Variations of certain geometrical objects in the kinematics of hypersurfaces

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June 2013

Abstract

Using concepts of continuum mechanics we prove formulas for the variation of:

- Principal curvatures
- Mean and Gauss curvature
- Affine connection
- Riemann curvature tensor

of a hypersurface moving in an ambient Riemannian manifold.

Outline

- 1. A brief on Continuum Mechanics
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1. A brief on Continuum Mechanics

A **continuum** or a **material body** is a submanifold $\mathcal{B}$ of a differentiable manifold which is viewed in an ambient Riemannian manifold $(N, \overline{g})$ via a **reference configuration** of $\mathcal{B}$, i.e: an embedding

$$k : \mathcal{B} \to N, \ M = k(\mathcal{B})$$

Deformation of the body, an embedding $\phi : M \to N$.
Deformation gradient

$$F(X) = d\phi(X) : T_X M \to T_{\phi(X)} N,$$

describes the deformation in a neighborhood of $X \in M$.

In **Classical Continuum Mechanics** $N = \mathbb{R}^3$, body $M \subseteq \mathbb{R}^3$ with $\dim M = \dim N$.

Polar decomposition theorem (PDT): The deformation gradient is decomposed:

$$F(X) = R(X)U(X), \tag{1}$$

where $C(X) = U^2(X) = F^T(X)F(X) : T_X M \to T_X M$ is symmetric, positive and $R(X) = (F^T(X))^{-1}U(X) : T_X M \to T_{\phi(X)}N$ is orthogonal.

Principal deformations: the eigenvalues $\lambda_i > 0$ of $C(X)$.

Principal axes of deformations: the eigenvectors e_i of $C(X)$.

Then, $\{\mathbf{e}_i, \sqrt{\lambda_i}\}$ are the corresponding quantities of $U(X)$ and

$$U\mathbf{e}_i = \sqrt{\lambda_i}\mathbf{e}_i \tag{2}$$

$$F\mathbf{e}_i = RU\mathbf{e}_i = \sqrt{\lambda_i}R\mathbf{e}_i \tag{3}$$

i.e: U shrinks or expands the principal axes and R only rotates them. Thus, the deformation is analysed in a pure deformation U followed by a rotation R.

Motion of the continuum: $\phi_t : M \to N$, $t \in \mathbb{R}$, $x = \phi_t(X) = \phi(X,t)$, $\phi_0(M) = M$.
Current configuration: $\phi_t(M) = M_t$.
Trajectory: of the material point X is $\phi_X : \mathbb{R} \to N$ such that $\phi_X(t) = \phi_t(X)$ and
Velocity: $V_X(t) = \dot{\phi}_X(t)$.
Material velocity: $V(X,t) = V_X(t)$, **Spatial velocity**: $v_x(t) = V_t(\phi_t^{-1}(x))$.
Relative motion: motion relative to the present configuration M_t of the body

$$\phi_t(\tau) = \phi_\tau \circ \phi_t^{-1} : M_t \to M_\tau$$

Is the flow generated by the spatial velocity $v_t(x)$,
Relative deformation gradient:

$$F_t(\tau)(x) = d\phi_t(\tau)(x) : T_x M_t \to T_{\phi_t(\tau)(x)} N \tag{4}$$

From PDT :

$$F_t(\tau) = R_t(\tau) U_t(\tau) \tag{5}$$

Strain measures:

The tensor fields $F_t(\tau)$, $R_t(\tau)$, $U_t(\tau)$ are defined along the trajectory of x

Velocity gradient: $G(t) = \frac{\partial F_t(\tau)}{\partial \tau}|_{\tau=t} = dv$

Stretching: $\mathcal{D}(t) = \frac{\partial U_t(\tau)}{\partial \tau}|_{\tau=t}$, **symmetric**

Spin: $W(t) = \frac{\partial R_t(\tau)}{\partial \tau}|_{\tau=t} = \overline{\nabla}_v R_t(\tau)$, **antisymmetric**

From $F_t(\tau) = R_t(\tau)U_t(\tau)$ (5) and since $R_t(t)(x) = I_{T_xN}$, $U_t(t) = I_{T_xM}$ we get:

$$G = \mathcal{D} + W \tag{6}$$

Due to the complexity of the structure of new materials, modern expositions of the subject require more general geometry and more dimensions for the body and the space manifolds.

