Ewald's Method Revisited: Rapidly Convergent Series Representations of Certain Green's Functions

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Suggested Running Head: Ewald's Method Revisited

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ABSTRACT

We propose a justification of Ewald's method of obtaining rapidly convergent series for the Green's function of the 3-dimensional Helmholtz equation. Our point of view enables us to extend the method to Green's functions for the Helmholtz equation in certain domains of $\mathbb{R}^d$, with quite general boundary conditions.

Key words. Helmholtz equation, Green's function, Ewald's method.

1 Introduction

In 1921 P. P. Ewald published a sensational paper (see [1]). He presented a method that was used to transform the slowly convergent series of the Green's function of the 3-d Helmholtz equation, with Floquet (e.g. periodic) boundary conditions in the x and y directions, to a series that converges very rapidly.

Ewald's method gives spectacular computational results and hence it has been used extensively by physicist, engineers, and numerical analysts (as an example, see [2]). However, it is not transparent at all. The reader is usually left with the impression that it works by coincidence. The method is based on the formula

$$\frac{e^{iKR}}{R} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-R^2 z^2 + \frac{K^2}{4z^2}\right) dz,$$

which somehow comes "out of the blue".

In this note we propose a point of view that sheds some understanding to the underlying structure of Ewald's approach. This enables us to easily extend his method to any dimension and a variety of boundary conditions.

2 A Rapidly Convergent Series for the Green's Function

Let L be a self-adjoint operator on the space $L^2(D)$, where D is a domain in $\mathbb{R}^d$, such that $\inf \sigma(L) = \lambda_0$, where $\sigma(L)$ is the spectrum of L. If λ is a complex number with $\Re\{\lambda\} < \lambda_0$, then

$$(L-\lambda)^{-1} = \int_0^{\infty} e^{\lambda t} e^{-tL} dt.$$

In terms of integral kernels, the above equation can be written as

$$G(x, y; \lambda) = \int_0^{\infty} e^{\lambda t} p(t, x, y) dt, \tag{1}$$

where $G(x, y; \lambda)$ is the Green's function and $p(t, x, y)$ the heat kernel associated to L (thus, the Green's function is the Laplace transform of the heat kernel, viewed as a function of t). The integral in (1) converges if

$$\Re\{\lambda\} < \lambda_0$$

and diverges if $\Re\{\lambda\} > \lambda_0$ (if $\Re\{\lambda\} = \lambda_0$, convergence is possible). On the other hand, $G(x, y; \lambda)$ is analytic in λ, on $\mathbb{C} \setminus \sigma(L)$.

If $L = -\Delta$, acting on $\mathbb{R}^d$ (hence $\sigma(L) = [0, \infty)$; L is the "free" d-dimensional negative Laplacian operator), then its heat kernel is

$$p(t,x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{4t} \right).$$

Let $\Gamma_d(x,y;\lambda)$ be the corresponding Green's function, i.e. the integral kernel of $(L-\lambda)^{-1}$. Then (1) gives

$$\Gamma_d(x,y;\lambda) = \int_0^\infty \frac{1}{(4\pi t)^{d/2}} \exp\left(\lambda t - \frac{|x-y|^2}{4t} \right) dt \stackrel{\text{def}}{=} \Gamma_d(|x-y|\,;\lambda) \tag{2}$$

($\Gamma_d(x,x;\lambda) = \infty$, if $d \geq 2$). By deforming the contour of the above integral, we can get $\Gamma_d(x,y;\lambda)$, for $\Re\{\lambda\} \geq 0$, but one can do even better, since $\Gamma_d(|x-y|\,;\lambda)$ can be computed explicitly. If $d=1$, it is easy to see (by solving the associated ordinary differential equation for Γ_1) that

$$\Gamma_1(x,y;\lambda) = \Gamma_1(|x-y|\,;\lambda) = \frac{e^{-\sqrt{-\lambda}|x-y|}}{2\sqrt{-\lambda}}.$$

Furthermore, from (2) one observes that

$$\frac{d}{d\lambda}\Gamma_d(R;\lambda) = \frac{1}{4\pi}\Gamma_{d-2}(R;\lambda).$$

From this equation one can compute $\Gamma_d(x,y;\lambda)$ for all odd dimensions d:

$$\Gamma_3(x,y;\lambda) = \frac{e^{-\sqrt{-\lambda}|x-y|}}{4\pi\,|x-y|}, \qquad \text{etc.}$$

