STOCASTIC CALCULUS & APPLICATIONS
Michaelmas Term 2003

BROWNIAN MOTION

The physical Brownian motion is the movement of pollen grains suspended in a liquid. The motion of the grains is due to the large number of collisions with the (much smaller) liquid molecules. The physical theory, formulated by Einstein in 1905 suggests that the motion is random, the paths are continuous, and the displacements of the grain are stationary, Gaussian, and independent over different time intervals. In this lecture we will construct a mathematical model for Brownian motion, i.e. a stochastic process satisfying these properties.

Until further notice, we take $\mathbf{T} = [0, \infty)$. We prescribe the FDD for Brownian motion as follows. Suppose $F = (t_1, \ldots, t_k) \in [0, \infty)^k$. Then,

- μ_F is Gaussian.
- $\int x_i d\mu_F(x) = 0$, for all $i = 1, \ldots, k$.
- $\int x_i x_j d\mu_F(x) = \min\{t_i, t_j\} =: t_i \wedge t_j$, for all $i, j \in \{1, \ldots, k\}$.

Exercise: Check that the family $\{\mu_F\}$ described above is consistent.(Hint: the density of μ_F if $0 < t_1 < t_2 < \ldots < t_k$ is given by:

$$\prod_{j=1}^{k} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left(-\frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}\right),$$

with the convention $t_0 = 0, x_0 = 0$.)

By Kolmogorov's theorem, we can define a stochastic process, as a measure W on $(\mathbb{R}^{[0,\infty)}, \mathcal{K})$ with the given FDD. The following elementary properties of this process can be verified easily:

1. $W(\{\omega; \omega(0) = 0\}) = 1$
2. $\omega(t) - \omega(s)$ is independent of $\omega(r)$, so long as $s, t \geq r$.
3. For all $t \geq 0$ we have $\omega(t + s) - \omega(t) \sim N(0, s)$.
4. If $t > 0$ and A is a Borel subset of $\mathbb{R}$, then:

$$W(\omega(t) \in A) := W(\{\omega;\ \omega(t) \in A\}) = \frac{1}{\sqrt{2\pi t}} \int_A \exp(-\frac{x^2}{2t}) dx.$$

Thus, W has all the properties of the model we want to construct except maybe continuity of the paths. The naive approach would be to try to show that W assigns probability 1 to continuous paths. But the set of continuous functions is not in $\mathcal{K}$. The problem is essentially that whether a function is in a

set $A \in \mathcal{K}$ or not is determined by its values on a countable number of points, whereas for continuity we need to know its full set of values!
Exercise: Make a rigorous proof of this argument using the monotone class theorem to prove that if $A \in \mathcal{K}$, then there is a countable set $D = \{t_1, t_2, \ldots\}$ and a measurable function on $\mathbb{R}^{\mathbb{N}}$, such that:

$$1_A(\omega) = F(\omega(t_1), \omega(t_2), \ldots).$$

What we can hope for is showing that the set $\mathbf{Y}$ of continuous functions has W-outer measure one, i.e. if $A \in \mathcal{K}$ and $A \supset \mathbf{Y}$, then $W(A) = 1$. Then by the following lemma from measure theory we can restrict our measure on the space $(\mathbf{Y}, \mathbf{Y} \cap \mathcal{K})$.

Lemma: Let $(\mathbf{X}, \mathcal{K}, W)$ be a probability space and let $\mathbf{Y} \subset \mathbf{X}$ have W-outer measure 1. Then there is a unique probability measure W' on $(\mathbf{Y}, \mathbf{Y} \cap \mathcal{K})$ such that $W'(C \cap \mathbf{Y}) = W(C)$ for all $C \in \mathcal{K}$.
Proof: It is easy to verify that $\mathbf{Y} \cap \mathcal{K} := \{\mathbf{Y} \cap C;\ C \in \mathcal{K}\}$ is a σ-algebra. In order to prove existence we need to show that W' is well defined: if $A \cap \mathbf{Y} = B \cap \mathbf{Y}$ for some $A, B \in \mathcal{K}$, then $A \triangle B \subset \mathbf{X} \setminus \mathbf{Y}$. Since $\mathbf{Y}$ has outer measure 1, then $W(A \triangle B) = 0$, and thus $W(A) = W(B)$. Uniqueness is trivial since we prescribe the value of the measure on the whole σ-algebra.□

The following lemma provides a criterion on whether the set of continuous functions $\mathbf{Y}$ has P-outer measure one, when P is a probablity measure on $(\mathbf{X}, \mathcal{K})$.