2. Geometry of hypersyrfaces

$(N, \overline{g}, \overline{\nabla})$: $n+1$ - dimensional Riemannian manifold.
(M, g, ∇) oriented hypersuface of N with unit normal field n,
$\mathcal{X}(N)$ vector fields on N, $\mathcal{X}(M)$ vector fields on M
$j : M \hookrightarrow N$, $j(M) = \widetilde{M} \subset N$ the canonical inclusion of M.
$g(u, w) = \overline{g}(Ju, Jw)$, $\forall u, w \in \mathcal{X}(M)$ where $J_X = dj_X$
$\overline{\mathcal{X}}(M)$ the set of vector fields on M with values in N
If $u \in \mathcal{X}(M)$ then $\overline{u} = Ju \in \overline{\mathcal{X}}(M)$
If $\overline{u} \in \mathcal{X}(N)$ then $\overline{u} \circ j \in \overline{\mathcal{X}}(M)$ is the restriction of $\overline{u}$ on M.

Projections :

$$\pi_X : T_{j(X)}N \longrightarrow T_{j(X)}N, \quad \pi_X(W) = W - \overline{g}(W, n)n. \tag{7}$$

Since $\pi_X(W) \in T_{j(X)}\widetilde{M}$, it is the image under J_X of a vector $w \in T_X M$, that is, $\pi_X(W) = J_X w$. Then define the projection

$$\mathcal{P}_X : T_{j(X)}N \longrightarrow T_X M, \;\; J_X P_X W = \pi_X w \tag{8}$$

The following relations hold between π, $\mathcal{P}$ and J:

$$J_X \mathcal{P}_X = \pi_X : T_{j(X)}N \to T_{j(X)}N, \tag{9}$$

$$\mathcal{P}_X J_X = I_X : T_X M \to T_X M \tag{10}$$

$$\mathcal{P}_X n_X = 0 \tag{11}$$

The **Levi Civita connection** ∇ on M is

$$\nabla_u w = \mathcal{P}\overline{\nabla}_{Ju} Jw, \;\; \forall u, w \in \mathcal{X}(M) \tag{12}$$

Shape operator: $S_X : T_X M \to T_X M,\ S_X u = -P_X \overline{\nabla}_{J_X u} n$

Second fundamental form: $B(u,v) = g(Su,v)$

Third fundamental form: III(u,v)=g(Su,Sv)=B(Su,v)

Gauss curvature: $K = \det S$,

Mean curvature: $nH = trS$

Gauss equation:

$$\overline{\nabla}_{Ju} Jw = J\nabla_u w + B(u,w) \cdot n = J\nabla_u w + g(Su,w) \cdot n \qquad (13)$$

We often interchange linear maps with their associated tensor fields defined via the metric tensor field g.
For example, to any linear map $T : T_X M \to T_X M$ we associate a $(0,2)$ tensor $T^\flat$ field by setting:

$$T^\flat(u,v) = g(Tu,v) \tag{14}$$

It can be shown that the following hold:

$$\nabla_Z T^\flat = (\nabla_Z T)^\flat \tag{15}$$

$$\pounds_Z T^\flat(X,Y) = (\pounds_Z T)^\flat(X,Y) + (\pounds_Z g)(X,TY) \tag{16}$$

that is, the $^\flat$ operation commutes with ∇ but it does not with $\pounds$.

The curvature tensors R, $\overline{R}$ on M and N respectively are related by:

$$\overline{R}^{\flat}(Ju, Jv, Jw, Jz) = R^{\flat}(u, v, w, z) - B(u, z)B(v, w) + B(v, z)B(u, w) \qquad (17)$$

$$\overline{R}^{\flat}(Ju, Jv, Jw, n) = (\nabla_u B)(v, w) - (\nabla_v B)(u, w) \qquad (18)$$

Since $S^{\flat} = B$ the Codazzi equation becomes

$$(\nabla_{\mathbf{v}} S)u - (\nabla_{\mathbf{u}} S)v = P\overline{R}(Ju, Jv)n \qquad (19)$$

Finally, the Hessian of $f \in C^{\infty}(M)$, relative to g, is :

$$Hessf(u, w) = g\left(\nabla_u \nabla f,\ w\right). \qquad (20)$$

3. Kinematics of a hypersurfaces

Definition 1 **Motion** *of a M in N is a 1-parameter family of embeddings ϕ_t , $t \in I \subseteq T$, i.e:*

$$\phi : M \times I \to N, \; x = \phi(X,t) = \phi_t(X).$$

Let $\tilde{\phi}_t : M \to M_t = \phi_t(M)$ induced diffeomorphism, $j_t : M_t \to N$ the canonical injection, $J_t(x) = dj_t(x) : T_x M_t \to T_{j(x)} N$, $\mathcal{P}_t : T_{j(x)} N \to T_x M$ the projection. The **material velocity** and **spatial velocity** are defined as before.

