Lecture notes

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Chapter 1

Finite-type arithmetic

Higher types and higher-type entities (functionals) constitute a natural, and constructive, way of extending the expressive power of arithmetic without increasing its proof-theoretic strength. They also provide the syntactic means to express the effective information contained in proofs of arithmetical statements.

As a foundation for our subsequent proof-theoretic considerations, we present a basic theory $\mathbf{HA}^{\boldsymbol{\omega}}$ of intuitionistic finite-type arithmetic, together with an extensional variant $\mathbf{E\text{-}HA}^{\boldsymbol{\omega}}$ and an intensional one $\mathbf{I\text{-}HA}^{\boldsymbol{\omega}}$.

Based in part on Troelstra and van Dalen (1988).

1.1 Syntax

1.1.1 The language of finite-type arithmetic

The following types are present:

1. An atomic type $\mathcal{N}$ (the type of natural numbers),
2. a type $\sigma \times \tau$ for any two types σ and τ (product types),
3. a type τ^σ for any two types σ and τ (function types).

Notation. $(\tau^\sigma)^\rho$ is simplified to $\tau^{\rho\sigma}$, and $\tau^{\vec{\sigma}}$ is governed by a similar convention; hence, τ^{σ^ρ} denotes the other alternative.

Terms, and their types, are generated by

0. There is an inexhaustible supply (infinite set) of variables of each type.
1. $\mathbf{0}$ is a term of type $\mathcal{N}$; for any term t of type $\mathcal{N}$, $\boldsymbol{S}t$ is a term of type $\mathcal{N}$.
2. For any terms t of type τ, u of type $\tau^{\mathcal{N}\tau}$ and v of type $\mathcal{N}$, $\boldsymbol{R}tuv$ is a term of type τ.
3. For any terms t_l and t_r of types τ_l and τ_r, $\langle t_l, t_r\rangle$ is a term of type $\tau_l \times \tau_r$.
4. For any term t of type $\tau_l \times \tau_r$, $\boldsymbol{p}_s t$ is a term of type τ_s, for $s \in \{l, r\}$.
5. For any variable x of type σ and term t of type τ, $\lambda x\, t$ is a term of type τ^σ.

6. For any terms t of type τ^σ and u of type σ, tu is a term of type τ.

Notation. Lists of variables/terms are conveniently abbreviated $\vec{x}$, $\vec{t}$ etc., with the aid of the following conventions (where $\vec{t} \equiv t_1, \dots, t_n$ and $\vec{u} \equiv u_1, \dots, u_m$):

$$\begin{aligned} t\vec{u} &\equiv tu_1 \cdots u_m \equiv (\cdots(tu_1)\cdots)u_m, \\ \vec{t}\vec{u} &\equiv t_1\vec{u}, \dots, t_n\vec{u}, \\ \lambda x_1, \dots, x_m\ t &\equiv \lambda x_1 \cdots \lambda x_m\ t, \\ \lambda\vec{x}\ \vec{t} &\equiv \lambda\vec{x}\ t_1, \dots, \lambda\vec{x}\ t_n. \end{aligned}$$

Prime (or atomic) formulae are equations $t = u$ between terms of the same type. *Formulae* are formed from prime formulae by means of &, $\to$, $\forall$, and $\exists$. We will use the following abbreviations:

$$\begin{aligned} \mathbf{1} &\equiv \boldsymbol{S}\mathbf{0}, \\ \bot &\equiv \mathbf{0} = \mathbf{1}, \\ \neg\phi &\equiv \phi \to \bot, \\ t \neq u &\equiv \neg(t = u), \\ \phi \vee \psi &\equiv \exists z\ ((z = \mathbf{0} \to \phi)\ \&\ (z \neq \mathbf{0} \to \psi)). \end{aligned}$$

Prefix operators take precedence over infix ones, e.g., $\forall x\ \phi \to \psi$ is $(\forall x\ \phi) \to \psi$. Parentheses may also be omitted around function arguments and whenever the association is uniquely determined by the rules of typing/term formation.

1.1.2 Axioms and rules of inference

Besides the usual (natural deduction or other) axioms and rules of intuitionistic first-order logic for the logical constants present in the system, there are rules for *equality*

$$\frac{}{t = t} \qquad\qquad \frac{t = u \qquad \phi(t)}{\phi(u)}\ ,$$

β-conversion

$$\frac{}{\boldsymbol{R}tu\mathbf{0} = t} \qquad\qquad \frac{}{\boldsymbol{R}tu\boldsymbol{S}v = uvRtuv}\ ,$$

$$\frac{}{\boldsymbol{p}_i\langle t_l, t_r\rangle = t_i}\ ,\ \ i \in \{l, r\},$$

$$\frac{}{(\lambda x\ t)u = t[x := u]}\ ,$$

and *induction*

$$\frac{\phi(\mathbf{0}) \qquad \forall x\ (\phi(x) \to \phi(\boldsymbol{S}x))}{\phi(v)}\ .$$

The above axioms and rules constitute $\mathbf{HA}^{\boldsymbol{\omega}}$. Occasionally, we will be interested in the following variants of this theory. *Intensional finite-type arithmetic*,

I-HA$^{\omega}$, augments the language of **HA$^{\omega}$** with *equality functionals* $\boldsymbol{E}_\tau$, one for each type τ, subject to

$$\frac{}{\boldsymbol{E}_\tau tu = \mathbf{0} \leftrightarrow t = u} \qquad\qquad \frac{}{\boldsymbol{E}_\tau tu = \mathbf{1} \leftrightarrow t \neq u} \;.$$

