Birkhoff-James ε-orthogonality sets of vectors and vector-valued polynomials

Vasiliki Panagakou*, Panayiotis Psarrakos† and Nikos Yannakakis‡

May 5, 2017

Abstract

Consider a complex normed linear space $(\mathcal{X}, \|\cdot\|)$, and let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$. Motivated by recent works on rectangular matrices and on normed linear spaces, we study the Birkhoff-James ε-orthogonality set of χ with respect to ψ, give an alternative definition for this set, and explore its rich structure. We also introduce the Birkhoff-James ε-orthogonality set of polynomials in one complex variable whose coefficients are members of $\mathcal{X}$, and survey and record extensions of results on matrix polynomials to these vector-valued polynomials.

Key words: norm, vector-valued polynomial, Birkhoff-James orthogonality, Birkhoff-James ε-orthogonality, numerical range.

AMS Subject Classifications: 46B99, 47A12.

1 Introduction

Let $(\mathcal{A}, \|\cdot\|)$ (for simplicity, $\mathcal{A}$) be a unital normed algebra over $\mathbb{C}$, and let $\mathcal{A}^*$ be the dual space of $\mathcal{A}$, i.e., the Banach space of all continuous linear functionals of $\mathcal{A}$ (using the induced operator norm). The *numerical range* (also known as the *field of values*) of an element $\alpha \in \mathcal{A}$ is defined as

$$F(\alpha) = \{f(\alpha) : f \in \mathcal{A}^*, \, f(\mathbf{1}) = 1, \, \|f\| = 1\}. \tag{1}$$

*Department of Mathematics, National Technical University of Athens, Greece (vaspanagakou@gmail.com).

†Department of Mathematics, National Technical University of Athens, Greece (ppsarr@math.ntua.gr).

‡Department of Mathematics, National Technical University of Athens, Greece (nyian@math.ntua.gr).

This set has been studied extensively, and is useful in understanding matrices and operators; see [3, 4, 15, 30] and the references therein. Stampfli and Williams [30, Theorem 4], and later Bonsall and Duncan [4, Lemma 6.22.1], observed that the numerical range $F(\alpha)$ can be written in the form

$$F(\alpha) = \{\mu \in \mathbb{C} : \, \|\alpha - \lambda \mathbf{1}\| \geq |\mu - \lambda|, \; \forall \, \lambda \in \mathbb{C}\} \,.$$

This means that $F(\alpha)$ is an infinite intersection of closed (circular) disks

$$\mathcal{D}\left(\lambda, \|\alpha - \lambda \mathbf{1}\|\right) = \{\mu \in \mathbb{C} : \, |\mu - \lambda| \leq \|\alpha - \lambda \mathbf{1}\|\} \,, \quad \lambda \in \mathbb{C},$$

namely,

$$F(\alpha) \; = \; \bigcap_{\lambda \in \mathbb{C}} \{\mu \in \mathbb{C} : \, |\mu - \lambda| \leq \|\alpha - \lambda \mathbf{1}\|\} \; = \; \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \|\alpha - \lambda \mathbf{1}\|\right) . \qquad (2)$$

For two elements χ and ψ of a complex normed linear space $(\mathcal{X}, \|\cdot\|)$, χ is said to be *Birkhoff-James orthogonal* to ψ, denoted by $\chi \perp_{BJ} \psi$, if $\|\chi + \lambda \psi\| \geq \|\chi\|$ for all $\lambda \in \mathbb{C}$ [2, 18]. This orthogonality is homogeneous, but it is neither symmetric nor additive [18]. Moreover, for any $\varepsilon \in [0, 1)$, χ is called *Birkhoff-James ε-orthogonal* to ψ, denoted by $\chi \perp_{BJ}^{\varepsilon} \psi$, if $\|\chi + \lambda \psi\| \geq \sqrt{1 - \varepsilon^2} \, \|\chi\|$ for all $\lambda \in \mathbb{C}$ [5, 8]. It is worth mentioning that this relation is also homogeneous. In an inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$, with the standard orthogonality relation $\perp$, a $\chi \in \mathcal{X}$ is called *ε-orthogonal* to a $\psi \in \mathcal{X}$, denoted by $\chi \perp^{\varepsilon} \psi$, if $|\langle \chi, \psi \rangle| \leq \varepsilon \, \|\chi\| \, \|\psi\|$. Furthermore, for any $\varepsilon \in [0, 1)$, $\chi \perp^{\varepsilon} \psi$ if and only if $\chi \perp_{BJ}^{\varepsilon} \psi$ [5, 8].

Inspired by (2) and the above definition of Birkhoff-James ε-orthogonality, Chorianopoulos and Psarrakos [7] (for rectangular matrices), and Karamanlis and Psarrakos [20] (for elements of a normed linear space) introduced and studied the following set: For any $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$, and any $\varepsilon \in [0, 1)$, the *Birkhoff-James ε-orthogonality set of χ with respect to ψ* is defined and denoted by

$$F_{\|\cdot\|}^{\varepsilon}(\chi; \psi) = \{\mu \in \mathbb{C} : \, \psi \perp_{BJ}^{\varepsilon} (\chi - \mu \psi)\} \,.$$

The Birkhoff-James ε-orthogonality set is a direct generalization of the numerical range, and appears to have a rich structure and interesting geometrical properties [7, 20]. In this paper, motivated by (1), we introduce a new (equivalent) definition for the Birkhoff-James ε-orthogonality set, using continuous linear functionals. Based on this definition, in the next section, we obtain some basic properties of the set $F_{\|\cdot\|}^{\varepsilon}(\chi; \psi)$ such as subadditivity in χ. In Section 3, we introduce the Birkhoff-James ε-orthogonality set of vector-valued polynomials in one complex variable, and investigate its localization in the complex plane. In Sections 4, 5 and 6, we study the connected components of the Birkhoff-James ε-orthogonality set of vector-valued polynomials, the boundary of this set, and the local dimension of its points, respectively. The proof techniques are analogous to existing proofs [22, 24, 25, 27, 28], albeit modified and adapted to the new setting. The main contribution of this effort is a concise generalization to a new concept. Furthermore, the results indicate that the information on Birkhoff-James ε-orthogonality set is useful in understanding vector-valued polynomials.

2 Definition and basic properties

Consider a complex normed linear space $(\mathcal{X}, \|\cdot\|)$ (for simplicity, $\mathcal{X}$), and let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$. For any $\varepsilon \in [0,1)$, it is straightforward to see that

$$\begin{aligned}
F^{\varepsilon}_{\|\cdot\|}(\chi;\psi) &= \{\mu \in \mathbb{C} : \psi \perp^{\varepsilon}_{BJ} (\chi - \mu\psi)\} &(3)\\
&= \left\{\mu \in \mathbb{C} : \|\psi - \lambda(\chi - \mu\psi)\| \geq \sqrt{1-\varepsilon^2}\,\|\psi\|,\ \forall\,\lambda \in \mathbb{C}\right\}\\
&= \left\{\mu \in \mathbb{C} : \left\|\psi - \frac{1}{\lambda}(\chi - \mu\psi)\right\| \geq \sqrt{1-\varepsilon^2}\,\|\psi\|,\ \forall\,\lambda \in \mathbb{C}\setminus\{0\}\right\}\\
&= \left\{\mu \in \mathbb{C} : \frac{1}{|\lambda|}\,\|\lambda\psi - (\chi - \mu\psi)\| \geq \sqrt{1-\varepsilon^2}\,\|\psi\|,\ \forall\,\lambda \in \mathbb{C}\setminus\{0\}\right\}\\
&= \left\{\mu \in \mathbb{C} : \|\chi - (\mu - \lambda)\psi\| \geq \sqrt{1-\varepsilon^2}\,\|\psi\|\,|\lambda|,\ \forall\,\lambda \in \mathbb{C}\right\}\\
&= \left\{\mu \in \mathbb{C} : \|\chi - \lambda\psi\| \geq \sqrt{1-\varepsilon^2}\,\|\psi\|\,|\mu - \lambda|,\ \forall\,\lambda \in \mathbb{C}\right\} &(4)\\
&= \bigcap_{\lambda\in\mathbb{C}} \mathcal{D}\left(\lambda, \frac{\|\chi - \lambda\psi\|}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right). &(5)
\end{aligned}$$

Corollary 2.2 of [18] implies that $F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$ is always *non-empty* (see also Proposition 2.1 of [20]), and the defining formula (5) ensures that $F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$ is a *compact* and *convex* subset of $\mathbb{C}$ that lies in the closed disk $\mathcal{D}\left(0, \frac{\|\chi\|}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right)$. Moreover, it is apparent that for any $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, $F^{\varepsilon_1}_{\|\cdot\|}(\chi;\psi) \subseteq F^{\varepsilon_2}_{\|\cdot\|}(\chi;\psi)$. The Birkhoff-James ε-orthogonality set is a direct generalization of the standard numerical range. In particular, for $\mathcal{X} = \mathcal{A}$, $\chi = \alpha$, $\psi = \mathbf{1}$ and $\varepsilon = 0$, we have $F^{0}_{\|\cdot\|}(\alpha;\mathbf{1}) = F(\alpha)$; see (2) and (5).

Remark 2.1. Let $\chi, \psi \in \mathcal{X}$ be nonzero, with ψ not a scalar multiple of χ, and consider the distance from ψ to $\mathrm{span}\{\chi\}$, $\mathrm{dist}(\psi, \mathrm{span}\{\chi\}) = \inf_{\lambda\in\mathbb{C}} \|\psi - \lambda\chi\|$. Then, for any $\varepsilon \in [0,1)$, it follows

$$\begin{aligned}
0 \in F^{\varepsilon}_{\|\cdot\|}(\chi;\psi) &\iff \psi \perp^{\varepsilon}_{BJ} \chi\\
&\iff \|\psi - \lambda\chi\| \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\|,\ \ \forall\lambda \in \mathbb{C}\\
&\iff \inf_{\lambda\in\mathbb{C}} \|\psi - \lambda\chi\| \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\| \quad (\psi \notin \mathrm{span}\{\chi\})\\
&\iff \mathrm{dist}(\psi, \mathrm{span}\{\chi\}) \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\|.
\end{aligned}$$

Clearly, for $\varepsilon = 0$, $0 \in F^{0}_{\|\cdot\|}(\chi;\psi)$ if and only if $\mathrm{dist}(\psi, \mathrm{span}\{\chi\}) = \|\psi\|$. Moreover, if $0 \notin F^{0}_{\|\cdot\|}(\chi;\psi)$ (or equivalently, if $\mathrm{dist}(\psi, \mathrm{span}\{\chi\}) < \|\psi\|$), then by Theorems 3.1 and 3.5 of [20] (see also Properties (P_6) and (P_8) below), there is a *unique* number $\varepsilon_0 \in [0,1)$ such that the origin lies on the boundary $\partial F^{\varepsilon_0}_{\|\cdot\|}(\chi;\psi)$ and $\mathrm{dist}(\psi, \mathrm{span}\{\chi\}) = \sqrt{1-{\varepsilon_0}^2}\,\|\psi\|$. This number ε_0 is the smallest value of the parameter $\varepsilon \in [0,1)$ with $0 \in F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$.

We remark that in the remainder of the paper, the zero vector is always considered as a scalar multiple of ψ.

