for all $n \geq 0$. In particular, it follows that $(S_n^\infty)_{n\geq 0}$ is a supermartingale. Since $S_n^\infty \geq G_n$ P-a.s. we see that $(S_n^\infty)^- \leq G_n^- \leq \sup_{n\geq 0} G_n^-$ P-a.s. for all $n \geq 0$ from where by means of (1.1.3) we see that $((S_n^\infty)^-)_{n\geq 0}$ is uniformly integrable. Thus by the optional sampling theorem (page 60) we get

$$S_n^\infty \geq \mathsf{E}\,(S_\tau^\infty \mid \mathcal{F}_n) \tag{1.1.54}$$

for all $\tau \in \mathfrak{M}_n$. Moreover, since $S_k^\infty \geq G_k$ P-a.s. for all $k \geq n$, it follows that $S_\tau^\infty \geq G_\tau$ P-a.s. for all $\tau \in \mathfrak{M}_n$, and hence

$$\mathsf{E}\,(S_\tau^\infty \mid \mathcal{F}_n) \geq \mathsf{E}\,(G_\tau \mid \mathcal{F}_n) \tag{1.1.55}$$

for all $\tau \in \mathfrak{M}_n$. Combining (1.1.54) and (1.1.55) we see by (1.1.30) that $S_n^\infty \geq S_n$ P-a.s. for all $n \geq 0$. Since the reverse inequality holds in general as shown in (1.1.51) above, this establishes that $S_n^\infty = S_n$ P-a.s. for all $n \geq 0$. From this it also follows that $\tau_n^\infty = \tau_n$ P-a.s. for all $n \geq 0$. Finally, the third identity $V_n^\infty = V_n$ follows by the monotone convergence theorem. The proof of the theorem is complete. □

1.2. Markovian approach

In this subsection we will present basic results of optimal stopping when the time is discrete and the process is Markovian. (Basic definitions and properties of such processes are given in Subsections 4.1 and 4.2.)

1. Throughout we consider a time-homogeneous Markov chain $X = (X_n)_{n\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathsf{P}_x)$ and taking values in a measurable space $(E, \mathcal{B})$ where for simplicity we assume that $E = \mathbb{R}^d$ for some $d \geq 1$ and $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$. It is assumed that the chain X starts at x under P_x for $x \in E$. It is also assumed that the mapping $x \mapsto \mathsf{P}_x(F)$ is measurable for each $F \in \mathcal{F}$. It follows that the mapping $x \mapsto \mathsf{E}_x(Z)$ is measurable for each random variable Z. Finally, without loss of generality we assume that $(\Omega, \mathcal{F})$ equals the canonical space $(E^{\mathbb{N}_0}, \mathcal{B}^{\mathbb{N}_0})$ so that the shift operator $\theta_n : \Omega \to \Omega$ is well defined by $\theta_n(\omega)(k) = \omega(n+k)$ for $\omega = (\omega(k))_{k\geq 0} \in \Omega$ and $n, k \geq 0$. (Recall that $\mathbb{N}_0$ stands for $\mathbb{N} \cup \{0\}$.)

2. Given a measurable function $G : E \to \mathbb{R}$ satisfying the following condition (with $G(X_N) = 0$ if $N = \infty$):

$$\mathsf{E}_x\Big(\sup_{0\leq n\leq N} |G(X_n)|\Big) < \infty \tag{1.2.1}$$

for all $x \in E$, we consider the optimal stopping problem

$$V^N(x) = \sup_{0\leq \tau\leq N} \mathsf{E}_x G(X_\tau) \tag{1.2.2}$$

where $x \in E$ and the supremum is taken over all stopping times τ of X. The latter means that τ is a stopping time with respect to the natural filtration of X given by $\mathcal{F}_n^X = \sigma(X_k : 0 \le k \le n)$ for $n \ge 0$. Since the same results remain valid if we take the supremum in (1.2.2) over stopping times τ with respect to $(\mathcal{F}_n)_{n \ge 0}$, and this assumption makes final conclusions more powerful (at least formally), we will assume in the sequel that the supremum in (1.2.2) is taken over this larger class of stopping times. Note also that in (1.2.2) we admit that N can be $+\infty$ as well. In this case, however, we still assume that the supremum is taken over stopping times τ, i.e. over Markov times τ satisfying $\tau < \infty$ P-a.s. In this way any specification of $G(X_\infty)$ becomes irrelevant for the problem (1.2.2).

3. To solve the problem (1.2.2) in the case when $N < \infty$ we may note that by setting

$$G_n = G(X_n) \tag{1.2.3}$$

for $n \ge 0$ the problem (1.2.2) reduces to the problem (1.1.5) where instead of P and E we have P_x and E_x for $x \in E$. Introducing the expectation in (1.2.2) with respect to P_x under which $X_0 = x$ and studying the resulting problem by means of the mapping $x \mapsto V^N(x)$ for $x \in E$ constitutes a profound step which most directly aims to exploit the Markovian structure of the problem. (The same remark applies in the theory of optimal stochastic control in contrast to classical methods developed in calculus of variations.)

Having identified the problem (1.2.2) as the problem (1.1.5) we can apply the method of backward induction (1.1.6)–(1.1.7) which leads to a sequence of random variables $(S_n^N)_{0 \le n \le N}$ and a stopping time τ_n^N defined in (1.1.8). The key identity is

$$S_n^N = V^{N-n}(X_n) \tag{1.2.4}$$

for $0 \le n \le N$. This will be established in the proof of the next theorem. Once (1.2.4) is known to hold, the results of Theorem 1.2 translate immediately into the present setting and get a more transparent form as follows.

In the sequel we set

$$C_n = \{\, x \in E : V^{N-n}(x) > G(x) \,\}, \tag{1.2.5}$$

$$D_n = \{\, x \in E : V^{N-n}(x) = G(x) \,\} \tag{1.2.6}$$

for $0 \le n \le N$. We define

$$\tau_D = \inf\{\, 0 \le n \le N : X_n \in D_n \,\}. \tag{1.2.7}$$

Finally, the transition operator T of X is defined by

$$TF(x) = \mathsf{E}_x F(X_1) \tag{1.2.8}$$

for $x \in E$ whenever $F : E \to \mathbb{R}$ is a measurable function so that $F(X_1)$ is integrable with respect to P_x for all $x \in E$.