Relative deformation gradient

$$F_t(\tau)(x) = d\phi_t(\tau)(x) : T_{x_t} M_t \to T_{x_\tau} N \tag{21}$$

then, by interchanging space and time derivatives we obtain the **velocity gradient**

$$\frac{\partial F_t(\tau)}{\partial \tau}|_{\tau=t} = \frac{\partial d\phi_t(\tau)}{\partial \tau}|_{\tau=t} = dv = G \tag{22}$$

$$G(t)(x) : T_{x_t} M_t \to T_{x_\tau} N \tag{23}$$

For hypersurfaces the adapted version of the PDT ([8], [5])assumes the form

Theorem 2 *Let* $\phi_t(\tau) : M_t \to N$ *be a motion with* $n(t)$ *and* $n(\tau)$ *the unit normals for* M_t *and* M_τ *respectively, the relative deformation gradient* $F_t(\tau) = d\phi_t(\tau)$ *is decomposed as*

$$F_t(\tau) = R_t(\tau) J_t U_t(\tau) \tag{24}$$

$$R_t(\tau) n(t) = n(\tau) \tag{25}$$

$C_t(\tau) = U_t^2(\tau) : T_x M_t \to T_x M_t$ **relative right stretch tensor** $U_t(\tau)$,
$R_t(\tau) : T_{x_t} N \to T_{x_\tau} N$, **relative rotation tensor**.

The **stretching tensor field** $\mathcal{D}(t)$ is defined by:

$$\mathcal{D}(t) = \frac{\partial U_t(\tau)}{\partial \tau}|_{\tau=t} = \frac{1}{2}\frac{\partial C_t(\tau)}{\partial \tau}|_{\tau=t} : T_x M_t \to T_x M_t \tag{26}$$

Spin tensor field W is the antisymmetric field defined by the relations:

$$W(t) = \frac{\partial R_t(\tau)}{\partial \tau}|_{\tau=t} = \overline{\nabla}_v R_t(\tau) \tag{27}$$

Lemma 3 *Let* ϕ_t *be a motion of the hypersurface* M *in the Riemannian manifold* N *with velocity*

$$v = v_{||} + v_n n = Jv^{||} + v_n n$$

where $v_{||} \in \mathcal{X}(M)$. *Then*

$$G = J\mathcal{D} + WJ \tag{28}$$

$$C_t(\tau)^\flat = \phi_t^*(\tau)\overline{g} \tag{29}$$

$$2\mathcal{D} = PG + (PG)^T \tag{30}$$

$$Gu = (\nabla_u v^{||} - v_n Su) + (B(v^{||}, u) + Ju(v_n))n \tag{31}$$

$$PG = \nabla v^{||} - v_n S \tag{32}$$

$$\pounds_{v^{||}} g = (\nabla v^{||} + \nabla v^{||^T})^\flat \tag{33}$$

$$2\mathcal{D}^\flat = \pounds_{v^{||}} g - 2v_n B \tag{34}$$

$$2\mathcal{D} = \nabla v^{||} + \nabla v^{||^T} - 2v_n S \tag{35}$$

4. Variation of the geometrical objects of a moving hypersurface

On M_t we define a geometry induced by the motion $\phi_t(\tau)$:

