Local Frames in Euclidean Space - Time

By

P.G. Appleby and N. Kadianakis

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Introduction

In recent years there have been a number of papers concerned with the formulation of classical mechanics in a way which does not assume a privileged reference frame. In place of an Aristotelian space-time these theories assume a Euclidean space-time M incorporating the principle of absolute simultaneity, with a three-dimensional Euclidean metric in each instantaneous space. This space-time may be represented mathematically as an affine bundle (M, R, τ), where $\tau\colon M \to R$ is a proper time function, and the structure group of isometries of a Euclidean three-space.

Studies of Newtonian gravitation in a Euclidean space-time [Trantman, 1] indicate that a classical space-time should also incorporare an inertial principle, whereby inertial frames can be distinguished from rotating frames. Appleby [2] has shown that this can be achieved by endowing the event world with a principal bundle structure in which the structure group is the group of translations of a Euclidean vector space. In an inertial space-time, local inertial frames characterise a class of local covariant differentiations which are compatible with the principal bundle structure. In this paper we consider the more general class of local covariant differentions which are compatible with the affine bundle structue of Euclidean space-time, and show that they characterise local rotating frames. Global rotating frames are then characterised by sections of the bundle of local rotating frames. We also show that non-rigid local frames may be characterised by a class of local covariant differentiations satisfying a weaker compatibility condition. We use these non-rigid local frames to classify the global connections of arbitrary motions in space-time.

1. Euclidean Space-Time

In this section we review the basic features of Euclidean space-time (3)

Definition 1. A Euclidean spase-time is an affine bundle (M,T,τ), there M is a 4-dimensional differentiable manifold, T is a 1-dimensional totally ordered Euclidean space and the structure group of the bundle is the group of isometries of a 3-dimensional Euclidean space E. The projection map $\tau: M \to T$, called the time, characterises the principle of absolute simultaneity. The fibers

$$M_t = \{ \chi \in M : \tau(\chi) = t \},\ t \in T$$

are 3-dimensional affine Euclidean spaces called instantaneous spaces. The 1-form

$$\underline{\tau} = D\tau : TM \to \mathbb{R}\ ,\ u \to \underline{\tau}.u$$

is called the world space normal. For each $u \in TM$ we call $\underline{\tau}.u$ the time value of u. If $\underline{\tau}.u > 0$ we say that u is time- like. If $\underline{\tau}.u=0$ we say that u is space- like. For each $x \in M$ the set of space-like vectors in T_xM may be identified with the translation space V_t of the instantaneous space M_t, where $t=\tau(x)$. However, translation spaces at different instants t can not be identified. The set $\bigcup_{t\in T} V_t$ forms a vector bundle over T associated with the affine bundle (M,T,τ). Through the canonical injection

$$\otimes^p \big(\bigcup_{t \in T} V_t\big) \to \otimes^p TM$$

the bundle of P^{th} order space-like tensors on M may be identified with a subset of the bundle of P^{th} order contravariant tensors .
The space metric tensor of M is the space-like tensor field g on M whose value at each $x \in M$ is the metric tensor of the Euclidean space $M_{\tau(x)}$. The set of space-like vector fields on M is denoted by VM .

A world - lime in M is a cross- section of the bundle (M,T,τ) , that is, a differentiable curve $\xi:T \to M$ such that $\tau(\xi(t))=t$. The velocity of ξ is the time-like vector field $\dot{\xi}=T\xi$. Obviously $\underline{\tau}.\dot{\xi}=1$. A world line in M represents the trajectory of a particle in a frame independent way. The motion of a continuum (or simply a motion) is represented by a flow φ : $\mathbb{R}\times M \to M$, $(s,x) \to \varphi_s(x)$, such that for each $s\in \mathbb{R}$, φ_s is a diffeomorphism of M with $\tau(\varphi_s(x))=\tau(x)+s$. If $\dot{\phi}$ is the associated vector field, then $\underline{\tau}.\dot{\phi}=1$. We call $\dot{\phi}$ the velocity of the continuum. Generally, vector fields σ with $\underline{\tau}.\sigma=1$ are called velocity fields. We write J^1M for the set of velocity fields.