If d is even, $\Gamma_d(x,y;\lambda)$ is not an elementary function. One can find the radial solutions of the equation

$$-\Delta u = \lambda u + \delta(x), \qquad x \in \mathbb{R}^d,$$

to conclude that (for all d)

$$\Gamma_d(x,y;\lambda) = \frac{1}{(2\pi)^{d/2}} (-\lambda)^{(d/4)-(1/2)} |x-y|^{1-(d/2)} K_{(d/2)-1}\left(\sqrt{-\lambda}\,|x-y| \right),$$

where $K_\nu(\cdot)$ is the modified Bessel function of the 2nd kind, of order ν. Finally, we notice that $\Gamma_d(R;\lambda)$ decays exponentially, as $R \to \infty$. This is easy to establish if d is odd, whereas, if d is even, one can, for example, use (2) to get the estimate

$$|\Gamma_d(R;\lambda)| < |\Gamma_{d+1}(R;\lambda)| + |\Gamma_{d-1}(R;\lambda)|$$

(alternatively, one can do asymptotic analysis, as $|x-y| \to \infty$, to the integral in (2)).

Now consider the domain D in $\mathbb{R}^d$

$$D = (0, b_1) \times \cdots \times (0, b_r) \times \mathbb{R}^{d-r}$$

(r is some integer between 1 and d), and let $L_\alpha = -\Delta$ act on D with α-Floquet-Bloch boundary conditions. This means that there are numbers $\alpha_1, ..., \alpha_r$ such that, for u to be in the domain of L_α, we must have

$$\begin{aligned} u(x_1, ..., b_j, ..., x_d) &= e^{i\alpha_j b_j} u(x_1, ..., 0, ..., x_d), \\ \nabla u(x_1, ..., b_j, ..., x_d) &= e^{i\alpha_j b_j} \nabla u(x_1, ..., 0, ..., x_d), \end{aligned} \tag{3}$$

for all $j = 1, ..., r$. Without loss of generality we assume that

$$-\frac{\pi}{b_j} < \alpha_j \leq \frac{\pi}{b_j}, \qquad j = 1, ..., r.$$

The method of images gives that the Green's function of L_α is

$$G(x, y; \lambda) = \sum_{m \in \mathbb{Z}^r} e^{-i\alpha \cdot (mb)} \Gamma_d(x + mb, y; \lambda),$$

where

$$\alpha = (\alpha_1, ..., \alpha_r, 0, ..., 0) \qquad \text{and} \qquad mb = (m_1 b_1, ..., m_r b_r, 0, ..., 0)$$

are considered in $\mathbb{R}^d$. Using (2) in the above equation, we get

$$G(x, y; \lambda) = \sum_{m \in \mathbb{Z}^r} e^{-i\alpha \cdot (mb)} \int_0^\infty \frac{1}{(4\pi t)^{d/2}} \exp\left(\lambda t - \frac{|x - y + mb|^2}{4t}\right) dt, \tag{4}$$

where, for $\lambda < 0$ (or, more generally, $\Re\{\lambda\} < 0$), the series converges absolutely, provided $x \neq y$ (x and y are in D).

We now show how to obtain a series representation of $G(x, y; \lambda)$, $x \neq y$, that converges very rapidly, for all $\lambda \in \mathbb{C} \setminus N$, where N is a discrete (countable) set. First we write (following Ewald's approach)

$$G(x, y; \lambda) = G_1(x, y; \lambda) + G_2(x, y; \lambda),$$

where

$$G_1(x, y; \lambda) = \sum_{m \in \mathbb{Z}^r} e^{-i\alpha \cdot (mb)} \int_0^E \frac{1}{(4\pi t)^{d/2}} \exp\left(\lambda t - \frac{|x - y + mb|^2}{4t}\right) dt, \tag{5}$$

$$G_2(x,y;\lambda) = \sum_{m\in\mathbb{Z}^r} e^{-i\alpha\cdot(mb)} \int_E^{\infty} \frac{1}{(4\pi t)^{d/2}} \exp\left(\lambda t - \frac{|x-y+mb|^2}{4t}\right) dt, \tag{6}$$

and E is a positive number. Observe that $G_1(x,y;\lambda)$ and $G_2(x,y;\lambda)$ are the integral kernels of the operators

$$\int_0^E e^{-t(L_\alpha-\lambda)}dt \qquad \text{and} \qquad \int_E^\infty e^{-t(L_\alpha-\lambda)}dt = (L_\alpha-\lambda)^{-1} e^{-E(L_\alpha-\lambda)},$$

respectively.