Extensional finite-type arithmetic, **E-HA$^{\omega}$**, is obtained from **HA$^{\omega}$** by the addition of the *extensionality rules*

$$\frac{\boldsymbol{p}_l t = \boldsymbol{p}_l u \qquad \boldsymbol{p}_r t = \boldsymbol{p}_r u}{t = u}$$

for t, u of product type, and

$$\frac{\forall x\ (tx = ux)}{t = u}\ ,\ \ x \notin \mathrm{FV}(t,u)$$

for t, u of function type. *Classical (or Peano) finite-type arithmetic* **PA$^{\omega}$** is the extension of **HA$^{\omega}$** by the *principle of the excluded middle*

$$\frac{}{\neg\neg\phi \to \phi}\ (\mathbf{PEM})\ .$$

1.2 Exercises

1. Show that equality at type $\mathcal{N}$ is decidable, i.e., $x = y \vee x \neq y$.
2. Using your preferred logical formalism, show that the set of theorems of **HA$^{\omega}$** is closed under substitution, i.e., if $\vdash_{\mathbf{HA}^{\omega}} \phi$, then $\vdash_{\mathbf{HA}^{\omega}} \phi[x := t]$.
3. Prove that extensionality is equivalent to the set of equations

$$(\eta) \qquad \begin{array}{c} \langle \boldsymbol{p}_l t, \boldsymbol{p}_r t\rangle = t,\ \ t \text{ of product type,} \\ \lambda x\ (tx) = t,\ \ t \text{ of function type, } x \notin \mathrm{FV}(t). \end{array}$$

4. Extensional equality $t =_e u$ between terms t, u of the same type is inductively defined by

$$t =_e u \equiv \left\{ \begin{array}{cl} t = u & t,\ u \text{ of atomic type,} \\ \boldsymbol{p}_l t =_e \boldsymbol{p}_l u \ \&\ \boldsymbol{p}_r t =_e \boldsymbol{p}_r u & t,\ u \text{ of product type,} \\ \forall x\ (tx =_e ux) & t,\ u \text{ of function type.} \end{array} \right.$$

   Show that extensionality is equivalent to the schema

$$t =_e u \leftrightarrow t = u,$$

   and conclude that, in **E-HA$^{\omega}$**, atomic formulae are equivalent to purely universal formulae involving equality at type $\mathcal{N}$ only.

5. (Closure of **HA$^{\omega}$** under *mutual primitive recursion.*) Let $\vec{\tau} \equiv \tau_1, \ldots, \tau_n$ be a list of types, $\vec{t}$ a list of terms of types $\vec{\tau}$ (i.e., each t_i has type τ_i) and $\vec{u}$ a list of terms of types $\vec{\tau}^{\mathcal{N}\vec{\tau}}$ (i.e., each u_i has type ${\tau_i}^{\mathcal{N}\tau_1\cdots\tau_n}$). Construct terms $\vec{r} \equiv \vec{r}(z)$, z fresh, with the properties

$$\vec{r}(\mathbf{0}) = \vec{t},$$
$$\vec{r}(\boldsymbol{S}v) = \vec{u}v\vec{r}(v).$$

Chapter 2

Modified realizability

The name *realizability* refers to any one of a family of translations that may be seen as formalizations of the BHK interpretation of the logical constants; for a more complete description of the BHK interpretation, the reader may consult Troelstra and van Dalen (1988).

Modified realizability is a variant of realizability introduced in Kreisel (1959) for the purpose of showing that Markov's principle is not derivable in intuitionistic logic. It could as well be called *typed realizability* because it uses functionals instead of numbers as realizing objects. This notion of realizability is well adapted to the study of typed theories; it will be our first, and simplest, example of term extraction.

2.1 Definition

To each formula ϕ in the language of finite-type arithmetic we associate its *modified realizability interpretation* ϕ^{mr}, which is a formula of the form

$$\exists \vec{x}\ \phi_{\mathrm{mr}}(\vec{x})$$

with the same free variables as ϕ, where $\phi_{\mathrm{mr}}(\vec{x})$ ($\vec{x}$ *modified realizes* ϕ, alternative notation: $\vec{x}\mathbf{mr}\phi$) is an $\exists$-free formula and $\vec{x}$ a possibly empty list of variables. The associations $(\ )_{\mathrm{mr}}$ and $(\ )^{\mathrm{mr}}$ are defined by the following induction:

$$\begin{aligned}
\phi^{\mathrm{mr}} &\equiv \phi \ \text{ for } \phi \text{ atomic},\\
(\phi \ \&\ \psi)^{\mathrm{mr}} &\equiv \exists \vec{x}, \vec{y}\ \Big[\phi_{\mathrm{mr}}(\vec{x}) \ \&\ \psi_{\mathrm{mr}}(\vec{y})\Big],\\
(\phi \to \psi)^{\mathrm{mr}} &\equiv \exists \vec{Y}\ \Big[\forall \vec{x}\ (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{Y}\vec{x}))\Big],\\
(\forall z\ \phi(z))^{\mathrm{mr}} &\equiv \exists \vec{X}\ \Big[\forall z\ (\phi(z)_{\mathrm{mr}}(\vec{X}z))\Big],\\
(\exists z\ \phi(z))^{\mathrm{mr}} &\equiv \exists z, \vec{x}\ \Big[\phi(z)_{\mathrm{mr}}(\vec{x})\Big],
\end{aligned}$$

where, in each case, the $\exists$-free kernel is delimited by brackets.

Remark on notation. Expressions like $\phi(v)_{\mathrm{mr}}(\vec{x})$ are unambiguous, since the interpretation commutes with substitution:

$$\phi[z := v]_{\mathrm{mr}}(\vec{x}) \equiv \phi_{\mathrm{mr}}(\vec{x})[z := v].$$

Proposition 2.1. *Let* $\phi^{\mathrm{mr}} \equiv \exists \vec{x}\ \phi_{\mathrm{mr}}(\vec{x})$.

1. $\phi_{\mathrm{mr}}(\vec{x})$ *is* $\exists$*-free, and if* ϕ *is* $\exists$*-free, then* $\vec{x}$ *is empty and* $\phi^{\mathrm{mr}} \equiv \phi_{\mathrm{mr}} \equiv \phi$.
2. *If* ψ *is* $\exists$*-free, then* $(\exists \vec{y}\ \psi)^{\mathrm{mr}} \equiv \exists \vec{y}\ \psi$*; in particular,* $(\phi^{\mathrm{mr}})^{\mathrm{mr}} \equiv \phi^{\mathrm{mr}}$.