Finally we note that the metric tensor $\underline{g}$ does not result in a unique covariant derivative on M. Nevertheless there exists a unique covariant derivative on the fibers, that is, applied only on space-like vector fields:

$$\nabla \,|_x : VM \times VM \to V_{\tau(x)} \;, \quad \nabla_{\underline{v}}\, \underline{u}\, |_x = \frac{d}{d\lambda}\, \underline{u}_{x+\lambda \underline{v}_x}$$

such that $\nabla \underline{g} = 0$. This fact allous the introduction of various compatible covariant derivaties on M, whose restriction on space-like fields will coincide with ∇.

Acovariant derivative D on M allous us to measure the acceleration $D_{\dot{\phi}}\, \dot{\phi}$ and the velocity gradiant $D_v\, \dot{\phi}$ of a motion φ . If such a covariant derivative is compatible with the structure of space-time M, we will call it a "frame". Our object is to classily the compatible covariant derivatives on M in a way in which frames are classified in classical Mechanics, according to their relative spin and their relative acceleration. We consider "local frames" first and then generalise to "global frames" in the next sections.

2. Local Euclidean Frames

Definition 2. Let JM be the set of vector fields on M .

A local Euclidean frame at $x \in M$ is a symmetric local covariant differentiation at $x \in M$

$$\gamma : JM \times JM \to T_x M \ , \ (\rho,\sigma) \to {}_\gamma D_\rho \ \sigma$$

which is compatible with the structure of M in the sense that

$${}_\gamma D \ \underline{\tau} = 0 \quad , \quad {}_\gamma D \ \underline{g} = 0 \ . \tag{2.1}$$

From the compatibility conditions (2.1) we get

$${}_\gamma D_{\underline{v}} \ \underline{u} = \nabla_{\underline{v}} \ \underline{u} \quad , \qquad \underline{u} \ , \ \underline{v} \in VM \tag{2.2}$$

$$\underline{\tau} . D_\sigma \ v = 0 \quad , \qquad \sigma \in JM \tag{2.3}$$

$$\underline{\tau} . D_\sigma \ \sigma = 0 \quad , \qquad \sigma \in J^1 M \ . \tag{2.4}$$

Equation (2.2) says that ${}_\gamma D$ coincides with ∇ when restricted to space-like vector fields , Equation (2.3) says that covariant derivatives of space-like vector fields, are space-like, and from (2.4) we deduce that accelerations are space-like vector fields.

Let γ be a local Euclidean frame at x. If $\sigma \in J^1M$, then we call the space-like vector

$$\underset{\gamma}{A}(\sigma) = {}_\gamma D_\sigma \ \sigma \tag{2.5}$$

the acceleration of σ with respect to γ . The map

$$\underset{\gamma}{A} : J^1M \to V_t \ , \ \sigma \to \underset{\gamma}{A}(\sigma)$$

will be called the acceleration map of γ. Further , the second order space-like tensor

$$\underset{\gamma}{L}(\sigma) : V_t \to V_t , \ \underset{\gamma}{L}(\sigma) v = {}_\gamma D_{\underline{v}} \ \sigma \ , \quad t=\tau(x) \tag{2.6}$$

is called the gradient of σ with respect to γ ,

The map $\underset{\gamma}{L} : J^1M \to \overset{2}{\times} V_t$ will simmilarly be called the gradient

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The map $\underset{\gamma}{L} : J^1M \to \overset{2}{\times}\ V_t$ will simmilarly be called the gradient map of γ. We show the following.

Proposition 1. A local Euclidean frame γ is completely defined by either one of the following :

(i) The acceleration map $\underset{\gamma}{A}$.

(ii) The gradient map $\underset{\gamma}{L}$, and the value of $\underset{\gamma}{A}$ for just one σ, that is, the space-like vector $\underset{\gamma}{A}(\sigma)$.

Proof. (i) We note that for any $\sigma,\rho \in J^1M$ there exists a space-like vector field $\underline{v}$ such that $\rho=\sigma+v$.