$$
\begin{aligned}
g_t(\tau)(u,w) &= \overline{g}(F_t(\tau)u, F_t(\tau)w)\\
F_t(\tau)S_t(\tau)u &= -\overline{\nabla}_{F_t(\tau)u}n(\tau)\\
B_t(\tau)(u,w) &= g_t(\tau)(S_t(\tau)u,w),\\
III_t(\tau)(u,w) &= g_t(\tau)(S_t^2(\tau)u,w)\\
K_t(\tau) &= \det S_t(\tau), \;\; H_t(\tau) = \frac{1}{n}\mathrm{tr} S_t(\tau),\\
F_t(\tau)\nabla_t(\tau)(u,w) &= \pi_t\bar{\nabla}_{F_t(\tau)u}F_t(\tau)w\\
&= \bar{\nabla}_{F_t(\tau)u}F_t(\tau)w - \bar{g}\left(\bar{\nabla}_{F_t(\tau)u}F_t(\tau)w, n(\tau)\right)n(\tau),\\
\bar{R}(J_tu, J_tw)J_tz &= J_tR_t(\tau)(u,w)z + g_t(\tau)(S_t(\tau)u,z)J_tS_t(\tau)w\\
&- g_t(\tau)(S_t(\tau)w,z)J_tS_t(\tau)u\\
&+ \{(\nabla_t(\tau)_uB_t(\tau))(w,z) - (\nabla_t(\tau)_wB_t(\tau))(u,z)\}\, n(\tau).
\end{aligned}
$$

Since $F_t(t) = J_t$, this geometry coincides for $t = \tau$ with the one induced by the injection j.

The variation of any of these quantities, say $Q_t(\tau)$, is defined by

$$\delta Q = \frac{\partial Q_t(\tau)}{\partial \tau}|_{\tau=t} \tag{36}$$

Further, when a vector or tensor quantity is defined on the hypersurface but takes values in the ambient manifold N, then we define its variation as the covariant derivative in the direction of the velocity field of the motion.

As an example, if $L \in \overline{\mathcal{X}}(M)$, define:

$$\delta L = \overline{\nabla}_v L \tag{37}$$

which is the *time derivative* along the trajectories of the motion.

Thus, for the normal $n(\tau)$ given by $R_t(\tau)n(t) = n(\tau)$, its variation is

$$\delta n = \frac{\partial n(\tau)}{\partial t}|_{\tau=t} = \overline{\nabla}_v \overline{n}, \;\; \overline{n} \in \mathcal{X}(N), \, \overline{n}(\phi_t(\tau)) = n(\tau) \tag{38}$$

5. Some older Results

Theorem 4 *Let ϕ_t be a motion of the hypersurface M in the Riemannian manifold N, with strain $\mathcal{D}$, spin W and velocity $v = v_{||} + v_n n = Jv^{||} + v_n n$ where $v_{||} \in \mathcal{X}(M)$. Then we have the following formulas:*

Variation of the metric

$$\delta g = 2\mathcal{D}^\flat = -2v_n B + \pounds_{\ v^{||}} g \tag{39}$$

Variation of the unit normal

$$\delta n = Wn = -J\nabla v_n - JS\ v^{||} \tag{40}$$

Variation of the shape operator

$$(\delta S)u = -PGSu - \mathcal{P}\overline{\nabla}_{Ju}Wn - \mathcal{P}\overline{R}(v, Ju)n \tag{41}$$

$$(\delta S)u = v_n S^2(u) + \nabla_{\mathbf{u}}\nabla v_n + (\pounds_{\mathbf{v}^{||}}S)u - v_n\mathcal{P}\overline{R}(n, Ju)n \tag{42}$$

Variation of the second fundamental form

$$\delta B = (2\mathcal{D}S + \delta S)^\flat \tag{43}$$

$$\begin{aligned}\delta B(u, w) &= Hess_{v_n}(u, w) - v_n III(u, w) + (\pounds_{\mathbf{v}^{||}}B)(u, w)\\ &\quad - v_n\overline{g}(\overline{R}(n, Ju)n, Jw)\end{aligned} \tag{44}$$

Variation of the third fundamental form

$$\delta III = \{2S\mathcal{D}S + S\delta S + \delta SS\}^\flat . \tag{45}$$

Remark 5 *Equations (39), (40), (41) and (43) expressing the variation using kinematical quantities are simpler than the rest and reduce to the corresponding equations for surfaces in Euclidean space proved by Kadianakis (***2009***) in [4].*

5.1 New Results

Variation of principal curvatures

$$\delta k_i = g(\delta S e_i, e_i) \tag{46}$$

$$= \mathrm{Hess}_{v_n}(e_i, e_i) + v_n k_i^2 + v^{||}(k_i) - v_n g(\mathcal{P}\bar{R}(n, Je_i)n, e_i). \tag{47}$$

