Stochastic Analysis for Jump Processes

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Lecture course @ TU Berlin, WS 2012/13

Important information

- The course takes place every Tuesday 12–14 @ MA 744
- The website of the course is:
  `http://page.math.tu-berlin.de/~papapan/StochasticAnalysisJP.html`
  and contains a course description, recommended literature, and other material related to the course (e.g. links, videos)
- Lecture notes will be posted on the website during the semester (weekly basis).
- My e-mail is `papapan@math.tu-berlin.de`
  and my office is MA 703
- Office hours: Monday 11:00–12:00
- There are 5 ECTS-points for the course (oral examination)

Overview of the course

Part I: Introduction to Lévy processes

- Definition and preliminary examples of Lévy processes
- infinitely divisible laws
- the Lévy–Khintchine formula and the Lévy–Itô decomposition
- elementary operations on Lévy processes (time-change, projection)
- moments and martingales

Part II: Stochastic calculus for jump processes

- Stochastic integration wrt semimartingales
- Itô's formula for general semimartingales
- measure transformations and Girsanov's theorem
- stochastic differential equations driven by jump processes

Part III: Applications

- mathematical finance: modeling, pricing, hedging, utility maximization $\rightsquigarrow$ Seminar "Mathematical Finance"

Literature

Part I: Introduction to Lévy processes

David Applebaum
Lévy Processes and Stochastic Calculus.
Cambridge University Press, 2nd ed., 2009.

Andreas Kyprianou
Introductory Lectures on Fluctuations of Lévy Processes with Applications.
Springer, 2006.

Ken-iti Sato
Lévy Processes and Infinitely Divisible Distributions.
Cambridge University Press, 1999.

Literature

Part II: Stochastic calculus for jump processes

J. Jacod and A. N. Shiryaev.
Limit Theorems for Stochastic Processes (2nd ed.).
Springer, 2003.

P. Protter
Stochastic Integration and Differential Equations (3rd ed.).
Springer, 2004.

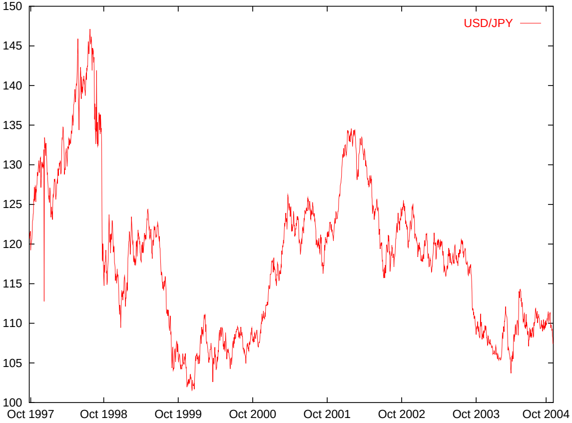
Part III: Applications

R. Cont and P. Tankov
Financial Modelling with Jump Processes.
Chapman & Hall/CRC, 2004.

Motivation

Empirical facts from finance I: asset prices ...

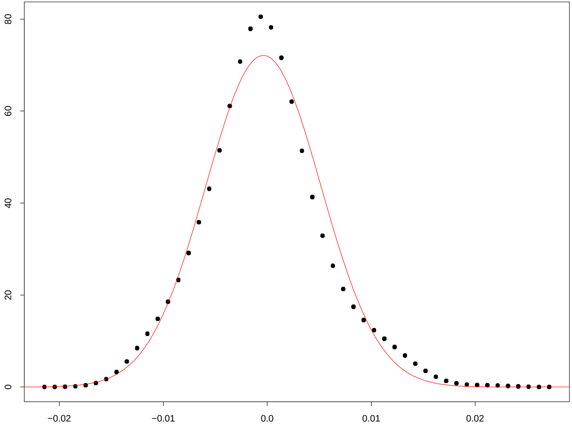
... do not evolve continuously, they exhibit jumps or spikes!



