Computational Finance

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Lecture course @ TU Berlin, SS 2013

Important information

- The course takes place every
  - Monday 12:00–14:00 @ MA 751
  - Friday 10:00–12:00 @ MA 848
- The website of the course is: `http://www.math.tu-berlin.de/~papapan/ComputationalFinance.html`
- contains: course description, recommended literature, and other material related to the course
- Lecture notes are available on the website
- E-mails: `Christian.Bayer@wias-berlin.de` `papapan@math.tu-berlin.de`
- Office: MA 703
- Office hours: Tuesday 11-12

Structure of the course

- Teaching (per week):
  - 3h Theory
  - 1h Computational practice (Scilab)
- Exam:
  - 3 Computational exercises
  - Oral examination
- Credit points: 10

Key points of the course

1. Review of stochastic analysis and mathematical finance
2. Monte Carlo simulation
   - Random number generation
   - Monte Carlo method
   - Quasi Monte Carlo method
3. Discretization of SDEs
   - Generating sample paths
   - Euler scheme
   - Advanced methods (Milstein)
4. PDE methods (finite differences, finite elements)
5. Lévy and affine processes
6. Fourier methods
7. Pricing American options with Monte Carlo

Books

- P. Glasserman
  Monte Carlo Methods in Financial Engineering
  Springer, 2003
- R. Seydel
  Tools for Computational Finance
  Springer, 2009
- S. Shreve
  Stochastic Calculus for Finance II
  Springer, 2004
- M. Musiela, M. Rutkowski
  Martingale Methods in Financial Modeling
  Springer, 2nd ed., 2005
- D. Filipović
  Term-structure Models: A Graduate Course
  Springer, 2009

Options

European options

- “plain vanilla” options
  - call $(S_T - K)^+$
  - digital $1_{\{S_T > B\}}$
- exotic options
  - barrier $(S_T - K)^+ 1_{\{\max_{t \le T} S_t > B\}}$
  - one-touch $1_{\{\max_{t \le T} S_t > B\}}$
  - Asian $(\frac{1}{n} \sum_{i=1}^{n} S_{T_i} - K)^+$
- options on several assets
  - basket call $(\sum_{i=1}^{d} S_T^i - K)^+$
  - best-of call $(S_T^1 \wedge \cdots \wedge S_T^d - K)^+$

American options

- call $(S_\tau - K)^+$
- τ: stopping time

Decomposition of options

Payoff function:
map $f : \mathbb{R}^d \to \mathbb{R}_+$

- $f(x) = (x - K)^+$
- $f(x) = 1_{\{x>B\}}$
- $f(x) = (x_1 + \cdots + x_d - K)^+$
- ...

Underlying process:
random variable X on the path space $\mathbb{D}([0, T]; \mathbb{R}^d)$

- $X = S_T$
- $X = \max_{t \leq T} S_t$
- $X = (S_T^1, \ldots, S_T^d)$
- ...

Thus, for suitable f and X, any European option can be thought of as

$$f(X)$$

Stochastic Analysis

Important topics from stochastic analysis (FiMa II)

- Stochastic Integration
- Itô processes
- Quadratic Variation and Covariation
- Itô's Formula
- Stochastic Differential Equations
- Stochastic Exponential
- Markov processes
- Girsanov's theorem

Arbitrage and option pricing

Definition

An arbitrage is a self-financing trading strategy satisfying

$$V(0) = 0 \quad \text{and} \quad V(T) \geq 0 \quad \text{and} \quad \mathbb{P}[V(T) > 0] > 0,$$

for some $T > 0$.

Definition

An equivalent (local) martingale measure (E(L)MM) $\mathbb{Q} \sim \mathbb{P}$ has the property that the (discounted) price processes S^i are $\mathbb{Q}$-(local) martingales for all $1 \leq i \leq d$.

Theorem (FTAP I)

A model is arbitrage-free, in the sense that there exists no admissible arbitrage strategy, if and only if there exists an ELMM $\mathbb{Q}$.

Reference: [Filipović, 2009, Ch. 4]

Arbitrage and option pricing II

Moral

The price of an option with payoff $f(X)$ is provided by the (discounted) expected payoff under a martingale measure $\mathbb{Q}$

$$\mathbb{E}_{\mathbb{Q}}[f(X)]$$

Aim of this course

How to compute numerically

$$\mathbb{E}_{\mathbb{Q}}[f(X)]$$

Black–Scholes model

In the Black–Scholes model, the risky asset satisfies the SDE

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \tag{1}$$

under the martingale measure $\mathbb{Q}$. The solution is the stochastic exponential

$$S_t = S_0\mathcal{E}(X)_t, \tag{2}$$

where X is an Itô process

$$X_t = \int_0^t r\mathrm{d}s + \int_0^t \sigma\mathrm{d}W_s. \tag{3}$$

Hence, S follows a geometric Brownian motion

$$S_t = S_0 \exp\left(\sigma W_t + \left(r - \tfrac{1}{2}\sigma^2\right)t\right). \tag{4}$$

