Kinematics of hypersurfaces in Riemannian manifolds

Nikolaos Kadianakis and Fotios Travlopanos

Department of Mathematics

National Technical University of Athens

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A brief on Continuum Mechanics

A **continuum** or a **material body** is a submanifold $\mathcal{B}$ of a certain differentiable manifold $\mathcal{M}$.

Ambient space is a Riemannian manifold (S, g).

Reference configuration of $\mathcal{B}$ is an embedding $\phi : \mathcal{B} \to S$. Write $B = \phi(\mathcal{B})$ then $\phi : \mathcal{B} \to B$ is a diffeomprphism.

The additional geometry on $\mathcal{B}$ and S depends on the particular physical situation.

The points $X \in \mathcal{B}$ are considered as **material points**.

A **deformation** of the body is an embedding $\phi : B \to S$.

The **deformation gradient** $F(X) : T_X B \to T_{\phi(X)} S$ describes the deformation in a neighborhood of $X \in B$.

The embedding of the body in the space enables the measurement of physical properties of its deformations.

The **polar decomposition theorem** (PDT) allows us to analyze the deformation near a material point X:

$$F(X) = R(X)U(X) \tag{1}$$

where $U(X)$ is a symmetric, positive definite, $R(X)$ is an orthogonal transformation such that $R^T(T)R(X) = I_{T_X(B)}$, $R(X)R^T(X) = I_{T_{\phi(X)}S}$ and

$$C(X) = U^2(X) = F^T(X)F(X) : T_X B \to T_X B$$
$$R(X) = (F^T(X))^{-1}U(X) : T_X B \to T_{\phi(X)} S$$

The $\mathbf{e}_i$, $\lambda_i > 0$ orthogonal eigenvectors and positive eigenvalues of the symmetric operator $C(X)$ shall be called **principal axes of deformation** and **principal deformations** respectively. Then, the $\mathbf{e}_i$, $\sqrt{\lambda_i}$ are the corresponding eigenvectors and eigenvalue of $U(X)$ and

$$U\mathbf{e}_i = \sqrt{\lambda_i}\mathbf{e}_i \tag{2}$$
$$F\mathbf{e}_i = RU\mathbf{e}_i = \sqrt{\lambda_i}R\mathbf{e}_i \tag{3}$$

i.e: U preserves the principal directions and their orthogonality and it only shrinks or expands their lenghts and R only rotates them. Thus, the deformation is analysed in a pure deformation and a rotation.

Motion of the continuum $\phi_t : B \to S$, $t \in \mathbb{R}$ with $\phi_0(B) = B$ and write $x = \phi_t(X) = \phi(X, t)$.

Current configuration: $\phi_t(B) = B_t$.

Trajectory of the material point X is the curve $\phi_X : \mathbb{R} \to S$ su ch that $\phi_X(t) = \phi_t(X)$ and its **velocity** is $V_X(t) = \dot{\phi}_X(t)$.

In classical treatments $S = \mathbb{R}^3$, the body B is a 3 dimensional submanifold of it and the operator $R(X)$ is a rotation in $\mathbb{R}^3$.
Studying membranes or rods one assumes that $\mathrm{dimB} < \mathrm{dim}S$ and in this case the PDT becomes

$$F(X) = R(X)J(X)U(X) \tag{4}$$

Statement of the problem

Let $(N, \overline{g})$ be a Riemannian manifold and $j : M \hookrightarrow N$ a hypersurface, j the inclusion mapping.
Let $\phi_t : M \times \mathbb{R} \rightarrow N$ be a motion of M in N with $\phi_0 = \phi(\cdot, 0) = j$ and velocity vector field v.
We study evolution equations for geometric objects on M using both geometric and kinematical quantities.
Kinematical quantities stem from a generalized version of the classical p.d.t. traditionally used in classical mechanics.
The above p.d.t is an adapted version of a special polar decomposition result proved by Chi-Sing Man and H. Cohen in [6](1986) for surfaces in $\mathbb{R}^3$ and used to derive evolution formulae for surfaces in [3](2009).

