\psi)$ is non-empty and closed, it follows readily that $W_\varepsilon^{\varepsilon}(P(z); \psi)$ is also non-empty and closed. Moreover, for any $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, $W_\varepsilon^{\varepsilon}(P(z); \psi) \subseteq W_{\varepsilon_2}^{\varepsilon}(P(z); \psi)$.

Remark 3.1. Consider a vector-valued polynomial $P(z)$ as in (6), a nonzero vector $\psi \in \mathcal{X}$ with $F_0^0(\chi_m; \psi) \neq \{0\}$, and a $\mu \in \mathbb{C}$ such that $P(\mu)$ is not a scalar multiple of $\psi$. For any $\varepsilon \in (0,1)$,

$$\mu \in W_\varepsilon^{\varepsilon}(P(z); \psi) \iff ||P(\mu) - \lambda \psi|| \geq \sqrt{1 - \varepsilon^2} ||\lambda||, \forall \lambda \in \mathbb{C}$$

$$\iff \frac{1}{\lambda^2} ||P(\mu) - \psi|| \geq \sqrt{1 - \varepsilon^2} ||\psi||, \forall \lambda \in \mathbb{C} \setminus \{0\}$$

$$\iff ||\psi - \lambda P(\mu)|| \geq \sqrt{1 - \varepsilon^2} ||\psi||, \forall \lambda \in \mathbb{C}$$

$$\iff \inf_{\lambda \in \mathbb{C}} ||\psi - \lambda P(\mu)|| \geq \sqrt{1 - \varepsilon^2} ||\psi|| \left( \psi \notin \text{span}\{P(\mu)\} \right)$$

$$\iff \text{dist}(\psi, \text{span}\{P(\mu)\}) \geq \sqrt{1 - \varepsilon^2} ||\psi||.$$ 

As in the case of $F_0^0(\chi; \psi)$, $\mu$ lies in the region $W_\varepsilon^{\varepsilon}(P(z); \psi)$ if and only if $\text{dist}(\psi, \text{span}\{P(\mu)\}) = ||\psi||$. Moreover, if $\mu \notin W_\varepsilon^{\varepsilon}(P(z); \psi)$ (or equivalently, if $\text{dist}(\psi, \text{span}\{P(\mu)\}) < ||\psi||$), then there is a number $\varepsilon_0 \in [0,1)$ such that $\mu \in \partial W_\varepsilon^{\varepsilon}(P(z); \psi)$ and $\text{dist}(\psi, \text{span}\{P(\mu)\}) = \sqrt{1 - \varepsilon_0^2} ||\psi||$. This number $\varepsilon_0$ can be chosen to be the smallest value of the parameter $\varepsilon \in [0,1)$ with $\mu \in W_\varepsilon^{\varepsilon}(P(z); \psi)$.

It is easy to verify the next three properties.

$(P_{12})$ For any scalar $a \in \mathbb{C} \setminus \{0\}$, $W_\varepsilon^{\varepsilon}(aP(z); \psi) = W_\varepsilon^{\varepsilon}(P(z); \psi)$, $W_\varepsilon^{\varepsilon}(P(az); \psi) = a^{-1}W_\varepsilon^{\varepsilon}(P(z); \psi)$ and $W_\varepsilon^{\varepsilon}(P(z + a); \psi) = W_\varepsilon^{\varepsilon}(P(z); \psi) - a$.

$(P_{13})$ If $R(z) = \chi_0 z^m + \chi_1 z^{m-1} + \cdots + \chi_{m-1} z + \chi_m = z^m P(z^{-1})$ is the reverse vector-valued polynomial of $P(z)$, then

$$W_\varepsilon^{\varepsilon}(R(z); \psi) \setminus \{0\} = \left\{ \mu \in \mathbb{C} : \mu^{-1} \in W_\varepsilon^{\varepsilon}(P(z); \psi) \setminus \{0\} \right\}.$$
If there exists a continuous linear functional \( f \in L_\epsilon(\psi) \) such that \( f(\chi_m) = f(\chi_{m-1}) = \cdots = f(\chi_0) = 0 \), then \( W^\varepsilon_{\| \cdot \|}(P(z); \psi) = \mathbb{C} \).

For the remainder of the paper, it is necessary to introduce the following radii.

**Definition 3.2.** Let \( \chi, \psi \in \mathcal{X} \), with \( \psi \) nonzero. For any \( \varepsilon \in [0, 1) \), the *Birkhoff-James \( \varepsilon \)-orthogonality inner radius of \( \chi \) with respect to \( \psi \) is defined as

\[
\hat{r}_{\| \cdot \|}^\varepsilon(\chi; \psi) = \min \left\{ |z| : z \in F^\varepsilon_{\| \cdot \|}(\chi; \psi) \right\},
\]

and the *Birkhoff-James \( \varepsilon \)-orthogonality outer radius of \( \chi \) with respect to \( \psi \) is defined as

\[
r_{\| \cdot \|}^\varepsilon(\chi; \psi) = \max \left\{ |z| : z \in F^\varepsilon_{\| \cdot \|}(\chi; \psi) \right\} \leq \frac{\| \chi \| \| \psi \|}{\sqrt{1 - \varepsilon^2}}.
\]

**Theorem 3.1.** (For rectangular matrix polynomials, see Theorem 12 in [7], and for the standard numerical range of square matrix polynomials, see Theorem 2.3 in [24].) Let \( P(z) \) be a vector-valued polynomial as in (6), \( \varepsilon \in [0, 1) \), and \( \psi \in \mathcal{X} \) be a nonzero vector such that \( F^\varepsilon_{\| \cdot \|}(\chi_m; \psi) \neq \{0\} \). Then, the set \( W^\varepsilon_{\| \cdot \|}(P(z); \psi) \) is bounded if and only if \( 0 \notin F^\varepsilon_{\| \cdot \|}(\chi_m; \psi) \).