Theorem 1.7. (Finite horizon: The time-homogeneous case) *Consider the optimal stopping problem* (1.2.2) *upon assuming that the condition* (1.2.1) *holds. Then the value function* V^n *satisfies the* Wald–Bellman equations

$$V^n(x) = \max(G(x), TV^{n-1}(x)) \qquad (x \in E) \tag{1.2.9}$$

for $n = 1, \ldots, N$ *where* $V^0 = G$ *. Moreover, we have*:

The stopping time τ_D *is optimal in* (1.2.2). (1.2.10)

If τ_* *is an optimal stopping time in* (1.2.2) *then* $\tau_D \le \tau_*$ P_x*-a.s. for every* $x \in E$. (1.2.11)

The sequence $(V^{N-n}(X_n))_{0 \le n \le N}$ *is the smallest supermartingale which dominates* $(G(X_n))_{0 \le n \le N}$ *under* P_x *for* $x \in E$ *given and fixed.* (1.2.12)

The stopped sequence $(V^{N-n\wedge\tau_D}(X_{n\wedge\tau_D}))_{0 \le n \le N}$ *is a martingale under* P_x *for every* $x \in E$. (1.2.13)

Proof. To verify (1.2.4) recall from (1.1.10) that

$$S_n^N = \mathsf{E}_x\big(G(X_{\tau_n^N}) \mid \mathcal{F}_n\big) \tag{1.2.14}$$

for $0 \le n \le N$. Since $S_k^{N-n} \circ \theta_n = S_{n+k}^N$ we get that τ_n^N satisfies

$$\tau_n^N = \inf\{\, n \le k \le N : S_k^N = G(X_k)\,\} = n + \tau_0^{N-n} \circ \theta_n \tag{1.2.15}$$

for $0 \le n \le N$. Inserting (1.2.15) into (1.2.14) and using the Markov property we obtain

$$\begin{aligned} S_n^N &= \mathsf{E}_x\big(G(X_{n+\tau_0^{N-n}\circ\theta_n}) \mid \mathcal{F}_n\big) = \mathsf{E}_x\big(G(X_{\tau_0^{N-n}}) \circ \theta_n \mid \mathcal{F}_n\big) \\ &= \mathsf{E}_{X_n} G(X_{\tau_0^{N-n}}) = V^{N-n}(X_n) \end{aligned} \tag{1.2.16}$$

where the final equality follows by (1.1.9)–(1.1.10) which imply

$$\mathsf{E}_x S_0^{N-n} = \mathsf{E}_x G(X_{\tau_0^{N-n}}) = \sup_{0 \le \tau \le N-n} \mathsf{E}_x G(X_\tau) = V^{N-n}(x) \tag{1.2.17}$$

for $0 \le n \le N$ and $x \in E$. Thus (1.2.4) holds as claimed.

To verify (1.2.9) note that (1.1.7) using (1.2.4) and the Markov property reads as follows:

$$\begin{aligned} V^{N-n}(X_n) &= \max\big(G(X_n), \mathsf{E}_x\big(V^{N-n-1}(X_{n+1}) \mid \mathcal{F}_n\big)\big) \\ &= \max\big(G(X_n), \mathsf{E}_x\big(V^{N-n-1}(X_1) \circ \theta_n \mid \mathcal{F}_n\big)\big) \\ &= \max\big(G(X_n), \mathsf{E}_{X_n}\big(V^{N-n-1}(X_1)\big)\big) \\ &= \max\big(G(X_n), TV^{N-n-1}(X_n)\big) \end{aligned} \tag{1.2.18}$$

for all $0 \le n \le N$. Letting $n = 0$ and using that $X_0 = x$ under P_x we see that (1.2.18) yields (1.2.9).

The remaining statements of the theorem follow directly from Theorem 1.2 above. The proof is complete. □

4. The Wald–Bellman equations (1.2.9) can be written in a more compact form as follows. Introduce the operator Q by setting

$$QF(x) = \max(G(x), TF(x)) \tag{1.2.19}$$

for $x \in E$ where $F : E \to \mathbb{R}$ is a measurable function for which $F(X_1) \in L^1(\mathsf{P}_x)$ for $x \in E$. Then (1.2.9) reads as follows:

$$V^n(x) = Q^n G(x) \tag{1.2.20}$$

for $1 \le n \le N$ where Q^n denotes the n-th power of Q. The recursive relations (1.2.20) form a constructive method for finding V^N when $\text{Law}(X_1 \mid \mathsf{P}_x)$ is known for $x \in E$.

5. Let us now discuss the case when X is a time-inhomogeneous Markov chain. Setting $Z_n = (n, X_n)$ for $n \ge 0$ one knows that $Z = (Z_n)_{n \ge 0}$ is a time-homogeneous Markov chain. Given a measurable function $G : \{0, 1, \ldots, N\} \times E \to \mathbb{R}$ satisfying the following condition:

$$\mathsf{E}_{n,x}\Big(\sup_{0 \le k \le N-n} |G(n+k, X_{n+k})|\Big) < \infty \tag{1.2.1$'$}$$

for all $0 \le n \le N$ and $x \in E$, the optimal stopping problem (1.2.2) therefore naturally extends as follows:

$$V^N(n, x) = \sup_{0 \le \tau \le N-n} \mathsf{E}_{n,x} G(n+\tau, X_{n+\tau}) \tag{1.2.2$'$}$$

where the supremum is taken over stopping times τ of X and $X_n = x$ under $\mathsf{P}_{n,x}$ with $0 \le n \le N$ and $x \in E$ given and fixed.