The series for $G_1(x,y;\lambda)$, given in (5), is already rapidly convergent, <u>for all $\lambda\in\mathbb{C}$</u> (as long as $x\neq y$), since its general term decays (in m) like $C|mb|^{-2}\exp\left(-|mb|^2/4E\right)$.

In the case $r=d$, $G_2(x,y;\lambda)$, being the integral kernel of $(L_\alpha-\lambda)^{-1}e^{-E(L_\alpha-\lambda)}$, has the eigenfunction expansion

$$G_2(x,y;\lambda) = \frac{1}{|D|}\sum_{m\in\mathbb{Z}^d} \frac{e^{-\left[|\alpha|^2+4\pi^2|m/b|^2+4\pi\alpha\cdot(m/b)-\lambda\right]E}}{|\alpha|^2+4\pi^2|m/b|^2+4\pi\alpha\cdot(m/b)-\lambda} e^{i[2\pi(m/b)+\alpha]\cdot(x-y)}, \tag{7}$$

where $|D| = b_1 b_2\cdots b_d$ is the volume of D and

$$m/b = (m_1/b_1, ..., m_d/b_d).$$

Formula (7) is valid for all $\lambda\neq|\alpha|^2+4\pi^2|m/b|^2+4\pi\alpha\cdot(m/b)$, i.e. all $\lambda\notin\sigma(L_\alpha)$, at which $G_2(x,y;\lambda)$ has simple poles. The terms of the series above decay (in m) like

$$C|m/b|^{-2}\exp\left(-4\pi^2|m/b|^2 E\right),$$

for any fixed $\lambda\in\mathbb{C}\setminus\sigma(L_\alpha)$.

Now, let us discuss the case $r<d$. For the series in (6) we first assume that $\Re\{\lambda\}<0$. In this case we can interchange summation and integration and get

$$G_2(x,y;\lambda) = \int_E^\infty \frac{e^{\lambda t-|\overline{x}^r-\overline{y}^r|^2/4t}}{(4\pi t)^{d/2}} \left(\sum_{m\in\mathbb{Z}^r} e^{-|x^r-y^r+mb|^2/4t-i\alpha\cdot(mb)}\right) dt, \tag{8}$$

where, for $x=(x_1,...,x_d)$, we have introduced

$$x^r = (x_1,...,x_r,0,...,0) \qquad \text{and} \qquad \overline{x}^r = (0,...,0,x_{r+1},...,x_d).$$

Next (following the spirit of Ewald's approach) we invoke a lemma.

Lemma 1. If $s > 0$ and $z, \gamma \in \mathbb{C}$, then

$$\sum_{n\in\mathbb{Z}} e^{-s^2(z+n)^2 - i\gamma n} = \frac{\sqrt{\pi}}{s} e^{i\gamma z - \gamma^2/4s^2} \sum_{n\in\mathbb{Z}} e^{-\pi^2 n^2/s^2 - \pi\gamma n/s^2 + 2\pi i z n}.$$

Proof. Poisson Summation Formula (see, e.g. [3]) applied to the function

$$f(x) = e^{-Ax^2+Bx}, \qquad A > 0, \quad B \in \mathbb{C}.$$

(in fact, the formula is also a theta function identity). ■

The lemma implies immediately

$$\sum_{m\in\mathbb{Z}^r} e^{-|x^r - y^r + mb|^2/4t - i\alpha\cdot(mb)} =$$
$$\frac{(4\pi t)^{r/2}}{b_1\cdots b_r} e^{i\alpha\cdot(x^r-y^r)-|\alpha|^2 t} \sum_{m\in\mathbb{Z}^r} e^{-\left[4\pi^2|m/b|^2 + 4\pi\alpha\cdot(m/b)\right]t + 2\pi i(x^r-y^r)\cdot(m/b)},$$

where

$$m/b = (m_1/b_1, ..., m_r/b_r, 0, ..., 0).$$

Thus (8) becomes

$G_2(x, y; \lambda) =$

$$\sum_{m\in\mathbb{Z}^r} \frac{e^{i[2\pi(m/b)+\alpha]\cdot(x^r-y^r)}}{b_1\cdots b_r} \int_E^\infty e^{-\left[|\alpha|^2 + 4\pi^2|m/b|^2 + 4\pi\alpha\cdot(m/b) - \lambda\right]t - |\overline{x}^r - \overline{y}^r|^2/4t} \frac{dt}{(4\pi t)^{(d-r)/2}}. \tag{9}$$