**Proof.** Let \( 0 \notin F^\varepsilon_{\| \cdot \|}(\chi_m; \psi) \), or equivalently, \( \hat{r}_{\| \cdot \|}^\varepsilon(\chi_m; \psi) > 0 \). We will obtain that \( W^\varepsilon_{\| \cdot \|}(P(z); \psi) \) is bounded; in particular, we will prove that \( W^\varepsilon_{\| \cdot \|}(P(z); \psi) \subseteq \mathcal{D}(0, M) \), where

\[
M = 1 + \frac{\max_{0 \leq j \leq m-1} r_{\| \cdot \|}^\varepsilon(\chi_j; \psi)}{\hat{r}_{\| \cdot \|}^\varepsilon(\chi_m; \psi)}.
\]

Since \( M \geq 1 \), we consider a scalar \( \mu \in W^\varepsilon_{\| \cdot \|}(P(z); \psi) \) with \( |\mu| \geq 1 \). Then, there exists a continuous linear functional \( f \in L_\epsilon(\psi) \) such that

\[
f(\chi_m) \mu^m + f(\chi_{m-1}) \mu^{m-1} + \cdots + f(\chi_1) \mu + f(\chi_0) = 0.
\]

As a consequence,

\[
|\mu|^m = \left| \sum_{j=0}^{m-1} f(\chi_j) \mu^j \right| \leq \frac{\sum_{j=0}^{m-1} |f(\chi_j)| |\mu|^j}{|f(\chi_m)|} \leq \frac{\max_{0 \leq j \leq m-1} r_{\| \cdot \|}^\varepsilon(\chi_j; \psi) |\mu|^m - 1}{|\mu| - 1} \leq \frac{|\mu|^m - 1}{|\mu| - 1}.
\]

Thus,

\[
|\mu| - 1 \leq \frac{\max_{0 \leq j \leq m-1} r_{\| \cdot \|}^\varepsilon(\chi_j; \psi)}{\hat{r}_{\| \cdot \|}^\varepsilon(\chi_m; \psi)} |\mu|^m \leq \frac{\max_{0 \leq j \leq m-1} r_{\| \cdot \|}^\varepsilon(\chi_j; \psi)}{\hat{r}_{\| \cdot \|}^\varepsilon(\chi_m; \psi)} |\mu|^m.
\]

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and hence, $|\mu| \leq M$.

For the converse, we assume that $W^\varepsilon_{||} (P(z); \psi)$ is bounded and $0 \in F^\varepsilon_{||} (\chi_m; \psi)$. Then there is a continuous linear functional $f \in L_\varepsilon(\psi)$ such that $f(\chi_m) = 0$. Since $W^\varepsilon_{||} (P(z); \psi) \neq \mathbb{C}$, Property (P14) implies that $f(\chi_s) \neq 0$ for some $s \in \{0, 1, 2, \ldots, m - 1\}$. Moreover, since $F^\varepsilon_{||} (\chi_m; \psi) \neq \{0\}$, there exists a sequence of continuous linear functionals $\{f_1, f_2, \ldots\} \subset L_\varepsilon(\psi)$ such that $f_j(\chi_m) \neq 0$, $j = 1, 2, \ldots$, and $f_j(\chi_m) \to 0$ as $j \to +\infty$. We consider now the scalar polynomials

$$f_j(P(z)) = f_j(\chi_m) z^m + f_j(\chi_{m-1}) z^{m-1} + \cdots + f_j(\chi_1) z + f_j(\chi_0), \quad j = 1, 2, \ldots$$

It is clear that $\frac{f_j(\chi_m)}{f_j(\chi_m)} \to \infty$ as $j \to +\infty$; this is a contradiction because we have assumed that $W^\varepsilon_{||} (P(z); \psi)$ is bounded, and hence, all the roots and the elementary symmetric functions of the scalar polynomials $f_j(P(z))$, $j = 1, 2, \ldots$, are bounded.

**Theorem 3.2.** (For the standard numerical range of square matrix polynomials, see Theorem 3.1 in [27].) Consider a nonzero vector $\psi \in X$, an $\varepsilon \in [0, 1]$, and the vector-valued polynomial $P(z) = \psi z^m + \chi_{m-1} z^{m-1} + \cdots + \chi_1 z + \chi_0$ (i.e., $\chi_m = \psi$). Then, for every $\mu \in W^\varepsilon_{||} (P(z); \psi)$, it holds

$$\frac{1}{\tilde{r}_\varepsilon(\chi_0; \psi)} + \max_{1 \leq j \leq m} r^\varepsilon_{||} (\chi_j; \psi) \leq |\mu| \leq 1 + \max_{0 \leq j \leq m-1} r^\varepsilon_{||} (\chi_j; \psi).$$

**Proof.** Since $F^\varepsilon_{||} (\psi; \psi) \neq \{1\}$ does not contain the origin, the set $W^\varepsilon_{||} (P(z); \psi)$ is bounded.

Let $\mu \in W^\varepsilon_{||} (P(z); \psi)$. By definition, there exists a continuous linear functional $f \in L_\varepsilon(\psi)$ such that $f(\psi) \mu^m + f(\chi_{m-1}) \mu^{m-1} + \cdots + f(\chi_1) \mu + f(\chi_0) = 0$. Since the lower bound of the theorem is less than or equal to 1, for the first inequality, we may assume that $|\mu| < 1$. Then, we have that

$$f(\chi_0) = - (f(\psi) \mu^m + f(\chi_{m-1}) \mu^{m-1} + \cdots + f(\chi_1) \mu),$$

or

$$|f(\chi_0)| = |f(\psi) \mu^m + f(\chi_{m-1}) \mu^{m-1} + \cdots + f(\chi_1)\mu|.$$

Hence,

$$\tilde{r}^\varepsilon_{||} (\chi_0, \psi) \leq \frac{|f(\psi) \mu^m + f(\chi_{m-1}) \mu^{m-1} + \cdots + f(\chi_1)\mu|}{\sqrt{1 - \varepsilon^2 \|\psi\|}} \leq \frac{|f(\psi)| |\mu| + f(\chi_{m-1}) |\mu| + \cdots + f(\chi_1) |\mu|}{\sqrt{1 - \varepsilon^2 \|\psi\|}} \leq \frac{|\mu|}{1 - |\mu|} \max_{1 \leq j \leq m} r^\varepsilon_{||} (\chi_j; \psi),$$

which yields the first inequality.