As above one verifies that

$$S^N_{n+k} = V^N(n+k, X_{n+k}) \tag{1.2.21}$$

under $\mathsf{P}_{n,x}$ for $0 \le n \le N - n$. Moreover, inserting this into (1.1.7) and using the Markov property one finds

$$\begin{aligned} &V^N(n+k, X_{n+k}) \qquad\qquad (1.2.22)\\ &= \max\Big(G(n+k, X_{n+k}), \mathsf{E}_{n,x}\big(V^N(n+k+1, X_{n+k+1}) \mid \mathcal{F}_{n+k}\big)\Big)\\ &= \max\Big(G(Z_{n+k}), \mathsf{E}_z\big(V^N(Z_{n+k+1}) \mid \mathcal{F}_{n+k}\big)\Big)\\ &= \max\Big(G(Z_{n+k}), \mathsf{E}_z\big(V^N(Z_1) \circ \theta_{n+k} \mid \mathcal{F}_{n+k}\big)\Big)\\ &= \max\Big(G(Z_{n+k}), \mathsf{E}_{Z_{n+k}}\big(V^N(Z_1)\big)\Big) \end{aligned}$$

for $0 \le k \le N-n-1$ where $z = (n,x)$ with $0 \le n \le N$ and $x \in E$. Letting $k = 0$ and using that $Z_n = z = (n,x)$ under P_z, one gets

$$V^N(n,x) = \max\big(G(n,x), TV^N(n,x)\big) \tag{1.2.23}$$

for $n = N-1, \ldots, 1, 0$ where $TV^N(N-1,x) = \mathsf{E}_{N-1,x} G(N, X_N)$ and T is the transition operator of Z given by

$$TF(n,x) = \mathsf{E}_{n,x} F(n+1, X_{n+1}) \tag{1.2.24}$$

for $0 \le n \le N$ and $x \in E$ whenever the right-hand side in (1.2.24) is well defined (finite).

The recursive relations (1.2.23) are the *Wald–Bellman equations* corresponding to the time-inhomogeneous problem $(1.2.2')$. Note that when X is time-homogeneous (and $G = G(x)$ only) we have $V^N(n,x) = V^{N-n}(x)$ and (1.2.23) reduces to (1.2.9). In order to present a reformulation of the property (1.2.12) in Theorem 1.7 above we will proceed as follows.

6. The following definition plays a fundamental role in finding a solution to the optimal stopping problem $(1.2.2')$.

Definition 1.8. A measurable function $F : \{0, 1, \ldots, N\} \times E \to \mathbb{R}$ is said to be *superharmonic* if

$$TF(n,x) \le F(n,x) \tag{1.2.25}$$

for all $(n,x) \in \{0, 1, \ldots, N\} \times E$.

It is assumed in (1.2.25) that $TF(n,x)$ is well defined i.e. that $F(n+1, X_{n+1}) \in L^1(\mathsf{P}_{n,x})$ for all (n,x) as above. Moreover, if $F(n+k, X_{n+k}) \in L^1(\mathsf{P}_{n,x})$ for all $0 \le k \le N-n$ and all (n,x) as above, then one verifies directly by the Markov property that the following stochastic characterization of superharmonic functions holds:

F is superharmonic if and only if $(F(n+k, X_{n+k}))_{0 \le k \le N-n}$ is a supermartingale under $\mathsf{P}_{n,x}$ for all $(n,x) \in \{0, 1, \ldots, N-1\} \times E$. (1.2.26)

The proof of this fact is simple and will be given in a more general case following (1.2.40) below.

Introduce the continuation set

$$C = \big\{(n,x) \in \{0, 1, \ldots, N\} \times E : V(n,x) > G(n,x)\big\} \tag{1.2.27}$$

and the stopping set

$$D = \big\{(n,x) \in \{0, 1, \ldots, N\} \times E : V(n,x) = G(n,x)\big\}. \tag{1.2.28}$$

Introduce the first entry time τ_D into D by setting

$$\tau_D = \inf\{\, n \le k \le N : (n+k, X_{n+k}) \in D \,\} \tag{1.2.29}$$

under $\mathsf{P}_{n,x}$ where $(n,x) \in \{0, 1, \ldots, N\} \times E$.

The preceding considerations may now be summarized in the following extension of Theorem 1.7.

Theorem 1.9. (Finite horizon: The time-inhomogeneous case) *Consider the optimal stopping problem* (1.2.2′) *upon assuming that the condition* (1.2.1′) *holds. Then the value function* V^N *satisfies the* Wald–Bellman equations

$$V^N(n,x) = \max\big(G(n,x), TV^N(n,x)\big) \tag{1.2.30}$$

for $n = N-1,\ldots,1,0$ *where* $TV^N(N-1,x) = \mathsf{E}_{N-1,x}G(N,X_N)$ *and* $x \in E$. *Moreover, we have*:

The stopping time τ_D *is optimal in* (1.2.2′). (1.2.31)

If τ_* *is an optimal stopping time in* (1.2.2′) *then* $\tau_D \le \tau_*$ $\mathsf{P}_{n,x}$*-a.s. for every* $(n,x) \in \{0,1,\ldots,N\}\times E$. (1.2.32)

The value function V^N *is the smallest superharmonic function which dominates the gain function* G *on* $\{0,1,\ldots,N\}\times E$. (1.2.33)

The stopped sequence $\big(V^N((n+k)\wedge\tau_D, X_{(n+k)\wedge\tau_D})\big)_{0\le k\le N-n}$ *is a martingale under* $\mathsf{P}_{n,x}$ *for every* $(n,x) \in \{0,1,\ldots,N\}\times E$. (1.2.34)

Proof. The proof is carried out in exactly the same way as the proof of Theorem 1.7 above. The key identity linking the problem (1.2.2′) to the problem (1.1.5) is (1.2.21). This yields (1.2.23) i.e. (1.2.30) as shown above. Note that (1.2.33) refines (1.2.12) and follows by (1.2.26). The proof is complete. □

7. Consider the optimal stopping problem (1.2.2) when $N = \infty$. Recall that (1.2.2) reads as follows:

$$V(x) = \sup_\tau \mathsf{E}_x G(X_\tau) \tag{1.2.35}$$

where τ is a stopping time of X and $\mathsf{P}_x(X_0 = x) = 1$ for $x \in E$.