Thus we have

$$ {}_{\gamma}D_{\rho}\,\rho = {}_{\gamma}D_{\sigma}\,\sigma + 2\,{}_{\gamma}D_{\underline{v}}\,\sigma + [\,\sigma,\underline{v}\,] + \nabla_{\underline{v}}\underline{v} \tag{2.7} $$

where $|\sigma,\underline{v}|$ is the usual Lie bracket.

Therefore when $\underset{\gamma}{A}$ is given the frame γ can be completely defined by the following:

$$ {}_{\gamma}D_{\sigma}\,\sigma = \underset{\gamma}{A}(\sigma) $$

$$ {}_{\gamma}D_{\underline{v}}\,\sigma = \frac{1}{2}\{A(\sigma+\underline{v}) - A(\sigma) - [\sigma,\underline{v}] - \nabla_{\underline{v}}\underline{v}\} \tag{2.8} $$

$$ {}_{\gamma}D_{v}\,\underline{u} = \nabla_{\underline{v}}\,\underline{u}\,. $$

(ii) Equation (2.7) can be written as

$$ \underset{\gamma}{A}(\rho) = \underset{\gamma}{A}(\sigma) + 2\,\underset{\gamma}{L}(\sigma)\underline{v} + [\sigma,\underline{v}] + \nabla_{\underline{v}}\underline{v} \tag{2.9} $$

Therefore $\underset{\gamma}{A}$ is known for every velocity field σ (and hence γ is defined) if and only if $\underset{\gamma}{A}$ is known for one velocity field and $\underset{\gamma}{L}$ is given for any σ.

Next we show that although the second order tensor $\underset{\gamma}{L}(\sigma)$ is defined by (2.6) in terms of the frame γ , its symmetric part $\frac{1}{2}[\,\underset{\gamma}{L}(\sigma) + \underset{\gamma}{L}(\sigma)^T\,]$ is independent of γ.

Proposition 2: For any local Euclidean frame γ and any velocity field σ

$$\underset{\gamma}{L}(\sigma) + \underset{\gamma}{L}^T(\sigma) = £_\sigma \underline{g} \qquad (2.10)$$

where $£_\sigma g$ is the Lie derivative of the space metric $\underline{g}$ relative to σ .

Proof. For any $\underline{u}$, $\underline{v} \in VM$ their inner product is a function : $\underline{u}.\underline{v} : M \to \mathbb{R}$ But

$${}_\gamma D_\sigma (\underline{u}.\underline{v}) = \underline{u}.{}_\gamma D_\sigma \underline{v} + \underline{v}.{}_\gamma D_\sigma \underline{u} + ({}_\gamma D_\sigma \underline{g})(\underline{u},\underline{v})$$

$$= \underline{u}.{}_\gamma D_{\underline{v}}\sigma + \underline{v}.{}_\gamma D_{\underline{u}}\sigma + \underline{u}.\, £_\sigma \underline{v} + \underline{v}.\, £_\sigma \underline{u}$$

$$= \underline{u}.\underset{\gamma}{L}(\sigma)\underline{v} + \underline{v}.\underset{\gamma}{L}(\sigma)\underline{u} + \underline{u}.\, £_\sigma \underline{v} + \underline{v}.\, £_\sigma \underline{u}$$

$$= [\, \underset{\gamma}{L}(\sigma) + \underset{\gamma}{L}^T(\sigma) \,] (\underline{u},\underline{v}) + \underline{u}.\, £_\sigma \underline{v} + \underline{v}.\, £_\sigma \underline{u}$$

Since we also have

$${}_\gamma D_\sigma (\underline{u}.\underline{v}) = £_\sigma (\underline{u}.\underline{v}) = \underline{u}.\, £_\sigma \underline{v} + \underline{v}.\, £_\sigma \underline{u} + (£_\sigma \underline{g})(\underline{u},\underline{v}),$$

(2.10) follous.

From proposition 2 it follous that a local Euclidem frame is completely defined by $\underset{\gamma}{A}(\sigma)$ for one $\sigma \in J^1M$ and the second order antisymmetric space-like tensor

$$\underset{\gamma}{R}(\sigma) = \frac{1}{2} [\, \underset{\gamma}{L}(\sigma) - \underset{\gamma}{L}^T(\sigma) \,] \qquad (2.11)$$

We call $\underset{\gamma}{R}(\sigma)$ the <u>spin of σ with respect to γ</u> . The map $\underset{\gamma}{R} : J^1M \to \Lambda^2 V_t$, $\sigma \to \underset{\gamma}{R}(\sigma)$ is called accordingly, the <u>spin map</u> of γ.