So far we have assumed $\Re\{\lambda\} < 0$. However, the terms of the series above decay (in m) like

$$C\,|m/b|^{-2} \exp\left(-4\pi^2\,|m/b|^2\,E\right),$$

for any fixed $\lambda \in \mathbb{C}$. Thus, in the tail of the series, we can take λ to be any complex number. There seems to be a problem though with the first few terms of the series, where we may have $\Re\{\lambda\} \geq |\alpha|^2 + 4\pi^2\,|m/b|^2 + 4\pi\alpha\cdot(m/b)$ (so that the corresponding integrals diverge). The following lemma shows how to overcome this problem.

Lemma 2. Let a, c, E be positive constants, and n a positive integer. For $\Re\{\lambda\} > 0$ we set

$$f(\lambda) = \int_E^\infty e^{-(a-\lambda)t - c^2/4t} \frac{dt}{(4\pi t)^{n/2}}.$$

Then the only singularity of $f(\lambda)$ on $\mathbb{C}$ is a branch point at $\lambda = a$.

Proof. Define

$$f_0(\lambda) = \int_0^E e^{-(a-\lambda)t - c^2/4t} \frac{dt}{(4\pi t)^{n/2}}.$$

Notice that $f_0(\lambda)$ is entire in λ and

$$f(\lambda) + f_0(\lambda) = \frac{1}{(2\pi)^{n/2}} (a-\lambda)^{(n/4)-(1/2)} c^{1-(n/2)} K_{(n/2)-1}\left(c\sqrt{a-\lambda}\right),$$

where $K_\nu(\cdot)$ is the modified Bessel function of the 2nd kind, of order ν. Thus, the exact type of the branch point of $f(\lambda)$ at $\lambda = a$ is known. ∎

Thus, if $r < d$, the lemma implies that the only singularities of $G_2(x, y; \lambda)$, viewed as a function of λ, are branch points at $\lambda = |\alpha|^2 + 4\pi^2 |m/b|^2 + 4\pi\alpha \cdot (m/b)$. These are also the singularities of $G(x, y; \lambda)$. Notice that, if $d-r > 2$, then $G(x, y; \lambda)$ stays finite at these values of λ.

3 The Main Result

We summarize the main result established in the previous section (see (5) and (9)):

Theorem. Consider the domain D in $\mathbb{R}^d$

$$D = (0, b_1) \times \cdots \times (0, b_r) \times \mathbb{R}^{d-r}$$

(r is some integer between 1 and d), and let $L_\alpha = -\Delta$ act on D with α-Floquet-Bloch boundary conditions (see (3)). Then, for any $E > 0$, the Green's function $G(x, y; \lambda)$ of L_α has the following rapidly convergent series representation

$$G(x, y; \lambda) = G_1(x, y; \lambda) + G_2(x, y; \lambda),$$

$$G_1(x, y; \lambda) = \sum_{m \in \mathbb{Z}^r} e^{-i\alpha\cdot(mb)} \int_0^E \frac{1}{(4\pi t)^{d/2}} \exp\left(\lambda t - \frac{|x-y+mb|^2}{4t}\right) dt,$$

$G_2(x, y; \lambda) =$

$$\sum_{m \in \mathbb{Z}^r} \frac{e^{i[2\pi(m/b)+\alpha]\cdot(x^r - y^r)}}{b_1 \cdots b_r} \int_E^\infty e^{-\left[|\alpha|^2 + 4\pi^2 |m/b|^2 + 4\pi\alpha\cdot(m/b) - \lambda\right]t - |\overline{x}^r - \overline{y}^r|^2/4t} \frac{dt}{(4\pi t)^{(d-r)/2}},$$

where $\alpha = (\alpha_1, ..., \alpha_r, 0, ..., 0)$, $mb = (m_1 b_1, ..., m_r b_r, 0, ..., 0)$, $m/b = (m_1/b_1, ..., m_r/b_r, 0, ..., 0)$, $x^r = (x_1, ..., x_r, 0, ..., 0)$, and $\overline{x}^r = (0, ..., 0, x_{r+1}, ..., x_d)$.