Proof. Exercise 3. □

2.2 Soundness & term extraction

Theorem 2.2 (soundness)**.** *Let* $\boldsymbol{H}$ *be any one of* $\boldsymbol{HA^\omega}$, $\boldsymbol{E\text{-}HA^\omega}$, $\boldsymbol{I\text{-}HA^\omega}$*, and let* $\boldsymbol{H}{-}\exists$ *be the* $\exists$*-free part of* $\boldsymbol{H}$*. If* $\vdash_{\boldsymbol{H}} \phi$*, then* $\vdash_{\boldsymbol{H}-\exists} \phi_{\mathrm{mr}}(\vec{t})$ *for a suitable list* $\vec{t}$ *of terms satisfying* $\mathrm{FV}(\vec{t}) \subseteq \mathrm{FV}(\phi)$.

Proof. We are going to apply induction on the proofs of **H**, for the purpose of which we will need the (superficially) stronger statement

> If $\Phi \vdash_{\mathbf{H}} \phi$, then $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H}-\exists} \phi_{\mathrm{mr}}(\vec{t})$, where all free variables of $\vec{t}$ are among those free in ϕ and those free in Φ_{mr}.

where Φ is an arbitrary (finite) set of formulae and $\Phi_{\mathrm{mr}} = \{\phi_{\mathrm{mr}} \mid \phi \in \Phi\}$. Of the axioms and rules of **H**, those that are $\exists$-free are *self-realizing* and don't need any further examination; this includes the "extras" of $\mathbf{E\text{-}HA^\omega}$ and $\mathbf{I\text{-}HA^\omega}$. For most of the others, a deduction will be furnished that may be combined with the induction hypotheses in an obvious way to yield the required conclusion. Exception: $\exists$-rules.

Natural deduction

$\phi \in \Phi$: Then $\phi_{\mathrm{mr}}(\vec{x}) \in \Phi_{\mathrm{mr}}$, whence $\Phi_{\mathrm{mr}} \vdash \phi_{\mathrm{mr}}(\vec{x})$.

$$\frac{\phi \qquad \psi}{\phi \,\&\, \psi} : \qquad \frac{\phi_{\mathrm{mr}}(\vec{t}) \qquad \psi_{\mathrm{mr}}(\vec{u})}{\phi_{\mathrm{mr}}(\vec{t}) \,\&\, \psi_{\mathrm{mr}}(\vec{u}) \equiv (\phi \,\&\, \psi)_{\mathrm{mr}}(\vec{t},\vec{u})}$$

$$\frac{\phi \,\&\, \psi}{\phi} : \qquad (\phi \,\&\, \psi)_{\mathrm{mr}}(\vec{t},\vec{u}) \equiv \frac{\phi_{\mathrm{mr}}(\vec{t}) \,\&\, \psi_{\mathrm{mr}}(\vec{u})}{\phi_{\mathrm{mr}}(\vec{t})}$$

$$\frac{\begin{matrix}[\phi]\\ \psi\end{matrix}}{\phi \to \psi} : \qquad \cfrac{\cfrac{\begin{matrix}[\phi_{\mathrm{mr}}(\vec{x})]\\ \psi_{\mathrm{mr}}(\vec{u})\end{matrix}}{\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{u})}}{\forall \vec{x}\ (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{u})) \leftrightarrow (\phi \to \psi)_{\mathrm{mr}}(\lambda \vec{x}\ \vec{u})}$$

$$\frac{\phi \to \psi \qquad \phi}{\psi} : \qquad (\phi \to \psi)_{\mathrm{mr}}(\vec{t}) \equiv \cfrac{\cfrac{\forall \vec{x}\ (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{t}\vec{x}))}{\phi_{\mathrm{mr}}(\vec{u}) \to \psi_{\mathrm{mr}}(\vec{t}\vec{u})} \qquad \phi_{\mathrm{mr}}(\vec{u})}{\psi_{\mathrm{mr}}(\vec{t}\vec{u})}$$

$$\frac{\phi(z)}{\forall z\ \phi(z)} : \qquad \frac{\phi(z)_{\mathrm{mr}}(\vec{t})}{\forall z\ (\phi(z)_{\mathrm{mr}}(\vec{t})) \leftrightarrow (\forall z\ \phi(z))_{\mathrm{mr}}(\lambda z\ \vec{t})}$$

$$\frac{\forall z\ \phi(z)}{\phi(v)} : \qquad (\forall z\ \phi(z))_{\mathrm{mr}}(\vec{t}) \equiv \frac{\forall z\ (\phi(z)_{\mathrm{mr}}(\vec{t}z))}{\phi(v)_{\mathrm{mr}}(\vec{t}v)}$$

$\dfrac{\phi(v)}{\exists z\,\phi(z)}$: Nothing to prove; the conclusion coincides with the induction hypothesis (this is because the interpretation of $\exists$ is "trivial", in the sense that it merely converts the existentially quantified variable into a realizing variable).

$\dfrac{\exists z\,\phi(z) \qquad \begin{matrix}[\phi(z)]\\ \psi\end{matrix}}{\psi}$: By hypothesis, there are deductions $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H-\exists}} \phi(v)_{\mathrm{mr}}(\vec{t})$ and $\Phi_{\mathrm{mr}}, \phi(z)_{\mathrm{mr}}(\vec{x}) \vdash_{\mathbf{H-\exists}} \psi_{\mathrm{mr}}(\vec{u})$, whence $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H-\exists}} \psi_{\mathrm{mr}}(\vec{u}[\vec{x} := \vec{t}])$.

Equality

$\dfrac{t = u \qquad \phi(t)}{\phi(u)}$: $\qquad \dfrac{t = u \qquad \phi(t)_{\mathrm{mr}}(\vec{v})}{\phi(u)_{\mathrm{mr}}(\vec{v})}$

(using the fact that modified realizability commutes with substitution).

