Computational Finance

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Lecture course @ TU Berlin, SS 2016

Important information

- The course takes place every
  - Monday 10:00–12:00 @ MA 742
  - Wednesday 10:00–12:00 @ MA 742
- The website of the course is:
  `http://www.math.tu-berlin.de/~papapan/ComputationalFinance.html`
- contains: course description, recommended literature, and other material related to the course
- Lecture notes are available on the website
- E-mails:
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- Office: MA 703 (AP)
- Office hours: Wednesday 13-14 (AP)

Structure of the course

- Teaching (per week):
  - 3h Theory
  - 1h Computational practice (Python)
- Exam:
  - 4 Computational exercises
  - Oral examination
- Credit points: 10

Key points of the course

1. Review of stochastic analysis and mathematical finance
2. Monte Carlo simulation
   - Random number generation
   - Monte Carlo method
   - Quasi Monte Carlo method
3. Discretization of SDEs
   - Generating sample paths
   - Euler scheme
   - Advanced methods (Milstein)
4. PDE methods (finite differences, finite elements)
5. Lévy and affine processes
6. Fourier methods
7. Pricing American options with Monte Carlo

Books

P. Glasserman
Monte Carlo Methods in Financial Engineering
Springer, 2003

R. Seydel
Tools for Computational Finance
Springer, 2009

S. Shreve
Stochastic Calculus for Finance II
Springer, 2004

M. Musiela, M. Rutkowski
Martingale Methods in Financial Modeling
Springer, 2nd ed., 2005

D. Filipović
Term-structure Models: A Graduate Course
Springer, 2009

Options

European options

- "plain vanilla" options
  - call $(S_T - K)^+$
  - digital $1_{\{S_T > B\}}$
- exotic options
  - barrier $(S_T - K)^+ 1_{\{\max_{t \le T} S_t > B\}}$
  - one-touch $1_{\{\max_{t \le T} S_t > B\}}$
  - Asian $(\frac{1}{n} \sum_{i=1}^n S_{T_i} - K)^+$
- options on several assets
  - basket call $(\sum_{i=1}^d S_T^i - K)^+$
  - best-of call $(S_T^1 \wedge \cdots \wedge S_T^d - K)^+$

American options

- call $(S_\tau - K)^+$
- τ: stopping time

Decomposition of options

Payoff function:
map $f : \mathbb{R}^d \to \mathbb{R}_+$

- $f(x) = (x - K)^+$
- $f(x) = 1_{\{x>B\}}$
- $f(x) = (x_1 + \cdots + x_d - K)^+$
- ...

Underlying process:
random variable X on the path space $\mathbb{D}([0, T]; \mathbb{R}^d)$

- $X = S_T$
- $X = \max_{t \leq T} S_t$
- $X = (S_T^1, \ldots, S_T^d)$
- ...

Thus, for suitable f and X, any European option can be thought of as

$$f(X)$$

Stochastic Analysis

Important topics from stochastic analysis (FiMa II)

- Stochastic Integration
- Itô processes
- Quadratic Variation and Covariation
- Itô's Formula
- Stochastic Differential Equations
- Stochastic Exponential
- Markov processes
- Girsanov's theorem

Arbitrage and option pricing

Definition

An arbitrage is a self-financing trading strategy satisfying

$$V(0) = 0 \quad \text{and} \quad V(T) \geq 0 \quad \text{and} \quad \mathbb{P}[V(T) > 0] > 0,$$

for some $T > 0$.

Definition

An equivalent (local) martingale measure (E(L)MM) $\mathbb{Q} \sim \mathbb{P}$ has the property that the (discounted) price processes S^i are $\mathbb{Q}$-(local) martingales for all $1 \leq i \leq d$.

Theorem (FTAP I)

A model is arbitrage-free, in the sense that there exists no admissible arbitrage strategy, if and only if there exists an ELMM $\mathbb{Q}$.

Reference: [Filipović, 2009, Ch. 4]

Arbitrage and option pricing II

Moral

The price of an option with payoff $f(X)$ is provided by the (discounted) expected payoff under a martingale measure $\mathbb{Q}$

$$\mathbb{E}_{\mathbb{Q}}[f(X)]$$

Aim of this course

How to compute numerically

$$\mathbb{E}_{\mathbb{Q}}[f(X)]$$

Black–Scholes model

In the Black–Scholes model, the risky asset satisfies the SDE

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \tag{1}$$

under the martingale measure $\mathbb{Q}$. The solution is the stochastic exponential

$$S_t = S_0\mathcal{E}(X)_t, \tag{2}$$

where X is an Itô process

$$X_t = \int_0^t r\mathrm{d}s + \int_0^t \sigma\mathrm{d}W_s. \tag{3}$$

Hence, S follows a geometric Brownian motion

$$S_t = S_0 \exp\left(\sigma W_t + \left(r - \tfrac{1}{2}\sigma^2\right)t\right). \tag{4}$$