Remarks. (a) If $n = 1$, then

$$f(\lambda) = \int_E^\infty e^{-(a-\lambda)t - c^2/4t} \frac{dt}{\sqrt{4\pi t}} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E}}^\infty e^{-(a-\lambda)s^2 - c^2/4s^2} ds.$$

It follows (differentiate with respect to $\sqrt{E}$) that

$$\begin{aligned} f(\lambda) &= \frac{e^{c\sqrt{a-\lambda}}}{4\sqrt{a-\lambda}} \text{erfc}\left(\sqrt{E\,(a-\lambda)} + \frac{c}{2\sqrt{E}}\right) \\ &\quad + \frac{e^{-c\sqrt{a-\lambda}}}{4\sqrt{a-\lambda}} \text{erfc}\left(\sqrt{E\,(a-\lambda)} - \frac{c}{2\sqrt{E}}\right), \end{aligned}$$

where

$$\text{erfc}(z) = 1 - \text{erf}(z) = 1 - \left(2/\sqrt{\pi}\right) \int_0^z \exp\left(-s^2\right) ds = \left(2/\sqrt{\pi}\right) \int_z^\infty \exp\left(-s^2\right) ds$$

(a well known entire function).

Thus, if $r = d - 1$, then we have a more explicit representation of the integrals in the series for $G_2(x, y; \lambda)$, in terms of the special function $\text{erfc}(z)$.

(b) Lemma 2 implies that the representation of $G_2(x, y; \lambda)$, is valid for all $\lambda \neq |\alpha|^2 + 4\pi^2 |m/b|^2 + 4\pi\alpha \cdot (m/b)$, even though $\sigma(L_\alpha) = [\lambda_0, \infty)$, where $\lambda_0 = |\alpha|^2$.

(c) If we want to balance the decays of the terms of the series for $G_1(x, y; \lambda)$ and for $G_2(x, y; \lambda)$, a reasonable choice is

$$E = \frac{1}{4\pi} \left[\frac{b_1^2 + b_2^2 + \cdots + b_r^2}{b_1^{-2} + b_2^{-2} + \cdots + b_r^{-2}} \right]^{1/2}.$$

(d) The case of Dirichlet or Neumann boundary conditions can be treated similarly, since the corresponding Green's function can be written as a sum (alternating sum in the Dirichlet case) of "free" Green's functions (method of images) so we again have a formula very similar to (4).

(e) Let us take $d = 3$. If $D = (0, b_1) \times (0, b_2) \times (0, b_3)$ and $\alpha = (0, 0, 0)$, we have

$$G(x, y; \lambda) = \frac{1}{4\pi} \sum_{m \in \mathbb{Z}^3} \frac{e^{-\sqrt{-\lambda}|x-y+mb|}}{|x-y+mb|},$$

$$G(x, y; \lambda) = G_1(x, y; \lambda) + G_2(x, y; \lambda),$$

where

$$G_1(x, y; \lambda) = \sum_{m \in \mathbb{Z}^3} \int_0^E \frac{1}{(4\pi t)^{3/2}} \exp\left(\lambda t - \frac{|x-y+mb|^2}{4t}\right) dt,$$

and, by (7),

$$G_2(x,y;\lambda) = \frac{1}{b_1 b_2 b_3} \sum_{m\in\mathbb{Z}^3} \frac{e^{-\left(4\pi^2 |m/b|^2 - \lambda\right)E}}{4\pi^2 \left|m/b\right|^2 - \lambda} e^{2i\pi m\cdot(x-y)}.$$

(f) Again, let $d = 3$. Observe that (by substituting $t = s^{-2}$)

$$\int_0^E \exp\left(\lambda t - \frac{c^2}{4t}\right) \frac{dt}{t^{3/2}} = 2\int_{1/\sqrt{E}}^{\infty} e^{-\left(c^2/4\right)s^2 + \lambda/s^2} ds.$$