The upper bound of the theorem coincides with the upper bound $M$ in (8), and the proof is complete. \blackslug
Suppose that the norm \( \| \cdot \| \) is induced by an inner product \( \langle \cdot, \cdot \rangle \). Then by Property \( P_{11} \) (see also Proposition 5.1 in [20]), the Birkhoff-James \( \varepsilon \)-orthogonality set of \( \chi \) with respect to \( \psi \neq 0 \) is a closed disk, namely,

\[
F_{\| \cdot \|}(\chi; \psi) = \mathcal{D}\left( \frac{\langle \chi, \psi \rangle}{\| \psi \|^2}, \frac{\epsilon}{\sqrt{1 - \epsilon^2 \| \psi \|^2}} \right).
\]

Let \( P(z) \) be a vector-valued polynomial as in (6), \( \varepsilon \in [0, 1) \), and \( \psi \in X \) be a nonzero vector such that \( F_{\| \cdot \|}(\chi_m; \psi) \neq \{0\} \). Then, by (7), we have

\[
W_{\| \cdot \|}(P(z); \psi) = \{ \mu \in \mathbb{C} : \psi \perp_{BJ} P(\mu) \} \\
= \{ \mu \in \mathbb{C} : \psi \perp_{\varepsilon} P(\mu) \} \\
= \{ \mu \in \mathbb{C} : |\langle P(\mu), \psi \rangle| \leq \varepsilon \| \psi \| \| P(\mu) \| \} \\
= \{ \mu \in \mathbb{C} : |\langle P(\mu), \psi \rangle|^2 \leq \varepsilon^2 \| \psi \|^2 \| P(\mu) \|^2 \} \\
= \{ \mu \in \mathbb{C} : \langle P(\mu), \psi \| P(\mu) \rangle \leq \varepsilon^2 \| \psi \|^2 \| P(\mu) \|^2 \} \\
= \mu \in \mathbb{C} : \sum_{i=0}^m \chi_i \mu^i, \psi \sum_{j=0}^m \mu^j \chi_j \leq \varepsilon^2 \| \psi \|^2 \sum_{i=0}^m \chi_i \mu^i, \sum_{j=0}^m \chi_j \mu^j \}
\]

\[
= \mu \in \mathbb{C} : \sum_{i,j=0}^m \langle \chi_i, \psi \rangle \langle \psi, \chi_j \rangle \mu^i \mu^j - \varepsilon^2 \| \psi \|^2 \sum_{i,j=0}^m \langle \chi_i, \chi_j \rangle \mu^i \mu^j \leq 0
\]

Figure 3: Birkhoff-James \( \varepsilon \)-orthogonality sets of \( P(z) \) (left part) and \( R(z) \) (right part).

Example 3.1. Consider the four-dimensional quadratic vector-valued polynomial

\[
P(z) = \begin{bmatrix} 1 & 0 & i \\ 0 & 0.8 & -1 \\ i & -0.5 & 0 \\ 1 & 0 & -i \\ \end{bmatrix} z^2 + \begin{bmatrix} i \\ -1 \\ -3 \\ 0 \\ \end{bmatrix} z + \begin{bmatrix} 2 \\ -3 \\ -0.1 \\ -1 \\ \end{bmatrix}
\]
its reverse vector-valued polynomial

\[ R(z) = \begin{bmatrix} 2 & -3 \\ -1 & -0.1 \end{bmatrix} z^2 + \begin{bmatrix} i & -1 \\ 0.5 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}, \]

and the vector \( \psi = \begin{bmatrix} 0.6 & 0 & 0.9 & 0.2 \end{bmatrix}^T \). For the euclidean norm (which is induced by the standard inner product), we have drawn the boundaries of the \( \varepsilon \)-orthogonality sets \( W^\varepsilon_{\|\cdot\|_2}(P(z); \psi) \), \( \varepsilon = 0.3, 0.5, 0.7, 0.73, \) and \( W^\varepsilon_{\|\cdot\|_2}(R(z); \psi) \), \( \varepsilon = 0.2, 0.25, 0.26, 0.265, 0.27 \), in the left and the right part of Figure 3, respectively. As expecting by Theorem 3.1, the origin lies in \( W^\varepsilon_{\|\cdot\|_2}(P(z); \psi) \) (or equivalently, the origin lies in \( F^\varepsilon_{\|\cdot\|_2}(\chi_0; \psi) \)) if and only if \( W^\varepsilon_{\|\cdot\|_2}(R(z); \psi) \) is unbounded.

### 4 Connected components

In this section, we study the connected components of the Birkhoff-James \( \varepsilon \)-orthogonality set \( W^\varepsilon_{\|\cdot\|_2}(P(z); \psi) \), when this set is bounded. The following lemma is necessary for our analysis.

**Lemma 4.1.** Let \( P(z) \) be a vector-valued polynomial as in (6), and let \( L \) be a non-empty, closed and convex subset of \( \mathcal{X}^* \) such that \( f(\chi_m) \neq 0 \) for all \( f \in L \). Then, the roots of the scalar polynomial \( f(P(z)) = f(\chi_m)z^m + f(\chi_{m-1})z^{m-1} + \cdots + f(\chi_1)z + f(\chi_0) \) are continuous with respect to \( f \in L \).

**Proof.** It is well known that the roots of a scalar polynomial are continuous functions of the coefficients of the polynomial, as long as the leading coefficient is nonzero; see Appendix D in [14]. The vector coefficients \( \chi_0, \chi_1, \ldots, \chi_m \in \mathcal{X} \) of the vector-valued polynomial \( P(z) = \chi_m z^m + \chi_{m-1} z^{m-1} + \cdots + \chi_1 z + \chi_0 \) are constant, and hence, the coefficients \( f(\chi_0), f(\chi_1), \ldots, f(\chi_m) \) of the scalar polynomial \( f(P(z)) \) depend only on \( f \in L \). If \( \{f_1, f_2, \ldots\} \subset L \) is a sequence of continuous linear functionals that converges to \( f \in L \) (i.e., \( \|f_k - f\| \to 0 \), as \( k \to +\infty \)), then for any \( j = 0, 1, \ldots, m \), it holds

\[
\|f(\chi_j) - f_k(\chi_j)\| \leq \|(f - f_k)(\chi_j)\| \leq \|f - f_k\| \|\chi_j\|, \quad k = 1, 2, \ldots,
\]

and the proof is complete. \( \square \)

**Theorem 4.2.** (For the standard numerical range of square matrix polynomials, see Theorem 2.2 in [24].) Let \( P(z) \) be a vector-valued polynomial as in (6), \( \varepsilon \in [0, 1] \), and \( \psi \in \mathcal{X} \) be a nonzero vector such that \( 0 \notin F^\varepsilon_{\|\cdot\|_2}(\chi_m; \psi) \) (or equivalently, \( W^\varepsilon_{\|\cdot\|_2}(P(z); \psi) \) is bounded). Suppose that \( W^\varepsilon_{\|\cdot\|_2}(P(z); \psi) \) has \( r \) connected components. If \( \kappa \) is the minimum number of distinct zeros of the scalar polynomial \( f(P(z)) = f(\chi_m)z^m + f(\chi_{m-1})z^{m-1} + \cdots + f(\chi_1)z + f(\chi_0) \) over all \( f \in L_\varepsilon(\psi) \), then \( r \leq \kappa \leq m \).
Definition 4.1. Let $\kappa \leq m$ distinct roots. Let also $f_2 \in L_\varepsilon(\psi)$. Since $0 \notin F_\|\|_1(\chi_m; \psi)$, both scalars $f_1(\chi_m)$ and $f_2(\chi_m)$ are nonzero. Moreover, by the convexity of the set $L_\varepsilon(\psi)$ and the region $F_\|\|_1(\chi_m; \psi)$ (keeping in mind that $0 \notin F_\|\|_1(\chi_m; \psi)$), every continuous linear functional 