USD/JPY daily exchange rate, October 1997 – October 2004.

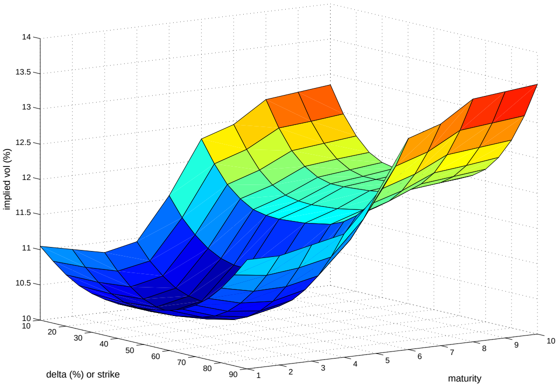
Empirical facts from finance II: asset log-returns ...

... are not normally distributed, they are fat-tailed and skewed!



Empirical distribution of daily log-returns on the GBP/USD rate and fitted Normal.

... are constant neither across strike, nor across maturity!



Implied volatilities of vanilla options on the EUR/USD rate, 5 November 2001.

Empirical facts from finance IV: “subprime” crisis ...

During the recent crisis:

- “The Normal copula model required implied correlations up to 120% to match market prices”.
  (Wim Schoutens’ talk @ GOCPS 2008)
- “Before the collapse, Carnegie Mellon’s alumni in the industry were telling me that the level of complexity in the mortgage-backed securities market had exceeded the limitations of their models”.
  (Steven Shreve “Don’t Blame The Quants” @ forbes.com)
- Dependence, and tail dependence, risk where completely underestimated.

Lévy processes in finance

Lévy processes provide a convenient framework to model the empirical phenomena from finance, since

1. the sample paths can have jumps
2. the generating distributions can be fat-tailed and skewed
3. the implied volatilities can have a "smile" shape
4. their dependence structure goes beyond correlation.

Lévy processes serve as

1. models themselves $\rightsquigarrow$ exponential Lévy models
2. building blocks for models, e.g. time-changed Lévy models and affine stochastic volatility models.

Lévy processes in other fields

Lévy processes appear also in:

1. Physics
2. Biology
3. Insurance mathematics
4. Telecommunications

Extensions or applications of Lévy processes:

1. Hilbert and Banach spaces, LCA and Lie groups
2. Quantum Mechanics and Free Probability
3. Lévy-type processes and pseudo-differential operators
4. Branching processes and fragmentation theory

Definition and toy example

Definition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete stochastic basis.

Definition

A càdlàg, adapted, real valued stochastic process $X = (X_t)_{t\geq 0}$, with $X_0 = 0$ a.s., is called a Lévy process if:

- (L1): X has independent increments, i.e.
  $X_t - X_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t \leq T$.
- (L2): X has stationary increments, i.e.
  for any $s, t \geq 0$ the distribution of $X_{t+s} - X_t$ does not depend on t.
- (L3): X is stochastically continuous, i.e.
  for every $t \geq 0$ and $\epsilon > 0$:
  $\lim_{s\to t} P(|X_t - X_s| > \epsilon) = 0$.

An equivalent definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition

A real valued stochastic process $X = (X_t)_{t \geq 0}$, with $X_0 = 0$ a.s., is called a Lévy process if:

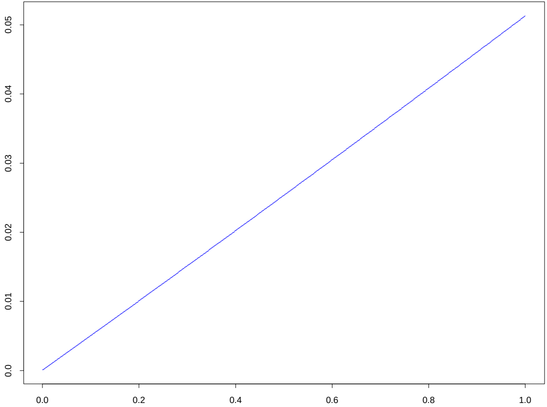
(L1): X has independent increments, i.e.
the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent, for any $n \geq 1$.

