COMPUTATIONAL FINANCE

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Exercise 2

Set-up. Consider the Heston stochastic volatility model (under a risk neutral measure), provided by the SDEs

$$\begin{aligned} \mathrm{d}X_t &= \big(r - \frac{1}{2}V_t\big)\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, \quad X_0 = x, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \eta\sqrt{V_t}\mathrm{d}\overline{W}_t, \quad V_0 = v. \end{aligned} \tag{1}$$

The parameters satisfy $r \in \mathbb{R}$, $\kappa, \theta, \eta \in \mathbb{R}_+$, the initial values are $x \in \mathbb{R}, v \in \mathbb{R}_+$, and the Brownian motions $W, \overline{W}$ are correlated with parameter $\rho \in [-1, 1]$.

An Euler discretization of this SDE is provided by

$$\begin{aligned} \overline{X}_{i+1} &= \overline{X}_i + \big(r - \frac{1}{2}\overline{V}_i\big)\Delta t_i + \sqrt{\overline{V}_i^+}\Delta W_i, \quad \overline{X}_0 = x, \\ \overline{V}_{i+1} &= \overline{V}_i + \kappa(\theta - \overline{V}_i)\Delta t_i + \eta\sqrt{\overline{V}_i^+}\Delta \overline{W}_i, \quad \overline{V}_0 = v, \end{aligned} \tag{2}$$

where $a^+ = \max\{a, 0\}$.

The process (V, X) is an affine process on $\mathbb{R}_+ \times \mathbb{R}$ and the characteristic function is provided by

$$\mathbb{E}_{v,x}\left[\mathrm{e}^{u_1 V_t + u_2 X_t}\right] = \exp\big(\phi(t, u_1, u_2) + \psi_1(t, u_1, u_2) \cdot v + \psi_2(t, u_1, u_2) \cdot x\big), \tag{3}$$

where (ϕ, ψ_1, ψ_2) are solutions of the system of Riccati equations

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t, u_1, u_2) &= F(\psi_1(t, u_1, u_2), \psi_2(t, u_1, u_2)), \quad \phi(0, u_1, u_2) = 0 \\ \frac{\partial}{\partial t}\psi_1(t, u_1, u_2) &= R(\psi_1(t, u_1, u_2), \psi_2(t, u_1, u_2)), \quad \psi_1(0, u_1, u_2) = u_1 \\ \psi_2(t, u_1, u_2) &= u_2, \end{aligned} \tag{4}$$

with

$$\begin{aligned} F(u_1, u_2) &= \kappa\theta u_1 + r u_2 \\ R(u_1, u_2) &= -\kappa u_1 - \frac{1}{2}u_2 + \frac{1}{2}u_2^2 + \frac{1}{2}\eta^2 u_1^2 + \eta\rho u_1 u_2. \end{aligned} \tag{5}$$

Tasks.

(1) Solve the system of Riccati equations (4) and thus determine the characteristic function (3) of the Heston model.
(Hint: see Lemma 5.2 in Filipović & Mayerhofer "Affine diffusion processes: theory and applications".)

(2) Compute the Fourier transform of the payoff function $f(x) = (K - \mathrm{e}^x)^+$ corresponding to the put option and determine the set $\mathcal{I}$ where the dampened payoff function $g(x) = \mathrm{e}^{-Rx}f(x)$ satisfies $g \in L^1_{\mathrm{bc}}(\mathbb{R})$ and $\widehat{g} \in L^1(\mathbb{R})$.

(3) Compute the price of a European put option $(K - S_T)^+$, where $X = \log(S)$ is provided by (1), using Fourier methods for option pricing.
(What is the range $\mathcal{I} \cap \mathcal{J}$ for R?).

(4) Compare these results with the put option prices determined by the Euler-Monte-Carlo method using (2), i.e. compute the option prices with Euler-MC and the 95% confidence intervals.
(5) Study empirically the convergence of the Euler-MC scheme.

Data.

- Spot price $S_0 = 100$, interest rate $r = 0\%$.
- Maturity $T = 5$, strike prices $K = \{80, 100, 120\}$.
- Heston parameters: $\kappa = 1$, $\theta = v = 9\%$, $\eta = 1$, $\rho = -0.3$.

Submit.

- The source code (in `scilab`/`matlab`/`C`/...).
- A PDF file explaining how the code was developed and discussing the results (preferably written in LaTeX).
- Submit everything per e-mail in a zip file called: `Exercise_2_Name_Surname`.