$$g_\varepsilon = (1 - t)f_1 + tf_2 \in L_\varepsilon(\psi), \quad t \in [0, 1],$$

satisfies the condition $g_\varepsilon(\chi_m) \neq 0$. Thus, by Lemma 4.1, the roots of the scalar polynomial 

$$g_\varepsilon(P(z)) = g_\varepsilon(\chi_m)z^m + g_\varepsilon(\chi_{m-1})z^{m-1} + \cdots + g_\varepsilon(\chi_1)z + g_\varepsilon(\chi_0), \quad t \in [0, 1]$$

are continuous functions of $t$. Hence, the $\kappa$ roots of the scalar polynomial $f_1(P(z))$ are connected with continuous curves in $W_\|\|_1(\xi; \psi)$ with the roots of $f_2(P(z))$. Consequently, the number of the connected components of $W_\|\|_1(\xi; \psi)$ is less than or equal to $\kappa$.

Proof. Consider a continuous linear functional $f_1 \in L_\varepsilon(\psi)$ such that the scalar polynomial $f_1(P(z)) = f_1(\chi_m)z^m + f(\chi_{m-1})z^{m-1} + \cdots + f_1(\chi_1)z + f_1(\chi_0)$ has exactly $\kappa$ distinct roots. Consider two continuous linear functionals $f_1, f_2 \in L_\varepsilon(\psi)$ and a $t \in [0, 1]$. Then we have that $f_1(\chi) \mu = f_2(\chi)\mu$. As a consequence, 

$$|tf_1 + (1 - t)f_2(\chi)| = \mu,$$

and $tf_1 + (1 - t)f_2$ also lies in $S_{\chi, \psi}(\mu)$. 

Lemma 4.3. Let $\chi, \psi \in X$, with $\psi \neq 0$, and consider a complex number $\mu \in F_\|\|_1(\chi_0; \chi \psi)$. We define the set 

$$S_{\chi, \psi}(\mu) = \left\{ f \in L_\varepsilon(\psi) : \mu = \frac{f(\chi)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} \subseteq L_\varepsilon(\psi) \right\}.$$

Moreover, for the vector-valued polynomial $P(z)$, we define the set 

$$S_{P(\psi)}(\mu) = \{ f \in L_\varepsilon(\psi) : f(P(\mu)) = 0 \} = S_{P(\mu), \psi}(0).$$

Definition 4.1. Let $\chi, \psi \in X$, with $\psi \neq 0$, and consider a complex number $\mu \in F_\|\|_1(\chi_0; \chi \psi)$. We define the set 

$$S_{\chi, \psi}(\mu) = \left\{ f \in L_\varepsilon(\psi) : \mu = \frac{f(\chi)}{\sqrt{1 - \varepsilon^2 \|\psi\|}} \subseteq L_\varepsilon(\psi) \right\}.$$
Theorem 4.4. (For operator polynomials, see Theorem 1 in [25].) Let \( P(z) \) be a vector-valued polynomial as in (6), \( \varepsilon \in [0,1) \), and \( \psi \in X \) be a nonzero vector such that \( 0 \notin F^\varepsilon_{\|\|}(X_m; \psi) \) (or equivalently, \( W^\varepsilon_{\|\|}(P(z); \psi) \) is bounded). Suppose that for every \( f \in L_\varepsilon(\psi) \), the scalar polynomial \( f(P(z)) = f(x_m)z^m + f(x_{m-1})z^{m-1} + \cdots + f(x_1)z + f(x_0) \) has exactly \( m \) simple roots. Then, \( W^\varepsilon_{\|\|}(P(z); \psi) \) has exactly \( m \) connected components.

Proof. We consider the images of the root functions \( \rho_1, \rho_2, \ldots, \rho_m \) in (9),

\[
W_i = \rho_i(L_\varepsilon(\psi)) \subseteq W^\varepsilon_{\|\|}(P(z); \psi), \quad i = 1, 2, \ldots, m.
\]

These sets are connected and satisfy

\[
W^\varepsilon_{\|\|}(P(z); \psi) = \bigcup_{1 \leq i \leq m} W_i.
\]

We need to prove that \( W_i \cap W_j = \emptyset \) for all \( i \neq j \).

Without loss of generality, assume that \( W_1 \cap W_2 \neq \emptyset \). Then there exists a \( \mu \in \mathbb{C} \) such that

\[
\rho_1(f_1) = \mu = \rho_2(f_2)
\]

for some functionals \( f_1, f_2 \in L_\varepsilon(\psi) \). Then both \( f_1 \) and \( f_2 \) lie in \( S_{P(z), \psi}(\mu) \). Moreover, it holds

\[
S_{P(z), \psi}(\mu) = \bigcup_{1 \leq i \leq m} \{ f \in L_\varepsilon(\psi) : \mu = \rho_i(f) \},
\]

i.e., \( S_{P(z), \psi}(\mu) \) is the union of

\[
S_1 = \{ f \in L_\varepsilon(\psi) : \mu = \rho_1(f) \} \quad \text{and} \quad S_2 = \bigcup_{2 \leq i \leq m} \{ f \in L_\varepsilon(\psi) : \mu = \rho_i(f) \}.
\]

Obviously, \( f_1 \in S_1 \) and \( f_2 \in S_2 \), and the sets \( S_1 \) and \( S_2 \) are not empty. The sets \( S_1 \) and \( S_2 \) are closed as pre-images of continuous maps. Since the set \( S_{P(z), \psi}(\mu) \) is convex, it is also connected, and hence, \( S_1 \cap S_2 \neq \emptyset \). Thus, there exists a functional \( f \) such that \( \rho_1(f) = z = \rho_i(f) \) for some \( i \geq 2 \); this is a contradiction because we have assumed that the roots are simple.

\( \square \)

5 Boundary

Since the Birkhoff-James \( \varepsilon \)-orthogonality set \( W^\varepsilon_{\|\|}(P(z); \psi) \) is closed, its boundary is of special interest. In the following two theorems, we describe the strong connection between a boundary point \( z_0 \) of \( W^\varepsilon_{\|\|}(P(z); \psi) \) and the origin as a boundary point of the region \( F^\varepsilon_{\|\|}(P(z_0); \psi) \).