Let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$. Next, for convenience, we summarize the results of [20] (see also [6, 7] for rectangular matrices), describing basic properties of the Birkhoff-James ε-orthogonality set.

(P_1) For any $a, b \in \mathbb{C}$ and any $\varepsilon \in [0,1)$, $F^{\varepsilon}_{\|\cdot\|}(a\chi + b\psi; \psi) = a\, F^{\varepsilon}_{\|\cdot\|}(\chi; \psi) + b$.

(P_2) For any nonzero $b \in \mathbb{C}$ and any $\varepsilon \in [0,1)$, $F^{\varepsilon}_{\|\cdot\|}(\chi; b\psi) = \frac{1}{b}\, F^{\varepsilon}_{\|\cdot\|}(\chi; \psi)$.

(P_3) If χ is a nonzero element of $\mathcal{X}$, then for any $\varepsilon \in [0,1)$,

$$\left\{ \mu^{-1} \in \mathbb{C} :\ \mu \in F^{\varepsilon}_{\|\cdot\|}(\chi; \psi),\ |\mu| \geq \frac{\|\chi\|}{\|\psi\|} \right\} \subseteq F^{\varepsilon}_{\|\cdot\|}(\psi; \chi).$$

(P_4) Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two equivalent norms acting in $\mathcal{X}$, i.e., there exist two real numbers $C, c > 0$ such that $c\,\|\zeta\|_\alpha \leq \|\zeta\|_\beta \leq C\,\|\zeta\|_\alpha$ for all $\zeta \in \mathcal{X}$. Then for any $\varepsilon \in [0,1)$, it holds that $F^{\varepsilon}_{\|\cdot\|_\alpha}(\chi; \psi) \subseteq F^{\varepsilon'}_{\|\cdot\|_\beta}(\chi; \psi)$, where $\varepsilon' = \sqrt{1 - \frac{c^2(1-\varepsilon^2)}{C^2}}$.

(P_5) $\chi = a\psi$ for some $a \in \mathbb{C}$ if and only if $F^{\varepsilon}_{\|\cdot\|}(\chi; \psi) = \{a\}$ for every $\varepsilon \in [0,1)$.

(P_6) If χ is not a scalar multiple of ψ, then for any $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, $F^{\varepsilon_1}_{\|\cdot\|}(\chi; \psi)$ lies in the interior of $F^{\varepsilon_2}_{\|\cdot\|}(\chi; \psi)$.

(P_7) If χ is not a scalar multiple of ψ, then for any $\varepsilon \in (0,1)$, $F^{\varepsilon}_{\|\cdot\|}(\chi; \psi)$ has a non-empty interior.

(P_8) If χ is not a scalar multiple of ψ, then for any bounded region $\Omega \subset \mathbb{C}$, there is an $\varepsilon_\Omega \in [0,1)$ such that $\Omega \subseteq F^{\varepsilon_\Omega}_{\|\cdot\|}(\chi; \psi)$. (This means that if χ is not a scalar multiple of ψ, then $F^{\varepsilon}_{\|\cdot\|}(\chi; \psi)$ can be arbitrarily large for ε sufficiently close to 1.)

(P_9) Let $\mu_0 \in F^{\varepsilon}_{\|\cdot\|}(\chi; \psi)$ for some $\varepsilon \in [0,1)$.

(i) The scalar μ_0 lies on the boundary $\partial F^{\varepsilon}_{\|\cdot\|}(\chi; \psi)$ if and only if

$$\inf_{\lambda \in \mathbb{C}} \left\{ \|\chi - \lambda\psi\| - \sqrt{1-\varepsilon^2}\, \|\psi\|\, |\mu_0 - \lambda| \right\} = 0.$$

(ii) If $\varepsilon > 0$, then $\mu_0 \in \partial F^{\varepsilon}_{\|\cdot\|}(\chi; \psi)$ if and only if

$$\min_{\lambda \in \mathbb{C}} \left\{ \|\chi - \lambda\psi\| - \sqrt{1-\varepsilon^2}\, \|\psi\|\, |\mu_0 - \lambda| \right\} = 0,$$

or equivalently, if and only if $\|\chi - \lambda_0\psi\| = \sqrt{1-\varepsilon^2}\, \|\psi\|\, |\mu_0 - \lambda_0|$ for some $\lambda_0 \in \mathbb{C}$.

(P_{10}) For any $\varepsilon \in (0,1)$,

$$\mathrm{Int}\left[F^{\varepsilon}_{\|\cdot\|}(\chi; \psi) \right] = \left\{ \mu \in \mathbb{C} :\ \|\chi - \lambda\psi\| > \sqrt{1-\varepsilon^2}\, \|\psi\|\, |\mu - \lambda|,\ \forall\, \lambda \in \mathbb{C} \right\}.$$

(P_{11}) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then for any $\varepsilon \in [0,1)$,

$$F^{\varepsilon}_{\|\cdot\|}(\chi; \psi) = \mathcal{D}\left(\frac{\langle \chi, \psi \rangle}{\|\psi\|^2}\, ,\ \left\| \chi - \frac{\langle \chi, \psi \rangle}{\|\psi\|^2}\, \psi \right\| \frac{\varepsilon}{\sqrt{1-\varepsilon^2}\, \|\psi\|} \right).$$

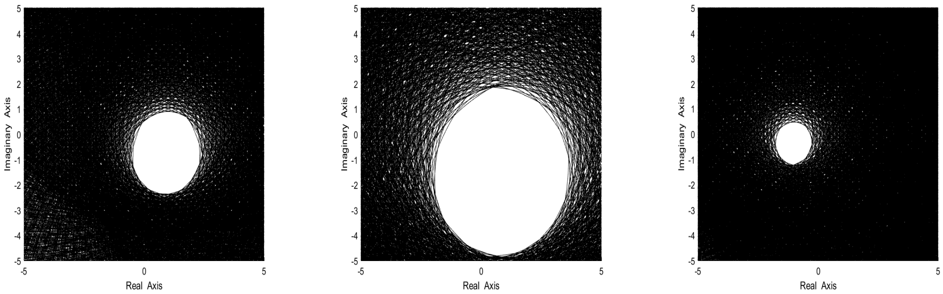

Figure 1: The sets $F^{0.5}_{\|\cdot\|_1}(\chi;\psi)$ (left), $F^{0.65}_{\|\cdot\|_1}(\chi;\psi)$ (middle), and $F^{0.5}_{\|\cdot\|_1}(\chi - 3\psi; 2\psi)$ (right).

Example 2.1. Consider the 2×4 complex matrices $\chi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2+\mathrm{i} & 0 & -\mathrm{i} & -11\,\mathrm{i} \end{bmatrix}$ and $\psi = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$. The Birkhoff-James ε-orthogonality sets $F^{0.5}_{\|\cdot\|_1}(\chi;\psi)$, $F^{0.65}_{\|\cdot\|_1}(\chi;\psi)$ and $F^{0.5}_{\|\cdot\|_1}(\chi - 3\psi; 2\psi) = \dfrac{1}{2}\, F^{0.5}_{\|\cdot\|_1}(\chi;\psi) - \dfrac{3}{2}$ are estimated by the unshaded regions in the left, middle and right parts of Figure 2, respectively. Each estimation results from having drawn 1000 circles of the form $\partial\mathcal{D}\left(\lambda, \frac{\|\chi-\lambda\psi\|}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right)$; see the defining formula (5). The compactness and the convexity of the sets are apparent, and Properties (P_1), (P_2), (P_6) and (P_7) are verified.

Let $\mathcal{X}^*$ denote the dual space of $\mathcal{X}$, i.e., the complex normed linear space of all continuous linear functionals of $\mathcal{X}$ (using the induced operator norm).

Definition 2.1. Let $\chi, \psi \in \mathcal{X}$ with $\psi \neq 0$. For any $\varepsilon \in [0,1)$, define the sets

$$L_\varepsilon(\psi) = \left\{ f \in \mathcal{X}^* : \, f(\psi) = \sqrt{1-\varepsilon^2}\,\|\psi\| \text{ and } \|f\| \leqslant 1 \right\}$$

and

$$\Omega_\varepsilon(\chi;\psi) \; = \; \left\{ \frac{f(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \, : \, f \in L_\varepsilon(\psi) \right\}.$$

Lemma 2.1. *For any nonzero vector $\psi \in \mathcal{X}$ and any $\varepsilon \in [0,1)$, the set $L_\varepsilon(\psi)$ is non-empty, closed and convex.*

Proof. Consider an element $\chi \in \mathcal{X}$ which is not a scalar multiple of ψ. From Corollary 2.1 in [20], the Birkhoff-James ε-orthogonality set $F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$ is not empty. So, there exists at least one complex number μ in the set $F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$. In the 2-dimensional vector subspace $\mathcal{Y} = \operatorname{span}\{\chi,\psi\}$, we define the linear functional $f_0 \in \mathcal{Y}^*$ such that

$$f_0(z_1\chi + z_2\psi) = z_1\mu\sqrt{1-\varepsilon^2}\,\|\psi\| + z_2\sqrt{1-\varepsilon^2}\,\|\psi\|, \quad z_1, z_2 \in \mathbb{C}.$$

Then $f_0(\chi) = \mu\sqrt{1-\varepsilon^2}\,\|\psi\|$ and $f_0(\psi) = \sqrt{1-\varepsilon^2}\,\|\psi\|$. Since $\mu \in F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$, we have that for every $\lambda \in \mathbb{C}$,

$$\begin{aligned}
\|\chi - \lambda\psi\| &\geqslant \sqrt{1-\varepsilon^2}\,\|\psi\|\,|\mu - \lambda| \\
&= |\,\sqrt{1-\varepsilon^2}\,\|\psi\|\,\mu - \sqrt{1-\varepsilon^2}\,\|\psi\|\,\lambda\,| \\
&= |f_0(\chi) - \lambda f_0(\psi)| \\
&= |f_0(\chi - \lambda\psi)|,
\end{aligned}$$

and $\|f_0\| \leqslant 1$ (as a continuous linear functional defined in the 2-dimensional subspace $\mathcal{Y}$). Applying the Hahn-Banach extension theorem, there is an extension of f_0, say $f \in \mathcal{X}^*$, such that

$$f(\chi) = \mu\sqrt{1-\varepsilon^2}\,\|\psi\|, \quad f(\psi) = \sqrt{1-\varepsilon^2}\,\|\psi\| \quad \text{and} \quad \|f\| = \|f_0\| \leqslant 1.$$

Then, $f \in L_\varepsilon(\psi)$, and the set $L_\varepsilon(\psi)$ is non-empty.

For the closedness of the set $L_\varepsilon(\psi)$, it is enough to see that the set $\mathcal{X}^*\backslash L_\varepsilon(\psi)$ is open. Indeed, if a linear functional $f \in \mathcal{X}^*$ does not belong to $L_\varepsilon(\psi)$, then

$$f(\psi) \neq \sqrt{1-\varepsilon^2}\,\|\psi\| \quad \text{or} \quad \|f\| > 1.$$

Consequently, by the continuity of the norm, there is a neighborhood $\mathcal{G}_f \subset \mathcal{X}^*$ of f such that for any $g \in \mathcal{G}_f$,

$$g(\psi) \neq \sqrt{1-\varepsilon^2}\,\|\psi\| \quad \text{or} \quad \|g\| > 1,$$

and so $\mathcal{G}_f \subset \mathcal{X}^*\backslash L_\varepsilon(\psi)$.