Introduce the continuation set

$$C = \{\, x \in E : V(x) > G(x) \,\} \tag{1.2.36}$$

and the stopping set

$$D = \{\, x \in E : V(x) = G(x) \,\}. \tag{1.2.37}$$

Introduce the first entry time τ_D into D by setting

$$\tau_D = \inf\{\, t \ge 0 : X_t \in D \,\}. \tag{1.2.38}$$

8. The following definition plays a fundamental role in finding a solution to the optimal stopping problem (1.2.35). Note that Definition 1.8 above may be viewed as a particular case of this definition.

Definition 1.10. A measurable function $F : E \to \mathbb{R}$ is said to be *superharmonic* if

$$TF(x) \le F(x) \tag{1.2.39}$$

for all $x \in E$.

It is assumed in (1.2.39) that $TF(x)$ is well defined by (1.2.8) above i.e. that $F(X_1) \in L^1(\mathsf{P}_x)$ for all $x \in E$. Moreover, if $F(X_n) \in L^1(\mathsf{P}_x)$ for all $n \ge 0$ and all $x \in E$, then the following stochastic characterization of superharmonic functions holds (recall (1.2.26) above):

F is superharmonic if and only if $(F(X_n))_{n\ge 0}$ is a supermartingale under P_x for every $x \in E$. (1.2.40)

The proof of this equivalence relation is simple. Suppose first that F is superharmonic. Then (1.2.39) holds for all $x \in E$ and therefore by the Markov property we get

$$\begin{aligned} TF(X_n) &= \mathsf{E}_{X_n}(F(X_1)) = \mathsf{E}_x(F(X_1)\circ\theta_n \mid \mathcal{F}_n) \\ &= \mathsf{E}_x(F(X_{n+1}) \mid \mathcal{F}_n) \le F(X_n) \end{aligned} \tag{1.2.41}$$

for all $n \ge 0$ proving the supermartingale property of $(F(X_n))_{n\ge 0}$ under P_x for every $x \in E$. Conversely, if $(F(X_n))_{n\ge 0}$ is a supermartingale under P_x for every $x \in E$, then the final inequality in (1.2.41) holds for all $n \ge 0$. Letting $n = 0$ and taking E_x on both sides gives (1.2.39). Thus F is superharmonic as claimed.

9. In the case of infinite horizon (i.e. when $N = \infty$ in (1.2.2) above) it is not necessary to treat the time-inhomogeneous case separately from the time-homogeneous case as we did it for clarity in the case of finite horizon (i.e. when $N < \infty$ in (1.2.2) above). This is due to the fact that the state space E may be general anyway (two-dimensional) and the passage from the time-inhomogeneous process $(X_n)_{n\ge 0}$ to the time-homogeneous process $(n, X_n)_{n\ge 0}$ does not affect the time set in which the stopping times of X take values (by altering the remaining time).

Theorem 1.11. (Infinite horizon) *Consider the optimal stopping problem* (1.2.35) *upon assuming that the condition* (1.2.1) *holds. Then the value function V satisfies the* Wald–Bellman equation

$$V(x) = \max(G(x), TV(x)) \tag{1.2.42}$$

for $x \in E$. Assume moreover when required below that

$$\mathsf{P}_x(\tau_D < \infty) = 1 \tag{1.2.43}$$

for all $x \in E$. Then we have:

The stopping time τ_D is optimal in (1.2.35). (1.2.44)

If τ_* is an optimal stopping time in (1.2.35) *then $\tau_D \le \tau_*$ P_x-a.s. for every $x \in E$.* (1.2.45)

The value function V is the smallest superharmonic function which dominates the gain function G on E. (1.2.46)

The stopped sequence $(V(X_{n\wedge\tau_D}))_{n\ge 0}$ is a martingale under P_x for every $x \in E$. (1.2.47)

Finally, if the condition (1.2.43) *fails so that* $\mathsf{P}_x(\tau_D = \infty) > 0$ *for some* $x \in E$ *, then there is no optimal stopping time* (*with probability* 1) *in* (1.2.35).

Proof. The key identity in reducing the problem (1.2.35) to the problem (1.1.29) is

$$S_n = V(X_n) \tag{1.2.48}$$

for $n \geq 0$. This can be proved by passing to the limit for $N \to \infty$ in (1.2.4) and using the result of Theorem 1.6 above. In exactly the same way one derives (1.2.42) from (1.2.9). The remaining statements follow from Theorem 1.4 above. Note also that (1.2.46) refines (1.1.38) and follows by (1.2.40). The proof is complete. □

Corollary 1.12. (Iterative method) *Under the initial hypothesis of Theorem* 1.11 *we have*

$$V(x) = \lim_{n\to\infty} Q^n G(x) \tag{1.2.49}$$

for all $x \in E$ *.*

Proof. It follows from (1.2.9) and Theorem 1.6 above. □

The relation (1.2.49) offers a constructive method for finding the value function V . (Note that $n \mapsto Q^n G(x)$ is increasing on $\{0, 1, 2, \ldots\}$ for every $x \in E$.)

10. We have seen in Theorem 1.7 and Theorem 1.9 that the Wald–Bellman equations (1.2.9) and (1.2.30) characterize the value function V^N when the horizon N is finite (i.e. these equations cannot have other solutions). This is due to the fact that V^N equals G in the “end of time” N . When the horizon N is infinite, however, this characterization is no longer true for the Wald–Bellman equation (1.2.42). For example, if G is identically equal to a constant c then any other constant C larger than c will define a function solving (1.2.42). On the other hand, it is evident from (1.2.42) that every solution of this equation is superharmonic and dominates G . By (1.2.46) we thus see that a minimal solution of (1.2.42) will coincide with the value function. This “minimality condition” (over all points) can be replaced by a single condition as the following theorem shows. From the standpoint of finite horizon such a “boundary condition at infinity” is natural.

