STOCASTIC CALCULUS & APPLICATIONS
Michaelmas Term 2003

STOCHASTIC PROCESSES

There are several equivalent ways we can think of a stochastic process. We can start with an underlying probability space (Ω, Σ, P) and a (real valued) stochastic process can be defined as a collection of random variables $\{X(t,\omega)\}_{t\in\mathbf{T}}$, indexed by a parameter set $\mathbf{T}$, i.e. for each $t \in \mathbf{T}$, $X(t,\cdot)$ is a measurable map from (Ω, Σ) into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The parameter set usually represents time and will often be a time interval in these lectures.

Then, we can view a stochastic process as a random function of $t \in \mathbf{T}$. We equip the space $\mathbf{X}$ of real valued functions defined on $\mathbf{T}$, ($\mathbf{X} := \mathbb{R}^{\mathbf{T}}$) with the σ-algebra $\mathcal{K}$ generated by projections $\{\pi_t\}_{t\in\mathbf{T}}$, where $\pi_t(f) = f(t)$ (the so called Kolmogorov σ-algebra). A stochastic process can be defined as a measurable map from (Ω, Σ) into $(\mathbf{X}, \mathcal{K})$. It is only natural to demand measurability with respect to this σ-algebra since we will be interested in answering such questions as "what is the probability that $X(t) \in A$?", for Borel sets A.
Definition: A cylinder set in $\mathbf{X}$ is a set of the form:

$$C = \{\omega \in \mathbf{X}; (\omega(t_1), \ldots, \omega(t_k)) \in A)\}, \tag{1}$$

where $k \in \mathbb{N}$, $t_i \in \mathbf{T}$, and A is a Borel set in $\mathbb{R}^k$. Check that the totality $\mathcal{C}$ of cylinder sets is an algebra of sets and that it generates $\mathcal{K}$.

The mapping $\omega \mapsto X(\cdot,\omega)$ naturally induces a measure on $(\mathbf{X}, \mathcal{K})$, by

$$Q(H) = P(\{\omega; X(\cdot,\omega) \in H\}).$$

As the underlying probability space is irrelevant, we can replace it by the so-called canonical model $(\mathbf{X}, \mathcal{K}, Q)$, with the choice $X(\cdot, f) = f(\cdot)$, for $f \in \mathbf{X}$. Hence, a third way to view a stochastic process is as a probability measure on $(\mathbf{X}, \mathcal{K})$. This is the same as regarding a random variable as a probability measure (its distribution) on $\mathbb{R}$. This approach offers the benefit that we may take advantage of the extra structure our probability space $\mathbf{X}$ has. (In most interesting cases Q is supported on a metrisable subset of $\mathbf{X}$- more on that later in the lectures).
Exercise: Let $\mathbf{Y}$ stand for the set of real valued continuous functions defined on $[0,1]$. Recall from a course in real variables that the space $\mathbf{Y}$ equipped with the distance

$$d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|$$

is a complete metric space. The metric d induces a topology on $\mathbf{Y}$ and let $\mathcal{B}(\mathbf{Y})$ stand for the Borel σ-algebra on $\mathbf{Y}$ (i.e. the σ-algebra generated by the open sets). Prove that

$$\mathcal{B}(\mathbf{Y}) = \mathcal{K} \cap \mathbf{Y}.$$

Another point of view is to focus on the *finite dimensional distributions* (FDD) of the process, i.e. the distibutions of the vectors $(X(t_1,\omega),\dots,X(t_k,\omega))$, for all $k \in \mathbb{N}$, and all k-tuples $F = (t_1,\dots,t_k)$ in $\mathbf{T}^k$. These can be specified as Borel probability measures μ_F on $\mathbb{R}^k$. Of course, they cannot be totally arbitrary.

1. If $F' = (t_{i_1},\dots,t_{i_k})$ is a permutation of F then

$$\mu_{F'} = \mu_F \circ \pi^{-1}, \qquad \textit{where } \pi(x_1,\dots,x_k) = (x_{i_1},\dots,x_{i_k}).$$

On the other hand,

2. if $F' = (t_1,\dots,t_k,t_{k+1})$ then we must have:

$$\mu_{F'}(A \times \mathbb{R}) = \mu_F(A), \qquad \forall A \in \mathcal{B}(\mathbb{R}^k).$$

We are interested in the extent to which we can follow the reverse procedure. That is, prescribe the finite dimensional distributions a priori, and prove the existence of a process with the prescribed FDD.

Definition: A family $\{\mu_F\}$, where F ranges over all finite (ordered) subsets of $\mathbf{T}$, is called consistent if it satisfies the conditions 1 and 2 above.