(L2): X has stationary increments, i.e.
for any $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

(L3): X is stochastically continuous, i.e.
for every $t \geq 0$ and $\epsilon > 0$:
$\lim_{s \to t} P(|X_t - X_s| > \epsilon) = 0$.

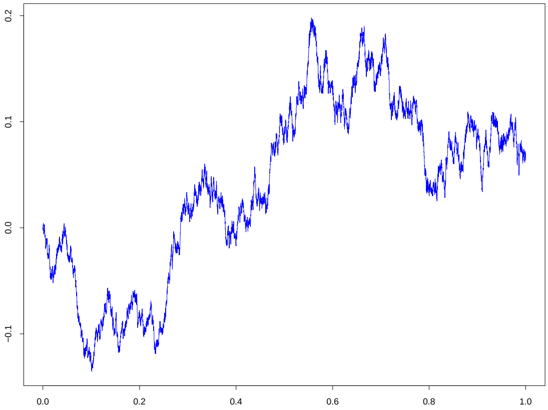
Example 1: linear drift

$$X_t = bt, \qquad \varphi_{X_t}(u) = \exp\left(iubt\right)$$



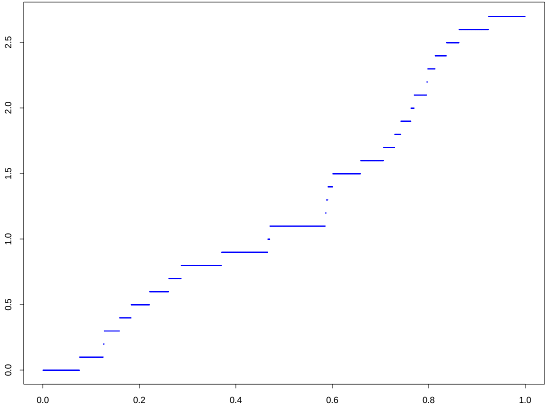
Example 2: Brownian motion

$$X_t = \sigma W_t, \qquad \varphi_{X_t}(u) = \exp\left(-\frac{u^2\sigma^2}{2}t\right)$$



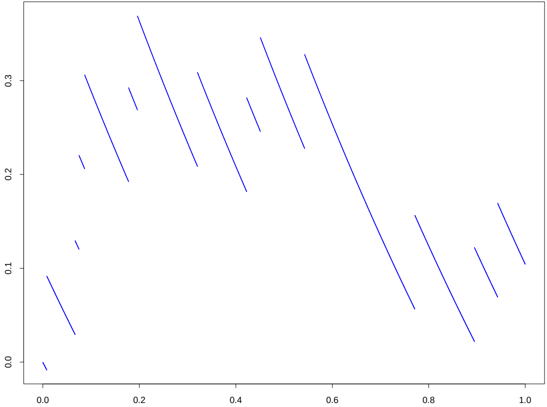
Example 3: Poisson process

$$X_t = \sum_{k=1}^{N_t} J_k,\ J_k \equiv 1 \qquad \varphi_{X_t}(u) = \exp\left(t\lambda(e^{iu} - 1)\right)$$



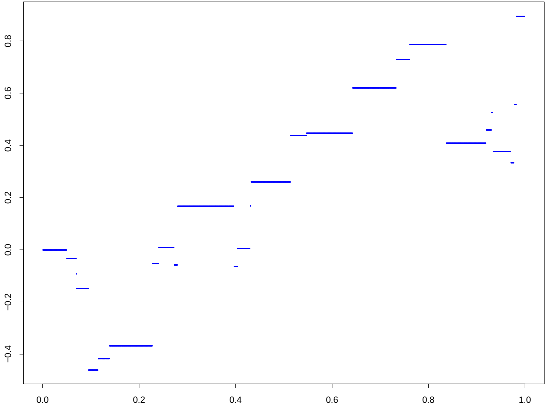
Example 4: compensated Poisson process (martingale!)