Theorem 5.1. (For rectangular matrix polynomials, see Theorem 19 (i) in [7], and for the standard numerical range of square matrix polynomials, see Theorem 1.1
in [27].) Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0, 1)$, and $\psi \in X$ be a nonzero vector such that $F^c_{\|\cdot\|}(\chi_m; \psi) \neq \{0\}$. If $z_0 \in \partial W^c_{\|\cdot\|}(P(z); \psi)$, then $0 \in \partial F^c_{\|\cdot\|}(P(z_0); \psi)$.

**Proof.** Since $z_0 \in \partial W^c_{\|\cdot\|}(P(z); \psi) \subseteq W^c_{\|\cdot\|}(P(z); \psi)$, there is a continuous linear functional $f_0 \in L_2(\psi)$ such that $f_0(P(z_0)) = 0$. So, $0 \in F^c_{\|\cdot\|}(P(z_0); \psi)$, and it is sufficient to prove that the origin does not belong to the interior of $F^c_{\|\cdot\|}(P(z_0); \psi)$.

Let $\{z_1, z_2, \ldots \} \subset \mathbb{C} \setminus W^c_{\|\cdot\|}(P(z); \psi)$ be a sequence of complex numbers converging to $z_0$, and assume that $0$ lies in the interior of $F^c_{\|\cdot\|}(P(z_0); \psi)$. Then, there is a real number $\delta > 0$ such that $D(0, \delta) \subseteq F^c_{\|\cdot\|}(P(z_0); \psi)$. Moreover, there exist $f_{\delta,1}, f_{\delta,2}, f_{\delta,3} \in L_2(\psi)$ such that the triangle with vertices $\frac{f_{\delta,1}(P(z_0))}{\sqrt{1 - \varepsilon^2 \|\psi\|}}$, $\frac{f_{\delta,2}(P(z_0))}{\sqrt{1 - \varepsilon^2 \|\psi\|}}$ and $\frac{f_{\delta,3}(P(z_0))}{\sqrt{1 - \varepsilon^2 \|\psi\|}}$ contains the origin in its interior and lies in the disk $D(0, \delta/2)$. Continuity yields

$$
\lim_{n \to +\infty} \frac{f_{\delta,i}(P(z_0))}{\sqrt{1 - \varepsilon^2 \|\psi\|}} = \frac{f_{\delta,i}(P(z_0))}{\sqrt{1 - \varepsilon^2 \|\psi\|}}, \quad i = 1, 2, 3,
$$

and as a consequence, there is a positive integer $n_0$ such that $0 \in F^c_{\|\cdot\|}(P(z_n); \psi)$ for every $n \geq n_0$. Hence, for every positive integer $n \geq n_0$, $z_n \in W^c_{\|\cdot\|}(P(z); \psi)$; this is a contradiction. 

For the remainder, we need to consider the vector-valued polynomial

$$
P'(z) = m_1 \chi_{m_2} z^{m_2 - 1} + (m - 1) \chi_{m_1} z^{m_1 - 1} + \cdots + 2 \chi_2 z + \chi_1.
$$

**Theorem 5.2.** (For rectangular matrix polynomials, see Theorem 19 (ii) in [7], and for the standard numerical range of square matrix polynomials, see Theorem 2 in [22].) Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0, 1)$, and $\psi \in X$ be a nonzero vector such that $L^c_{\|\cdot\|}(\chi_m; \psi) \neq \{0\}$. Let also $z_0 \in W^c_{\|\cdot\|}(P(z); \psi)$ such that $F^c_{\|\cdot\|}(P(z_0); \psi) \neq \{0\}$ and $0 \notin F^c_{\|\cdot\|}(P(z); \psi)$. If $0 \in \partial F^c_{\|\cdot\|}(P(z_0); \psi)$, then $z_0 \in \partial W^c_{\|\cdot\|}(P(z); \psi)$.

**Proof.** Let $0 \in \partial F^c_{\|\cdot\|}(P(z_0); \psi)$, and assume that $z_0$ is an interior point of the set $W^c_{\|\cdot\|}(P(z); \psi)$. Then, there exists a $\delta > 0$ such that $D(z_0, \delta) \subseteq W^c_{\|\cdot\|}(P(z); \psi)$. Hence, for any $z \in D(z_0, \delta) \setminus \{z_0\}$, there is a $f_z \in L_2(\psi)$ such that $f_z(P(z)) = 0$. Moreover,

$$
0 = f_z(P(z)) = f_z((z - z_0)P(z_0) + (z_0)P'(z_0)) + (z - z_0)R(z, z_0)
$$

where $R(z, z_0)$ is a vector-valued polynomial in $z_0$ and $z$, such that $\|R(z, z_0)\| \to 0$ as $|z - z_0| \to 0$. Since $0 \notin F^c_{\|\cdot\|}(P'(z_0); \psi)$, by the subadditivity of Proposition 2.3, the
positive number $\delta$ can be chosen small enough such that for every $z \in \mathcal{D}(z_0, \delta) \setminus \{z_0\}$,

$$0 \notin \mathcal{F}^*_{\|\|}(P'(z_0) + R(z, z_0); \psi) \subseteq \mathcal{F}^*_{\|\|}(P'(z_0); \psi) + \mathcal{D}\left(0, \frac{\|R(z, z_0)\|}{\sqrt{1 - \varepsilon^2 \|\psi\|}}\right)$$

and

$$z - z_0 = -\frac{f_z(P(z_0))}{f_z(P'(z_0) + R(z, z_0))}.$$  \hspace{1cm} (10)

By the convexity of $\mathcal{F}^*_{\|\|}(P'(z_0) + R(z, z_0); \psi)$, there exist angles $\theta_1, \theta_2, \theta_3$ such that $0 < \theta_2 - \theta_1 \leq \theta_3 < \pi$ and

$$\mathcal{F}^*_{\|\|}(P'(z_0) + R(z, z_0); \psi) \subset \{w \in \mathbb{C} : \theta_1 \leq \arg(w) \leq \theta_2\}, \quad \forall z \in \mathcal{D}(z_0, \delta) \setminus \{z_0\}.
$$

Also, $\mathcal{F}^*_{\|\|}(P(z_0); \psi) \neq \{0\}$ and $0 \in \partial \mathcal{F}^*_{\|\|}(P(z_0); \psi)$. Therefore, by the convexity of $\mathcal{F}^*_{\|\|}(P(z_0); \psi)$, there exist angles $\phi_1, \phi_2$ such that $0 < \phi_2 - \phi_1 \leq \pi$ and

$$\mathcal{F}^*_{\|\|}(P(z_0); \psi) \subset \{w \in \mathbb{C} : \phi_1 \leq \arg(w) \leq \phi_2\}.
$$

Consequently, the angular of the right hand-side of (10) cannot take all the values in $[0, 2\pi)$. This is a contradiction, since the left hand-side is not constrained. \qed

Next, we consider the isolated points of the Birkhoff-James $\varepsilon$-orthogonality set $W^*_{\|\|}(P(z); \psi)$.