Black–Scholes model: call option

The price of a call option with payoff $(S_T - K)^+$ is

$$\mathbb{E}[(S_T - K)^+] = \mathbb{E}[S_T 1_{\{S_T > K\}}] - K\mathbb{E}[1_{\{S_T > K\}}]. \tag{5}$$

We can use that

$$\begin{aligned}\{S_T > K\} &= \{\log S_0 + \sigma W_T + \left(r - \tfrac{1}{2}\sigma^2\right) T > \log K\} \\ &= \left\{W_T > \frac{\log(\frac{K}{S_0}) - \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma}\right\}\end{aligned}$$

and the fact that $W_T \sim \mathcal{N}(0, \sqrt{T})$, and $1 - \Phi(x) = \Phi(-x)$ to deduce

$$K\mathbb{E}[1_{\{S_T > K\}}] = K\Phi\left(\frac{\log(\frac{S_0}{K}) + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right). \tag{6}$$

Applying also Girsanov's theorem, we arrive at the Black–Scholes equation

$$\pi = S_0\Phi\left(\frac{\log(\frac{S_0}{K}) + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right) - Ke^{-rT}\Phi\left(\frac{\log(\frac{S_0}{K}) + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right). \tag{7}$$

Black–Scholes model: barrier option

The price of an up-and-out barrier call option with payoff

$$(S_T - K)^+ 1_{\{\max_{0\le t\le T} S_t \le B\}}$$

is provided by

$$\begin{aligned}\pi = S_0\Big[\Phi\big(d_+(\tfrac{S_0}{K})\big) - \Phi\big(d_+(\tfrac{S_0}{B})\big)\Big] - K\mathrm{e}^{-rT}\Big[\Phi\big(d_-(\tfrac{S_0}{K})\big) - \Phi\big(d_-(\tfrac{S_0}{B})\big)\Big] \\ - B(\tfrac{S_0}{B})^{-\frac{2r}{\sigma^2}}\Big[\Phi\big(d_+(\tfrac{B^2}{S_0K})\big) - \Phi\big(d_+(\tfrac{B}{S_0})\big)\Big] \\ + K\mathrm{e}^{-rT}(\tfrac{S_0}{B})^{-\frac{2r}{\sigma^2}+1}\Big[\Phi\big(d_-(\tfrac{B^2}{S_0K})\big) - \Phi\big(d_-(\tfrac{B}{S_0})\big)\Big],\end{aligned}$$

where

$$d_\pm(x) = \frac{\log(x) + \left(r \mp \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

[Shreve, 2004, Ch. 7]

Aim of this course

How to compute numerically

$$\mathbb{E}[f(X)]$$

Why bother? Black–Scholes is easy!

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How to compute numerically

$$\mathbb{E}[f(X)]$$

Why bother? Black–Scholes is easy!

- Many options don't have closed form solutions
- The Black–Scholes model does not describe the reality
- Many other models are interesting and relevant:
  - Lévy and affine models
  - local and stochastic volatility models
- Relevant for other applications
- It is interesting mathematics!

Monte Carlo simulation

Assume we can generate $(X_i)_{i \in \mathbb{N}}$ independent copies of X. The strong law of large numbers implies

$$\frac{1}{M} \sum_{i=1}^{M} f(X_i) \xrightarrow[M \to \infty]{} \mathbb{E}[f(X)] \tag{8}$$

- How to generate independent samples?
- Error control
- Variance reduction techniques
- Quasi Monte Carlo
- How to generate sample paths? (BM, Lévy)
- Euler discretization of SDEs
- Advanced methods (stochastic Taylor expansion)

PDE methods

Assuming that the option price is "Markovian" it satisfies

$$u(t,x) = \mathbb{E}[f(S_T)|S_t = x]. \tag{9}$$

Applying Itô's formula yields

$$\begin{aligned} \mathrm{d}u(t,S_t) = \frac{\partial}{\partial t}u(t,S_t)\mathrm{d}t + \frac{\partial}{\partial x}u(t,S_t)\sigma S_t \mathrm{d}W_t + \frac{\partial}{\partial x}u(t,S_t) r S_t \mathrm{d}t \\ + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,S_t)\sigma^2 S_t^2 \mathrm{d}t. \end{aligned}$$

By no-arbitrage arguments, we deduce that $u(t,x)$ satisfies the PDE

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + \frac{\partial}{\partial x}u(t,x)rx + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x)\sigma^2 x^2 = 0, \\ u(T,x) = f(x). \end{cases} \tag{10}$$

- When can we relate an expectation with a PDE?
- How to solve the PDE numerically?
- Finite difference methods (explicit, Crank-Nicolson)