3. Relative spin an Relative acceleration

Although we have defined, in the previous paragraph, the concept of spin $\underset{\gamma}{R}(\sigma)$ of any velocity field (and consequently any motion φ) relative to a local frame γ , we can not assign a spin to γ itself, in an absolute way. In order to do this one needs extra structure on the space-time manifold M, which will correspond to an additional physical principle. Nevertheless we can define the spin of a local frame γ relative to another local frame γ'. We first show.

Proposition 3: For any two local Euclidean frames γ and γ', the second order antisymmetric tensor $\underset{\gamma'}{R}(\sigma) - \underset{\gamma}{R}(\sigma)$ is independent of σ, $\sigma \in J^1M$.

Prof : From (2.10) we have

$$\underset{\gamma'}{L}(\sigma) - \underset{\gamma}{L}(\sigma) = \frac{1}{2}\, \pounds_\sigma\, g + \underset{\gamma'}{R}(\sigma) - \frac{1}{2}\, \pounds_\sigma\, g - \underset{\gamma}{R}(\sigma)$$

$$= \underset{\gamma'}{R}(\sigma) - \underset{\gamma}{R}(\sigma)$$

If $\rho = \sigma + \underline{v}$, $\underline{v} \in VM$, is another velocity field, then

$$\left[\underset{\gamma'}{L}(\rho) - \underset{\gamma}{L}(\rho) \right] \underline{u} = \left[\underset{\gamma'}{L}(\sigma+\underline{v}) - \underset{\gamma}{L}(\sigma+\underline{v}) \right] \underline{u}$$

$$= {}_{\gamma'}D_{\underline{u}}\, \sigma+\underline{v} - {}_{\gamma}D_{\underline{u}}\, \sigma+\underline{v} = {}_{\gamma'}D_{\underline{u}}\sigma - {}_{\gamma}D_{\underline{u}}\sigma$$

$$= \left[\underset{\gamma'}{L}(\sigma) - \underset{\gamma}{L}(\sigma) \right] \underline{u}$$

Hence $\underset{\gamma'}{L}(\rho) - \underset{\gamma}{L}(\rho) = \underset{\gamma'}{L}(\sigma) - \underset{\gamma}{L}(\sigma)$.

Therefore

$$\underset{\gamma'}{R}(\rho) - \underset{\gamma}{R}(\rho) = \underset{\gamma'}{R}(\sigma) - \underset{\gamma}{R}(\sigma) .$$

Definition 3 : The second order antisymmetric space-like tensor

$$r = \underset{\gamma'}{R}(\sigma) - \underset{\gamma}{R}(\sigma) \qquad (3.1)$$

is called the spin of γ relative to γ' . It is independont of r and depends only on γ and γ'.

Let φ be a motion of a continuum with velocity $\dot{\phi}$. The velocity gradients of φ relative to the frames γ and γ' are then related by $\underset{\gamma'}{L}(\dot{\phi}) = \underset{\gamma}{L}(\dot{\phi})+r$, equivalently :

$$_{\gamma'}D_{\underline{v}}\,\dot{\phi} = {}_{\gamma}D_{\underline{v}}\,\dot{\phi} + r\,\underline{v} \quad , \underline{v} \in VM \qquad (3.2)$$

From equation (2.9) we get

$$\underset{\gamma'}{A}(\rho) - \underset{\gamma}{A}(\rho) = \underset{\gamma'}{A}(\sigma) - \underset{\gamma}{A}(\sigma) + 2(\underset{\gamma'}{L}(\sigma) - \underset{\gamma}{L}(\sigma))\,v$$

$$= \underset{\gamma'}{A}(\sigma) - \underset{\gamma}{A}(\sigma) + 2\,r\,\underline{v} \quad . \qquad (3.3)$$