**Proposition 5.3.** (For the standard numerical range of square matrix polynomials, see Theorem 2.1 in [27].) Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0, 1]$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $0 \notin \mathcal{F}^*_{\|\|}(\chi_m; \psi)$ (or equivalently, $W^*_{\|\|}(P(z); \psi)$ is bounded). If $z_0$ is an isolated point of $W^*_{\|\|}(P(z); \psi)$, then $\mathcal{F}^*_{\|\|}(P(z_0); \psi) = \{0\}$.

**Proof.** Suppose that the singleton $\{z_0\}$ is a connected component of $W^*_{\|\|}(P(z); \psi)$. Then, there is a continuous linear functional $f_0 \in L_z(\psi)$ such that

$$f_0(P(z_0)) = f_0(\chi_m)z_0^m + f_0(\chi_{m-1})z_0^{m-1} + \cdots + f_0(\chi_1)z_0 + f_0(\chi_0) = 0.
$$

Since $0 \notin \mathcal{F}^*_{\|\|}(\chi_m; \psi)$, the convexity of $L_z(\psi)$ and the continuity of the roots of the scalar polynomial $f(P(z))$ with respect to $f \in L_z(\psi)$ imply that the roots of the scalar polynomial $f_0(P(z))$ are connected to the roots of any scalar polynomial $f(P(z))$, with $f \in L_z(\psi)$, by continuous curves in $W^*_{\|\|}(P(z); \psi)$ (see also the proof of Theorem 4.2). As a consequence, for any $f \in L_z(\psi)$, $z_0$ is a root of the scalar polynomial $f(P(z))$. Thus, $f(P(z_0)) = 0$ for every $f \in L_z(\psi)$, and hence, $\mathcal{F}^*_{\|\|}(P(z_0); \psi) = \{0\}$. Furthermore, if $\varepsilon > 0$, then Properties $(P_{f_0})$ and $(P_{f_1})$ yield $P(z_0) = 0$. \qed
6 Local dimension

Let $\Omega$ be a closed subset of $\mathbb{C}$. A recursive definition of the \textit{topological dimension} of $\Omega$, denoted by $\dim \{ \Omega \}$, is the following \cite{13, 16}: If $\Omega$ is an empty set, then $\dim \{ \Omega \} = -1$. If $\Omega$ is a non-empty set, then $\dim \{ \Omega \}$ is the least integer number $k \in \{0, 1, 2\}$ for which every point of $\Omega$ has arbitrarily small neighborhoods in $\Omega$ whose boundaries are of topological dimension less than $k$. Clearly, if $\Omega$ is countable, then $\dim \{ \Omega \} = 0$, and if $\Omega$ is a (non-degenerate) curve, then $\dim \{ \Omega \} = 1$.

Consider a point $z_0 \in \Omega$. The \textit{local dimension} of $z_0$ in $\Omega$ is defined as the limit $\lim_{\epsilon \to 0} \dim \{ \Omega \cap (z_0, \epsilon) \}$, $h \in (0, +\infty)$. In particular, the local dimension of $z_0$ in $\Omega$ is equal to

\begin{itemize}
  \item[0] if and only if $z_0$ is an isolated point of $\Omega$,
  \item[1] if and only if $z_0$ is a non-isolated point of $\Omega$ which does not lie in the closure of the interior of $\Omega$, and
  \item[2] if and only if $z_0$ lies in the closure of the interior of $\Omega$.
\end{itemize}

As in the case of the boundary, the local dimension of a point $z_0$ in $\mathcal{W}^\varepsilon(P(z); \psi)$ is strongly connected to the local dimension of the origin in the set $F^\varepsilon(P(z_0); \psi)$.

**Theorem 6.1.** (For the standard numerical range of square matrix polynomials, see Theorem 1 in \cite{28}.) Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0, 1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^\varepsilon(\mathfrak{X}; \psi) \neq \{0\}$. Let also $z_0 \in \mathcal{W}^\varepsilon(P(z); \psi)$ with local dimension in $\mathcal{W}^\varepsilon(P(z); \psi)$ equal to 1, such that $F^\varepsilon(P(z_0); \psi) \neq \{0\}$, the origin is a differentiable point of $\partial F^\varepsilon(P(z_0); \psi)$ and $0 \notin F^\varepsilon(P'(z_0); \psi)$. Then, the local dimension of 0 in $F^\varepsilon(P(z_0); \psi)$ is 1.

**Proof.** Since the local dimension of $z_0$ in $\mathcal{W}^\varepsilon(P(z); \psi)$ is equal to 1, it follows that $z_0 \in \partial \mathcal{W}^\varepsilon(P(z); \psi)$, $z_0$ is not an isolated point of $\mathcal{W}^\varepsilon(P(z); \psi)$, and there is a real number $r > 0$ such that $\mathcal{W}^\varepsilon(P(z); \psi) \cap D(z_0, r) \subseteq \partial \mathcal{W}^\varepsilon(P(z); \psi)$. For the sake of contradiction, assume that the local dimension of the origin in $F^\varepsilon(P(z_0); \psi)$ is equal to 2 (i.e., the convex set $F^\varepsilon(P(z_0); \psi)$ has a non-empty interior).

By Theorem 5.1, for every $z \in D(z_0, r)$, it holds that $0 \in \partial F^\varepsilon(P(z); \psi)$. Moreover, the origin is a differentiable point of $\partial F^\varepsilon(P(z_0); \psi)$, and hence, there is a unique tangent line of $\partial F^\varepsilon(P(z_0); \psi)$ at the origin, which defines a closed half-plane $\mathcal{H}_1$ and an open half-plane $\mathcal{H}_2 = \mathbb{C} \setminus \mathcal{H}_1$, such that $F^\varepsilon(P(z_0); \psi) \subset \mathcal{H}_1$.