Finally, for the convexity, we consider two linear functionals $f, g \in L_\varepsilon(\psi)$. It is easy to see that for any $t \in [0,1]$,

$$[(1-t)f + t\,g](\psi) = (1-t)f(\psi) + t\,g(\psi) = \sqrt{1-\varepsilon^2}\,\|\psi\|$$

and

$$\|(1-t)f + t\,g\| \leqslant (1-t)\|f\| + t\,\|g\| \leqslant 1,$$

and hence, $(1-t)f + t\,g$ lies in $L_\varepsilon(\psi)$. □

We have proved that for $\psi \neq 0$, the set $L_\varepsilon(\psi)$ is non-empty. As a consequence, the region $\Omega_\varepsilon(\chi;\psi)$ is non-empty. Moreover, the set $\Omega_\varepsilon(\chi;\psi)$ coincides with the Birkhoff-James ε-orthogonality set $F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$.

Theorem 2.2. *Let $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$. For every $\varepsilon \in [0,1)$, it holds that*

$$\Omega_\varepsilon(\chi;\psi) = F^{\varepsilon}_{\|\cdot\|}(\chi;\psi).$$

Proof. Let $\mu \in \Omega_\varepsilon(\chi;\psi)$. Then, $\mu = \dfrac{f_\mu(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|}$ for some linear functional $f_\mu \in L_\varepsilon(\psi)$. For every $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\sqrt{1-\varepsilon^2}\,\|\psi\|\,|\mu-\lambda| &= \left|\sqrt{1-\varepsilon^2}\,\|\psi\|\,\frac{f_\mu(\chi)-\sqrt{1-\varepsilon^2}\,\|\psi\|\,\lambda}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right| \\
&= |f_\mu(\chi-\lambda\psi)| \\
&\leqslant \|f_\mu\|\,\|\chi-\lambda\psi\| \\
&\leqslant \|\chi-\lambda\psi\|.
\end{aligned}$$

Thus, $\mu \in F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$, and clearly, $\Omega_\varepsilon(\chi;\psi) \subseteq F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$.

For the converse, we consider two cases:

(i) Suppose that $\chi = c\psi$ for a constant $c \in \mathbb{C}$. Then, by Property (P_5), $F^{\varepsilon}_{\|\cdot\|}(\chi;\psi) = F^{\varepsilon}_{\|\cdot\|}(c\psi;\psi) = \{c\}$. Also,

$$\frac{f(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} = \frac{f(c\psi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} = \frac{cf(\psi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} = c, \quad \forall\, f \in L_\varepsilon(\psi),$$

and hence, $\Omega_\varepsilon(c\psi;\psi) = \{c\}$.

(ii) Suppose that $\chi,\psi \in \mathcal{X}$ are nonzero and linearly independent, and consider a scalar $\mu \in F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$. By the proof of Lemma 2.1, there exists a continuous linear functional $f_\mu \in L_\varepsilon(\psi)$ such that $f_\mu(\chi) = \mu\sqrt{1-\varepsilon^2}\,\|\psi\|$. Thus, $\mu \in \Omega_\varepsilon(\chi;\psi)$, and the proof is complete. ◻

The above alternative definition of the Birkhoff-James ε-orthogonality set yields readily the subadditivity of $F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$ in χ, which is necessary for the proofs of our results in Sections 5 and 6.

Proposition 2.3. *Let $\chi_1,\chi_2,\psi \in \mathcal{X}$, with $\psi \neq 0$. Then, it holds that*

$$F^{\varepsilon}_{\|\cdot\|}(\chi_1+\chi_2;\psi) \subseteq F^{\varepsilon}_{\|\cdot\|}(\chi_1;\psi) + F^{\varepsilon}_{\|\cdot\|}(\chi_2;\psi).$$

Proof. It is easy to see that

$$\begin{aligned}
F^{\varepsilon}_{\|\cdot\|}(\chi_1+\chi_2;\psi) &= \Omega_\varepsilon(\chi_1+\chi_2;\psi) \\
&= \left\{\frac{f(\chi_1+\chi_2)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} : f \in L_\varepsilon(\psi)\right\} \\
&= \left\{\frac{f(\chi_1)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} + \frac{f(\chi_2)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} : f \in L_\varepsilon(\psi)\right\} \\
&\subseteq \left\{\frac{f(\chi_1)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} : f \in L_\varepsilon(\psi)\right\} + \left\{\frac{g(\chi_2)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} : g \in L_\varepsilon(\psi)\right\} \\
&= \Omega_\varepsilon(\chi_1;\psi) + \Omega_\varepsilon(\chi_2;\psi) \\
&= F^{\varepsilon}_{\|\cdot\|}(\chi_1;\psi) + F^{\varepsilon}_{\|\cdot\|}(\chi_2;\psi).
\end{aligned}$$

◻

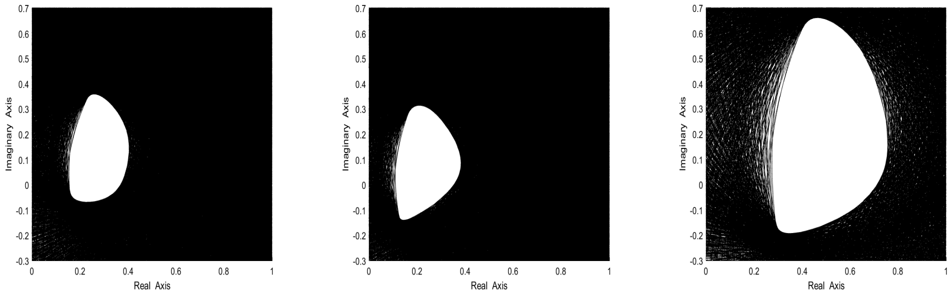

Figure 2: The sets $F^{0.4}_{\|\cdot\|_1}(\chi_1;\psi)$ (left), $F^{0.4}_{\|\cdot\|_1}(\chi_2;\psi)$ (middle), and $F^{0.4}_{\|\cdot\|_1}(\chi_1+\chi_2;\psi)$ (right).

Example 2.2. Consider the sequences $\chi_1 = \left\{1, \dfrac{1}{2-\mathrm{i}}, \dfrac{1}{(2-\mathrm{i})^2}, \dfrac{1}{(2-\mathrm{i})^3}, \dots\right\}$, $\chi_2 = \left\{1, \dfrac{1}{1-2\mathrm{i}}, \dfrac{1}{(1-2\mathrm{i})^2}, \dfrac{1}{(1-2\mathrm{i})^3}, \dots\right\}$ and $\psi = \left\{1, \dfrac{1}{1+\mathrm{i}}, \dfrac{1}{(1+\mathrm{i})^2}, \dfrac{1}{(1+\mathrm{i})^3}, \dots\right\}$ of the complex normed linear space ℓ^1. The Birkhoff-James ε-orthogonality sets $F^{0.4}_{\|\cdot\|_1}(\chi_1;\psi)$, $F^{0.4}_{\|\cdot\|_1}(\chi_2;\psi)$ and $F^{0.4}_{\|\cdot\|_1}(\chi_1+\chi_2;\psi)$ are estimated by the unshaded regions in the left, middle and right parts of Figure 2, respectively. Each estimation results from having drawn 500 circles; see the defining formula (5). The compactness and the convexity of the sets, Property (P_7), and the subadditivity of Proposition 2.3 are verified.

Proposition 2.4. *Let $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$, χ not a scalar multiple of ψ, and $\varepsilon \in [0,1)$. If $\mu \in \partial F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$, then for every continuous linear functional $f_\mu \in L_\varepsilon(\psi)$ such that $\mu = \frac{f_\mu(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|}$, it holds that $\|f_\mu\| = 1$.*

Proof. Let $\mu \in \partial F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)$. Then, by Property (P_9),

$$\inf_{\lambda\in\mathbb{C}} \left\{ \|\chi-\lambda\psi\| - \sqrt{1-\varepsilon^2}\|\psi\|\,|\mu-\lambda| \right\} = 0.$$

For every $f_\mu \in L_\varepsilon(\psi)$ with $\mu = \frac{f_\mu(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|}$, we have

$$\begin{aligned}
0 &= \inf_{\lambda\in\mathbb{C}} \left\{ \|\chi-\lambda\psi\| - \left| \sqrt{1-\varepsilon^2}\,\|\psi\|\,\frac{f_\mu(\chi)-\sqrt{1-\varepsilon^2}\,\|\psi\|\,\lambda}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \right| \right\} \\
&= \inf_{\lambda\in\mathbb{C}} \{ \|\chi-\lambda\psi\| - |f_\mu(\chi)-\lambda f_\mu(\psi)| \} \\
&= \inf_{\lambda\in\mathbb{C}} \{ \|\chi-\lambda\psi\| - |f_\mu(\chi-\lambda\psi)| \} \\
&= -\sup_{\lambda\in\mathbb{C}} \{ |f_\mu(\chi-\lambda\psi)| - \|\chi-\lambda\psi\| \} \\
&= -\sup_{\lambda\in\mathbb{C}} \left\{ \frac{|f_\mu(\chi-\lambda\psi)|}{\|\chi-\lambda\psi\|} - 1 \right\},
\end{aligned}$$

and we conclude that $\|f_\mu\| = 1$. ☐

Proposition 2.5. *Let $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$, χ not a scalar multiple of ψ, and $\varepsilon \in [0,1)$. Then, it holds that*

$$\max\left\{\mathrm{Re}\mu :\ \mu \in F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)\right\} \leqslant \inf_{a>0} \frac{1}{a}\left\{\frac{\|\psi + a\chi\|}{\sqrt{1-\varepsilon^2}\,\|\psi\|} - 1\right\}.$$

Proof. Consider a continuous linear functional $f \in L_{\varepsilon}(\psi)$. Then, for any $a > 0$, we have

$$\begin{aligned}
\frac{f(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} &= \frac{1}{a}\left[\frac{f(\psi + a\chi - \psi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right] \\
&= \frac{1}{a}\left[\frac{f(\psi + a\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} - \frac{f(\psi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right] \\
&= \frac{1}{a}\left[\frac{f(\psi + a\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} - 1\right].
\end{aligned}$$

Hence,

$$\mathrm{Re}\,\frac{f(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} = \mathrm{Re}\,\frac{1}{a}\left[\frac{f(\psi + a\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} - 1\right] = \frac{1}{a}\left[\mathrm{Re}\,\frac{f(\psi + a\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} - 1\right],$$

and consequently,

$$\mathrm{Re}\,\frac{f(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} + \frac{1}{a} = \frac{1}{a}\,\mathrm{Re}\,\frac{f(\psi + a\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \leqslant \frac{1}{a}\left[\frac{|f(\psi + a\chi)|}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right].$$

Thus, for any $a > 0$,

$$\mathrm{Re}\,\frac{f(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \leqslant \frac{1}{a}\left[\frac{|f(\psi + a\chi)|}{\sqrt{1-\varepsilon^2}\,\|\psi\|} - 1\right] \leqslant \frac{1}{a}\left[\frac{\|\psi + a\chi\|}{\sqrt{1-\varepsilon^2}\,\|\psi\|} - 1\right],$$

and the proof is complete. □

3 Vector-valued polynomials

Consider a vector-valued polynomial

$$P(z) = \chi_m z^m + \chi_{m-1} z^{m-1} + \cdots + \chi_1 z + \chi_0, \tag{6}$$

with vector coefficients $\chi_i \in \mathcal{X}$ $(i = 0, 1, \ldots, m)$, $\chi_m \neq 0$, and a scalar variable $z \in \mathbb{C}$. Vector-valued polynomials appear in the approximation of vector-valued functions [1, 29]. Moreover, special cases of vector-valued polynomials such as square matrix polynomials [9, 10, 11, 19, 21], rectangular matrix polynomials [9, 19] and operator polynomials [12, 17, 23, 26], appear in many applications like systems of differential-algebraic equations, linear system theory, control theory, vibration analysis of structural systems, and acoustics.