Polar decomposition theorem

Theorem 1 *Let V be a finite dimensional Euclidean and $F \in \mathcal{L}(V)$ a linear transformation in V. Then, there exist uniquely defined transformations, an orthogonal $R \in \mathrm{Orth}(V)$ and a symmetric, positive definite one $U \in \mathrm{Sym}^{+}(V)$ such that the following decomposition holds*

$$F = RU \tag{5}$$

where $U^2 = F^T F$.

Example 2 *Let $F \in \mathcal{L}(\mathbb{R}^3)$ with matrice $M_F = \begin{bmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then*

$$U = \sqrt{F^T F} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 3+\sqrt{3} & 3-\sqrt{3} & 0 \\ 3-\sqrt{3} & 1+3\sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{6}$$

and

$$R = (F^T)^{-1} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1+\sqrt{3} & \sqrt{3}-1 & 0 \\ 1-\sqrt{3} & 1+\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix} \tag{7}$$

Geometry of hypersyrfaces

$(N, \overline{g}, \overline{\nabla})$ is a $n+1$ - dimensional Riemannian manifold.
(M, g, ∇) hypersuface in N, unit normal n, g, ∇ the **induced** metric and connection
$j : M \hookrightarrow N$, $j(M) = \widetilde{M} \subset N$ **natural injection** $J(X) = dj(X) : T_X M \hookrightarrow T_{j(X)} N$
$\mathcal{X}(M)$: **vector fields** on M, $\overline{\mathcal{X}}(M)$ **vector fields** on M with values in N.

$$u \in \mathcal{X}(M) \Rightarrow Ju \in \overline{\mathcal{X}}(M)$$

$$\overline{w} \in \mathcal{X}(N) \Rightarrow \overline{w} \circ j \in \overline{\mathcal{X}}(M)$$

$$g(u, w) = \overline{g}(Ju, Jw), \ \overline{g}(Ju, n) = 0$$

The normal projection along $\mathbf{n}$ is

$$\pi_X : T_{j(X)} N \longrightarrow T_{j(X)} N, \quad \pi_X(W) = W - \overline{g}(W, n)n. \tag{8}$$

Since $\pi_X(W) \in T_{j(X)} \widetilde{M}$, it is the image under J_X of a vector $w \in T_X M$, that is, $\pi_X(W) = J_X w$. We call w the projection of W to $T_X M$ and the map

$$P_X : T_{i(X)} N \longrightarrow T_X M, \ J_X P_X W = \pi_X w \tag{9}$$

Then following relations hold between π, $\mathcal{P}$ and J:

$$J_X P_X = \pi_X : T_{i(X)}N \to T_{i(X)}N, \tag{10}$$

$$P_X J_X = I_X : T_X M \to T_X M \tag{11}$$

$$P_X n_X = 0 \tag{12}$$

Shape operator is defined by

$$S_X : T_X M \to T_X M,\ S_X u = -P_X \nabla_{J_X u} n \tag{13}$$

Second fundamental form is defined by

$$B(u, v) = g(Su, v) \tag{14}$$

Third fundamental form is defined by

$$III(u, v) = g(Su, Sv) = B(Su, v) \tag{15}$$

Gauss and **Mean** curvature are:

$$K = \det S, \quad nH = trS$$

Gauss Equation:

$$\overline{\nabla}_{Ju}Jw = J\nabla_u w + B(u,w)\cdot n = J\nabla_u w + g(Su,w)\cdot n \tag{16}$$

Using g to each $u \in \mathcal{X}(M)$ and to each 1-form we associate

$$u^\flat(v) = g(u,v) \tag{17}$$

$$g(\xi^\sharp, v) = \xi(v) \tag{18}$$

Further, to any linear map $T : T_X M \to T_X M$ we associate

$$T^1(\alpha, u) = \alpha(Tu) \tag{19}$$

and using g the

$$T^\flat(u,v) = g(Tu,v) = T^1(u^\flat, v). \tag{20}$$

For any linear T as before, we define

$$(\nabla_X T^1)(\alpha, Y) = \alpha((D_X T)Y)$$

and

$$(\pounds_X T^1)(\alpha, Y) = \alpha((\pounds_X T)Y)$$

For any $X, Y, Z \in \mathcal{X}(M)$, hold

$$(\nabla_X T)Y = \nabla_X TY - TD_X Y, \tag{21}$$

$$(\pounds_X T)Y = \pounds_X TY - T\pounds_X Y. \tag{22}$$

and further :

$$\nabla_Z T^\flat = (\nabla_Z T)^\flat \tag{23}$$

$$\pounds_Z T^\flat(X,Y) = (\pounds_Z T)^\flat(X,Y) + (\pounds_Z g)(X,TY) \tag{24}$$