Hence the difference between two local Euclidean frames γ' and γ may be characterised by the linear map (equivalently a $\binom{1}{1}$ tensor)

$$h: T_xM \to V_t$$

defined by

$$\begin{aligned} h(\sigma) &= \underset{\gamma'}{A}(\sigma) - \underset{\gamma}{A}(\sigma) \\ h(\underline{v}) &= 2r\,\underline{v} \end{aligned} \qquad (3.4)$$

The third order tensor $\gamma'-\gamma$ is then written as

$$\gamma' - \gamma = \frac{1}{2}\,(\underline{\tau} \otimes h + h \otimes \underline{\tau}) \qquad (3.5)$$

Let γ and γ' be two local Euclidean frames having zero relative spin i.e $r=0$. It follous from (3.3) that

$$\underset{\gamma'}{A}(\rho) - \underset{\gamma}{A}(\rho) = \underset{\gamma'}{A}(\sigma) - \underset{\gamma}{A}(\sigma)$$

That is, $\underset{\gamma'}{A}(\sigma) - \underset{\gamma}{A}(\sigma)$ is independent of σ.

<u>Definition 4</u>: For any two local Euclidean frames with zero relative spin, we call the space-like vector field

$$\underline{f} = \underset{\gamma'}{A}(\sigma) - \underset{\gamma}{A}(\sigma) \ , \qquad (3.6)$$

the <u>acceleration of γ relative to γ'</u> .

When γ and γ' have zero relative spin the linear map h is given by

$$h = \underline{f} \otimes \underline{\tau} \,, \tag{3.7}$$

and therefore the third order tensor $\gamma' - \gamma$ is given by

$$\gamma' - \gamma = \underline{\tau} \otimes \underline{\tau} \otimes \underline{f} \,. \tag{3.8}$$

Hence, the difference between two frames with zero relative spin, is characterised by their relative acceleration only.

4. Non-Rigid Local Frames

In this section we study local covariant differentiations which satisfy weaker compatibility conditions than (2.1). In fact we allow covariant differentiations D for which $D_\sigma\, \underline{g} \neq 0$, where σ is a velocity field.

Definition 5 : A non-rigid local frame at $x \in M$ is a local covariant differentiation

$$\delta:\ JM\times JM \to V_{\tau(x)}\ ,\qquad (\rho,\sigma) \to {}_\delta D_\rho\, \sigma$$

which satisfies the campatibility conditions

$$ {}_\delta D\, \underline{\tau} = 0\ ,\quad {}_\delta D_{\underline{v}}\, \underline{g} = 0 \qquad (4.1)$$

for any space-like vector field $\underline{v}$.

It follous that (2.2) , (2.3) and (2.4) hold for non-rigid local frames as well. The acceleration map and the gradient map of δ are defined in the same way. Since (2.7) is true for a non-rigid local frame, proposition 1 holds in this case too.

If $\sigma \in J^1M$ and $\underline{v} \in VM$, we have

$$ {}_\delta D_{\sigma+\underline{v}}\, \underline{g} = {}_\delta D_\sigma\, \underline{g} + {}_\delta D_{\underline{v}}\, \underline{g} = {}_\delta D_\sigma\, \underline{g}\ . \qquad (4.2)$$

Hence for any $p,\sigma \in J^1M$, ${}_\delta D_p\, \underline{g} = {}_\delta D_\sigma\, \underline{g}$ i.e ${}_\delta D_\sigma\, \underline{g}$ is independent of σ .

Definition 6 : For any non-rigid local frame δ , the second order symmetric space-like tensor

$$ \underset{\delta}{\underline{d}} = {}_\delta D_\sigma\, \underline{g}\ ,\quad \sigma \in J^1M \qquad (4.3)$$

is called the deformation of δ

We note that the space-time structure does not allow a definition of spin for a frame, but it does allow an intrinsic definition of its deformation (which is zero when the frame is Euclidean).