For any $\varepsilon \in [0,1)$, and any nonzero vector $\psi \in \mathcal{X}$ such that $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$, we can define the Birkhoff-James ε-orthogonality set of $P(z)$ with respect to ψ.

Definition 3.1. Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$. The *Birkhoff-James ε-orthogonality set of $P(z)$ with respect to ψ* is defined and denoted by

$$\begin{aligned} W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) &= \left\{\mu \in \mathbb{C} : \, 0 \in F^{\varepsilon}_{\|\cdot\|}(P(\mu);\psi)\right\} \\ &= \{\mu \in \mathbb{C} : \, f(P(\mu)) = 0, \; f \in L_{\varepsilon}(\psi)\} \\ &= \left\{\mu \in \mathbb{C} : \, f(\chi_m)\mu^m + f(\chi_{m-1})\mu^{m-1} + \cdots + f(\chi_1)\mu + f(\chi_0) = 0, \; f \in L_{\varepsilon}(\psi)\right\} \\ &= \{\mu \in \mathbb{C} : \, \psi \perp^{\varepsilon}_{BJ} P(\mu)\} \qquad (7) \\ &= \left\{\mu \in \mathbb{C} : \, \|P(\mu) - \lambda\psi\| \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\|\,|\lambda|, \; \forall \lambda \in \mathbb{C}\right\}. \end{aligned}$$

Note that for $\chi_m \neq 0$ and $\varepsilon \in (0,1)$, the condition $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$ is always satisfied; see Properties (P_5) and (P_7).

Since the set $L_{\varepsilon}(\psi)$ is non-empty and closed, it follows readily that $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is also non-empty and closed. Moreover, for any $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, $W^{\varepsilon_1}_{\|\cdot\|}(P(z);\psi) \subseteq W^{\varepsilon_2}_{\|\cdot\|}(P(z);\psi)$.

Remark 3.1. Consider a vector-valued polynomial $P(z)$ as in (6), a nonzero vector $\psi \in \mathcal{X}$ with $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$, and a $\mu \in \mathbb{C}$ such that $P(\mu)$ is not a scalar multiple of ψ. For any $\varepsilon \in [0,1)$,

$$\begin{aligned} \mu \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) &\iff \|P(\mu) - \lambda\psi\| \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\|\,|\lambda|, \;\; \forall \lambda \in \mathbb{C} \\ &\iff \left\|\frac{1}{\lambda}P(\mu) - \psi\right\| \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\|, \;\; \forall \lambda \in \mathbb{C} \setminus \{0\} \\ &\iff \|\psi - \lambda P(\mu)\| \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\|, \;\; \forall \lambda \in \mathbb{C} \\ &\iff \inf_{\lambda \in \mathbb{C}} \|\psi - \lambda P(\mu)\| \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\| \quad (\psi \notin \operatorname{span}\{P(\mu)\}) \\ &\iff \operatorname{dist}(\psi, \operatorname{span}\{P(\mu)\}) \geqslant \sqrt{1-\varepsilon^2}\,\|\psi\|. \end{aligned}$$

As in the case of $F^{0}_{\|\cdot\|}(\chi;\psi)$, μ lies in the region $W^{0}_{\|\cdot\|}(P(z);\psi)$ if and only if $\operatorname{dist}(\psi, \operatorname{span}\{P(\mu)\}) = \|\psi\|$. Moreover, if $\mu \notin W^{0}_{\|\cdot\|}(P(z);\psi)$ (or equivalently, if $\operatorname{dist}(\psi, \operatorname{span}\{P(\mu)\}) < \|\psi\|$), then there is a number $\varepsilon_0 \in [0,1)$ such that $\mu \in \partial W^{\varepsilon_0}_{\|\cdot\|}(P(z);\psi)$ and $\operatorname{dist}(\psi, \operatorname{span}\{P(\mu)\}) = \sqrt{1-{\varepsilon_0}^2}\,\|\psi\|$. This number ε_0 can be chosen to be the smallest value of the parameter $\varepsilon \in [0,1)$ with $\mu \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$.

It is easy to verify the next three properties.

(P_{12}) For any scalar $a \in \mathbb{C}\backslash\{0\}$, $W^{\varepsilon}_{\|\cdot\|}(aP(z);\psi) = W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$, $W^{\varepsilon}_{\|\cdot\|}(P(az);\psi) = a^{-1}W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ and $W^{\varepsilon}_{\|\cdot\|}(P(z+a);\psi) = W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) - a$.

(P_{13}) If $R(z) = \chi_0 z^m + \chi_1 z^{m-1} + \cdots + \chi_{m-1}z + \chi_m = z^m P(z^{-1})$ is the *reverse vector-valued polynomial* of $P(z)$, then

$$W^{\varepsilon}_{\|\cdot\|}(R(z);\psi) \setminus \{0\} = \left\{\mu \in \mathbb{C} : \;\; \mu^{-1} \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) \setminus \{0\}\right\}.$$

(P_{14}) If there exists a continuous linear functional $f \in L_\varepsilon(\psi)$ such that $f(\chi_m) = f(\chi_{m-1}) = \cdots = f(\chi_0) = 0$, then $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) = \mathbb{C}$.

For the remainder of the paper, it is necessary to introduce the following radii.

Definition 3.2. Let $\chi, \psi \in \mathcal{X}$, with ψ nonzero. For any $\varepsilon \in [0,1)$, the *Birkhoff-James ε-orthogonality inner radius of χ with respect to ψ* is defined as

$$\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi;\psi) = \min\left\{|z| : z \in F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)\right\},$$

and the *Birkhoff-James ε-orthogonality outer radius of χ with respect to ψ* is defined as

$$r^{\varepsilon}_{\|\cdot\|}(\chi;\psi) = \max\left\{|z| : z \in F^{\varepsilon}_{\|\cdot\|}(\chi;\psi)\right\} \quad \left(\leq \frac{\|\chi\|}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right).$$

Theorem 3.1. (For rectangular matrix polynomials, see Theorem 12 in [7], and for the standard numerical range of square matrix polynomials, see Theorem 2.3 in [24].) *Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$. Then, the set $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is bounded if and only if $0 \notin F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)$.*

Proof. Let $0 \notin F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)$, or equivalently, $\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) > 0$. We will obtain that $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is bounded; in particular, we will prove that $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) \subseteq \mathcal{D}(0,M)$, where

$$M = 1 + \frac{\max\limits_{0\leqslant j\leqslant m-1} r^{\varepsilon}_{\|\cdot\|}(\chi_j;\psi)}{\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)}. \tag{8}$$

Since $M \geq 1$, we consider a scalar $\mu \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ with $|\mu| \geq 1$. Then, there exists a continuous linear functional $f \in L_\varepsilon(\psi)$ such that

$$f(\chi_m)\mu^m + f(\chi_{m-1})\mu^{m-1} + \cdots + f(\chi_1)\mu + f(\chi_0) = 0.$$

As a consequence,

$$\begin{aligned}
|\mu|^m &= \frac{\left|\sum\limits_{j=0}^{m-1} f(\chi_j)\mu^j\right|}{|f(\chi_m)|} \leqslant \frac{\sum\limits_{j=0}^{m-1} |f(\chi_j)|\,|\mu|^j}{|f(\chi_m)|} \\
&\leqslant \frac{\max\limits_{0\leqslant j\leqslant m-1} r^{\varepsilon}_{\|\cdot\|}(\chi_j;\psi)}{\frac{|f(\chi_m)|}{\sqrt{1-\varepsilon^2}\,\|\psi\|}} \, \frac{|\mu|^m - 1}{|\mu| - 1} \\
&\leqslant \frac{\max\limits_{0\leqslant j\leqslant m-1} r^{\varepsilon}_{\|\cdot\|}(\chi_j;\psi)}{\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)} \, \frac{|\mu|^m - 1}{|\mu| - 1}.
\end{aligned}$$

Thus,

$$|\mu| - 1 \leqslant \frac{\max\limits_{0\leqslant j\leqslant m-1} r^{\varepsilon}_{\|\cdot\|}(\chi_j;\psi)}{\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)} \, \frac{|\mu|^m - 1}{|\mu|^m} \leqslant \frac{\max\limits_{0\leqslant j\leqslant m-1} r^{\varepsilon}_{\|\cdot\|}(\chi_j;\psi)}{\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)},$$

and hence, $|\mu| \leqslant M$.

For the converse, we assume that $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is bounded and $0 \in F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)$. Then there is a continuous linear functional $f \in L_\varepsilon(\psi)$ such that $f(\chi_m) = 0$. Since $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) \neq \mathbb{C}$, Property (P_{14}) implies that $f(\chi_s) \neq 0$ for some $s \in \{0, 1, 2, \ldots, m-1\}$. Moreover, since $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$, there exists a sequence of continuous linear functionals $\{f_1, f_2, \ldots\} \subset L_\varepsilon(\psi)$ such that $f_j(\chi_m) \neq 0$, $j = 1, 2, \ldots$, and $f_j(\chi_m) \to 0$ as $j \to +\infty$. We consider now the scalar polynomials

$$f_j(P(z)) = f_j(\chi_m)z^m + f_j(\chi_{m-1})z^{m-1} + \cdots + f_j(\chi_1)z + f_j(\chi_0), \quad j = 1, 2, \ldots$$

It is clear that $\dfrac{f_j(\chi_s)}{f_j(\chi_m)} \to \infty$ as $j \to +\infty$; this is a contradiction because we have assumed that $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is bounded, and hence, all the roots and the elementary symmetric functions of the scalar polynomials $f_j(P(z))$, $j = 1, 2, \ldots$, are bounded. □

Theorem 3.2. (For the standard numerical range of square matrix polynomials, see Theorem 3.1 in [27].) *Consider a nonzero vector $\psi \in \mathcal{X}$, an $\varepsilon \in [0, 1)$, and the vector-valued polynomial $P(z) = \psi z^m + \chi_{m-1}z^{m-1} + \cdots + \chi_1 z + \chi_0$ (i.e., $\chi_m = \psi$). Then, for every $\mu \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$, it holds*

$$\frac{\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi_0;\psi)}{\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi_0;\psi) + \max\limits_{1\leqslant j\leqslant m} r^{\varepsilon}_{\|\cdot\|}(\chi_j;\psi)} \leqslant |\mu| \leqslant 1 + \max_{0\leqslant j\leqslant m-1} r^{\varepsilon}_{\|\cdot\|}(\chi_j;\psi).$$

Proof. Since $F^{\varepsilon}_{\|\cdot\|}(\psi;\psi) = \{1\}$ does not contain the origin, the set $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is bounded.