Black–Scholes model: call option

The price of a call option with payoff $(S_T - K)^+$ is

$$\mathbb{E}[(S_T - K)^+] = \mathbb{E}[S_T 1_{\{S_T > K\}}] - K\mathbb{E}[1_{\{S_T > K\}}]. \tag{5}$$

We can use that

$$\begin{aligned}\{S_T > K\} &= \{\log S_0 + \sigma W_T + \left(r - \tfrac{1}{2}\sigma^2\right)T > \log K\} \\ &= \left\{W_T > \frac{\log(\frac{K}{S_0}) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma}\right\}\end{aligned}$$

and the fact that $W_T \sim \mathcal{N}(0, \sqrt{T})$, and $1 - \Phi(x) = \Phi(-x)$ to deduce

$$K\mathbb{E}[1_{\{S_T > K\}}] = K\Phi\left(\frac{\log(\frac{S_0}{K}) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right). \tag{6}$$

Applying also Girsanov's theorem, we arrive at the Black–Scholes equation

$$\pi = S_0\Phi\left(\frac{\log(\frac{S_0}{K}) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log(\frac{S_0}{K}) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right). \tag{7}$$

Black–Scholes model: barrier option

The price of an up-and-out barrier call option with payoff

$$(S_T - K)^+ 1_{\{\max_{0 \le t \le T} S_t \le B\}}$$

is provided by

$$\begin{aligned}
\pi = S_0 &\Big[\Phi\big(d_+(\tfrac{S_0}{K})\big) - \Phi\big(d_+(\tfrac{S_0}{B})\big)\Big] - K\mathrm{e}^{-rT}\Big[\Phi\big(d_-(\tfrac{S_0}{K})\big) - \Phi\big(d_-(\tfrac{S_0}{B})\big)\Big] \\
&- B(\tfrac{S_0}{B})^{-\frac{2r}{\sigma^2}}\Big[\Phi\big(d_+(\tfrac{B^2}{S_0 K})\big) - \Phi\big(d_+(\tfrac{B}{S_0})\big)\Big] \\
&+ K\mathrm{e}^{-rT}(\tfrac{S_0}{B})^{-\frac{2r}{\sigma^2}+1}\Big[\Phi\big(d_-(\tfrac{B^2}{S_0 K})\big) - \Phi\big(d_-(\tfrac{B}{S_0})\big)\Big],
\end{aligned}$$

where

$$d_\pm(x) = \frac{\log(x) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

[Shreve, 2004, Ch. 7]

Aim of this course

How to compute numerically

$$\mathbb{E}[f(X)]$$

Why bother? Black–Scholes is easy!

Aim of this course

How to compute numerically

$$\mathbb{E}[f(X)]$$

Why bother? Black–Scholes is easy!

- Many options don't have closed form solutions
- The Black–Scholes model does not describe the reality
- Many other models are interesting and relevant:
  - Lévy and affine models
  - local and stochastic volatility models
- Relevant for other applications
- It is interesting mathematics!

Monte Carlo simulation

Assume we can generate $(X_i)_{i\in\mathbb{N}}$ independent copies of X. The strong law of large numbers implies

$$\frac{1}{M}\sum_{i=1}^{M} f(X_i) \xrightarrow[M\to\infty]{} \mathbb{E}[f(X)] \tag{8}$$

- How to generate independent samples?
- Error control
- Variance reduction techniques
- Quasi Monte Carlo
- How to generate sample paths? (BM, Lévy)
- Euler discretization of SDEs
- Advanced methods (stochastic Taylor expansion)

PDE methods

Assuming that the option price is "Markovian" it satisfies

$$u(t,x) = \mathbb{E}[f(S_T)|S_t = x]. \tag{9}$$

Applying Itô's formula yields

$$\begin{aligned} \mathrm{d}u(t,S_t) = {} & \frac{\partial}{\partial t}u(t,S_t)\mathrm{d}t + \frac{\partial}{\partial x}u(t,S_t)\sigma S_t \mathrm{d}W_t + \frac{\partial}{\partial x}u(t,S_t) r S_t \mathrm{d}t \\ & + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,S_t)\sigma^2 S_t^2 \mathrm{d}t. \end{aligned}$$

By no-arbitrage arguments, we deduce that $u(t,x)$ satisfies the PDE

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) + \dfrac{\partial}{\partial x}u(t,x) r x + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}u(t,x)\sigma^2 x^2 = 0, \\ \hfill u(T,x) = f(x). \end{cases} \tag{10}$$

- When can we relate an expectation with a PDE?
- How to solve the PDE numerically?
- Finite difference methods (explicit, Crank-Nicolson)

Fourier methods

We can express the option price as follows:

$$\mathbb{E}[f(S_T)] = \int f(x) p_{S_T}(x) \mathrm{d}x. \tag{11}$$

Let $\widehat{f}$ denote the Fourier transform. Assuming that f is "nice" enough, then

$$f(x) = \frac{1}{2\pi} \int \mathrm{e}^{iux} \widehat{f}(u) \mathrm{d}u. \tag{12}$$