For every $r \in [0, r]$ and $\theta \in [0, 2\pi]$, $z_0 + re^{i\theta}$ is either a boundary point or an exterior point of the set $\mathcal{W}^\varepsilon(P(z); \psi)$. As a consequence, for every $r \in [0, r]$ and $\theta \in [0, 2\pi]$, the origin is either a boundary point or an exterior point of the convex set $F^\varepsilon(P(z_0 + re^{i\theta}); \psi)$. Moreover, it holds

$$P(z_0 + re^{i\theta}) = P(z_0) + re^{i\theta} P'(z_0) + re^{i\theta} R(z_0, re^{i\theta}),$$
where \( R(z_0, \rho e^{i\theta}) \) is a vector-valued polynomial in \( z_0 \) and \( \rho e^{i\theta} \), such that \( \| R(z_0, \rho e^{i\theta}) \| \to 0 \) as \( \rho \to 0 \). Since \( 0 \notin F^\varepsilon_{\| \cdot \|} (P(z_0); \psi) \), subadditivity implies that for small enough \( r \), there exists a convex cone

\[
\mathcal{K}(z_0, r) = \{ z \in \mathbb{C} : \theta_1 \leq \arg(z) \leq \theta_2, \ 0 < \theta_2 - \theta_1 \leq \theta_3 < \pi \},
\]

such that for every \( \rho \in [0, r] \) and \( \theta \in [0, 2\pi] \),

\[
F^\varepsilon_{\| \cdot \|} (P'(z_0) + R(z_0, \rho e^{i\theta}); \psi) \subset \mathcal{K}(z_0, r) \setminus \{ 0 \}.
\]

For suitable \( \theta \in [0, 2\pi] \),

\[
e^{i\theta} F^\varepsilon_{\| \cdot \|} (P'(z_0) + R(z_0, \rho e^{i\theta}); \psi) \subset e^{i\theta} \mathcal{K}(z_0, r) \setminus \{ 0 \} \subset \mathcal{H}_2.
\]

Then, for every linear functional \( f \in L_\varepsilon(\psi) \),

\[
f(P(z_0 + \rho e^{i\theta})) = \frac{f(P(z_0))}{\sqrt{1 - \varepsilon^2 \| \psi \|^2}} + \frac{\rho e^{i\theta} f(P'(z_0) + R(z_0, \rho e^{i\theta}))}{\sqrt{1 - \varepsilon^2 \| \psi \|^2}}.
\]

where

\[
\frac{f(P(z_0))}{\sqrt{1 - \varepsilon^2 \| \psi \|^2}} \in F^\varepsilon_{\| \cdot \|} (P(z_0); \psi) \subset \mathcal{H}_1
\]

and

\[
\frac{\rho e^{i\theta} f(P'(z_0) + R(z_0, \rho e^{i\theta}))}{\sqrt{1 - \varepsilon^2 \| \psi \|^2}} \in e^{i\theta} \mathcal{K}(z_0, r) \setminus \{ 0 \} \subset \mathcal{H}_2.
\]

Consequently, as \( \rho \) takes values from 0 to \( r \), a part of \( F^\varepsilon_{\| \cdot \|} (P(z_0 + \rho e^{i\theta}); \psi) \), in a neighborhood of the origin, is moving continuously into the half-plane \( \mathcal{H}_2 \). Thus, there is an \( r_0 \in (0, r] \) such that the origin lies in the interior of \( F^\varepsilon_{\| \cdot \|} (P(z_0) + r_0 e^{i\theta}; \psi) \) and Property \( (P_2) \) imply that the local dimension of 0 in \( F^\varepsilon_{\| \cdot \|} (P(z_0); \psi) \) is equal to 2. As a consequence, we have the following corollary.

**Corollary 6.2.** Let \( P(z) \) be a vector-valued polynomial as in (6), \( \varepsilon \in (0, 1) \), and \( \psi \in \mathcal{X} \) be a nonzero vector such that \( F^\varepsilon_{\| \cdot \|} (\chi_m; \psi) \neq \{ 0 \} \). Let also \( z_0 \) be a non-isolated boundary point of \( W^\varepsilon_{\| \cdot \|} (P(z); \psi) \) such that \( F^\varepsilon_{\| \cdot \|} (P(z_0); \psi) \neq \{ 0 \} \), the origin is a differentiable point of \( \partial F^\varepsilon_{\| \cdot \|} (P(z_0); \psi) \) and \( 0 \notin F^\varepsilon_{\| \cdot \|} (P'(z_0); \psi) \). Then the local dimension of \( z_0 \) in \( W^\varepsilon_{\| \cdot \|} (P(z); \psi) \) is equal to 2.

The case \( \varepsilon = 0 \) is considered in the next result.

**Theorem 6.3.** (For the standard numerical range of square matrix polynomials, see Theorem 2 in [28].) Let \( P(z) \) be a vector-valued polynomial as in (6) and \( \psi \in \mathcal{X} \) be a nonzero vector such that \( F^0_{\| \cdot \|} (\chi_m; \psi) \neq \{ 0 \} \). Let also \( z_0 \) be an interior point of \( W^0_{\| \cdot \|} (P(z); \psi) \) or a differentiable point of \( \partial W^0_{\| \cdot \|} (P(z); \psi) \) with local dimension in \( W^0_{\| \cdot \|} (P(z); \psi) \) equal to 2, such that \( F^0_{\| \cdot \|} (P(z_0); \psi) \neq \{ 0 \} \) and \( 0 \notin F^0_{\| \cdot \|} (P'(z_0); \psi) \). Then, the local dimension of the origin in \( F^0_{\| \cdot \|} (P(z_0); \psi) \) is equal to 2.
Proof. If \( z_0 \) is an interior point of \( W^0_{\|\|}(P(z); \psi) \), then by Theorem 5.2, the origin is also an interior point of \( F^0_{\|\|}(P(z_0); \psi) \). In this case, the local dimension of \( z_0 \) in \( W^0_{\|\|}(P(z); \psi) \) and the local dimension of 0 in \( F^0_{\|\|}(P(z_0); \psi) \) are both equal to 2.

Let \( z_0 \in \partial W^0_{\|\|}(P(z); \psi) \). Since \( z_0 \) is a differentiable point of \( \partial W^\varepsilon_{\|\|}(P(z); \psi) \) and has local dimension 2 in \( W^0_{\|\|}(P(z); \psi) \), there exists a \( \phi_0 \in [0, 2\pi] \) such that for every \( \phi \in (\phi_0, \phi_0 + \pi) \), there is an arbitrarily small \( r_\phi > 0 \) with \( z_0 + r_\phi e^{i\phi} \) lying in the interior of \( W^0_{\|\|}(P(z); \psi) \). For the sake of contradiction, we assume that the origin has local dimension 1 in \( F^0_{\|\|}(P(z_0); \psi) \). Then, by the convexity of the set \( F^0_{\|\|}(P(z_0); \psi) \neq \{0\} \), it follows that \( F^0_{\|\|}(P(z_0); \psi) \) is a (non-degenerate) line segment passing through the origin.