Fundamental equations for hypersurfaces

$$\overline{R}^\flat(Ju, Jv, Jw, Jz) = R^\flat(u,v,w,z) - B(u,z)B(v,w) + B(v,z)B(u,w) \tag{25}$$

$$\overline{R}^\flat(Ju, Jv, Jw, n) = (\nabla_u B)(v,w) - (\nabla_v B)(u,w) \tag{26}$$

$$(\nabla_{\mathbf{v}} S)u - (\nabla_{\mathbf{u}} S)v = P\overline{R}(Ju, Jv)n \tag{27}$$

and finally the Hessian of $f \in C^\infty(M)$, relative to g, is given by:

$$Hessf(u,w) = g\left(\nabla_u \nabla f,\ w\right). \tag{28}$$

Kinematics

Definition 3 *A* **motion** *of a M in N is a 1-parameter family of deformations ϕ_t , $t \in I \subseteq T$, i.e:*

$$\phi : M \times I \to N, \; x = \phi(X, t) = \phi_t(X).$$

The velocity of the material point X at time t is the velocity $V(X, t)$ of the curve $\phi_X : I \to N, \;\; \phi_X(t) = \phi(X, t)$ i.e

$$V(X, t) = \frac{\partial}{\partial t} \phi_X$$

that is, for any differentiable function $g : N \to \mathbb{R}$,

$$V(X, t)(g) = \frac{\partial}{\partial t} (g \circ \phi)(X, t)$$

Velocity field is the map $V(\cdot, t) : M \to TN$, i.e $V \in \bar{\mathcal{X}}(M)$. *Spatial velocity* at $x = \chi(X, t)$ is $v(\cdot, t) : M_t \to TN$, given by,

$$v(x,t) = V(\tilde{\phi}^{-1}(x), t) \quad \text{i.e} \quad v(\tilde{\phi}(X,t), t) = V(X,t). \tag{29}$$

Gradient of $w \in \mathcal{X}(N)$, is defined at each $x \in N$ as the linear map

$$\overline{\nabla} w : T_x N \to T_x N, \quad (\overline{\nabla} w)u = \overline{\nabla}_u w$$

while the gradient of vector field $W \in \bar{\mathcal{X}}(M)$, is defined at each $x \in M$ as the linear map

$$\overline{\nabla} W : T_X M \to T_x N, \quad (\overline{\nabla} W)Z = \overline{\nabla}_{JZ} W$$

For the velocity field v the map,

$$G(x) = dv : T_x M_t \to T_{j(x)} N, \quad G(x)u = dv(u) = \overline{\nabla}_{Ju} v = \overline{\nabla} v(Ju) \qquad (30)$$

is the *velocity gradient* of the motion.

Variation concept

Let $\alpha : I \subset \mathbb{R} \to N$ a C^∞ curve, $W(t) \in \mathcal{X}(\alpha)$ then define

$$W'(t_0) = \overline{\nabla}_{\alpha'(t_0)} \overline{W} \tag{31}$$

and (31) is independent of the extension.
Let $\phi_t : M \to N$ with $X \to \phi(X,t) = x$, $\phi(X,0) = j(X)$ be a motion, then the **trajectory** of X is

$$\phi_X : \mathbb{R} \ni t \to \phi_X(t) = \phi(X,t) \in N \tag{32}$$

and its differential

$$F(t) : T_X M \to T_x N \tag{33}$$

Let $u \in T_X M$ then $W(t) = F(t)u$ is a vector field along the trajectory ϕ_X and also $n(t)$ can be viewed as a vector field along the same trajectory. Thus, from 31 we can write

$$\frac{\partial}{\partial t}|_{t=t_0} F(t)u = \overline{\nabla}_v \overline{W}, \; \overline{W}(\phi_X(t)) = W(t) \tag{34}$$

and

$$\frac{\partial}{\partial t}|_{t=t_0} n(t) = \overline{\nabla}_v \overline{n},\ \overline{n}(\phi_X(t)) = n(t) \tag{35}$$