Theorem 1.13. (Uniqueness in the Wald–Bellman equation)
Under the hypothesis of Theorem 1.11 *suppose that* $F : E \to \mathbb{R}$ *is a function solving the* Wald–Bellman equation

$$F(x) = \max(G(x), TF(x)) \tag{1.2.50}$$

for $x \in E$ *.* (*It is assumed that* F *is measurable and* $F(X_1) \in L^1(\mathsf{P}_x)$ *for all* $x \in E$ *.*) *Suppose moreover that* F *satisfies*

$$\mathsf{E}\Big(\sup_{n\geq 0} F(X_n)\Big) < \infty. \tag{1.2.51}$$

Then F equals the value function V if and only if the following "boundary condition at infinity" holds:

$$\limsup_{n\to\infty} F(X_n) = \limsup_{n\to\infty} G(X_n) \qquad \mathsf{P}_x\text{-a.s.} \tag{1.2.52}$$

for every $x \in E$. (In this case the $\limsup$ *on the left-hand side of* (1.2.52) *equals the* $\liminf$*, i.e. the sequence $(F(X_n))_{n\geq 0}$ is convergent P_x-a.s. for every $x \in E$.)*

Proof. If $F = V$ then by (1.2.46) we know that F is the smallest superharmonic function which dominates G on E. Let us show (the fact of independent interest) that any such function F must satisfy (1.2.52). Note that the condition (1.2.51) is not needed for this implication.

Since $F \geq G$ we see that the left-hand side in (1.2.52) is evidently larger than the right-hand side. To prove the reverse inequality, consider the function $H : E \to \mathbb{R}$ defined by

$$H(x) = \mathsf{E}_x\Big(\sup_{n\geq 0} G(X_n)\Big) \tag{1.2.53}$$

for $x \in E$. Then the key property of H stating that

$$H \text{ is superharmonic} \tag{1.2.54}$$

can be verified as follows. By the Markov property we have

$$\begin{aligned} TH(x) &= \mathsf{E}_x H(X_1) = \mathsf{E}_x\Big(\mathsf{E}_{X_1}\Big(\sup_{n\geq 0} G(X_n)\Big)\Big) \\ &= \mathsf{E}_x\Big(\mathsf{E}_x\Big(\sup_{n\geq 0} G(X_n)\circ\theta_1 \,\Big|\, \mathcal{F}_1\Big)\Big) = \mathsf{E}_x\Big(\sup_{n\geq 0} G(X_{n+1})\Big) \\ &\leq H(x) \end{aligned} \tag{1.2.55}$$

for all $x \in E$ proving (1.2.54). Moreover, since $X_0 = x$ under P_x we see that $H(x) \geq G(x)$ for all $x \in E$. Hence $F(x) \leq H(x)$ for all $x \in E$ by assumption. By the Markov property it thus follows that

$$\begin{aligned} F(X_n) &\leq H(X_n) = \mathsf{E}_{X_n}\Big(\sup_{k\geq 0} G(X_k)\Big) = \mathsf{E}_x\Big(\sup_{k\geq 0} G(X_k)\circ\theta_n \,\Big|\, \mathcal{F}_n\Big) \\ &= \mathsf{E}_x\Big(\sup_{k\geq 0} G(X_{k+n}) \,\Big|\, \mathcal{F}_n\Big) \leq \mathsf{E}_x\Big(\sup_{l\geq m} G(X_l) \,\Big|\, \mathcal{F}_n\Big) \end{aligned} \tag{1.2.56}$$

for any $m \leq n$ given and fixed where $x \in E$. The final expression in (1.2.56) defines a (generalized) martingale for $n \geq 1$ under P_x which is known to converge P_x-a.s. as $n \to \infty$ for every $x \in E$ with the limit satisfying the following inequality:

$$\lim_{n\to\infty} \mathsf{E}_x\Big(\sup_{l\geq m} G(X_l) \,\Big|\, \mathcal{F}_n\Big) \leq \mathsf{E}_x\Big(\sup_{l\geq m} G(X_l) \,\Big|\, \mathcal{F}_\infty\Big) = \sup_{l\geq m} G(X_l) \tag{1.2.57}$$

where the final identity follows from the fact that $\sup_{l \geq m} G(X_l)$ is $\mathcal{F}_\infty$-measurable. Letting $n \to \infty$ in (1.2.56) and using (1.2.57) we find

$$\limsup_{n\to\infty} F(X_n) \leq \sup_{l \geq m} G(X_l) \qquad \mathsf{P}_x\text{-a.s.} \tag{1.2.58}$$

for all $m \geq 0$ and $x \in E$. Letting finally $m \to \infty$ in (1.2.58) we end up with (1.2.52). This ends the first part of the proof.

Conversely, suppose that F satisfies (1.2.50)–(1.2.52) and let us show that F must then be equal to V. For this, first note that (1.2.50) implies that F is superharmonic and that $F \geq G$. Hence by (1.2.46) we see that $V \leq F$. To show that $V \geq F$ consider the stopping time

$$\tau_{D_\varepsilon} = \inf\{\, n \geq 0 : F(X_n) \leq G(X_n) + \varepsilon \,\} \tag{1.2.59}$$

where $\varepsilon > 0$ is given and fixed. Then by (1.2.52) we see that $\tau_{D_\varepsilon} < \infty$ P_x-a.s. for $x \in E$. Moreover, we claim that $\big(F(X_{\tau_{D_\varepsilon} \wedge n})\big)_{n \geq 0}$ is a martingale under P_x for $x \in E$. For this, note that the Markov property and (1.2.50) imply