Variation of Gauss curvature

$$\delta K = m v_n H K + v^{||}(K) + \sum_{i=1}^{m} \hat{K}_i \mathrm{Hess}_{v_n}(e_i, e_i)$$

$$- v_n \sum_{i=1}^{m} \hat{K}_i g(\mathcal{P}\bar{R}(n, Je_i)n, e_i). \tag{48}$$

Variation of mean curvature

$$\delta H = \Delta v_n + m v^{||}(H) + v_n \sum_{i=1}^{m} k_i^2 + v_n \bar{\mathrm{Ric}}(n, n). \tag{49}$$

Variation of the Levi - Civita connection

$$(\delta\nabla)(u,w) = -\mathcal{P}G\nabla_u w + \mathcal{P}\bar{\nabla}_{Ju}Gw - B(u,w)\mathcal{P}Wn + \mathcal{P}\bar{R}(v,Ju)\bar{w}. \tag{50}$$

In geometrical terms:

$$\begin{aligned}(\delta\nabla)(u,w) = (\pounds_{v^{||}}\nabla)(u,w) - v_n(\nabla_u S)w - \{u(v_n)Sw + w(v_n)Su\} \\ + B(u,w)\nabla v_n + v_n\mathcal{P}\bar{R}(n,Ju)Jw.\end{aligned} \tag{51}$$

Variation of the Riemann curvature tensor

$$\begin{aligned}(\delta R)(u,w)z = \{-2\mathcal{D}S - \delta S - S\}^\flat(u,z)Sw + \{2\mathcal{D}S + \delta S + S\}^\flat(w,z)Su \\ + \{(\nabla_u B)(w,z) - (\nabla_w B)(u,z)\}\,\mathcal{P}Wn.\end{aligned} \tag{52}$$

6. Special motions

We show that the above formulas capture some known results in special cases.

6.1. Parallel motion

In the case of a parallel motion ($\nabla v_n = 0$ and $v^{||} = 0$) the above equations reduce to:

$$\delta k_i = v_n k_i^2 - v_n g(\mathcal{P}\bar{R}(n, Je_i)n, e_i). \tag{53}$$

$$\delta K = m v_n H K - v_n \sum_{i=1}^{m} \hat{K}_i g(\mathcal{P}\bar{R}(n, Je_i)n, e_i). \tag{54}$$

$$\delta H = v_n \sum_{i=1}^{m} k_i^2 + v_n \bar{\mathrm{Ric}}(n, n). \tag{55}$$

$$\delta \nabla(u, w) = -v_n (\nabla_u S) w + v_n \mathcal{P}\bar{R}(n, Ju) Jw. \tag{56}$$

If the ambient N^{m+1} is Euclidean then the above relations 53 - 56 are further simplified by ignoring the last term concerning the Riemann curvature.

In this case, parallel motion preserving certain geometric quantities of the original hypersurface, induces restrictions on the geometry of the hypersurface.

Corollary 6 *In a parallel motion of a hypersurface in Euclidean space:*

- $\delta k_i = 0$ *then* $k_i = 0$ *and consequently* $K = 0$*.*
- $\delta K = 0$ *if and only if* $K = 0$ *or* M *is minimal in* E^{m+1}*.*
- $\delta H = 0$ *then* $B \equiv 0$ *(*M *totally geodesic in* E^{m+1}*).*
- $\delta \nabla = 0$ *if and only if* S *is covariantly constant (parallel).*

Note

Starting with the definition of the parallel motion:

$$\phi(X,t) = j(X) + \epsilon(t)n(X),\ \epsilon(0) = 0.$$

one can deduce the kinematical and geometric quantities:

$$F(X,t) = J(I - \epsilon(t)S)(X),\ U(X,t) = (I - \epsilon(t)S)(X),$$
$$R(X,t) = I_{V^3},\ g(t) = g - 2\epsilon(t)B + \epsilon^2(t)III,$$
$$S(t) = (I - \epsilon(t)S)^{-1}S$$

and then, by differentiating we get all the variations.