For $\phi_t(\cdot, \tau) : M_t \to N$ with $\phi(X, \tau) = \phi_t(\phi(X, t), \tau)$ let $F_t\tau : T_x M_t \to T_{\phi_t(x,\tau)} N$ and the trajectory $\phi_t(x) : I \subset \mathbb{R} \ni \tau \to \phi_t(x, \tau) \in N$ and the **spatial** velocity of x is

$$v_x = \frac{\partial}{\partial \tau}|_{\tau=t} \phi_t(x, \tau) \tag{36}$$

Each $u \in T_x M_t$ defines $\overline{u}$ along the trajectory $\phi_t(x)$ by

$$\overline{u}(t) = F_t(\tau) u \tag{37}$$

Then, by means of (34) the time rate of u under the motion is

$$u'(t) = \overline{\nabla}_v \overline{u},\ \overline{u}(\phi_t(x, \tau)) = \overline{u}(\tau) \tag{38}$$

and similarly

$$n'(t) = \overline{\nabla}_v \overline{n},\ \overline{n}(\phi_t(x, \tau)) = \overline{n}(\tau) \tag{39}$$

where the unit normal fields along the trajectory at the instants t and τ are related via the rotation mapping $n(\tau) = R_t(\tau) n(t)$.

Polar decomposition for hypersurfaces

Theorem 4 *Let $\phi : M \to N$ be a deformation of M, with $\phi(X) = x$. Then, at each $X \in M$ there exists a unique orthogonal $R(X) : T_{j(X)}N \to T_{\phi(X)}N$ such that*

$$F(X) = R(X)J_X U(X) \tag{40}$$

where $U^2(X) = \widetilde{F}^T(X)\widetilde{F}(X) = F^T(X)F(X) : T_X M \to T_X M$ is a positive, symmetric and $J(X) : T_X M \to T_{j(X)}N$ is the differential of the canonical inclusion $j : M \hookrightarrow N$.

Using (40) the $F_t(\tau) : T_{x_t}M_t \to T_{x_\tau}N$, where $x_\tau = \phi_t(x_t, \tau)$ is written as

$$F_t(\tau) = R_t(\tau)J_t U_t(\tau), \tag{41}$$

where $C_t(\tau) = U_t^2(\tau) = F_t^T(\tau)F_t(\tau) : T_x M_t \to T_x M_t$, $U_t(\tau)$ is the **relative right stretch tensor** and $R_t(\tau) : T_{x_t}N \to T_{x_\tau}N$ is the **relative rotation tensor**.

Kinematical tensor fields

The *stretching tensor field* is *defined by*

$$\mathcal{D}(t) = \frac{\partial}{\partial \tau} U_t(\tau)|_{\tau=t} = \frac{1}{2}\frac{\partial}{\partial \tau} C_t(\tau)|_{\tau=t} : T_x M_t \to T_x M_t \ . \tag{42}$$

The following formulas are true:

$$C_t(\tau)^\flat = \phi_t^*(\tau)\overline{g} \tag{43}$$

$$2\mathcal{D} = PG + (PG)^T \tag{44}$$

$$Gu = (\nabla_u v^{||} - v_n S u) + (B(v^{||}, u) + Ju(v_n))n \tag{45}$$

$$PG = \nabla v^{||} - v_n S \tag{46}$$

$$\pounds_{v^{||}} g = (\nabla v^{||} + \nabla {v^{||}}^T)^\flat \tag{47}$$

$$2\mathcal{D}^\flat = \pounds_{v^{||}} g - 2 v_n B \tag{48}$$

$$2\mathcal{D} = \nabla v^{||} + \nabla {v^{||}}^T - 2 v_n S \tag{49}$$

Variation concept

τ-dependent geometry on M_t defined by τ-dependent metric $g(\tau)$ and $n(\tau)$ on M_τ.
Start with shape operator

$$S_t(\tau)u = -\tilde{F}_t^{-1}(\tau)P_\tau\overline{\nabla}_{F_t(\tau)u}n(\tau). \tag{50}$$