Induction

$\dfrac{\phi(\mathbf{0}) \qquad \forall z\,(\phi(z) \to \phi(\boldsymbol{S}z))}{\phi(v)}$:

$$\dfrac{\phi(\mathbf{0})_{\mathrm{mr}}(\vec{t}) \qquad \dfrac{\forall z, \vec{x}\,(\phi(z)_{\mathrm{mr}}(\vec{x}) \to \phi(\boldsymbol{S}z)_{\mathrm{mr}}(\vec{u}z\vec{x}))}{\forall z\,(\phi(z)_{\mathrm{mr}}(\vec{w}(z)) \to \phi(\boldsymbol{S}z)_{\mathrm{mr}}(\vec{u}z\vec{w}(z)))}}{\phi(v)_{\mathrm{mr}}(\vec{w}(v))}$$

where $\vec{w} \equiv \vec{w}(z)$ is a list of terms such that

$$\vec{w}(\mathbf{0}) = \vec{t},$$
$$\vec{w}(\boldsymbol{S}z) = \vec{u}z\vec{w}(z).$$

One way to guarantee the existence of $\vec{w}(z)$ is by formulating the system with *mutual primitive recursion*. In the presence of product types, however, mutual primitive recursion is reducible to ordinary primitive recursion; e.g., $\vec{w}(z)$ may be constructed as follows: Assuming $\vec{t} \equiv t_1, \ldots, t_n$ and $\vec{u} \equiv u_1, \ldots, u_n$, define

$$\boldsymbol{t} \equiv \langle \vec{t} \rangle \equiv \langle t_1, \ldots, t_n \rangle,$$
$$\boldsymbol{u} \equiv \lambda z, y\, \langle \vec{u}z(\vec{q}y) \rangle,$$

where $\langle\ \rangle$ denotes an arbitrary representation of n-tuples (using pairing), with corresponding projections $\vec{q}y \equiv q_1 y, \ldots, q_n y$. Then, the terms

$$\vec{w}(z) \equiv \vec{q}\boldsymbol{R}\boldsymbol{t}\boldsymbol{u}z$$

have the required properties. □

2.3 Axiomatization

Lemma 2.3. *For each instance θ of one of the schemata*

(AC) $\qquad \forall \vec{x}\,\exists \vec{y}\,\phi(\vec{x}, \vec{y}) \to \exists \vec{Y}\,\forall \vec{x}\,\phi(\vec{x}, \vec{Y}\vec{x})$,

($\mathrm{IP}^{\omega}_{\mathrm{ef}}$) $\qquad (\psi \to \exists \vec{x}\,\phi) \to \exists \vec{x}\,(\psi \to \phi)$, ψ $\exists$-*free*,

there are terms $\vec{t}$ such that $\vdash_{\boldsymbol{HA}^{\omega}-\exists} \theta_{\mathrm{mr}}(\vec{t})$.

Proof. Exercise 4. □

Theorem 2.4. *The following are identical theories:*

1. $\boldsymbol{HA^\omega} + \{\phi^{\mathrm{mr}} \leftrightarrow \phi\}$,
2. $\boldsymbol{HA^\omega} + \{\phi^{\mathrm{mr}} \to \phi\}$,
3. $\{\phi \mid \vdash_{\boldsymbol{HA^\omega}} \phi^{\mathrm{mr}}\}$,
4. $\{\phi \mid \vdash_{\boldsymbol{HA^\omega}-\exists} \phi_{\mathrm{mr}}(\vec{t})$ *with* $\vec{t}$ *as in theorem 2.2*$\}$,
5. $\boldsymbol{HA^\omega} + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}}$,
6. *the preceding one, with* AC *and* $\mathrm{IP}^\omega_{\mathrm{ef}}$ *restricted to* $\exists$*-free* ϕ.

Proof. We show that the last theory implies (i.e. includes) the first; the other implications are easy. We proceed by induction on ϕ:

$$\phi^{\mathrm{mr}} \leftrightarrow \phi \text{ for } \phi \text{ atomic.}$$

$$\begin{aligned}
(\phi \,\&\, \psi)^{\mathrm{mr}} &\leftrightarrow \exists \vec{x}, \vec{y}\, (\phi_{\mathrm{mr}}(\vec{x}) \,\&\, \psi_{\mathrm{mr}}(\vec{y})) \\
&\leftrightarrow (\exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})) \,\&\, (\exists \vec{y}\, \psi_{\mathrm{mr}}(\vec{y})) \\
&\leftrightarrow \phi^{\mathrm{mr}} \,\&\, \psi^{\mathrm{mr}} \\
&\leftrightarrow \phi \,\&\, \psi.
\end{aligned}$$

$$\begin{aligned}
(\phi \to \psi)^{\mathrm{mr}} &\leftrightarrow \exists \vec{Y}\, \forall \vec{x}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{Y}\vec{x})) \\
&\leftrightarrow \forall \vec{x}\, \exists \vec{y}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{y})) && [\mathrm{AC}] \\
&\leftrightarrow \forall \vec{x}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \exists \vec{y}\, \psi_{\mathrm{mr}}(\vec{y})) && [\mathrm{IP}^\omega_{\mathrm{ef}}] \\
&\leftrightarrow (\exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})) \to (\exists \vec{y}\, \psi_{\mathrm{mr}}(\vec{y})) \\
&\leftrightarrow \phi^{\mathrm{mr}} \to \psi^{\mathrm{mr}} \\
&\leftrightarrow \phi \to \psi.
\end{aligned}$$

$$\begin{aligned}
(\forall z\, \phi(z))^{\mathrm{mr}} &\leftrightarrow \exists \vec{X}\, \forall z\, (\phi(z)_{\mathrm{mr}}(\vec{X}z)) \\
&\leftrightarrow \forall z\, \exists \vec{x}\, (\phi(z)_{\mathrm{mr}}(\vec{x})) && [\mathrm{AC}] \\
&\leftrightarrow \forall z\, \phi(z)^{\mathrm{mr}} \\
&\leftrightarrow \forall z\, \phi(z).
\end{aligned}$$

$$\begin{aligned}
(\exists z\, \phi(z))^{\mathrm{mr}} &\leftrightarrow \exists z, \vec{x}\, (\phi(z)_{\mathrm{mr}}(\vec{x})) \\
&\leftrightarrow \exists z\, (\phi(z)^{\mathrm{mr}}) \\
&\leftrightarrow \exists z\, \phi(z).
\end{aligned}$$

□

Corollary 2.5. $\boldsymbol{HA^\omega} + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}}$ *has the existence property.*

Proof. Assume $\exists z\, \phi(z) \in \mathbf{HA^\omega} + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}} = \{\phi \mid \vdash_{\mathbf{HA^\omega}-\exists} \phi_{\mathrm{mr}}(\vec{t})\}$. Then, $\phi(v)_{\mathrm{mr}}(\vec{t}) \in \mathbf{HA^\omega}-\exists$ for suitable terms $v, \vec{t}$, whence $\phi(v)^{\mathrm{mr}} \in \mathbf{HA^\omega}$, and finally $\phi(v) \in \mathbf{HA^\omega} + \{\phi^{\mathrm{mr}} \to \phi\} = \mathbf{HA^\omega} + \mathrm{AC} + \mathrm{IP}^\omega_{\mathrm{ef}}$. □

2.4 Exercises

1. Prove the following:
   (a) $(\forall \vec{z}\ \phi(\vec{z}))^{\mathrm{mr}} \equiv \exists \vec{X}\ \forall \vec{z}\ (\phi(\vec{z})_{\mathrm{mr}}(\vec{X}\vec{z}))$.
   (b) $(\exists \vec{z}\ \phi(\vec{z}))^{\mathrm{mr}} \equiv \exists \vec{z}, \vec{x}\ (\phi(\vec{z})_{\mathrm{mr}}(\vec{x}))$.