6.2. Zero spin motion

A zero spin motion (or a **pure strain motion**) is a motion such that $W \equiv 0$, then:

$$\delta S u = -\mathcal{D} S u - \mathcal{P}\bar{R}(v, Ju)n$$

hence

$$\delta S u = v_n S^2 + (\pounds_{v^{||}} S)u - \nabla_u S v^{||} - v_n \mathcal{P}\bar{R}(n, Ju)\bar{n}. \tag{57}$$

and thus the variation of the principal curvature k_i is given by:

$$\delta k_i = v_n k_i^2 + v^{||}(k_i) - g(\nabla_{e_i} S v^{||}, e_i) - v_n g(\mathcal{P}\bar{R}(n, Je_i)\bar{n}, e_i) \tag{58}$$

and the variation of the mean curvature is

$$\delta H = v_n \sum_{i=1}^{m} k_i^2 + m v^{||}(H) - \bar{\mathrm{Ric}}(n, n) - \mathrm{div}_M S v^{||}. \tag{59}$$

Fo the variation of the Gaussian curvature we get:

$$\delta K = m v_n H K + v^{||}(K) - \sum_{i=1}^{m} \widehat{K}_i g(\nabla_{e_i} S v^{||}, e_i) - \sum_{i=1}^{m} \widehat{K}_i g(\mathcal{P}\bar{R}(n, J e_i) n, e_i). \tag{60}$$

Since $W = 0$ the variation of the Levi Civita connection becomes:

$$\begin{aligned} \delta\nabla(u,w) &= -\mathcal{P} G \nabla_u w + \mathcal{P} \bar{\nabla}_{Ju} G w + \mathcal{P}\bar{R}(v, Ju)\bar{w} \\ &= (\nabla_u \mathcal{D}) w + \mathcal{P}\bar{R}(v, Ju)\bar{w}. \end{aligned} \tag{61}$$

and for an Euclidean ambient space:

$$\delta\nabla(u,w) = (\nabla_u \mathcal{D}) w. \tag{62}$$

The following characterization of pure strain motions holds:

Corollary 7 *A motion is a* **pure strain motion** *if and only if the following conditions*

$$\nabla v^{||} = \nabla^T v^{||}, \; S v^{||} = -\nabla v_n. \tag{63}$$

hold true.

It is shown also that

Corollary 8 *If a motion is of zero spin then the variation of its third fundamental form vanishes*

$$\delta III = 0$$

(see: [14]).

This implies that:
a normal motion is a pure strain motion if and only if it is a parallel motion.

6.3. Motions preserving the unit normal field

The previous motions are special cases of **motions preserving the unit normal field**($\delta n = 0$).
It is clear that $\delta n = 0$ if and only if the tangential part of the valocity and the normal component, satisfy:

$$Sv^{||} = -\nabla v_n. \tag{64}$$

In case $k_i \neq 0$ for all $i = 1, 2, \ldots, m$ this gives:

$$v^{||} = -\sum_{i=1}^{m} \frac{1}{k_i} e_i(v_n) e_i, \tag{65}$$

Further,

Corollary 9 *A normal motion preserves the unit normal field if and only if is a parallel motion.*

6.4. Infinitesimal isometries and pure spin motions

A motion is said to be **infinitesimally isometric** if and only if $\delta g = 0$.

A motion is said to be a **pure spin** motion if and only if $\mathcal{D} = 0$.

Since $\delta g = 2\mathcal{D}^\flat$ it follows:

Corollary 10 *A motion is an infinitesimal isometry if and only if is a pure spin motion.*

Also

Corollary 11 *A hypersurface admits a normal infinitesimal isometric motion if and only if is totally geodesic in N (see: [6]).*
A hypersurface admits a tangential ($v = Jv^{||}$) infinitesimal isometry if and only if $v^{||}$ is a Killing vector field for g.

6.5. Infinitesimally area preserving motions

A motion is said to be **infinitesimally area preserving** if and only if $\delta\omega = 0$. Since

$$\delta\omega = \mathrm{div}_M v^{||} - m v_n H,$$

it follows:

Corollary 12 *A motion is infinitesimally area preserving if and only if*

$$\mathrm{div}_M v^{||} - m v_n H = 0. \tag{66}$$

For a normal and a tangential motion the following hold:

Corollary 13 *A normal motion is infinitesimally area preserving if and only if $H = 0$, i.e: iff M is minimal in N.*
A tangential motion with $v = Jv^{||}$ is infinitesimally area preserving if and only if $v^{||}$ is incompressible.

7. Application

7.1. A curve moving on a surface

Let γ be a curve moving, with unit speed, on 2 dimensional manifold $(M, \bar{g})$ (a surface).

