Stochastic Analysis for Jump Processes

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The course takes place every **Tuesday 12–14 @ MA 744**

The website of the course is:
http://page.math.tu-berlin.de/~papapan/
StochasticAnalysisJP.html
and contains a course description, recommended literature, and other material related to the course (e.g. links, videos)

Lecture notes will be posted on the website during the semester (weekly basis).

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Office hours: Monday 11:00–12:00

There are **5 ECTS-points** for the course (oral examination)
Overview of the course

**Part I: Introduction to Lévy processes**
- Definition and preliminary examples of Lévy processes
- infinitely divisible laws
- the Lévy–Khintchine formula and the Lévy–Itô decomposition
- elementary operations on Lévy processes (time-change, projection)
- moments and martingales

**Part II: Stochastic calculus for jump processes**
- Stochastic integration wrt semimartingales
- Itô’s formula for general semimartingales
- measure transformations and Girsanov’s theorem
- stochastic differential equations driven by jump processes

**Part III: Applications**
- mathematical finance: modeling, pricing, hedging, utility maximization
Part I: Introduction to Lévy processes

David Applebaum
*Lévy Processes and Stochastic Calculus.*

Andreas Kyprianou
*Introductory Lectures on Fluctuations of Lévy Processes with Applications.*

Ken-iti Sato
*Lévy Processes and Infinitely Divisible Distributions.*
Part II: Stochastic calculus for jump processes

J. Jacod and A. N. Shiryaev.
Limit Theorems for Stochastic Processes (2nd ed.).

P. Protter
Stochastic Integration and Differential Equations (3rd ed.).

Part III: Applications

R. Cont and P. Tankov
Financial Modelling with Jump Processes.
Motivation
Empirical facts from finance I: asset prices …

… do not evolve continuously, they exhibit jumps or spikes!

Empirical facts from finance II: asset log-returns ...

... are not normally distributed, they are fat-tailed and skewed!

Empirical distribution of daily log-returns on the GBP/USD rate and fitted Normal.
Empirical facts from finance III: implied volatilities ... 

... are constant neither across strike, nor across maturity!

During the recent crisis:

- “The Normal copula model required implied correlations up to 120% to match market prices”. (Wim Schoutens’ talk @ GOCPS 2008)
- “Before the collapse, Carnegie Mellon’s alumni in the industry were telling me that the level of complexity in the mortgage-backed securities market had exceeded the limitations of their models”. (Steven Shreve “Don’t Blame The Quants” @ forbes.com)
- Dependence, and tail dependence, risk where completely underestimated.
Lévy processes in finance

Lévy processes provide a convenient framework to model the empirical phenomena from finance, since

1. the sample paths can have jumps
2. the generating distributions can be fat-tailed and skewed
3. the implied volatilities can have a “smile” shape
4. their dependence structure goes beyond correlation.

Lévy processes serve as

1. models themselves \(\leadsto\) exponential Lévy models
2. building blocks for models, e.g. time-changed Lévy models and affine stochastic volatility models.
Lévy processes appear also in:

1. Physics
2. Biology
3. Insurance mathematics
4. Telecommunications

Extensions or applications of Lévy processes:

1. Hilbert and Banach spaces, LCA and Lie groups
2. Quantum Mechanics and Free Probability
3. Lévy-type processes and pseudo-differential operators
4. Branching processes and fragmentation theory
Definition and toy example
Definition

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a complete stochastic basis.

Definition

A càdlàg, adapted, real valued stochastic process \(X = (X_t)_{t \geq 0}\), with \(X_0 = 0\) a.s., is called a Lévy process if:

(L1): \(X\) has independent increments, i.e. \(X_t - X_s\) is independent of \(\mathcal{F}_s\) for any \(0 \leq s < t \leq T\).

(L2): \(X\) has stationary increments, i.e. for any \(s, t \geq 0\) the distribution of \(X_{t+s} - X_t\) does not depend on \(t\).