2. Show that modified realizability commutes with substitution,

$$\phi[x := t]^{\mathrm{mr}} \equiv \phi^{\mathrm{mr}}[x := t].$$

3. Prove proposition 2.1.

4. Prove lemma 2.3.

5. *Harrop* formulae are defined by the induction
   (a) atomic formulae are Harrop,
   (b) if ϕ and ψ are Harrop, then $\phi\ \&\ \psi$ is Harrop,
   (c) if ψ is Harrop, then $\phi \to \psi$ is Harrop (ϕ any formula),
   (d) if ϕ is Harrop, then $\forall x\ \phi$ is Harrop.

   Prove that a formula ϕ is Harrop if and only if ϕ^{mr} is $\exists$-free, i.e., if in $\phi^{\mathrm{mr}} \equiv \exists \vec{x}\ \phi_{\mathrm{mr}}(\vec{x})$, $\vec{x}$ is the empty list.

6. Show that for any instance θ of schema

   $(\mathrm{IP}^{\omega}_{\mathrm{Harrop}})$ $\qquad (\phi \to \exists x\ \psi) \to \exists x\ (\phi \to \psi), \ \ \phi$ Harrop

   there are terms $\vec{t}$ such that $\vdash_{\mathbf{HA}^{\omega}-\exists} \theta_{\mathrm{mr}}(\vec{t})$.

7. Expand $(\neg\neg\phi \to \phi)^{\mathrm{mr}}$ and show that it is provable in $\mathbf{PA}^{\omega}$. Conclude that if $\vdash_{\mathbf{PA}^{\omega}} \phi$, then $\vdash_{\mathbf{PA}^{\omega}} \phi^{\mathrm{mr}}$ (soundness for $\mathbf{PA}^{\omega}$).

Chapter 3

Functional interpretation

This chapter is loosely based on Diller and Nahm (1974).

3.1 Definition and elementary properties

3.1.1 Bounded universal quantification

Bounded universal quantification is generally a finitistic operation on formulae, in contrast to its usual definition,

$$\forall z{<}v\ \phi(z) \equiv \forall z\ (z < v \to \phi(z)), \tag{3.1}$$

which employs unrestricted quantification. For the purpose of making sense of bounded universal quantification in quantifier-free settings below, we will treat the bounded universal quantifier as a primitive logical constant, with introduction rules

$$\frac{}{\forall z{<}\mathbf{0}\ \phi(z)} \qquad\qquad \frac{\forall z{<}v\ \phi(z) \qquad \phi(v)}{\forall z{<}\boldsymbol{S}v\ \phi(z)} \tag{3.2}$$

and elimination rule

$$\frac{\forall z{<}v\ \phi(z) \qquad \psi(\mathbf{0}) \qquad \begin{matrix}[\phi(z)]\ \ [\psi(z)]\\ \psi(\boldsymbol{S}z)\end{matrix}}{\psi(v)}\quad, \tag{3.3}$$

where, in the last rule, z may not occur in any open assumptions. Accordingly, a formula is *quantifier-free* if it does not contain any *unbounded* quantifiers.

3.1.2 The interpretation

We let $\mathbf{T}$ be the quantifier-free fragment of $\mathbf{HA}^{\boldsymbol{\omega}}$ (with the induction rule adapted as appropriate), and we define $\mathbf{T}_\wedge$ to be $\mathbf{T}$ augmented with bounded universal quantification. The *Diller-Nahm interpretation* $\phi^\wedge$ of a formula ϕ in the language of $\mathbf{HA}^{\boldsymbol{\omega}}$ is a formula of the form

$$\exists\vec{x}\ \forall\vec{y}\ \phi_\wedge(\vec{x},\vec{y})$$

with the same free variables as ϕ, where $\phi_\wedge(\vec{x},\vec{y})$ is a formula of $\mathbf{T}_\wedge$ and $\vec{x}$, $\vec{y}$ are possibly empty lists of variables. The associations $(\ )^\wedge$ and $(\ )_\wedge$ are inductively defined by

$$\phi^\wedge \equiv \phi \text{ for } \phi \text{ atomic},$$

$$(\phi \,\&\, \phi')^\wedge \equiv \exists \vec{x}\vec{x}'\, \forall \vec{y}\vec{y}' \left[\phi_\wedge(\vec{x},\vec{y}) \,\&\, \phi'_\wedge(\vec{x}',\vec{y}')\right],$$

$$(\phi \to \phi')^\wedge \equiv \begin{cases} \exists Z\vec{X}\vec{Y}\, \forall \vec{x}\vec{y} \left[\forall z{<}Z\vec{x}\vec{y}\, \phi_\wedge(\vec{x},\vec{Y}\vec{x}\vec{y}z) \to \phi'_\wedge(\vec{X}\vec{x},\vec{y})\right], & \vec{Y} \text{ non-nil},\\ \exists \vec{X}\, \forall \vec{x}\vec{y} \left[\phi_\wedge(\vec{x},) \to \phi'_\wedge(\vec{X}\vec{x},\vec{y})\right], & \text{otherwise}, \end{cases}$$

$$(\forall z\, \phi(z))^\wedge \equiv \exists \vec{X}\, \forall \vec{y}z \left[\phi(z)_\wedge(\vec{X}z,\vec{y})\right],$$

$$(\exists z\, \phi(z))^\wedge \equiv \exists z\vec{x}\, \forall \vec{y} \left[\phi(z)_\wedge(\vec{x},\vec{y})\right].$$