- Geometry of γ:

$$g(T,T) = 1,\ \bar{g}(JT, n) = 0,$$

$$\bar{\nabla}_{JT} JT = kn,\ \bar{\nabla}_{JT} n = -kJT,$$

$$\nabla_T T = 0,\ ST = kT,$$

$$B(T,T) = k,\ III(T,T) = k^2.$$

- Let $v = v_T JT + v_n n$ - Kinematics of γ:

$$GT = (v_T' - kv_n)JT + (v_T k + v_n')n,$$
$$\mathcal{P}GT = (v_T' - kv_n)T = \mathcal{D}T,$$
$$Wn = -(kv_T + v_n')JT,\ WJT = (kv_T + v_T')n$$

- Evolution equations: since $K_N = \bar{g}(\bar{R}(JT, n)\bar{n}, JT)$ then

$$\delta g = 2(v_T' - kv_n)g, \tag{67}$$
$$\delta n = -(kv_T + v_n')JT, \tag{68}$$
$$\delta S = \{k'v_T + k^2 v_n + v_n'' + v_n K_N\}I, \tag{69}$$
$$\delta k = k'v_T + k^2 v_n + v_n'' + v_n K_N. \tag{70}$$

and

$$(\delta\nabla)(T, T) = \{v_T'' - (v_n k)'\}\, T. \tag{71}$$

If the motion is normal (i.e: $v = v_n n$), then the variation of the geodesic curvature reduces to an ordinary Ricatti equation:

$$\delta k = k^2 v_n + v_n'' + v_n K_N, \tag{72}$$

and if v_n does not depend on the point of the curve but on the evolution parameter only(**offset curves on a surface**) then $v_n'' = 0$ and

$$\delta k = k^2 v_n + v_n K_N. \tag{73}$$

With no loss of the generality, suppose that $v_n = 1$, then we reduce further to the **Jacobi's geodesic** equation:

$$\delta k = k^2 + K_N. \tag{74}$$

Under the initial condition of being γ a geodesic, i.e: $k(0,s)=0$, and that $K_N = \mathrm{const}$ we deduce:

$$k(t,s)=\sqrt{K_N}\tan\sqrt{K_N}t,\ \text{if}\ K_N>0,$$

$$k(t,s)=-\sqrt{-K_N}\tan\sqrt{-K_N}t,\ \text{if}\ K_N<0.$$

For example, on a sphere of radius R, for the motion of a great circle, we get:

$$k(t,s)=\frac{1}{R}\tan\frac{t}{R}. \tag{75}$$

References

[1] Ben Andrews: *Contraction of convex hypersurfaces in Riemannian spaces* J. Differential Geometry, **39** (1994), 407–431.

[2] Barret O' Neil: *Semi Riemannian Geometry with applications to relativity* Academic Press (1983).

[3] M.P. Do Carmo: *Riemannian Geometry*

[4] Kadianakis N.: *Evolution of surfaces and kinematics of membranes* Journal of Elasticity, (2010).

[5] Kadianakis N and F. Travlopanos.: *Kinematics of hypersurfaces in Riemannian manifolds* Journal of Elasticity, (2012).

[6] R.A. Goldstein and P. J. Ryan: *Infinitesimal rigidity of Euclidean submanifolds* Journal of Differential Geometry, **10** (1975) 49-60.

[7] Noll, W.: A mathematical theory of the mechanical behavior of continuous media. Arch. Rational Mech. Anal. **2**, 197–226 (1958).

[8] Man, C.-S. , Cohen, H.: *A coordinate-free approach to the kinematics of membranes*

J. Elast. **16**, 97–104 (1986).

[9] Murdoch, A.I.: A coordinate-free approach to surface kinematics. Glasgow Math. J. **32**, 299–307 (19).

[10] Marsden, J.E., Hughes, T.J.R.: Mathematical Foundations of Elasticity. Prentice-Hall, Englewood Cliffs, New Jersey (1983).

[11] Yano, K.: Integral Formulas in Riemannian Geometry. Marcel Dekker, New York (1970).

[12] Kadianakis, N.: On the geometry of Lagrangian and Eulerian descriptions in continuum mechanics. Z. Angew. Math. Mech. **79**, 131–138 (1999).

[13] Spivak, M.: A Comprehensive Introduction to Differential Geometry, Volume 4. Publish or Perish, Boston (1979).

[14] Szwabowicz, M.L.:Pure strain Deformations of Surfaces. J. Elast. 92, 255-275 (2008).

[15] Truesdell, C.: A First Course in Rational Continuum Mechanics, Volume 1. Academic Press, New York (1977).