Lemma: $\mathbf{Y}$ has P-outer measure one, if and only if for every bounded, countable set $S \subset [0, \infty)$ we have:

$$P(\{\omega;\ \omega|_S \text{ is uniformly continuous}\}) = 1. \tag{1}$$

Proof: (Note that the set of functions appearing in (1) is in $\mathcal{K}$.) Necessity is trivial since continuous functions are uniformly continuous on bounded intervals. For sufficience suppose (1) holds and that $\mathbf{Y} \subset A \in \mathcal{K}$. By the previous exercise,

$$1_A(\omega) = F(\omega(t_1), \omega(t_2), \ldots)$$

for some countable set $D = t_1, t_2, \ldots$ and a measurable function F on $\mathbb{R}^{\mathbb{N}}$. Let now $S_N = D \cap [0, N]$. Then,

$$\bigcap_N \{\omega; \omega|_{S_N} \text{ is uniformly continuous}\} \subset A.$$

Indeed, if a function $\omega(\cdot)$ is uniformly continuous on every S_N then it can be extended to a continuous function $\omega_c(\cdot)$ on $\mathbb{R}$, such that $\omega(t) = \omega_c(t)$, for all $t \in D$. Then, since $\omega_c \in A$, we have $\omega \in A$. By continuity of measure we have $P(A) = 1$. □.

Checking condition (1) still looks like a formidable task. However, there is a celebrated criterion by Kolmogorov, which gives an easy to check sufficient condition for (1) to hold.

Theorem: (Kolmogorov's criterion) If for some constants $\alpha, \beta, C > 0$, we have:

$$\int |\omega(t) - \omega(s)|^{\alpha} dP(\omega) \leq C|t-s|^{1+\beta}, \qquad (2)$$

for all s, t, then (1) holds. Furthermore, the paths are a.s. γ-Hölder continuous with any exponent $\gamma < \beta/\alpha$.

The proof is omitted, but you can find it in your Advanced Probability notes. Note that condition (2) only involves the 2-dimensional distributions of the process, so it is often very easy to check as the following computation shows.

We have already seen that under the measure W we constructed earlier, $\omega(t) - \omega(s) \sim N(0, |t-s|)$. Therefore,

$$\begin{aligned}\int |\omega(t) - \omega(s)|^{\alpha} dW(\omega) &= \frac{1}{\sqrt{2\pi|t-s|}} \int_{\mathbb{R}} |x|^{\alpha} \exp\left(-\frac{x^2}{2|t-s|}\right) dx \\ &= E[|\chi|^{\alpha}] \times |t-s|^{\alpha/2},\end{aligned}$$

where χ is a standard normal r.v. Taking $\alpha > 2$ the condition in Kolmogorov's criterion is satisfied.

Hence, we have defined a measure (a stochastic process if you prefer) W' on the space $\mathbf{Y}=C([0,\infty);\mathbb{R})$, equipped with the σ-algebra $\mathbf{Y} \cap \mathcal{K}$ (note that from an earlier exercise this is the Borel σ-algebra of $\mathbf{Y}$ seen as a metric space with the uniform metric). This measure is usually referred to as the *Wiener measure* on $\mathbb{R}$, or the 1-dimansional Brownian motion starting from 0. To make notation simpler, from now on we will drop the prime from the Wiener measure. We can also define a Brownian motion starting from $x \in \mathbb{R}$ as $W^x = W \circ \tau_x^{-1}$, where τ_x is the transformation that maps $\omega(\cdot) \mapsto x + \omega(\cdot)$. Multi-dimensional Brownian motion is defined as the product of 1-dimensional ones (that is we take independent Brownian motions on each co-ordinate). You will learn more properties of Brownian motion in your Advanced Probability course.