For the bounded universal quantifier, one may optionally add

$$(\forall z{<}v\, \phi(z))^\wedge \equiv \exists \vec{X}\, \forall \vec{y} \left[\forall z{<}v\, \phi(z)_\wedge(\vec{X}z,\vec{y})\right].$$

This clause is consistent with definition (3.1); its sole purpose is to provide for bounded universal quantification as a primitive, so that $\mathbf{T}_\wedge$ may be construed as a subsystem of $\mathbf{HA}^{\boldsymbol{\omega}}$. Similarly, the two branches in the definition of $(\phi \to \phi')^\wedge$ are equivalent in case $\vec{Y}$ is the empty list, whence the first, more general one suffices for both cases, and we will tacitly assume this simpler definition. With this case distinction, however, the formulae of $\mathbf{T}_\wedge$ are translated onto themselves:

Proposition 3.1. *Let $\phi^\wedge \equiv \exists \vec{x}\, \forall \vec{y}\, \phi_\wedge(\vec{x},\vec{y})$.*

1. *$\phi_\wedge(\vec{x},\vec{y})$ is q.f., and if ϕ is q.f., then $\vec{x}$, $\vec{y}$ are empty and $\phi^\wedge \equiv \phi_\wedge \equiv \phi$.*
2. *If ψ is q.f., then $(\exists \vec{x}\, \forall \vec{y}\, \psi)^\wedge \equiv \exists \vec{x}\, \forall \vec{y}\, \psi$; in particular, $(\phi^\wedge)^\wedge \equiv \phi^\wedge$.*

Proof. Exercise 4. □

3.2 Soundness & term extraction

Theorem 3.2 (soundness). *If $\vdash_{\boldsymbol{HA^\omega}} \phi$, then $\vdash_{\boldsymbol{T}_\wedge} \phi_\wedge(\vec{t},\vec{y})$ for suitable terms $\vec{t}$ in which $\vec{y}$ do not occur.*

Proof. In the following, $\vdash$ will denote provability in $\mathbf{T}_\wedge$. For the purpose of applying induction on $\mathbf{HA}^{\boldsymbol{\omega}}$-derivations, we will prove that if $\{\phi^i\}_{i\in I} \vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \phi$, then $\{\forall w{<}V^i\, \phi^i_\wedge(\vec{x}^i,\vec{U}^i w)\}_{i\in I} \vdash \phi_\wedge(\vec{t},\vec{y})$ for suitable terms $(V^i)_{i\in I}, (\vec{U}^i)_{i\in I}, \vec{t}$, with $\vec{y}$ not occuring in $\vec{t}$.

Some preparation: Let $\mathcal{A}$ be the collection of assumption sets of the form $\{\forall w{<}V^i\, \phi^i_\wedge(\vec{x}^i,\vec{U}^i w)\}_{i\in I}$ for all possible choices of $(V^i)_{i\in I}, (\vec{U}^i)_{i\in I}$. The three properties of $\mathcal{A}$ stated in the following lemmata will allow an almost complete "algebraisation" of the proof.

Lemma 3.3. *$\mathcal{A}$ is closed under $\vec{y}$-substitution, i.e., $\Gamma \in \mathcal{A} \Rightarrow \Gamma[\vec{y} := \vec{u}] \in \mathcal{A}$.*

Lemma 3.4. *Given a formula $\psi(\vec{x})$, there are terms $v(z_1, z_2), \vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2)$ such that*

$$\forall w{<}v(z_1, z_2)\ \psi(\vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2)w) \vdash \forall w{<}z_i\ \psi(\vec{y}_i w),\ \ i = 1, 2.$$

In particular, assumption sets may be merged, i.e., for $\Gamma_1, \Gamma_2 \in \mathcal{A}$ there is $\Gamma \in \mathcal{A}$ such that $\Gamma \vdash \Gamma_1$ and $\Gamma \vdash \Gamma_2$.

[Proof hint: Let $v(z_1, z_2) \equiv z_1 + z_2$. $\vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2)$ may be defined as

$$\vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2) \equiv \lambda w \text{ if } z_1 \dot{-} w = \mathbf{0} \text{ then } \vec{y}_2(w \dot{-} z_1) \text{ else } \vec{y}_1 w,$$

or by the primitive recursion

$$\begin{aligned} \vec{u}(\vec{y}_1, \vec{y}_2, z_1, \mathbf{0}) &= \vec{y}_1, \\ \vec{u}(\vec{y}_1, \vec{y}_2, z_1, \boldsymbol{S}z_2) &= \lambda w \text{ if } w = v(z_1, z_2) \text{ then } \vec{y}_2 z_2 \text{ else } \vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2)w. \end{aligned}$$]

For an arbitrary set Φ of formulae, let $\forall z{<}v\ \Phi = \{\forall z{<}v\ \phi \mid \phi \in \Phi\}$.

Lemma 3.5. *Let j, j_1, j_2 satisty $\vdash j_i(j(x_1, x_2)) = x_i$ for $i = 1, 2$. Given terms v, v' there is a term b such that*

$$\forall w{<}b\ \phi(j_1 w, j_2 w) \vdash \forall w{<}v\ \forall w'{<}v'\ \phi(w, w').$$

In particular, for any term v of type $\mathcal{N}$ and $\Gamma \in \mathcal{A}$ there is $\Gamma' \in \mathcal{A}$ such that $\Gamma' \vdash \forall w{<}v\ \Gamma$.

[As concerns the applicability of the lemma, let us mention that there are well-known primitive recursive pairing functions, e.g. $\frac{1}{2}((x+y)^2 + 3x + y)$ or $2^x 3^y$. Proof hint: Define

$$\begin{aligned} t(\mathbf{0}) &= \mathbf{0}, \\ t(\boldsymbol{S}w) &= \max\{t(w), \boldsymbol{S}j(v, w)\} \end{aligned}$$

and

$$\begin{aligned} u(\mathbf{0}) &= \mathbf{0}, \\ u(\boldsymbol{S}w) &= \max\{u(w), t(v')\}. \end{aligned}$$

The required term is $u(v)$.]