$$X_t = \sum_{k=1}^{N_t} J_k - t\lambda = N_t - \lambda t, \qquad \varphi_{X_t}(u) = \exp\Big(t\lambda(e^{iu} - 1 - iu)\Big)$$



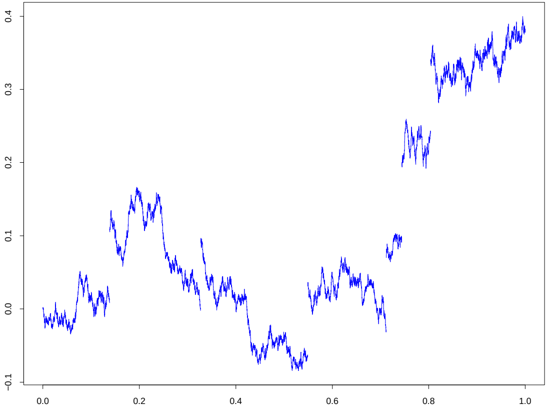
Example 5: compound Poisson process

$$X_t = \sum_{k=1}^{N_t} J_k, \qquad \varphi_{X_t}(u) = \exp\left(t\lambda(E[e^{iuJ} - 1])\right)$$



Example 6: Lévy jump-diffusion

$$X_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - \lambda t E[J]$$



Characteristic function of the Lévy jump-diffusion

$$
\begin{aligned}
E\left[e^{iuX_t}\right] &= \exp\left[iubt\right] E\Big[\exp\left(iu\sigma W_t\right)\Big] E\Big[\exp\Big(iu\sum_{k=1}^{N_t} J_k - iut\lambda E[J]\Big)\Big] \\
&= \exp\left[iubt\right] \exp\left[-\frac{1}{2}u^2\sigma^2 t\right] \exp\Big[\lambda t\big(E[e^{iuJ}-1] - iuE[J]\big)\Big] \\
&= \exp\left[iubt\right] \exp\left[-\frac{1}{2}u^2\sigma^2 t\right] \exp\Big[\lambda t \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right) F(dx)\Big] \\
&= \exp\Big[t\Big(iub - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux)\lambda F(dx)\Big)\Big].
\end{aligned}
$$

Observations:

1. Time and space factorize
2. Drift, diffusion and jumps separate
3. Jumps have the decomposition $\lambda \times F$

A basic question

Observations:

1. Time and space factorize
2. Drift, diffusion and jumps separate
3. Jumps have the decomposition $\lambda \times F$ ($\lambda = E[\#$ of jumps$]$)

Question

Are these observations always true?

Answers:

1. Yes $\rightsquigarrow$ stationary increments
2. Yes $\rightsquigarrow$ independent increments
3. No $\rightsquigarrow$ infinitely many jumps can occur (in $[0, t]$)

Aim: the connection between ...

1. Lévy processes
2. infinitely divisible laws
3. Lévy triplets

$$\begin{tikzcd}
(X_t)_{t\geq 0} \arrow[rr, shift left, "C"] \arrow[dr, shift left, "C"] & & \mathcal{L}(X_t) \arrow[ll, shift left, "LI"] \arrow[dl, Leftrightarrow, "LK"] \\
& (b, c, \nu) \arrow[ul, shift left, "LI"] &
\end{tikzcd}$$

Commutative diagram of the relationship between a Lévy process $(X_t)_{t\geq 0}$, the law of the infinitely divisible random variable $\mathcal{L}(X_t)$ and the Lévy triplet (b, c, ν), demonstrating the role of the Lévy–Khintchine formula and the Lévy–Itô decomposition.