for which:

$$F_t(\tau)S_t(\tau)u = -\overline{\nabla}_{F_t(\tau)u}n(\tau) \tag{51}$$

and also

$$B_t(\tau)(u,w) = g_t(\tau)(S_t(\tau)u,w). \tag{52}$$

$$III_t(\tau)(u,w) = g_t(\tau)(S_t(\tau)u, S_t(\tau)w) \tag{53}$$

$$K_t(\tau) = \det S_t(\tau), \qquad nH_t(\tau) = trS_t(\tau). \tag{54}$$

$$\omega_t(\tau)(u_1,...u_n) = \omega_N(F_t(\tau)u_1,...F_t(\tau)u_n,\ n(\tau)) \tag{55}$$

where ω_N is the volume form on the ambient space.
When $\tau = t$, these quantities coincide with the already existing ones on $M_{t.}$

Basic Results

Evolution equations of the geometry of a moving hypersurface in both kinematical and purely geometric terms.

Variation of the metric

$$\delta g = 2\mathcal{D}^\flat \tag{56}$$

$$\delta g = -2v_n B + \pounds_{\;v^{||}} g \tag{57}$$

Variation of the unit normal

$$\delta n = -J\nabla v_n - JS\, v^{||} \tag{58}$$

and in case the ambient manifold is Euclidean the following holds as well

$$\delta n = Wn \tag{59}$$

Variation of the shape operator

$$(\delta S)u = -PGSu - \mathcal{P}\overline{\nabla}_{Ju}\delta n + \mathcal{P}\overline{R}(v, Ju)n \tag{60}$$

$$(\delta S)u = v_n S^2(u) + \nabla_{\mathbf{u}}\nabla v_n + (\pounds_{\mathbf{v}^{||}} S)u + v_n \mathcal{P}\overline{R}(n, Ju)n \tag{61}$$

$$\begin{aligned} \delta S u &= v_n S^2(u) + v_n \mathcal{P}\overline{R}(n, Ju)n + \nabla_{\mathbf{u}}\nabla v_n \\ &+ \nabla_u S v^{||} - \nabla_{Su} v^{||} + \mathcal{P}\overline{R}(Jv^{||}, Ju)n \end{aligned} \tag{62}$$

Variation of the second fundamental form

$$\delta B = (2\mathcal{D}S + \delta S)^{\flat} \tag{63}$$

$$\begin{aligned} &= Hess_{v_n}(u, w) - v_n III(u, w) + (\pounds_{\mathbf{v}^{||}} B)(u, w) \\ &+ v_n \overline{g}(\overline{R}(n, Ju)n, Jw) \end{aligned} \tag{64}$$

The equations 57, 58 and 61 are generalizing the corresponding results included in the paper [1] whereas the 56, 59 and 60 are their kinematical analogues.

Applications

Let $j : \mathbb{R} \ni s \to (\cos s, \sin s)$ be a curve with $J = dj = \begin{bmatrix} -\sin s \\ \cos s \end{bmatrix}$ and consider the motion

$$\phi_t(s) = j(s) + tj'(s),\ \ \phi_0(s) = j(s) \tag{65}$$

which can also be written under the form

$$\phi_t(s) = j(s) + J\left(\frac{\partial}{\partial s}\right) = (\cos s - t\sin s, \sin s + t\cos s) \tag{66}$$

and its differential is

$$F(s,t) = \begin{bmatrix} -\sin s - t\cos s \\ \cos s - t\sin s \end{bmatrix} \tag{67}$$

Then

$$F_t(s) = R_t(s)J_t(s)U_t(s),\ U_t^2(s) = F_t^T(s)F_t(s) \tag{68}$$

where

$$R_t(s) = \frac{1}{\sqrt{f^2 + t^2 f^2}} \begin{bmatrix} -f_t(s) & tf_t(2) \\ -tf_t(s) & -f_t(s) \end{bmatrix},\ U_t(s) = \sqrt{f^2 + tf^2} \begin{bmatrix} -\sin s - t\cos s \end{bmatrix} \tag{69}$$

with $f_t(s) = \frac{1}{t\cos s + \sin s}$ and it becomes

$$R'(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{70}$$

and $R'(0)n = \delta n$.

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