To the induction. We will examine the more interesting cases, leaving the verification of the other ones as a (relatively trivial) exercise.

Case $\phi \equiv \phi^{i_0}$: Take $V^{i_0} \equiv \mathbf{1}$, $\vec{U}^{i_0} \equiv \lambda w\ \vec{y}$ and $\vec{t} \equiv \vec{x}^{i_0}$.

Case $\dfrac{\phi \quad \phi'}{\phi \ \&\ \phi'}$: Use lemma 3.4.

Case $\dfrac{\begin{matrix}[\phi] \\ \phi'\end{matrix}}{\phi \to \phi'}$: The induction hypothesis provides us with terms $V, \vec{U}, \vec{t}$ such that $\forall w{<}V\ \phi_\wedge(\vec{x}, \vec{U}w) \to \phi'_\wedge(\vec{t}, \vec{y})$, or, equivalently,

$$(\phi \to \phi')_\wedge(\lambda\vec{x}\ \lambda\vec{y}\ V, \lambda\vec{x}\ \vec{t}, \lambda\vec{x}\ \lambda\vec{y}\ \vec{U}; \vec{x}, \vec{y}).$$

Case $\dfrac{\phi \to \phi' \qquad \phi}{\phi'}$: The induction hypotheses are

$$\Gamma' \vdash \forall w{<}V\vec{x}\vec{y}'\ \phi_\wedge(\vec{x}, \vec{U}\vec{x}\vec{y}'\vec{w}) \to \phi'_\wedge(\vec{t}'\vec{x}, \vec{y}') \tag{3.4}$$

and

$$\Gamma'' \vdash \phi_\wedge(\vec{t}, \vec{y}). \tag{3.5}$$

Substituting $\vec{t}$ for $\vec{x}$ in (3.4), we obtain (using lemma 3.3)

$$\Gamma''' \vdash \forall w{<}V\vec{t}\vec{y}'\ \phi_\wedge(\vec{t}, \vec{U}\vec{t}\vec{y}'w) \to \phi'_\wedge(\vec{t}'\vec{t}, \vec{y}'). \tag{3.6}$$

Substituting $\vec{U}\vec{t}\vec{y}'w$ for $\vec{y}$ in (3.5) and quantifying both sides with $\forall w{<}V\vec{t}\vec{y}'$, we obtain (using lemmata 3.3 and 3.5)

$$\Gamma'''' \vdash \forall w{<}V\vec{t}\vec{y}'\ \phi_\wedge(\vec{t}, \vec{U}\vec{t}\vec{y}'w). \tag{3.7}$$

Merging Γ''' and Γ'''' into Γ, we eventually arrive at

$$\Gamma \vdash \phi'_\wedge(\vec{t}'\vec{t}, \vec{y}').$$

Case $\dfrac{\phi(\mathbf{0}) \qquad \forall z\ (\phi(z) \to \phi(\boldsymbol{S}z))}{\phi(v)}$: The induction hypotheses are

$$\Gamma^o \vdash \phi(\mathbf{0})_\wedge(\vec{t}^o, \vec{y}) \tag{3.8}$$

and

$$\Gamma^s \vdash \forall w{<}Vz\vec{x}\vec{y}\ \phi(z)_\wedge(\vec{x}, \vec{U}z\vec{x}\vec{y}w) \to \phi(\boldsymbol{S}z)_\wedge(\vec{t}^s z\vec{x}, \vec{y}). \tag{3.9}$$

It is advisable to develop a general intuition regarding the existence, and form, of the witnesses $\vec{t}$ in

$$\phi(v)_\wedge(\vec{t}, \vec{y}) \tag{3.10}$$

given (3.8) and (3.9), namely, $\vec{t} \equiv \vec{r}(v)$, where

$$\vec{r}(\mathbf{0}) = \vec{t}^o,$$
$$\vec{r}(\boldsymbol{S}z) = \vec{t}^s z\vec{r}(z).$$

The actual proof that these satisfy (3.10), while important to have, may be skipped at first reading.

By substituting $\vec{r}(z)$ for $\vec{x}$ in (3.9) and replacing equals with equals we obtain

$$\Gamma^o \vdash \phi(\mathbf{0})_\wedge(\vec{r}(\mathbf{0}), \vec{y}),$$
$$\Gamma^s[\vec{x} := \vec{r}(z)] \vdash \forall w{<}Vz\vec{r}(z)\vec{y}\ \phi(z)_\wedge(\vec{r}(z), \vec{U}z\vec{r}(z)\vec{y}w) \to \phi(\boldsymbol{S}z)_\wedge(\vec{r}(\boldsymbol{S}z), \vec{y}).$$

To simplify notation, let $\Gamma' \equiv \Gamma^s[\vec{x} := \vec{r}(z)]$, $\psi(z, \vec{y}) \equiv \phi(z)_\wedge(\vec{r}(z), \vec{y})$, $V' \equiv Vz\vec{r}(z)$ and $\vec{U}' \equiv \vec{U}z\vec{r}(z)$. Then,

$$\Gamma^o \vdash \psi(\mathbf{0}, \vec{y}), \tag{3.11}$$

$$\Gamma' \vdash \forall w{<}V'\vec{y}\ \psi(z, \vec{U}'\vec{y}w) \rightarrow \psi(\boldsymbol{S}z, \vec{y}). \tag{3.12}$$

Substituting $\vec{c}(z', \boldsymbol{S}z, w')$ for $\vec{y}$ in (3.12) and applying $\forall w'{<}d(z', \boldsymbol{S}z)$ to both sides ($\vec{c}, d$ to be defined later), we obtain (using lemmata 3.3, 3.4 and 3.5)

(3.13)

$$\Gamma'' \vdash \forall w{<}b\ \psi(z, \vec{U}\vec{c}(z', \boldsymbol{S}z, j_1 w)j_2 w) \rightarrow \forall w{<}d(z', \boldsymbol{S}z)\ \psi(\boldsymbol{S}z, \vec{c}(z', \boldsymbol{S}z, w))$$

for some $\Gamma'' \in \mathcal{A}$ and some term b. By defining

$$\begin{aligned} \vec{c}(\mathbf{0}, z, w) &= \vec{y}, \\ \vec{c}(\boldsymbol{S}z', z, w) &= \vec{U}\vec{c}(z', \boldsymbol{S}z, j_1 w)j_2 w, \end{aligned}$$

and

$$\begin{aligned} d(\mathbf{0}, z) &= \mathbf{1}, \\ d(\boldsymbol{S}z', z) &= b, \end{aligned}$$