Thus, by (a) above we get

$$\int_0^E \exp\left(\lambda t - \frac{c^2}{4t}\right) \frac{dt}{t^{3/2}}$$

$$= \frac{\sqrt{\pi}}{c}\left[e^{ic\sqrt{\lambda}}\mathrm{erfc}\left(\frac{c}{2\sqrt{E}} + i\sqrt{\lambda E}\right) + e^{-ic\sqrt{\lambda}}\mathrm{erfc}\left(\frac{c}{2\sqrt{E}} - i\sqrt{\lambda E}\right)\right]. \tag{10}$$

For $d = 3$, the formula for $G_1(x,y;\lambda)$ becomes

$$G_1(x,y;\lambda) = \sum_{m\in\mathbb{Z}^3} e^{-i\alpha\cdot(mb)} \int_0^E \frac{1}{(4\pi t)^{3/2}} \exp\left(\lambda t - \frac{\left|x - y + mb\right|^2}{4t}\right) dt,$$

and therefore, by (10), the integrals in the series terms can be computed explicitly, in terms of the function $\mathrm{erfc}(z)$.

(g) If $D = (0,b_1)\times(0,b_2)\times\mathbb{R}$ (hence $d = 3$ and $r = 2$), and $\alpha = (\alpha_1,\alpha_2,0)$, we obtain the case of the paper of Kirk Jordan et al. (see [2]). Notice that this is the only case where the terms of both $G_1(x,y;\lambda)$ and $G_2(x,y;\lambda)$ can be computed in terms of the function $\mathrm{erfc}(z)$.

(h) In the case $d > 1$ we have that $G(x,y;\lambda) \to \infty$, as $|x-y| \to 0$. All the singular behavior of $G(x,y;\lambda)$ is captured in the term of the series for $G_1(x,y;\lambda)$ that corresponds to $m = 0$.

APPENDIX

The "master" formula in Ewald's work [1],

$$\frac{e^{iKR}}{R} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-R^2 z^2 + \frac{K^2}{4z^2}\right) dz$$

(the formula is in the sense of analytic continuation with respect to K; strictly speaking it is valid in the region $\Re\{K^2\} \le 0$; alternatively we can take K real in the formula above and think of the integral as a contour integral where the contour is the entire x-axis except for a small interval containing 0 which can be replaced by a small semicircle avoiding 0—see also [2]), can be seen as a variant of (2) for $d = 3$. An alternative, more direct way to justify this formula is by applying the corollary of the (well-known) lemma, given below, to the function

$$f(x) = e^{-x^2}.$$

Lemma 3. If $f \in L^1(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

Proof. By using the substitution $u = -1/x$ we obtain

$$\int_{0}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{0} f\left(u - \frac{1}{u}\right) \frac{du}{u^2}$$

and

$$\int_{-\infty}^{0} f\left(x - \frac{1}{x}\right) dx = \int_{0}^{\infty} f\left(u - \frac{1}{u}\right) \frac{du}{u^2}.$$

Adding up and replacing the dummy variable u by x we get

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) \frac{dx}{x^2} = \frac{1}{2}\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right) dx.$$

Now, the substitution $u = x - 1/x$ gives

$$\int_{0}^{\infty} f\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right) dx = \int_{-\infty}^{0} f\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right) dx = \int_{-\infty}^{\infty} f(u)\, du$$

and the proof is completed. ■

Corollary. If $f \in L^1(\mathbb{R})$, $a > 0$, and $b \geq 0$, then

$$\int_{-\infty}^{\infty} f\left(ax - \frac{b}{x}\right) dx = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\, dx$$

(thus the first integral does not depend on b).

Proof. The case $b = 0$ is obvious. If $b > 0$, the substitution $u = (ab)^{1/2}\, x$ gives

$$\int_{-\infty}^{\infty} f\left(ax - \frac{b}{x}\right) dx = \left(\frac{b}{a}\right)^{1/2} \int_{-\infty}^{\infty} f\left[\sqrt{ab}\left(u - \frac{1}{u}\right)\right] du.$$

Thus, by the previous lemma

$$\int_{-\infty}^{\infty} f\left(ax - \frac{b}{x}\right) dx = \left(\frac{b}{a}\right)^{1/2} \int_{-\infty}^{\infty} f\left(u\sqrt{ab}\right) du$$

and the proof is finished by substituting $u = (ab)^{-1/2}\, x$ in the integral of the right hand side. ∎

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