Let $\mu \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$. By definition, there exists a continuous linear functional $f \in L_\varepsilon(\psi)$ such that $f(\psi)\mu^m + f(\chi_{m-1})\mu^{m-1} + \cdots + f(\chi_1)\mu + f(\chi_0) = 0$. Since the lower bound of the theorem is less than or equal to 1, for the first inequality, we may assume that $|\mu| < 1$. Then, we have that

$$f(\chi_0) = -\left(f(\psi)\mu^m + f(\chi_{m-1})\mu^{m-1} + \cdots + f(\chi_1)\mu\right),$$

or

$$|f(\chi_0)| = |f(\psi)\mu^m + f(\chi_{m-1})\mu^{m-1} + \cdots + f(\chi_1)\mu|.$$

Hence,

$$\begin{aligned}
\widehat{r}^{\varepsilon}_{\|\cdot\|}(\chi_0,\psi) &\leqslant \frac{|f(\psi)\mu^m + f(\chi_{m-1})\mu^{m-1} + \cdots + f(\chi_1)\mu|}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \\
&\leqslant \frac{|f(\psi)|\,|\mu|^m + |f(\chi_{m-1})|\,|\mu|^{m-1} + \cdots + |f(\chi_1)|\,|\mu|}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \\
&\leqslant \frac{|\mu|}{1-|\mu|} \max_{1\leqslant j\leqslant m} r^{\varepsilon}_{\|\cdot\|}(\chi_j;\psi),
\end{aligned}$$

which yields the first inequality.

The upper bound of the theorem coincides with the upper bound M in (8), and the proof is complete. □

Suppose that the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$. Then by Property (P_{11}) (see also Proposition 5.1 in [20]), the Birkhoff-James ε-orthogonality set of χ with respect to $\psi \neq 0$ is a closed disk, namely,

$$F^{\varepsilon}_{\|\cdot\|}(\chi;\psi) = \mathcal{D}\left(\frac{\langle\chi,\psi\rangle}{\|\psi\|^2}\,,\,\left\|\chi - \frac{\langle\chi,\psi\rangle}{\|\psi\|^2}\,\psi\right\|\frac{\varepsilon}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right).$$

Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$. Then, by (7), we have

$$\begin{aligned}
W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) &= \{\mu\in\mathbb{C}:\ \psi\perp^{\varepsilon}_{BJ} P(\mu)\}\\
&= \{\mu\in\mathbb{C}:\ \psi\perp^{\varepsilon} P(\mu)\}\\
&= \{\mu\in\mathbb{C}:\ |\langle P(\mu),\psi\rangle| \le \varepsilon\,\|\psi\|\,\|P(\mu)\|\}\\
&= \big\{\mu\in\mathbb{C}:\ |\langle P(\mu),\psi\rangle|^2 \le \varepsilon^2\|\psi\|^2\|P(\mu)\|^2\big\}\\
&= \big\{\mu\in\mathbb{C}:\ \langle P(\mu),\psi\rangle\langle\psi,P(\mu)\rangle \le \varepsilon^2\|\psi\|^2\langle P(\mu),P(\mu)\rangle\big\}\\
&= \left\{\mu\in\mathbb{C}:\ \langle\sum_{i=0}^{m}\chi_i\mu^i,\psi\rangle\langle\psi,\sum_{j=0}^{m}\chi_j\mu^j\rangle \le \varepsilon^2\|\psi\|^2\langle\sum_{i=0}^{m}\chi_i\mu^i,\sum_{j=0}^{m}\chi_j\mu^j\rangle\right\}\\
&= \left\{\mu\in\mathbb{C}:\ \sum_{i,j=0}^{m}\langle\chi_i,\psi\rangle\langle\psi,\chi_j\rangle\mu^i\overline{\mu}^j - \varepsilon^2\|\psi\|^2\sum_{i,j=0}^{m}\langle\chi_i,\chi_j\rangle\mu^i\overline{\mu}^j \le 0\right\}.
\end{aligned}$$

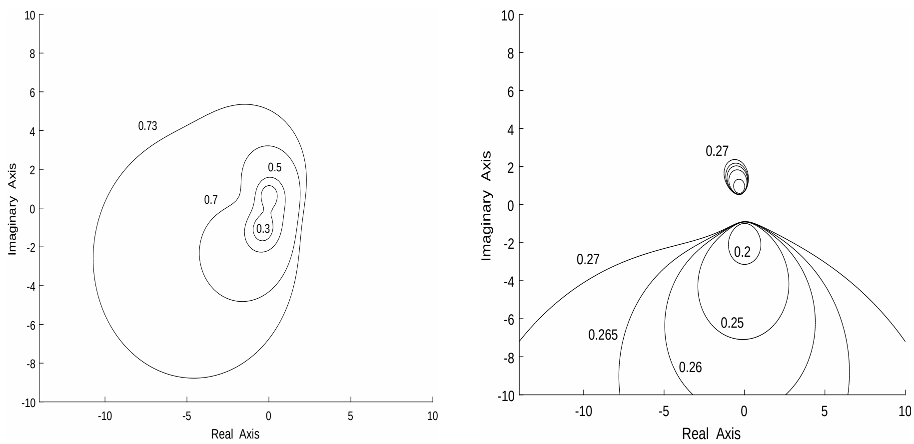

Figure 3: Birkhoff-James ε-orthogonality sets of $P(z)$ (left part) and $R(z)$ (right part).

Example 3.1. Consider the four-dimensional quadratic vector-valued polynomial

$$P(z) = \begin{bmatrix} 1\\ 0\\ 0.8\\ \mathrm{i}\end{bmatrix} z^2 + \begin{bmatrix} \mathrm{i}\\ -1\\ 0.5\\ 0\end{bmatrix} z + \begin{bmatrix} 2\\ -3\\ -0.1\\ -\mathrm{i}\end{bmatrix},$$

its reverse vector-valued polynomial

$$R(z) = \begin{bmatrix} 2 \\ -3 \\ -0.1 \\ -\mathrm{i} \end{bmatrix} z^2 + \begin{bmatrix} \mathrm{i} \\ -1 \\ 0.5 \\ 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0.8 \\ \mathrm{i} \end{bmatrix},$$

and the vector $\psi = \begin{bmatrix} 0.6 & 0 & 0.9 & 0.2 \end{bmatrix}^T$. For the euclidean norm (which is induced by the standard inner product), we have drawn the boundaries of the ε-orthogonality sets $W^{\varepsilon}_{\|\cdot\|_2}(P(z);\psi)$, $\varepsilon = 0.3, 0.5, 0.7, 0.73$, and $W^{\varepsilon}_{\|\cdot\|_2}(R(z);\psi)$, $\varepsilon = 0.2, 0.25, 0.26, 0.265, 0.27$, in the left and the right part of Figure 3, respectively. As expecting by Theorem 3.1, the origin lies in $W^{\varepsilon}_{\|\cdot\|_2}(P(z);\psi)$ (or equivalently, the origin lies in $F^{\varepsilon}_{\|\cdot\|_2}(\chi_0;\psi)$) if and only if $W^{\varepsilon}_{\|\cdot\|_2}(R(z);\psi)$ is unbounded.

4 Connected components

In this section, we study the connected components of the Birkhoff-James ε-orthogonality set $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$, when this set is bounded. The following lemma is necessary for our analysis.

Lemma 4.1. *Let $P(z)$ be a vector-valued polynomial as in (6), and let L be a non-empty, closed and convex subset of $\mathcal{X}^*$ such that $f(\chi_m) \neq 0$ for all $f \in L$. Then, the roots of the scalar polynomial $f(P(z)) = f(\chi_m)z^m + f(\chi_{m-1})z^{m-1} + \cdots + f(\chi_1)z + f(\chi_0)$ are continuous with respect to $f \in L$.*

Proof. It is well known that the roots of a scalar polynomial are continuous functions of the coefficients of the polynomial, as long as the leading coefficient is nonzero; see Appendix D in [14]. The vector coefficients $\chi_0, \chi_1, \dots, \chi_m \in \mathcal{X}$ of the vector-valued polynomial $P(z) = \chi_m z^m + \chi_{m-1}z^{m-1} + \cdots + \chi_1 z + \chi_0$ are constant, and hence, the coefficients $f(\chi_0), f(\chi_1), \dots, f(\chi_m)$ of the scalar polynomial $f(P(z))$ depend only on $f \in L$. If $\{f_1, f_2, \dots\} \subset L$ is a sequence of continuous linear functionals that converges to $f \in L$ (i.e., $\|f_k - f\| \to 0$, as $k \to +\infty$), then for any $j = 0, 1, \dots, m$, it holds

$$\|f(\chi_j) - f_k(\chi_j)\| \leqslant \|(f - f_k)(\chi_j)\| \leqslant \|f - f_k\| \, \|\chi_j\|, \quad k = 1, 2, \dots,$$

and the proof is complete. □

Theorem 4.2. (For the standard numerical range of square matrix polynomials, see Theorem 2.2 in [24].) *Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $0 \notin F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)$ (or equivalently, $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is bounded). Suppose that $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ has r connected components. If κ is the minimum number of distinct zeros of the scalar polynomial $f(P(z)) = f(\chi_m)z^m + f(\chi_{m-1})z^{m-1} + \cdots + f(\chi_1)z + f(\chi_0)$ over all $f \in L_{\varepsilon}(\psi)$, then $r \leqslant \kappa \leqslant m$.*

Proof. Consider a continuous linear functional $f_1 \in L_\varepsilon(\psi)$ such that the scalar polynomial $f_1(P(z)) = f_1(\chi_m)z^m + f(\chi_{m-1})z^{m-1} + \cdots + f_1(\chi_1)z + f_1(\chi_0)$ has exactly $\kappa\,(\leqslant m)$ distinct roots. Let also $f_2 \in L_\varepsilon(\psi)$. Since $0 \notin F^\varepsilon_{\|\cdot\|}(\chi_m;\psi)$, both scalars $f_1(\chi_m)$ and $f_2(\chi_m)$ are nonzero. Moreover, by the convexity of the set $L_\varepsilon(\psi)$ and the region $F^\varepsilon_{\|\cdot\|}(\chi_m;\psi)$ (keeping in mind that $0 \notin F^\varepsilon_{\|\cdot\|}(\chi_m;\psi)$), every continuous linear functional

$$g_t = (1-t)f_1 + tf_2 \in L_\varepsilon(\psi), \quad t \in [0,1],$$

satisfies the condition $g_t(\chi_m) \neq 0$. Thus, by Lemma 4.1, the roots of the scalar polynomial

$$g_t(P(z)) = g_t(\chi_m)z^m + g_t(\chi_{m-1})z^{m-1} + \cdots + g_t(\chi_1)z + g_t(\chi_0), \quad t \in [0,1]$$

are continuous functions of t. Hence, the κ roots of the scalar polynomial $f_1(P(z))$ are connected with continuous curves in $W^\varepsilon_{\|\cdot\|}(P(z);\psi)$ with the roots of $f_2(P(z))$. Consequently, the number of the connected components of $W^\varepsilon_{\|\cdot\|}(P(z);\psi)$ is less than or equal to κ. □