Alfhough the symmetric part of the gradient of a velocity field σ with respect to a local Euclidean frame is independent of the frame itself (eq.(2.10)), the same is not true uhen the frame is non-rigid. In fact if δ is a non-rigid local frame having deformation $\underset{\delta}{\underline{d}}$, then

$$\underset{\delta}{L}(\sigma) + \underset{\delta}{L}^T(\sigma) = \pounds_\sigma \underline{g} - \underset{\delta}{d} \,. \tag{4.4}$$

Indeed since

$$
\begin{aligned}
{}_\delta D_\sigma(\underline{u}\cdot\underline{v}) &= \underline{u}\cdot{}_\delta D_\sigma \underline{v} + \underline{v}\cdot{}_\delta D_\sigma \underline{u} + ({}_\delta D_\sigma \underline{g})(\underline{u},\underline{v}) \\
&= \underline{u}\cdot \underset{\delta}{L}(\sigma)\underline{v} + \underline{v}\cdot \underset{\delta}{L}(\sigma)\underline{u} + \underline{u}\cdot\pounds_\sigma \underline{v} + \underline{v}\cdot\pounds_\sigma \underline{u} + \underset{\delta}{d}(\underline{u},\underline{v}) \\
&= \left[\, \underset{\delta}{L}(\sigma) + \underset{\delta}{L}^T(\sigma) \,\right](\underline{u},\underline{v}) + \underline{u}\cdot \pounds_\sigma \underline{v} + \underline{v}\cdot \pounds_\sigma \underline{u} + \underset{\delta}{d}\,(\underline{u},\underline{v})
\end{aligned}
$$

and

$${}_\delta D_\sigma(\underline{u}\cdot\underline{v}) = \pounds_\sigma(\underline{u}\cdot\underline{v}) = \underline{u}\cdot \pounds_\sigma \underline{v} + \underline{v}\cdot\pounds_\sigma \underline{u} + (\pounds_\sigma g)(\underline{u},\underline{v}) \quad ,$$

equation (4.4) follows.

From proposition 2 and equation (4.4) we deduce that.

Proposition 5 : A non-rigid local frame δ is completely defined by the following

(i) Its spin map $\underset{\delta}{R} : J^1M \to V_{\tau(x)}$, $\underset{\delta}{R}(\sigma) = \frac{1}{2}\left[\, \underset{\delta}{L}(\sigma) - \underset{\delta}{L}^T(\sigma) \,\right]$

(ii) Its deformation tensor $\underset{\delta}{d}$

(iii) The space-like vector $\underset{\delta}{A}(\sigma)$ for just one $\sigma \in J^1M$.

Therefore, the additional information needed (compared to a local Euclidean frame) to specify a non-rigid local frame is the symmetric second order space-like tensor $\underset{\delta}{d}$. If δ is a non-rigid local frame, there is a unique local Euclidean frame γ such that

$$\underset{\gamma}{A} = \underset{\delta}{A} \qquad , \qquad \underset{\gamma}{R} = \underset{\delta}{R} \,. \tag{4.4}$$

We call γ the associated local Euclidean frame.

It follows that $\underset{\gamma}{L}(\sigma) = \underset{\delta}{L}(\sigma) + \underset{\delta}{d}$ and therefore

$$ {}_{\gamma}D_{\underline{v}}\sigma = {}_{\delta}D_{v}\sigma + \underset{\delta}{d}\,\underline{v} \tag{4.5}$$

for any $\sigma \in J^1M$ and $V^{\cdot} \in VM$.

One can proceed and express the difference between two non-rigid local frames δ, δ' in terms of the tensor h defined by the associated local Euclidean frames γ, γ', and the tensor $\underset{\delta'}{d} - \underset{\delta}{d}$.

5. Global Frames

A global Euclidean frame in M is a mapping $\Gamma : x \rightarrow \Gamma_x$, which associates with each point $x \in M$ a local Euclidean frame Γ_x at x, such that if ρ, σ are vector fields in M the mapping

$$ {}_{\Gamma}D_{\rho}\sigma : M \rightarrow TM \quad , \quad x \rightarrow {}_{\Gamma_x}D_{\rho}\sigma $$

is a vector field on M. All the concepts and results concerning local Euclidean frames can be carried over to global ones. In particular one defines the acceleration and gradient maps, as the fields $\underset{\Gamma}{A} : x \rightarrow \underset{\Gamma_x}{A}$ and $\underset{\Gamma}{L} : x \rightarrow \underset{\Gamma_x}{L}$ respectively. It follows that one can define the spin of a global Euclidean frame Γ relative to a Γ' as a second order antisymmetric space-like tensor field an M, given at each x by

$$ R_x = \underset{\Gamma'_x}{R}(\sigma) - \underset{\Gamma_x}{R}(\sigma) \quad . \qquad (5.1) $$