$$\begin{aligned}
&\mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}\wedge n}) \,|\, \mathcal{F}_{n-1}\big) \qquad (1.2.60)\\
&= \mathsf{E}_x\big(F(X_n) I(\tau_{D_\varepsilon} \geq n) \,|\, \mathcal{F}_{n-1}\big) + \mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}}) I(\tau_{D_\varepsilon} < n) \,|\, \mathcal{F}_{n-1}\big)\\
&= \mathsf{E}_x\big(F(X_n) \,|\, \mathcal{F}_{n-1}\big) I(\tau_{D_\varepsilon} \geq n) + \mathsf{E}_x\big(\textstyle\sum_{k=0}^{n-1} F(X_k) I(\tau_{D_\varepsilon} = k) \,|\, \mathcal{F}_{n-1}\big)\\
&= \mathsf{E}_{X_{n-1}}\big(F(X_1)\big)\, I(\tau_{D_\varepsilon} \geq n) + \textstyle\sum_{k=0}^{n-1} F(X_k)\, I(\tau_{D_\varepsilon} = k)\\
&= TF(X_{n-1})\, I(\tau_{D_\varepsilon} \geq n) + F(X_{\tau_{D_\varepsilon}})\, I(\tau_{D_\varepsilon} < n)\\
&= F(X_{n-1})\, I(\tau_{D_\varepsilon} \geq n) + F(X_{\tau_{D_\varepsilon}})\, I(\tau_{D_\varepsilon} < n)\\
&= F(X_{\tau_{D_\varepsilon}\wedge(n-1)})\, I(\tau_{D_\varepsilon} \geq n) + F(X_{\tau_{D_\varepsilon}\wedge(n-1)})\, I(\tau_{D_\varepsilon} < n)\\
&= F(X_{\tau_{D_\varepsilon}\wedge(n-1)})
\end{aligned}$$

for all $n \geq 1$ and $x \in E$ proving the claim. Hence

$$\mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}\wedge n})\big) = F(x) \tag{1.2.61}$$

for all $n \geq 0$ and $x \in E$. Next note that

$$\mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}\wedge n})\big) = \mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}})\, I(\tau_{D_\varepsilon} \leq n)\big) + \mathsf{E}_x\big(F(X_n)\, I(\tau_{D_\varepsilon} > n)\big) \tag{1.2.62}$$

for all $n \geq 0$. Letting $n \to \infty$, using (1.2.51) and (1.2.1) with $F \geq G$, we get

$$\mathsf{E}_x\big(F(X_{\tau_{D_\varepsilon}})\big) = F(x) \tag{1.2.63}$$

for all $x \in E$. This fact is of independent interest.

Finally, since V is superharmonic, we find using (1.2.63) that

$$V(x) \geq \mathsf{E}_x V(X_{\tau_{D_\varepsilon}}) \geq \mathsf{E}_x G(X_{\tau_{D_\varepsilon}}) \geq \mathsf{E}_x F(X_{\tau_{D_\varepsilon}}) - \varepsilon = F(x) - \varepsilon \tag{1.2.64}$$

for all $\varepsilon > 0$ and $x \in E$. Letting $\varepsilon \downarrow 0$ we get $V \geq F$ as needed and the proof is complete. □

11. Given $\alpha \in (0,1]$ and (bounded) measurable functions $g : E \to \mathbb{R}$ and $c : E \to \mathbb{R}_+$, consider the optimal stopping problem

$$V(x) = \sup_\tau \mathsf{E}_x \left(\alpha^\tau g(X_\tau) - \sum_{k=1}^{\tau} \alpha^{k-1} c(X_{k-1}) \right) \tag{1.2.65}$$

where τ is a stopping time of X and $\mathsf{P}_x(X_0 = x) = 1$.

The value $c(x)$ is interpreted as the cost of making the next observation of X when X equals x. The sum in (1.2.65) by definition equals 0 when τ equals 0.

The problem formulation (1.2.65) goes back to a problem formulation due to Bolza in classic calculus of variation (a more detailed discussion will be given in Chapter III below). Let us briefly indicate how the problem (1.2.65) can be reduced to the setting of Theorem 1.11 above.

For this, let $\widetilde{X} = (\widetilde{X}_n)_{n \geq 0}$ denote the Markov chain X killed at rate α. It means that the transition operator of $\widetilde{X}$ is given by

$$\widetilde{T} F(x) = \alpha T F(x) \tag{1.2.66}$$

for $x \in E$ whenever $F(X_1) \in L^1(\mathsf{P}_x)$. The problem (1.2.65) then reads

$$V(x) = \sup_\tau \mathsf{E}_x \left(g(\widetilde{X}_\tau) - \sum_{k=1}^{\tau} c(\widetilde{X}_{k-1}) \right) \tag{1.2.65$'$}$$

where τ is a stopping time of $\widetilde{X}$ and $\mathsf{P}_x(\widetilde{X}_0 = x) = 1$.

Introduce the sequence

$$\widetilde{I}_n = a + \sum_{k=1}^{n} c(\widetilde{X}_{k-1}) \tag{1.2.67}$$

for $n \geq 1$ with $\widetilde{I}_0 = a$ in $\mathbb{R}$. Then $\widetilde{Z}_n = (\widetilde{X}_n, \widetilde{I}_n)$ defines a Markov chain for $n \geq 0$ with $\widetilde{Z}_0 = (\widetilde{X}_0, \widetilde{I}_0) = (x, a)$ under P_x so that we may write $\mathsf{P}_{x,a}$ instead of P_x. (The latter can be justified rigorously by passage to the canonical probability space.) The transition operator of $\widetilde{Z} = (\widetilde{X}, \widetilde{I})$ equals

$$T_{\widetilde{Z}} F(x,a) = \mathsf{E}_{x,a} F(\widetilde{X}_1, \widetilde{I}_1) \tag{1.2.68}$$

for $(x,a) \in E \times \mathbb{R}$ whenever $F(\widetilde{X}_1, \widetilde{I}_1) \in L^1(\mathsf{P}_{x,a})$.

The problem $(1.2.65')$ may now be rewritten as follows:

$$W(x,a) = \sup_{\tau} \mathsf{E}_{x,a} G(Z_\tau) \tag{1.2.65''}$$

where we set

$$G(z) = g(x) - a \tag{1.2.69}$$

for $z = (x,a) \in E \times \mathbb{R}$. Obviously by subtracting a on both sides of $(1.2.65')$ we set that

$$W(x,a) = V(x) - a \tag{1.2.70}$$

for all $(x,a) \in E \times \mathbb{R}$.