Suppose that for every continuous linear functional $f \in L_\varepsilon(\psi)$, the scalar polynomial $f(P(z))$ has m simple roots (this means that $f(\chi_m)$ is always nonzero and $W^\varepsilon_{\|\cdot\|}(P(z);\psi)$ is bounded). Then, these m simple roots define m continuous maps

$$\rho_i : \; L_\varepsilon(\psi) \to \mathbb{C}, \quad i = 1, 2, \ldots, m. \tag{9}$$

Definition 4.1. Let $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$, and consider a complex number $\mu \in F^\varepsilon_{\|\cdot\|}(\chi;\psi)$. We define the set

$$S_{\chi,\psi}(\mu) = \left\{ f \in L_\varepsilon(\psi) : \; \mu = \frac{f(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \right\} \subseteq L_\varepsilon(\psi).$$

Moreover, for the vector-valued polynomial $P(z)$, we define the set

$$S_{P(z),\psi}(\mu) = \{ f \in L_\varepsilon(\psi) : \; f(P(\mu)) = 0 \} = S_{P(\mu),\psi}(0).$$

Lemma 4.3. *Let $\chi, \psi \in \mathcal{X}$, with $\psi \neq 0$, and consider a complex number $\mu \in F^\varepsilon_{\|\cdot\|}(\chi;\psi)$. Then, the set $S_{\chi,\psi}(\mu)$ is convex.*

Proof. Consider two continuous linear functionals $f_1, f_2 \in S_{\chi,\psi}(\mu)$ and a $t \in [0,1]$. Then we have that $\dfrac{f_1(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} = \mu = \dfrac{f_2(\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|}$. As a consequence,

$$\frac{[tf_1 + (1-t)f_2](\chi)}{\sqrt{1-\varepsilon^2}\,\|\psi\|} = \mu,$$

and $tf_1 + (1-t)f_2$ also lies in $S_{\chi,\psi}(\mu)$. □

Theorem 4.4. (For operator polynomials, see Theorem 1 in [25].) *Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $0 \notin F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi)$ (or equivalently, $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is bounded). Suppose that for every $f \in L_\varepsilon(\psi)$, the scalar polynomial $f(P(z)) = f(\chi_m)z^m + f(\chi_{m-1})z^{m-1} + \cdots + f(\chi_1)z + f(\chi_0)$ has exactly m simple roots. Then, $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ has exactly m connected components.*

Proof. We consider the images of the root functions $\rho_1, \rho_2, \ldots, \rho_m$ in (9),

$$W_i = \rho_i(L_\varepsilon(\psi)) \subseteq W^{\varepsilon}_{\|\cdot\|}(P(z);\psi), \quad i = 1, 2, \ldots, m.$$

These sets are connected and satisfy

$$W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) = \bigcup_{1\leqslant i\leqslant m} W_i.$$

We need to prove that $W_i \cap W_j = \emptyset$ for all $i \neq j$.

Without loss of generality, assume that $W_1 \cap W_2 \neq \emptyset$. Then there exists a $\mu \in \mathbb{C}$ such that

$$\rho_1(f_1) = \mu = \rho_2(f_2) \text{ for some functionals } f_1, f_2 \in L_\varepsilon(\psi).$$

Then both f_1 and f_2 lie in $S_{P(z),\psi}(\mu)$. Moreover, it holds

$$S_{P(z),\psi}(\mu) = \bigcup_{1\leqslant i\leqslant m} \{f \in L_\varepsilon(\psi) : \mu = \rho_i(f)\},$$

i.e., $S_{P(z),\psi}(\mu)$ is the union of

$$S_1 = \{f \in L_\varepsilon(\psi) : \mu = \rho_1(f)\} \quad \text{and} \quad S_2 = \bigcup_{2\leqslant i\leqslant m} \{f \in L_\varepsilon(\psi) : \mu = \rho_i(f)\}.$$

Obviously, $f_1 \in S_1$ and $f_2 \in S_2$, and the sets S_1 and S_2 are not empty. The sets S_1 and S_2 are closed as pre-images of continuous maps. Since the set $S_{P(z),\psi}(\mu)$ is convex, it is also connected, and hence, $S_1 \cap S_2 \neq \emptyset$. Thus, there exists a functional f such that $\rho_1(f) = z = \rho_i(f)$ for some $i \geqslant 2$; this is a contradiction because we have assumed that the roots are simple. □

5 Boundary

Since the Birkhoff-James ε-orthogonality set $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is closed, its boundary is of special interest. In the following two theorems, we describe the strong connection between a boundary point z_0 of $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ and the origin as a boundary point of the region $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$.

Theorem 5.1. (For rectangular matrix polynomials, see Theorem 19 (i) in [7], and for the standard numerical range of square matrix polynomials, see Theorem 1.1

in [27].) *Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$. If $z_0 \in \partial W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$, then $0 \in \partial F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$.*

Proof. Since $z_0 \in \partial W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) \subseteq W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$, there is a continuous linear functional $f_0 \in L_\varepsilon(\psi)$ such that $f_0(P(z_0)) = 0$. So, $0 \in F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$, and it is sufficient to prove that the origin does not belong to the interior of $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$.

Let $\{z_1, z_2, \dots\} \subset \mathbb{C} \setminus W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ be a sequence of complex numbers converging to z_0, and assume that 0 lies in the interior of $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$. Then, there is a real number $\delta > 0$ such that $\mathcal{D}(0,\delta) \subseteq F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$. Moreover, there exist $f_{\delta,1}, f_{\delta,2}, f_{\delta,3} \in L_\varepsilon(\psi)$ such that the triangle with vertices $\dfrac{f_{\delta,1}(P(z_0))}{\sqrt{1-\varepsilon^2}\,\|\psi\|}$, $\dfrac{f_{\delta,2}(P(z_0))}{\sqrt{1-\varepsilon^2}\,\|\psi\|}$ and $\dfrac{f_{\delta,3}(P(z_0))}{\sqrt{1-\varepsilon^2}\,\|\psi\|}$ contains the origin in its interior and lies in the disk $\mathcal{D}(0,\delta/2)$. Continuity yields

$$\lim_{n\to+\infty} \frac{f_{\delta,i}(P(z_n))}{\sqrt{1-\varepsilon^2}\,\|\psi\|} = \frac{f_{\delta,i}(P(z_0))}{\sqrt{1-\varepsilon^2}\,\|\psi\|}, \quad i = 1,2,3,$$

and as a consequence, there is a positive integer n_0 such that $0 \in F^{\varepsilon}_{\|\cdot\|}(P(z_n);\psi)$ for every $n \geqslant n_0$. Hence, for every positive integer $n \geqslant n_0$, $z_n \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$; this is a contradiction. □

For the remainder, we need to consider the vector-valued polynomial

$$P'(z) = m\chi_m z^{m-1} + (m-1)\chi_{m-1} z^{m-2} + \cdots + 2\chi_2 z + \chi_1.$$

Theorem 5.2. (For rectangular matrix polynomials, see Theorem 19 (ii) in [7], and for the standard numerical range of square matrix polynomials, see Theorem 2 in [22].) *Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$. Let also $z_0 \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ such that $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi) \neq \{0\}$ and $0 \notin F^{\varepsilon}_{\|\cdot\|}(P'(z_0);\psi)$. If $0 \in \partial F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$, then $z_0 \in \partial W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$.*

Proof. Let $0 \in \partial F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$, and assume that z_0 is an interior point of the set $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$. Then, there exists a $\delta > 0$ such that $\mathcal{D}(z_0,\delta) \subseteq W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$. Hence, for any $z \in \mathcal{D}(z_0,\delta)\backslash\{z_0\}$, there is a $f_z \in L_\varepsilon(\psi)$ such that $f_z(P(z)) = 0$. Moreover,

$$\begin{aligned} 0 &= f_z(P(z)) \;=\; f_z(P(z - z_0 + z_0)) \\ &= f_z(P(z_0) + (z-z_0)P'(z_0) + (z-z_0)R(z,z_0)) \\ &= f_z(P(z_0)) + (z-z_0) f_z(P'(z_0) + R(z,z_0)), \end{aligned}$$

where $R(z,z_0)$ is a vector-valued polynomial in z_0 and z, such that $\|R(z,z_0)\| \to 0$ as $|z - z_0| \to 0$. Since $0 \notin F^{\varepsilon}_{\|\cdot\|}(P'(z_0);\psi)$, by the subadditivity of Proposition 2.3, the positive number δ can be chosen small enough such that for every $z \in \mathcal{D}(z_0, \delta)\backslash\{z_0\}$,

$$0 \notin F^{\varepsilon}_{\|\cdot\|}(P'(z_0) + R(z, z_0); \psi) \quad \left(\subseteq F^{\varepsilon}_{\|\cdot\|}(P'(z_0); \psi) + \mathcal{D}\left(0, \frac{\|R(z, z_0)\|}{\sqrt{1-\varepsilon^2}\,\|\psi\|}\right) \right)$$

and

$$z - z_0 = -\frac{f_z(P(z_0))}{f_z(P'(z_0) + R(z, z_0))}. \tag{10}$$

By the convexity of $F^{\varepsilon}_{\|\cdot\|}(P'(z_0) + R(z, z_0); \psi)$, there exist angles $\theta_1, \theta_2, \theta_3$ such that $0 < \theta_2 - \theta_1 \leqslant \theta_3 < \pi$ and

$$F^{\varepsilon}_{\|\cdot\|}(P'(z_0) + R(z, z_0); \psi) \subset \{w \in \mathbb{C} : \theta_1 \leqslant \arg(w) \leqslant \theta_2\}, \quad \forall\, z \in \mathcal{D}(z_0, \delta)\backslash\{z_0\}.$$

Also, $F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi) \neq \{0\}$ and $0 \in \partial F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi)$. Therefore, by the convexity of $F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi)$, there exist angles ϕ_1, ϕ_2 such that $0 < \phi_2 - \phi_1 \leqslant \pi$ and

$$F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi) \subset \{w \in \mathbb{C} : \phi_1 \leqslant \arg(w) \leqslant \phi_2\}.$$

Consequently, the angular of the right hand-side of (10) cannot take all the values in $[0, 2\pi)$. This is a contradiction, since the left hand-side is not constrained. ☐

Next, we consider the isolated points of the Birkhoff-James ε-orthogonality set $W^{\varepsilon}_{\|\cdot\|}(P(z); \psi)$.