Fourier methods

We can express the option price as follows:

$$\mathbb{E}[f(S_T)] = \int f(x) p_{S_T}(x) \mathrm{d}x. \tag{11}$$

Let $\widehat{f}$ denote the Fourier transform. Assuming that f is "nice" enough, then

$$f(x) = \frac{1}{2\pi} \int \mathrm{e}^{iux} \widehat{f}(u) \mathrm{d}u. \tag{12}$$

Using Fubini's theorem, we arrive at (Plancherel's theorem)

$$\begin{aligned} \mathbb{E}[f(S_T)] &= \frac{1}{2\pi} \int \widehat{f}(u) \Big(\int \mathrm{e}^{iux} p_{S_T}(x) \mathrm{d}x \Big) \mathrm{d}u \\ &= \frac{1}{2\pi} \int \widehat{f}(u) \widehat{p}_{S_T}(u) \mathrm{d}u, \end{aligned} \tag{13}$$

where $\widehat{p}$ denotes the characteristic function of the measure p_{S_T}.

- When can we apply this method?
- Which models have a known characteristic function?
- How to implement with FFT?

Other applications

In mathematical finance

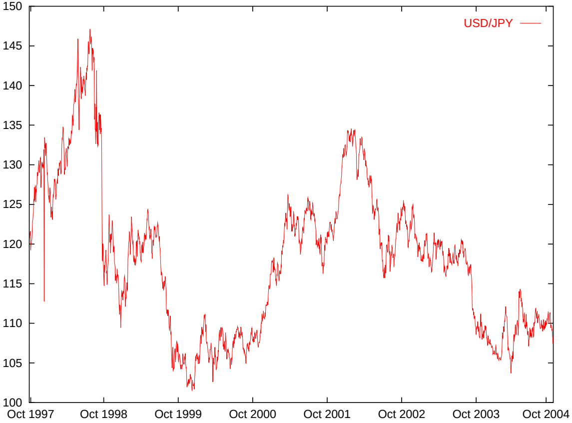
- Risk measurement and risk management
- Portfolio optimization
- Algorithmic trading
- . . .

In other sciences

- Filtering
- Statistical mechanics
- Particle pyhsics
- Computational chemistry
- Molecular dynamics
- Computational biology
- . . .

Empirical facts from finance I: asset prices ...

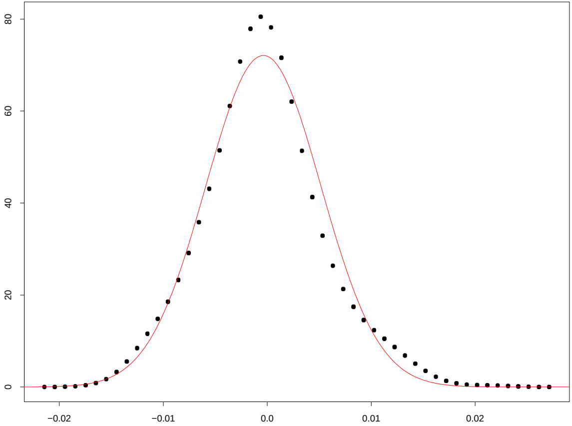
... do not evolve continuously, they exhibit jumps or spikes!



USD/JPY daily exchange rate, October 1997 – October 2004.

Empirical facts from finance II: asset log-returns ...

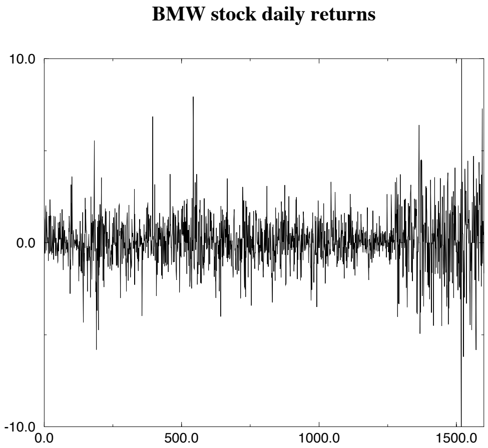
... are not normally distributed, they are fat-tailed and skewed!



Empirical distribution of daily log-returns on the GBP/USD rate and fitted Normal.

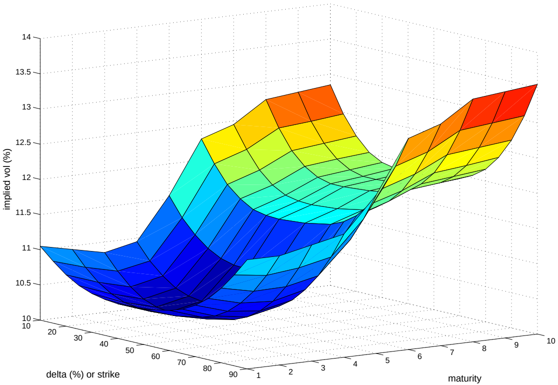
Empirical facts from finance III: volatilities ...

... are not constant over time!



Empirical facts from finance IV: implied volatilities ...

... are constant neither across strike, nor across maturity!



Implied volatilities of vanilla options on the EUR/USD rate, 5 November 2001.