The problem $(1.2.65'')$ is of the same type as the problem (1.2.35) above and thus Theorem 1.11 is applicable. To write down (1.2.42) more explicitly note that

$$\begin{aligned} T_{\widetilde{Z}} W(x,a) &= \mathsf{E}_{x,a} W(\widetilde{X}_1, \widetilde{I}_1) = \mathsf{E}_{x,a}\big(V(\widetilde{X}_1) - \widetilde{I}_1\big) \\ &= \mathsf{E}_x V(\widetilde{X}_1) - a - c(x) = \alpha T V(x) - a - c(x) \end{aligned} \tag{1.2.71}$$

so that (1.2.42) reads

$$V(x) - a = \max\big(g(x) - a\,,\ \alpha T V(x) - a - c(x)\big) \tag{1.2.72}$$

where we used (1.2.70), (1.2.69) and (1.2.71). Clearly a can be removed from (1.2.72) showing finally that the Wald–Bellman equation (1.2.42) takes the following form:

$$V(x) = \max\big(g(x)\,,\ \alpha T V(x) - c(x)\big) \tag{1.2.73}$$

for $x \in E$. Note also that (1.2.39) takes the following form:

$$\alpha T F(x) - c(x) \le F(x) \tag{1.2.74}$$

for $x \in E$. Thus F satisfies (1.2.74) if and only if $(x,a) \mapsto F(x) - a$ is superharmonic relative to the Markov chain $\widetilde{Z} = (\widetilde{X}, \widetilde{I})$. Having (1.2.73) and (1.2.74) set out explicitly the remaining statements of Theorem 1.11 and Corollary 1.12 are directly applicable and we shall omit further details. It may be noted above that $L = T - I$ is the generator of the Markov chain X. More general problems of this type (involving also the maximum functional) will be discussed in Chapter III below. We will conclude this section by giving an illustrative example.

12. The following example illustrates general results of optimal stopping theory for Markov chains when applied to a nontrivial problem in order to determine the value function and an optimal Markov time (in the class $\bar{\mathfrak{M}}$).

Example 1.14. Let $\xi, \xi_1, \xi_2, \dots$ be independent and identically distributed random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, with $\mathsf{E}\xi < 0$. Put $S_0 = 0$,

$S_n = \xi_1 + \cdots + \xi_n$ for $n \geq 1$; $X_0 = x$, $X_n = x + S_n$ for $n \geq 1$, and $M = \sup_{n\geq 0} S_n$. Let P_x be the probability distribution of the sequence $(X_n)_{n\geq 0}$ with $X_0 = x$ from $\mathbb{R}$. It is clear that the sequence $(X_n)_{n\geq 0}$ is a Markov chain started at x.

For any $n \geq 1$ define the gain function $G_n(x) = (x^+)^n$ where $x^+ = \max(x, 0)$ for $x \in \mathbb{R}$, and let

$$V_n(x) = \sup_{\tau \in \mathfrak{M}} \mathsf{E}_x G_n(X_\tau) \tag{1.2.75}$$

where the supremum is taken over the class $\mathfrak{M}$ of all Markov (stopping) times τ satisfying $\mathsf{P}_x(\tau < \infty) = 1$ for all $x \in \mathbb{R}$. Let us also denote

$$\bar{V}_n(x) = \sup_{\tau \in \bar{\mathfrak{M}}} \mathsf{E}_x G_n(X_\tau) I(\tau < \infty) \tag{1.2.76}$$

where the supremum is taken over the class $\bar{\mathfrak{M}}$ of all Markov times.

The problem of finding the value functions $V_n(x)$ and $\bar{V}_n(x)$ is of interest for the theory of American options because these functions represent arbitrage-free (fair, rational) prices of "Power options" under the assumption that any exercise time τ belongs to the class $\mathfrak{M}$ or $\bar{\mathfrak{M}}$ respectively. In the present case we have $V_n(x) = \bar{V}_n(x)$ for $n \geq 1$ and $x \in \mathbb{R}$, and it will be clear from what follows below that an optimal Markov time exists in the class $\bar{\mathfrak{M}}$ (but does not belong to the class $\mathfrak{M}$ of stopping times).

We follow [144] where the authors solved the formulated problems (see also [119]). First of all let us introduce the notion of the Appell polynomial which will be used in the formulation of the basic results.

Let $\eta = \eta(\omega)$ be a random variable with $\mathsf{E}\, e^{\lambda|\eta|} < \infty$ for some $\lambda > 0$. Consider the Esscher transform

$$x \rightsquigarrow \frac{e^{ux}}{\mathsf{E} e^{u\eta}} \qquad |u| \leq \lambda, \quad x \in \mathbb{R}, \tag{1.2.77}$$

and the decomposition

$$\frac{e^{ux}}{\mathsf{E} e^{u\eta}} = \sum_{k=0}^{\infty} \frac{u^k}{k!}\, Q_k^{(\eta)}(x). \tag{1.2.78}$$

Polynomials $Q_k^{(\eta)}(x)$ are called the *Appell polynomials* for the random variable η. (If $\mathsf{E}|\eta|^n < \infty$ for some $n \geq 1$ then the polynomials $Q_k^{(\eta)}(x)$ are uniquely defined for all $k \leq n$.)

The polynomials $Q_k^{(\eta)}(x)$ can be expressed through the semi-invariants $\varkappa_1$, $\varkappa_2, \ldots$ of the random variable η. For example,

$$\begin{aligned} &Q_0^{(\eta)}(x) = 1, && Q_2^{(\eta)}(x) = (x - \varkappa_1)^2 - \varkappa_2, \\ &Q_1^{(\eta)}(x) = x - \varkappa_1, && Q_3^{(\eta)}(x) = (x - \varkappa_1)^3 - 3\varkappa_2(x - \varkappa_1) - \varkappa_3, \end{aligned} \quad \ldots \tag{1.2.79}$$

where (as is well known) the semi-invariants $\varkappa_1, \varkappa_2, \ldots$ are expressed via the moments $\mu_1, \mu_2, \ldots$ of η:

$$\varkappa_1 = \mu_1, \quad \varkappa_2 = \mu_2 - \mu_1^2, \quad \varkappa_3 = 2\mu_1^3 - 3\mu_1\mu_2 + \mu_3, \quad \ldots. \tag{1.2.80}$$