Using Fubini's theorem, we arrive at (Plancherel's theorem)

$$\begin{aligned} \mathbb{E}[f(S_T)] &= \frac{1}{2\pi} \int \widehat{f}(u) \Big(\int \mathrm{e}^{iux} p_{S_T}(x) \mathrm{d}x \Big) \mathrm{d}u \\ &= \frac{1}{2\pi} \int \widehat{f}(u) \widehat{p}_{S_T}(u) \mathrm{d}u, \end{aligned} \tag{13}$$

where $\widehat{p}$ denotes the characteristic function of the measure p_{S_T}.

- When can we apply this method?
- Which models have a known characteristic function?
- How to implement with FFT?

Other applications

In mathematical finance

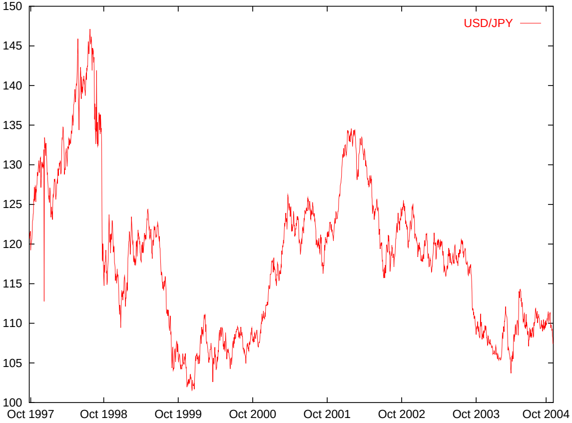
- Risk measurement and risk management
- Portfolio optimization
- Algorithmic trading
- ...

In other sciences

- Filtering
- Statistical mechanics
- Particle pyhsics
- Computational chemistry
- Molecular dynamics
- Computational biology
- ...

Empirical facts from finance I: asset prices ...

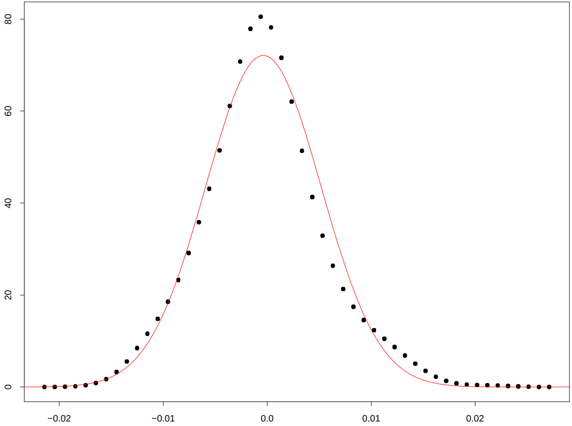
... do not evolve continuously, they exhibit jumps or spikes!



USD/JPY daily exchange rate, October 1997 – October 2004.

Empirical facts from finance II: asset log-returns ...

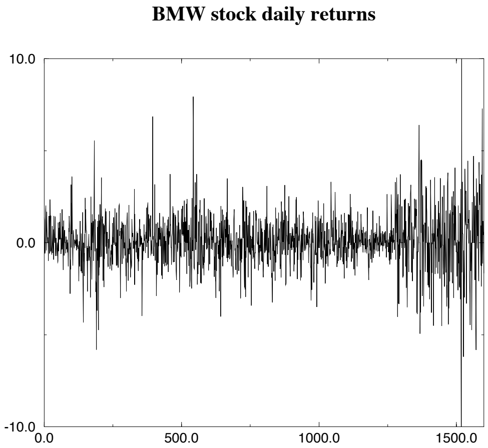
... are not normally distributed, they are fat-tailed and skewed!



Empirical distribution of daily log-returns on the GBP/USD rate and fitted Normal.

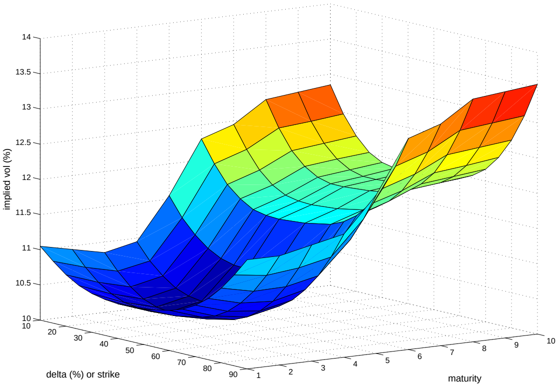
Empirical facts from finance III: volatilities ...

... are not constant over time!



Empirical facts from finance IV: implied volatilities ...

... are constant neither across strike, nor across maturity!



Implied volatilities of vanilla options on the EUR/USD rate, 5 November 2001.