Proposition 5.3. (For the standard numerical range of square matrix polynomials, see Theorem 2.1 in [27].) *Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0, 1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $0 \notin F^{\varepsilon}_{\|\cdot\|}(\chi_m; \psi)$ (or equivalently, $W^{\varepsilon}_{\|\cdot\|}(P(z); \psi)$ is bounded). If z_0 is an isolated point of $W^{\varepsilon}_{\|\cdot\|}(P(z); \psi)$, then $F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi) = \{0\}$. If, in addition, $\varepsilon > 0$, then $P(z_0) = 0$.*

Proof. Suppose that the singleton $\{z_0\}$ is a connected component of $W^{\varepsilon}_{\|\cdot\|}(P(z); \psi)$. Then, there is a continuous linear functional $f_0 \in L_{\varepsilon}(\psi)$ such that

$$f_0(P(z_0)) = f_0(\chi_m)z_0^m + f_0(\chi_{m-1})z_0^{m-1} + \cdots + f_0(\chi_1)z_0 + f_0(\chi_0) = 0.$$

Since $0 \notin F^{\varepsilon}_{\|\cdot\|}(\chi_m; \psi)$, the convexity of $L_{\varepsilon}(\psi)$ and the continuity of the roots of the scalar polynomial $f(P(z))$ with respect to $f \in L_{\varepsilon}(\psi)$ imply that the roots of the scalar polynomial $f_0(P(z))$ are connected to the roots of any scalar polynomial $f(P(z))$, with $f \in L_{\varepsilon}(\psi)$, by continuous curves in $W^{\varepsilon}_{\|\cdot\|}(P(z); \psi)$ (see also the proof of Theorem 4.2). As a consequence, for any $f \in L_{\varepsilon}(\psi)$, z_0 is a root of the scalar polynomial $f(P(z))$. Thus, $f(P(z_0)) = 0$ for every $f \in L_{\varepsilon}(\psi)$, and hence, $F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi) = \{0\}$. Furthermore, if $\varepsilon > 0$, then Properties (P_5) and (P_7) yield $P(z_0) = 0$. ☐

6 Local dimension

Let Ω be a closed subset of $\mathbb{C}$. A recursive definition of the *topological dimension* of Ω, denoted by $\dim\{\Omega\}$, is the following [13, 16]: If Ω is an empty set, then $\dim\{\Omega\} = -1$. If Ω is a non-empty set, then $\dim\{\Omega\}$ is the least integer number $k \in \{0, 1, 2\}$ for which every point of Ω has arbitrarily small neighborhoods in Ω whose boundaries are of topological dimension less than k. Clearly, if Ω is countable, then $\dim\{\Omega\} = 0$, and if Ω is a (non-degenerate) curve, then $\dim\{\Omega\} = 1$.

Consider a point $z_0 \in \Omega$. The *local dimension* of z_0 in Ω is defined as the limit $\lim_{h\to 0^+} \dim\{\Omega \cap D(z_0, h)\}$, $h \in (0, +\infty)$. In particular, the local dimension of z_0 in Ω is equal to

0 if and only if z_0 is an isolated point of Ω,

1 if and only if z_0 is a non-isolated point of Ω which does not lie in the closure of the interior of Ω,

2 if and only if z_0 lies in the closure of the interior of Ω.

As in the case of the boundary, the local dimension of a point z_0 in $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is strongly connected to the local dimension of the origin in the set $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$.

Theorem 6.1. (For the standard numerical range of square matrix polynomials, see Theorem 1 in [28].) *Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in [0, 1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{\varepsilon}_{\|\cdot\|}(\chi_m;\psi) \neq \{0\}$. Let also $z_0 \in W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ with local dimension in $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ equal to 1, such that $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi) \neq \{0\}$, the origin is a differentiable point of $\partial F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$ and $0 \notin F^{\varepsilon}_{\|\cdot\|}(P'(z_0);\psi)$. Then, the local dimension of 0 in $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$ is 1.*

Proof. Since the local dimension of z_0 in $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ is equal to 1, it follows that $z_0 \in \partial W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$, z_0 is not an isolated point of $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$, and there is a real number $r > 0$ such that $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi) \cap \mathcal{D}(z_0, r) \subseteq \partial W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$. For the sake of contradiction, assume that the local dimension of the origin in $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$ is equal to 2 (i.e., the convex set $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$ has a non-empty interior).

By Theorem 5.1, for every $z \in \mathcal{D}(z_0, r)$, it holds that $0 \in \partial F^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$. Moreover, the origin is a differentiable point of $\partial F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$, and hence, there is a unique tangent line of $\partial F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi)$ at the origin, which defines a closed half-plane $\mathcal{H}_1$ and an open half-plane $\mathcal{H}_2 = \mathbb{C}\backslash\mathcal{H}_1$, such that $F^{\varepsilon}_{\|\cdot\|}(P(z_0);\psi) \subset \mathcal{H}_1$.

For every $\rho \in [0, r]$ and $\theta \in [0, 2\pi]$, $z_0 + \rho e^{\mathrm{i}\theta}$ is either a boundary point or an exterior point of the set $W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$. As a consequence, for every $\rho \in [0, r]$ and $\theta \in [0, 2\pi]$, the origin is either a boundary point or an exterior point of the convex set $F^{\varepsilon}_{\|\cdot\|}(P(z_0 + \rho e^{\mathrm{i}\theta});\psi)$. Moreover, it holds

$$P(z_0 + \rho e^{\mathrm{i}\theta}) = P(z_0) + \rho e^{\mathrm{i}\theta} P'(z_0) + \rho e^{\mathrm{i}\theta} R(z_0, \rho e^{\mathrm{i}\theta}),$$

where $R(z_0, \rho e^{\mathrm{i}\theta})$ is a vector-valued polynomial in z_0 and $\rho e^{\mathrm{i}\theta}$, such that $\|R(z_0, \rho e^{\mathrm{i}\theta})\| \to 0$ as $\rho \to 0$. Since $0 \notin F^{\varepsilon}_{\|\cdot\|}(P'(z_0); \psi)$, subadditivity implies that for small enough r, there exists a convex cone

$$\mathcal{K}(z_0, r) = \{z \in \mathbb{C} : \theta_1 \leqslant \arg(z) \leqslant \theta_2,\ 0 < \theta_2 - \theta_1 \leqslant \theta_3 < \pi\},$$

such that for every $\rho \in [0, r]$ and $\theta \in [0, 2\pi]$,

$$F^{\varepsilon}_{\|\cdot\|}(P'(z_0) + R(z_0, \rho e^{\mathrm{i}\theta}); \psi) \subset \mathcal{K}(z_0, r) \setminus \{0\}.$$

For suitable $\theta \in [0, 2\pi]$,

$$e^{\mathrm{i}\theta} F^{\varepsilon}_{\|\cdot\|}(P'(z_0) + R(z_0, \rho e^{\mathrm{i}\theta}); \psi) \ \subset\ e^{\mathrm{i}\theta}\mathcal{K}(z_0, r) \setminus \{0\} \ \subset\ \mathcal{H}_2.$$

Then, for every linear functional $f \in L_\varepsilon(\psi)$,

$$\frac{f(P(z_0 + \rho e^{\mathrm{i}\theta}))}{\sqrt{1-\varepsilon^2}\,\|\psi\|} = \frac{f(P(z_0))}{\sqrt{1-\varepsilon^2}\,\|\psi\|} + \frac{\rho e^{\mathrm{i}\theta} f(P'(z_0) + R(z_0, \rho e^{\mathrm{i}\theta}))}{\sqrt{1-\varepsilon^2}\,\|\psi\|},$$

where

$$\frac{f(P(z_0))}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \in F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi) \subset \mathcal{H}_1$$

and

$$\frac{\rho e^{\mathrm{i}\theta} f(P'(z_0) + R(z_0, \rho e^{\mathrm{i}\theta}))}{\sqrt{1-\varepsilon^2}\,\|\psi\|} \ \in\ e^{\mathrm{i}\theta}\mathcal{K}(z_0, r) \setminus \{0\} \ \subset\ \mathcal{H}_2.$$

Consequently, as ρ takes values from 0 to r, a part of $F^{\varepsilon}_{\|\cdot\|}(P(z_0 + \rho e^{\mathrm{i}\theta}); \psi)$, in a neighborhood of the origin, is moving continuously into the half-plane $\mathcal{H}_2$. Thus, there is an $r_\theta \in (0, r]$ such that the origin lies in the interior of $F^{\varepsilon}_{\|\cdot\|}(P(z_0) + r_\theta e^{\mathrm{i}\theta}[P'(z_0) + R(z_0, \rho e^{\mathrm{i}\theta})]; \psi) = F^{\varepsilon}_{\|\cdot\|}(P(z_0 + r_\theta e^{\mathrm{i}\theta}); \psi)$; this contradicts the definition of r. □

If $z_0 \in W^{\varepsilon}_{\|\cdot\|}(P(z); \psi)$ such that $F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi) \neq \{0\}$, then $P(z_0)$ is not a scalar multiple of ψ. Hence, if $\varepsilon > 0$, then the convexity of $F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi)$ and Property (P_7) imply that the local dimension of 0 in $F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi)$ is equal to 2. As a consequence, we have the following corollary.

Corollary 6.2. *Let $P(z)$ be a vector-valued polynomial as in (6), $\varepsilon \in (0, 1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{\varepsilon}_{\|\cdot\|}(\chi_m; \psi) \neq \{0\}$. Let also z_0 be a non-isolated boundary point of $W^{\varepsilon}_{\|\cdot\|}(P(z); \psi)$ such that $F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi) \neq \{0\}$, the origin is a differentiable point of $\partial F^{\varepsilon}_{\|\cdot\|}(P(z_0); \psi)$ and $0 \notin F^{\varepsilon}_{\|\cdot\|}(P'(z_0); \psi)$. Then the local dimension of z_0 in $W^{\varepsilon}_{\|\cdot\|}(P(z); \psi)$ is equal to 2.*

The case $\varepsilon = 0$ is considered in the next result.

Theorem 6.3. (For the standard numerical range of square matrix polynomials, see Theorem 2 in [28].) *Let $P(z)$ be a vector-valued polynomial as in (6) and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{0}_{\|\cdot\|}(\chi_m; \psi) \neq \{0\}$. Let also z_0 be an interior point of $W^{0}_{\|\cdot\|}(P(z); \psi)$ or a differentiable point of $\partial W^{0}_{\|\cdot\|}(P(z); \psi)$ with local dimension in $W^{0}_{\|\cdot\|}(P(z); \psi)$ equal to 2, such that $F^{0}_{\|\cdot\|}(P(z_0); \psi) \neq \{0\}$ and $0 \notin F^{0}_{\|\cdot\|}(P'(z_0); \psi)$. Then, the local dimension of the origin in $F^{0}_{\|\cdot\|}(P(z_0); \psi)$ is equal to 2.*

Proof. If z_0 is an interior point of $W^0_{\|\cdot\|}(P(z);\psi)$, then by Theorem 5.2, the origin is also an interior point of $F^0_{\|\cdot\|}(P(z_0);\psi)$. In this case, the local dimension of z_0 in $W^0_{\|\cdot\|}(P(z);\psi)$ and the local dimension of 0 in $F^0_{\|\cdot\|}(P(z_0);\psi)$ are both equal to 2.

Let $z_0 \in \partial W^0_{\|\cdot\|}(P(z);\psi)$. Since z_0 is a differentiable point of $\partial W^{\varepsilon}_{\|\cdot\|}(P(z);\psi)$ and has local dimension 2 in $W^0_{\|\cdot\|}(P(z);\psi)$, there exists a $\phi_0 \in [0,2\pi]$ such that for every $\phi \in (\phi_0, \phi_0+\pi)$, there is an arbitrarily small $r_\phi > 0$ with $z_0 + r_\phi e^{\mathrm{i}\phi}$ lying in the interior of $W^0_{\|\cdot\|}(P(z);\psi)$. For the sake of contradiction, we assume that the origin has local dimension 1 in $F^0_{\|\cdot\|}(P(z_0);\psi)$. Then, by the convexity of the set $F^0_{\|\cdot\|}(P(z_0);\psi) \neq \{0\}$, it follows that $F^0_{\|\cdot\|}(P(z_0);\psi)$ is a (non-degenerate) line segment passing through the origin.