The straight line which is defined by the line segment \( F^0_{\|\|}(P(z_0); \psi) \) defines two closed half-planes \( H_1 \) and \( H_2 \). As in the proof of Theorem 6.1,

\[
P(z_0 + re^{i\phi}) = P(z_0) + re^{i\phi}P'(z_0) + re^{i\phi}R(z_0, re^{i\phi}),
\]

where \( \|R(z_0, re^{i\phi})\| \to 0 \) as \( r \to 0 \). Since \( 0 \notin F^0_{\|\|}(P(z_0); \psi) \), for small enough \( r \), there exists a convex cone

\[
\mathcal{K}(z_0, r) = \{ z \in \mathbb{C} : \theta_1 \leq \arg(z) \leq \theta_2, \ 0 < \theta_2 - \theta_1 \leq \theta_3 < \pi \},
\]

such that

\[
F^0_{\|\|}(P'(z_0) + R(z_0, re^{i\phi}); \psi) \subseteq \mathcal{K}(z_0, r) \setminus \{0\}.
\]

Also, there is a \( \theta \in (\phi_0, \phi_0 + \pi) \) such that the set \( e^{i\theta}F^0_{\|\|}(P'(z_0) + R(z_0, re^{i\phi}); \psi) \) lies in the interior of \( H_1 \) or \( H_2 \). Since

\[
F^0_{\|\|}(P(z_0 + r_\phi e^{i\phi}); \psi) \subseteq F^0_{\|\|}(P(z_0); \psi) + r_\phi e^{i\phi}F^0_{\|\|}(P'(z_0) + R(z_0, re^{i\phi}); \psi),
\]

\[
F^0_{\|\|}(P(z_0 + r_\phi e^{i\phi}); \psi) \subseteq \mathcal{K}(z_0, r) \setminus \{0\} \cup \{ \psi \}
\]

lies in the interior of \( H_1 \) or \( H_2 \). As a consequence, \( 0 \notin F^0_{\|\|}(P(z_0 + r_\phi e^{i\phi}); \psi) \); this is a contradiction because \( z_0 + r_\phi e^{i\phi} \in W^0_{\|\|}(P(z); \psi) \). \( \square \)

Finally, we obtain that bounded Birkhoff-James \( \varepsilon \)-orthogonality sets of linear vector-valued polynomials are simply connected.

**Theorem 6.4.** (For the standard numerical range of square matrix polynomials, see Theorem 4 in [28].) Let \( \chi_1z + \chi_0 \) be a linear vector-valued polynomial, \( \varepsilon \in [0, 1) \), and \( \psi \in \mathcal{X} \) be a nonzero vector such that \( F^0_{\|\|}(\chi_1; \psi) \neq \{0\} \). If the set \( W^\varepsilon_{\|\|}(\chi_1z + \chi_0; \psi) \) is bounded, then it is simply connected.

**Proof.** Suppose \( W^\varepsilon_{\|\|}(\chi_1z + \chi_0; \psi) \) is not simply connected. Then there is a complex number \( w_0 \notin W^\varepsilon_{\|\|}(\chi_1z + \chi_0; \psi) \) such that for every \( \phi \in [0, 2\pi] \), there exists an \( r_\phi > 0 \) such that \( w_0 + r_\phi e^{i\phi} \in W^\varepsilon_{\|\|}(\chi_1z + \chi_0; \psi) \). By Property (P12), for any scalar \( a \in \mathbb{C} \), it holds that \( W^\varepsilon_{\|\|}(\chi_1(z + a) + \chi_0; \psi) = W^\varepsilon_{\|\|}(\chi_1z + \chi_0; \psi) - a \). Thus, without loss of generality, we may assume that \( w_0 = 0 \).
By the boundedness of $W_{\varepsilon\|\cdot\|}(\chi_1 z + \chi_0; \psi)$ and the assumption that the origin does not lie in $W_{\varepsilon\|\cdot\|}(\chi_1 z + \chi_0; \psi)$, both convex sets $F_{\varepsilon\|\cdot\|}(\chi_1; \psi)$ and $F_{\varepsilon\|\cdot\|}(\chi_0; \psi)$ do not contain the origin. As a consequence, there exist two convex cones

$$K_1 = \left\{ z \in \mathbb{C} : \theta_1 \leq \arg(z) \leq \tilde{\theta}_1, \ 0 < \tilde{\theta}_1 - \theta_1 \leq \xi_1 < \pi \right\}$$

and

$$K_2 = \left\{ z \in \mathbb{C} : \theta_2 \leq \arg(z) \leq \tilde{\theta}_2, \ 0 < \tilde{\theta}_2 - \theta_2 \leq \xi_2 < \pi \right\},$$

such that $F_{\varepsilon\|\cdot\|}(\chi_1; \psi)$ lies in the interior of $K_1$ and $F_{\varepsilon\|\cdot\|}(\chi_0; \psi)$ lies in the interior of $K_2$. Hence, there exists a $\phi_0 \in [0, 2\pi]$ such that the convex regions $F_{\varepsilon\|\cdot\|}(r_{\phi_0} e^{i\phi_0} \chi_1; \psi) = r_{\phi_0} e^{i\phi_0} F_{\varepsilon\|\cdot\|}(\chi_1; \psi)$ and $F_{\varepsilon\|\cdot\|}(\chi_0; \psi)$ lie in the interior of the convex cone

$$K_0 = \left\{ z \in \mathbb{C} : \theta_0 \leq \arg(z) \leq \tilde{\theta}_0, \ 0 < \tilde{\theta}_0 - \theta_0 \leq \xi_0 < \pi \right\},$$

where $\max\{\xi_1, \xi_2\} \leq \xi_0$. Therefore, by the subadditivity of Proposition 2.3, the set

$$F_{\varepsilon\|\cdot\|}(\chi_1 r_{\phi_0} e^{i\phi_0} + \chi_0; \psi) \subseteq r_{\phi_0} e^{i\phi_0} F_{\varepsilon\|\cdot\|}(\chi_1; \psi) + F_{\varepsilon\|\cdot\|}(\chi_0; \psi)$$

lies in the interior of $K_0$, and it does not contain the origin; this is a contradiction. \qed

References