The third order tensor field $\Gamma' - \Gamma$ is then given by

$$ \Gamma' - \Gamma = \frac{1}{2}(\underline{\tau} \otimes H + H \otimes \underline{\tau}) \quad , \qquad (5.2) $$

where H is the second order tensor field for which

$$ H(\sigma) = \underset{\Gamma'}{A}(\sigma) - \underset{\Gamma}{A}(\sigma) \quad , \quad H(\underline{v}) = 2\ R\ \underline{v} \ . \qquad (5.3) $$

When the frames have zero relative spin $(R = 0)$ their difference is characterised by their relative acceleration $\underline{F}$ only, and

$$ \Gamma' - \Gamma = \underline{\tau} \otimes \underline{\tau} \otimes \underline{F} $$

Non-rigid global frames are introduced in the same way using non-rigid local frames. Then, it follows from proposition 5 that a non-rigid global frame Δ is completely de fined by: (i) its spin map $\underset{\Delta}{R}$, (ii) Its deformation $\underset{\Delta}{d}$ and (iii) the space-like vector field $\underset{\Delta}{A}(\sigma)$ for just one $\sigma \in J^1M$.

6. Affine Connections of a Motion

In Continuum Mechanics there are ofter employed variows kinds of derivatives that "follow" the motion. They may be seen, in this framework, as special kinds of compatible covariant derivatives usually called "affine connections of the motion".

Definition 7 :Let φ be a motion, a continuum in M having velocity $\dot{\varphi}$. An affine connection of the motion φ is a non-rigid global frame Ω such that

$$\underset{\Omega}{D}_{\dot{\varphi}}\,\dot{\varphi} = 0 \ . \qquad (6.1)$$

i.e $\underset{\Omega}{A}(\dot{\varphi})=0$.

From the section 5 we deduce that Ω is completely defined by its spin map $\underset{\Omega}{R}$ and its deformation $\underset{\Omega}{d}$.

Let C^{φ} be the set of affine connections of φ . An affine connection $\Gamma \in C^{\varphi}$ is called metric connection of φ if it is a global Euclidean frame. We write C^{φ}_{m} for the subset of metric connections of φ. Let $\Gamma \in C^{\varphi}_{m}$. Since $\underset{\Gamma}{A}(\dot{\varphi}) = 0$, Γ is completely defined by its spin map $\underset{\Gamma}{R}$.

Given an $\Omega \in C^{\varphi}$ one can construct (see section 4) a unique $\Gamma \in C^{\varphi}_{m}$ such that $\underset{\Gamma}{R} = \underset{\Omega}{R}$. Conversely, given a $\Gamma \in C^{\varphi}_{m}$ and a second order symmetric space-like field d , there is a unique $\Omega \in C^{\varphi}$ such that

$$\underset{\Omega}{R} = \underset{\Gamma}{R} \quad \text{and} \quad \underset{\Omega}{d} = d$$

Therefore, there is a 1-1 mapping

$$C^{\varphi} \to C^{\varphi}_{m} \times S^2M \ , \ \Omega \to (\ \Gamma, \underset{\Omega}{d}\ )$$

Where S^2M is the set of symmetric second order space-like tensor fields. Further if Ω ,$\Omega' \in C^{\varphi}$ correspond, through the above mapping, to $(\Gamma, \underset{\Omega}{d})$ and $(\Gamma', \underset{\Omega'}{d})$, respectively, their difference $\Omega' - \Omega$ is characterised by the tensors $\underset{\Omega'}{d} - \underset{\Omega}{d}$ and $\Gamma' - \Gamma$, equivalently by the tensors $\underset{\Omega'}{d} - \underset{\Omega}{d}$ and $R = \underset{\Gamma'}{R} - \underset{\Gamma}{R}$.