The straight line which is defined by the line segment $F^0_{\|\cdot\|}(P(z_0);\psi)$ defines two closed half-planes $\mathcal{H}_1$ and $\mathcal{H}_2$. As in the proof of Theorem 6.1,

$$P(z_0 + re^{\mathrm{i}\phi}) = P(z_0) + re^{\mathrm{i}\phi}P'(z_0) + re^{\mathrm{i}\phi}R(z_0, re^{\mathrm{i}\phi}),$$

where $\|R(z_0, re^{\mathrm{i}\phi})\| \to 0$ as $r \to 0$. Since $0 \notin F^0_{\|\cdot\|}(P'(z_0);\psi)$, for small enough r, there exists a convex cone

$$\mathcal{K}(z_0, r) = \{z \in \mathbb{C} : \theta_1 \leqslant \arg(z) \leqslant \theta_2,\ 0 < \theta_2 - \theta_1 \leqslant \theta_3 < \pi\},$$

such that

$$F^0_{\|\cdot\|}(P'(z_0) + R(z_0, re^{\mathrm{i}\phi});\psi) \subseteq \mathcal{K}(z_0, r) \setminus \{0\}.$$

Also, there is a $\theta \in (\phi_0, \phi_0 + \pi)$ such that the set $e^{\mathrm{i}\theta}F^0_{\|\cdot\|}(P'(z_0) + R(z_0, re^{\mathrm{i}\phi});\psi)$ lies in the interior of $\mathcal{H}_1$ or $\mathcal{H}_2$. Since

$$F^0_{\|\cdot\|}(P(z_0 + r_\theta e^{\mathrm{i}\theta});\psi) \subseteq F^0_{\|\cdot\|}(P(z_0);\psi) + r_\theta e^{\mathrm{i}\theta}F^0_{\|\cdot\|}(P'(z_0) + R(z_0, re^{\mathrm{i}\phi});\psi),$$

$F^0_{\|\cdot\|}(P(z_0 + r_\theta e^{\mathrm{i}\theta});\psi)$ lies in the interior of $\mathcal{H}_1$ or $\mathcal{H}_2$. As a consequence, $0 \notin F^0_{\|\cdot\|}(P(z_0 + r_\theta e^{\mathrm{i}\theta});\psi)$; this is a contradiction because $z_0 + r_\theta e^{\mathrm{i}\theta} \in W^0_{\|\cdot\|}(P(z);\psi)$. ☐

Finally, we obtain that bounded Birkhoff-James ε-orthogonality sets of linear vector-valued polynomials are simply connected.

Theorem 6.4. (For the standard numerical range of square matrix polynomials, see Theorem 4 in [28].) *Let $\chi_1 z + \chi_0$ be a linear vector-valued polynomial, $\varepsilon \in [0,1)$, and $\psi \in \mathcal{X}$ be a nonzero vector such that $F^{\varepsilon}_{\|\cdot\|}(\chi_1;\psi) \neq \{0\}$. If the set $W^{\varepsilon}_{\|\cdot\|}(\chi_1 z + \chi_0;\psi)$ is bounded, then it is simply connected.*

Proof. Suppose $W^{\varepsilon}_{\|\cdot\|}(\chi_1 z + \chi_0;\psi)$ is not simply connected. Then there is a complex number $w_0 \notin W^{\varepsilon}_{\|\cdot\|}(\chi_1 z + \chi_0;\psi)$ such that for every $\phi \in [0,2\pi]$, there exists an $r_\phi > 0$ such that $w_0 + r_\phi e^{\mathrm{i}\phi} \in W^{\varepsilon}_{\|\cdot\|}(\chi_1 z + \chi_0;\psi)$. By Property (P_{12}), for any scalar $a \in \mathbb{C}$, it holds that $W^{\varepsilon}_{\|\cdot\|}(\chi_1(z+a) + \chi_0;\psi) = W^{\varepsilon}_{\|\cdot\|}(\chi_1 z + \chi_0;\psi) - a$. Thus, without loss of generality, we may assume that $w_0 = 0$.

By the boundedness of $W^{\varepsilon}_{\|\cdot\|}(\chi_1 z + \chi_0; \psi)$ and the assumption that the origin does not lie in $W^{\varepsilon}_{\|\cdot\|}(\chi_1 z + \chi_0; \psi)$, both convex sets $F^{\varepsilon}_{\|\cdot\|}(\chi_1; \psi)$ and $F^{\varepsilon}_{\|\cdot\|}(\chi_0; \psi)$ do not contain the origin. As a consequence, there exist two convex cones

$$\mathcal{K}_1 = \left\{ z \in \mathbb{C} : \, \theta_1 \leqslant \arg(z) \leqslant \widetilde{\theta}_1, \; 0 < \widetilde{\theta}_1 - \theta_1 \leqslant \xi_1 < \pi \right\}$$

and

$$\mathcal{K}_2 = \left\{ z \in \mathbb{C} : \, \theta_2 \leqslant \arg(z) \leqslant \widetilde{\theta}_2, \; 0 < \widetilde{\theta}_2 - \theta_2 \leqslant \xi_2 < \pi \right\},$$

such that $F^{\varepsilon}_{\|\cdot\|}(\chi_1; \psi)$ lies in the interior of $\mathcal{K}_1$ and $F^{\varepsilon}_{\|\cdot\|}(\chi_0; \psi)$ lies in the interior of $\mathcal{K}_2$. Hence, there exists a $\phi_0 \in [0, 2\pi]$ such that the convex regions $F^{\varepsilon}_{\|\cdot\|}(r_{\phi_0} e^{\mathrm{i}\phi_0} \chi_1; \psi) = r_{\phi_0} e^{\mathrm{i}\phi_0} F^{\varepsilon}_{\|\cdot\|}(\chi_1; \psi)$ and $F^{\varepsilon}_{\|\cdot\|}(\chi_0; \psi)$ lie in the interior of the convex cone

$$\mathcal{K}_0 = \left\{ z \in \mathbb{C} : \, \theta_0 \leqslant \arg(z) \leqslant \widetilde{\theta}_0, \; 0 < \widetilde{\theta}_0 - \theta_0 \leqslant \xi_0 < \pi \right\},$$

where $\max\{\xi_1, \xi_2\} \leqslant \xi_0$. Therefore, by the subadditivity of Proposition 2.3, the set

$$F^{\varepsilon}_{\|\cdot\|}(\chi_1 r_{\phi_0} e^{\mathrm{i}\phi_0} + \chi_0; \psi) \subseteq r_{\phi_0} e^{\mathrm{i}\phi_0} F^{\varepsilon}_{\|\cdot\|}(\chi_1; \psi) + F^{\varepsilon}_{\|\cdot\|}(\chi_0; \psi)$$

lies in the interior of $\mathcal{K}_0$, and it does not contain the origin; this is a contradiction. □

References

[1] G.A. Anastassiou and S.G. Gal, On the best approximation of vector valued functions by polynomials with coefficients in vector spaces, *Ann. Mat. Pura Appl.*, **186** (2007), 251–265.

[2] G. Birkhoff, Orthogonality in linear metric spaces, *Duke Math. J.*, **1** (1935), 169–172.

[3] F.F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Note Ser., Cambridge University Press, New York, 1971.

[4] F.F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser., Cambridge University Press, New York, 1973.

[5] J. Chmielinski, On an ε-Birkhoff orthogonality, *J. Ineq. Pure and Appl. Math.*, **6** (2005), Article 79.

[6] Ch. Chorianopoulos, S. Karanasios and P. Psarrakos, A definition of numerical range of rectangular matrices, *Linear Multilinear Algebra*, **57** (2009), 459–475.

[7] Ch. Chorianopoulos and P. Psarrakos, Birkhoff-James approximate orthogonality sets and numerical ranges, *Linear Algebra Appl.*, **434** (2011), 2089–2108.

[8] S.S. Dragomir, On approximation of continuous linear functionals in normed linear spaces, *An. Univ. Timişoara Ser. Ştiinţ. Mat.*, **29** (1991), 51–58.

[9] F.R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing Company, New York, 1959.

[10] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.

[11] I. Gohberg, P. Lancaster and L. Rodman, Spectral analysis of selfadjoint matrix polynomials, *Ann. of Math.*, **112** (1980), 33–71.

[12] I. Gohberg, P. Lancaster and L. Rodman, Representation and divisibility of operator polynomials, *Canad. J. Math.*, **30** (1978), 1045–1069.

[13] J. Hocking and G. Young, *Topology*, Dover, New York, 1988.

[14] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1990.

[15] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[16] W. Hurewicz and H. Wallman, *Dimension Theory* (revised edition), Princeton Mathematical Series, Vol. 4, Princeton University Press, Princeton, 1996.

[17] N. Ito and H.K. Wimmer, Self-inversive Hilbert space operator polynomials with spectrum on the unit circle, *J. Math. Anal. Appl.*, **436** (2016), 683–691.

[18] R.C. James, Orthogonality and linear functionals in normed linear spaces, *Trans. Amer. Math. Soc.*, **61** (1947), 265–292.

[19] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, 1980.

[20] M. Karamanlis and P.J. Psarrakos, Birkhoff-James ϵ-orthogonality sets in normed linear spaces, *Textos Mat.*, University of Coimbra, **44** (2013), 81–92.

[21] P. Lancaster, *Lambda-matrices and Vibrating Systems*, Pergamon Press, Oxford, 1966.

[22] P. Lancaster and P. Psarrakos, Normal and seminormal eigenvalues of analytic matrix functions, *Integral Equations Operator Theory*, **41** (2001), 331–342.

[23] L. Lerer, L. Rodman and M. Tismenetsky, Bezoutian and Schur-Cohn problem for operator polynomials, *J. Math. Anal. Appl.*, **103** (1984), 83–102.

[24] C-K. Li and L. Rodman, Numerical range of matrix polynomials, *SIAM J. Matrix Anal. Appl.*, **15** (1994), 1256–1265.

[25] Y. Lyubich, Seperation of roots of matrix and operator polynomials, *Integral Equations Operator Theory*, **29** (1997), 52–62.

[26] A.S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Translations of Mathematical Monographs, Vol. 71, American Mathematical Society, Providence, 1988.

[27] J. Maroulas and P. Psarrakos, The boundary of numerical range of matrix polynomials, *Linear Algebra Appl.*, **267** (1997), 101–111.

[28] H. Nakazato and P. Psarrakos, On the shape of numerical range of matrix polynomials, *Linear Algebra Appl.*, **338** (2001), 105–123.

[29] I.B. Prolla, *Approximation of Vector Valued Functions*, Notas de Matematica, Vol. 62, North-Holland Publishing Company, Amsterdam, 1977.

[30] J.G. Stampfli and J.P. Williams, Growth conditions and the numerical range in a Banach algebra, *Tôhoku Math. Journ.*, **20** (1968), 417–424.