(L3): \(X\) is stochastically continuous, i.e. for every \(t \geq 0\) and \(\epsilon > 0\):
\[
\lim_{s \to t} P(|X_t - X_s| > \epsilon) = 0.
\]
Let \((\Omega, \mathcal{F}, P)\) be a probability space.

**Definition**

A real valued stochastic process \(X = (X_t)_{t \geq 0}\), with \(X_0 = 0\) a.s., is called a **Lévy process** if:

(L1): \(X\) has independent increments, i.e. the random variables \(X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}\) are independent, for any \(n \geq 1\).

(L2): \(X\) has stationary increments, i.e. for any \(0 \leq s \leq t\), \(X_t - X_s\) is equal in distribution to \(X_{t-s}\).

(L3): \(X\) is stochastically continuous, i.e. for every \(t \geq 0\) and \(\epsilon > 0\):
\[
\lim_{s \to t} P(|X_t - X_s| > \epsilon) = 0.
\]
Example 1: linear drift

\[ X_t = bt, \quad \varphi_{X_t}(u) = \exp(iubt) \]
Example 2: Brownian motion

\[ X_t = \sigma W_t, \quad \varphi_X(t, u) = \exp \left( -\frac{u^2 \sigma^2}{2} t \right) \]
Example 3: Poisson process

\[ X_t = \sum_{k=1}^{N_t} J_k, \quad J_k \equiv 1 \quad \varphi_X(t) = \exp \left( t\lambda(e^{iu} - 1) \right) \]
Example 4: compensated Poisson process (martingale!)

\[ X_t = \sum_{k=1}^{N_t} J_k - t \lambda = N_t - \lambda t, \quad \varphi_{X_t}(u) = \exp \left( t \lambda (e^{iu} - 1 - iu) \right) \]
Example 5: compound Poisson process

\[ X_t = \sum_{k=1}^{N_t} J_k, \quad \varphi_X(t) = \exp \left( t\lambda(E[e^{iuj} - 1]) \right) \]
Example 6: Lévy jump-diffusion

\[ X_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - \lambda t E[J] \]
Characteristic function of the Lévy jump-diffusion

\[
E\left[ e^{iux} \right] = \exp \left[ iubt \right] E\left[ \exp \left( iu\sigma W_t \right) \right] E\left[ \exp \left( iu \sum_{k=1}^{N_t} J_k - iut\lambda E[J] \right) \right]
\]

\[
= \exp \left[ iubt \right] \exp \left[ -\frac{1}{2} u^2 \sigma^2 t \right] \exp \left[ \lambda t (E[e^{iuJ} - 1] - iuE[J]) \right]
\]

\[
= \exp \left[ iubt \right] \exp \left[ -\frac{1}{2} u^2 \sigma^2 t \right] \exp \left[ \lambda t \int_\mathbb{R} (e^{iu} - 1 - iux) F(dx) \right]
\]

\[
= \exp \left[ t(iub - \frac{u^2\sigma^2}{2} + \int_\mathbb{R} (e^{iu} - 1 - iux) \lambda F(dx)) \right].
\]

Observations:

1. Time and space factorize
2. Drift, diffusion and jumps separate
3. Jumps have the decomposition \( \lambda \times F \)
**A basic question**

**Observations:**

1. Time and space factorize
2. Drift, diffusion and jumps separate
3. Jumps have the decomposition $\lambda \times F$ ($\lambda = E[\# \text{ of jumps}]$)

**Question**

*Are these observations always true?*

**Answers:**

1. Yes $\rightsquigarrow$ stationary increments
2. Yes $\rightsquigarrow$ independent increments
3. No $\rightsquigarrow$ infinitely many jumps can occur (in $[0, t]$)
Aim: the connection between ...

1. Lévy processes
2. infinitely divisible laws
3. Lévy triplets

Commutative diagram of the relationship between a Lévy process $(X_t)_{t \geq 0}$, the law of the infinitely divisible random variable $\mathcal{L}(X_t)$ and the Lévy triplet $(b, c, \nu)$, demonstrating the role of the Lévy–Khintchine formula and the Lévy–Itô decomposition.