Theorem:(Kolmogorov) If $\{\mu_F\}$ is a consistent family, then there exists a probability measure P on $(\mathbf{X},\mathcal{K})$, such that the FDD of the measure P are given by $\{\mu_F\}$. Hence if $F = (t_1,\dots,t_k)$, and $A \in \mathcal{B}(\mathbb{R}^k)$ then:

$$P\left(\{\omega; (\omega(t_1),\dots,\omega(t_k)) \in A\}\right) = \mu_F(A). \tag{2}$$

Proof: We can define P on the algebra of cylinder sets $\mathcal{C}$ by (2). The definition is consistent because of the consistency of the prescribed measures, and P is finitely additive. If we could prove that P is in fact countably additive on $\mathcal{C}$ then by Caratheodory's extension theorem we could extend the definition of P to $\mathcal{K} = \sigma[\mathcal{C}]$ and prove the assertion.

Countable additivity can be checked by proving that if $\{C_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$, then

$$C_n \downarrow \emptyset \Rightarrow P(C_n) \to 0.$$

Suppose that $C_n \downarrow \emptyset$, and assume that for some $\epsilon > 0$, we have $P(C_n) > \epsilon$ for all n. Because C_n are decreasing we may assume without loss of generality (convince yourseves) that there exists a sequence $t_1, t_2, \dots$ such that C_n are of the form:

$$C_n = \{\omega; (\omega(t_1),\dots,\omega(t_n)) \in A_n\},$$

for some $A_n \in \mathcal{B}(\mathbb{R}^n)$. Define $F_n = (t_1,\dots,t_n)$.

Now recall from a course in Real Analysis the concept of *regularity.* A Borel measure μ is called inner regular if for every Borel set A and every $\epsilon > 0$ there exists a compact set $K \subset A$ such that $\mu(A \setminus K) < \epsilon$. Probability Borel measures on $\mathbb{R}^d$ such as μ_F are always inner regular (because $\mathbb{R}^d$ is σ-compact.) Hence for each of the A_n above consider a compact set $K_n \subset A_n$ such that:

$$\mu_{F_n}(A_n \setminus K_n) < \frac{\epsilon}{2^n}.$$

Define also the compact sets $\tilde{K}_n \subset \mathbb{R}^n$ by:

$$\tilde{K}_n = \bigcap_{j=1}^{n} (K_j \times \mathbb{R}^{n-j})$$

and the sets $\tilde{E}_n \subset E_n \subset C_n$ by:

$$E_n = \{\omega; (\omega(t_1), \ldots, \omega(t_n)) \in K_n\},$$

with a similar definition for the tilded ones. Then,

$$\begin{aligned} \epsilon &< P(C_n) = P(\tilde{E}_n) + P(C_n \setminus \tilde{E}_n) = \mu_{F_n}(\tilde{K}_n) + P(\bigcup_{j=1}^{n}(C_n \setminus E_j)) \\ &\leq \mu_{F_n}(\tilde{K}_n) + \sum_{j=1}^{n} P(C_j \setminus E_j) = \mu_{F_n}(\tilde{K}_n) + \sum_{j=1}^{n} \mu_{F_j}(A_j \setminus K_j) \\ &\leq \mu_{F_n}(\tilde{K}_n) + \frac{\epsilon}{2}. \end{aligned}$$

Thus, $\mu_{F_n}(\tilde{K}_n) > 0$, and in particular $\tilde{K}_n$ are non-empty.

Now, for each n pick a point $(x_1^n, x_2^n, \ldots, x_n^n) \in \tilde{K}_n$. The sequence $\{x_1^n\}_{n \in \mathbb{N}}$ is contained in the compact set $\tilde{K}_1$, and therefore has a convergent subsequence with a limit $x_1 \in \tilde{K}_1$. Likewise, as $\{(x_1^n, x_2^n)\}_{n \in \mathbb{N}} \subset \tilde{K}_2$, the aforementioned subsequence has a further subsequence that converges to $(x_1, x_2) \in \tilde{K}_2$. We can proceed in this fashion and construct a sequence $\{x_n\}_{n \in \mathbb{N}}$, such that for every n, $(x_1, \ldots, x_n) \in \tilde{K}_n$. Consequently the set $S := \{\omega;\ \omega(t_j) = x_j,\ \forall j \in \mathbb{N}\}$ is contained in $\cap_n \tilde{E}_n = \cap_n E_n$, and hence in $\cap_n C_n$, contradicting the assumption that $C_n \downarrow \emptyset$. $\square$

The content of the above theorem is that so long as we can prescribe consistently the FDD of a stochastic process, then we can actually construct such a stochastic process on $(\mathbf{X}, \mathcal{K})$. We will now use this theorem to construct the Brownian motion.