Let us also mention the following property of the Appell polynomials: if $\mathsf{E}\,|\eta|^n < \infty$ then for $k \le n$ we have

$$\frac{d}{dx} Q_k^{(\eta)}(x) = k Q_{k-1}^{(\eta)}(x), \tag{1.2.81}$$

$$\mathsf{E}\, Q_k^{(\eta)}(x+\eta) = x^k. \tag{1.2.82}$$

For simplicity of notation we will use $\mathbb{Q}_k(s)$ to denote the polynomials $Q_k^{(M)}(x)$ for the random variable $M = \sup_{n \ge 0} S_n$. Every polynomial $\mathbb{Q}_k(x)$ has a unique positive root a_k^*. Moreover, $\mathbb{Q}_k(x) \le 0$ for $0 \le x < a_k^*$ and $\mathbb{Q}_k(x)$ increases for $x \ge a_k^*$.

In accordance with the characteristic property (1.2.46) recall that *the value function* $V_n(x)$ *is the smallest superharmonic* (*excessive*) *function which dominates the gain function* $G_n(x)$ *on* $\mathbb{R}$. Thus, one method to find $V_n(x)$ is to search for the smallest excessive majorant of the function $G_n(x)$. In [144] this method is realized as follows.

For every $a \ge 0$ introduce the Markov time

$$\tau_a = \inf\{n \ge 0 : X_n \ge a\} \tag{1.2.83}$$

and for each $n \ge 1$ consider the new optimal stopping problem:

$$\widehat{V}(x) = \sup_{a \ge 0} \mathsf{E}_x G_n(X_{\tau_a}) I(\tau_a < \infty). \tag{1.2.84}$$

It is clear that $G_n(X_{\tau_a}) = (X_{\tau_a}^+)^n = X_{\tau_a}^n$ (on the set $\{\tau_a < \infty\}$). Hence

$$\widehat{V}(x) = \sup_{a \ge 0} \mathsf{E}_x X_{\tau_a}^n I(\tau_a < \infty). \tag{1.2.85}$$

The identity (1.2.82) prompts that the following property should be valid: if $\mathsf{E}\,|M|^n < \infty$ then

$$\mathsf{E}\,\mathbb{Q}_n(x+M) I(x+M \ge a) = \mathsf{E}_x X_{\tau_a}^n I(\tau_a < \infty). \tag{1.2.86}$$

This formula and properties of the Appell polynomials imply that

$$\widehat{V}(x) = \sup_{a \ge 0} \mathsf{E}\,\mathbb{Q}_n(x+M) I(x+M \ge a) = \mathsf{E}\,\mathbb{Q}_n(x+M) I(x+M \ge a_n^*). \tag{1.2.87}$$

From this we see that $\tau_{a_n^*}$ is an optimal Markov time for the problem (1.2.84).

It is clear that $\bar{V}_n(x) \ge \widehat{V}_n(x)$. From (1.2.87) and properties of the Appell polynomials we obtain that $\widehat{V}_n(x)$ is an excessive majorant of the gain function ($\widehat{V}_n(x) \ge \mathsf{E}_x \widehat{V}_n(X_1)$ and $\widehat{V}_n(x) \ge G_n(x)$ for $x \in \mathbb{R}$). But $\bar{V}_n(x)$ is the *smallest* excessive majorant of $G_n(x)$. Thus $\bar{V}_n(x) \le \widehat{V}_n(x)$.

On the whole we obtain the following result (for further details see [144]):

Suppose that $\mathsf{E}\,(\xi^+)^{n+1} < \infty$ and a_n^* is the largest root of the equation $\mathbb{Q}_n(x) = 0$ for $n \ge 1$ fixed. Denote $\tau_n^* = \inf\{\, k \ge 0 : X_k \ge a_n^* \,\}$. Then the Markov time τ_n^* is optimal:

$$V_n(x) = \sup_{\tau \in \bar{\mathfrak{M}}} \mathsf{E}_x (X_\tau^+)^n I(\tau < \infty) = \mathsf{E}_x \big(X_{\tau_n^*}^+\big)^n I(\tau < \infty). \tag{1.2.88}$$

Moreover,

$$V_n(x) = \mathsf{E}\, \mathbb{Q}_n(x{+}M) I(x{+}M \ge a_n^*). \tag{1.2.89}$$

Remark 1.15. In the cases $n = 1$ and $n = 2$ we have

$$a_1^* = \mathsf{E}\, M \quad \text{and} \quad a_2^* = \mathsf{E}\, M + \sqrt{\mathsf{D} M}. \tag{1.2.90}$$

Remark 1.16. If $\mathsf{P}(\xi = 1) = p$, $\mathsf{P}(\xi = -1) = q$ and $p < q$, then $M := \sup_{n \ge 0} S_n$ (with $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$) has geometric distribution:

$$\mathsf{P}(M \ge k) = \Big(\frac{p}{q}\Big)^k \tag{1.2.91}$$

for $k \ge 0$. Hence

$$\mathsf{E}\, M = \frac{q}{q-p}. \tag{1.2.92}$$

2. Continuous time

The aim of the present section is to exhibit basic results of optimal stopping in the case of continuous time. We first consider a martingale approach (cf. Subsection 1.1 above). This is then followed by a Markovian approach (cf. Subsection 1.2 above).

2.1. Martingale approach

1. Let $G = (G_t)_{t \ge 0}$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathsf{P})$. We interpret G_t as the *gain* if the observation of G is stopped at time t . It is assumed that G is adapted to the filtration $(\mathcal{F}_t)_{t \ge 0}$ in the sense that each G_t is $\mathcal{F}_t$ -measurable. Recall that each $\mathcal{F}_t$ is a σ -algebra of subsets of Ω such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \le t$. Typically $(\mathcal{F}_t)_{t \ge 0}$ coincides with the natural filtration $(\mathcal{F}_t^G)_{t \ge 0}$ but generally may also be larger. We interpret $\mathcal{F}_t$ as the *information* available up to time t . All our decisions in regard to optimal