(3.13) becomes

$$\Gamma'' \vdash \theta(\boldsymbol{S}z', z) \rightarrow \theta(z', \boldsymbol{S}z) \tag{3.14}$$

where

$$\theta(z', z) \equiv \forall w{<}d(z', z)\ \psi(z, \vec{c}(z', z, w)).$$

For $z' := v \dot{-} \boldsymbol{S}z$, (3.14) implies

$$\Gamma''', \theta(\boldsymbol{S}(v \dot{-} \boldsymbol{S}z), z) \vdash \theta(v \dot{-} \boldsymbol{S}z, \boldsymbol{S}z) \tag{3.15}$$

Letting $\Delta \equiv \Gamma''' \cup \{v \dot{-} z = \boldsymbol{S}(v \dot{-} \boldsymbol{S}z)\}$ and merging $\forall z{<}v\ \Gamma'''$ and Γ^o into one assumption set $\Gamma \in \mathcal{A}$, everything may be put together into one big deduction:

$$\dfrac{\dfrac{\begin{matrix}\Gamma \\ \forall z{<}v\ \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\Gamma \\ \psi(\mathbf{0}, \vec{y})\end{matrix}}{\theta(v \dot{-} \mathbf{0}, \mathbf{0})} \qquad \dfrac{[\Gamma'''] \qquad \dfrac{[v \dot{-} z = \boldsymbol{S}(v \dot{-} \boldsymbol{S}z)] \qquad [\theta(v \dot{-} z, z)]}{\theta(\boldsymbol{S}(v \dot{-} \boldsymbol{S}z), z)}}{\theta(v \dot{-} \boldsymbol{S}z, \boldsymbol{S}z)}\ (3.15)}{\theta(v \dot{-} v, v)}\ (*)}{\psi(v, \vec{y}) \equiv \phi(v)_\wedge(\vec{r}(v), \vec{y})}$$

where horizontal lines may conceal several steps. Rule $(*)$ is a generalization of (3.3) where multiple occurences of the same bounded universal quantifier are eliminated at once (exercise 2). □

3.3 Axiomatization

A *purely universal formula* is a formula of the form $\forall \vec{x}\ \phi$ with ϕ quantifier-free.

Lemma 3.6. *For each instance θ of one of the schemata*

(AC) $\quad \forall \vec{x}\, \exists \vec{y}\, \phi(\vec{x}, \vec{y}) \to \exists \vec{Y}\, \forall \vec{x}\, \phi(\vec{x}, \vec{Y}\vec{x})$,

($\mathrm{IP}^{\omega}_{\forall}$) $\quad (\psi \to \exists \vec{x}\, \phi) \to \exists \vec{x}\, (\psi \to \phi)$, ψ *purely universal*,

($\mathrm{M}_{\wedge}$) $\quad (\forall \vec{x}\, \phi(\vec{x}) \to \psi) \to \exists Z\, \exists \vec{X}\, (\forall z{<}Z\, \phi(\vec{X}z) \to \psi)$, ϕ, ψ *q.f.*,

there are terms $\vec{t}$ such that $\vdash_{\boldsymbol{T}_{\wedge}} \theta_{\wedge}(\vec{t}, \vec{y})$. (Schema $\mathrm{M}_{\wedge}$ *is a version of* Markov's principle.*)*

Proof. Exercise 5. □

Theorem 3.7 (axiomatization of $^{\wedge}$)**.** *The following are identical theories:*

1. $\boldsymbol{HA^{\omega}} + \{\phi^{\wedge} \leftrightarrow \phi\}$,
2. $\boldsymbol{HA^{\omega}} + \{\phi^{\wedge} \to \phi\}$,
3. $\{\phi \mid \vdash_{\boldsymbol{HA^{\omega}}} \phi^{\wedge}\}$,
4. $\{\phi \mid \vdash_{\boldsymbol{T}_{\wedge}} \phi_{\wedge}(\vec{t}, \vec{y})$ *with $\vec{t}$ as in theorem 3.2*$\}$,
5. $\boldsymbol{HA^{\omega}} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\forall} + \mathrm{M}_{\wedge}$,
6. *the preceding one, with* AC *and* $\mathrm{IP}^{\omega}_{\forall}$ *restricted to purely universal ϕ.*

Proof. We show that the last theory implies (i.e. includes) the first; the other inclusions are easy. We proceed by induction on ϕ, the base case being obvious.

$$\begin{aligned} \phi \,\&\, \phi' &\leftrightarrow \exists \vec{x}\, \forall \vec{y}\, \phi_{\wedge}(\vec{x}, \vec{y}) \,\&\, \exists \vec{x}'\, \forall \vec{y}'\, \phi'_{\wedge}(\vec{x}', \vec{y}') && \text{[induction hypothesis]} \\ &\leftrightarrow (\phi \,\&\, \phi')^{\wedge}. \end{aligned}$$

$$\begin{aligned} \phi \to \phi' &\leftrightarrow \exists \vec{x}\, \forall \vec{y}\, \phi_{\wedge}(\vec{x}, \vec{y}) \to \exists \vec{x}'\, \forall \vec{y}'\, \phi'_{\wedge}(\vec{x}', \vec{y}') && \text{[induction hypothesis]} \\ &\leftrightarrow \forall \vec{x}\, (\forall \vec{y}\, \phi_{\wedge}(\vec{x}, \vec{y}) \to \exists \vec{x}'\, \forall \vec{y}'\, \phi'_{\wedge}(\vec{x}', \vec{y}')) \\ &\leftrightarrow \forall \vec{x}\, \exists \vec{x}'\, (\forall \vec{y}\, \phi_{\wedge}(\vec{x}, \vec{y}) \to \forall \vec{y}'\, \phi'_{\wedge}(\vec{x}', \vec{y}')) && [\mathrm{IP}^{\omega}_{\forall}] \\ &\leftrightarrow \forall \vec{x}\, \exists \vec{x}'\, \forall \vec{y}'\, (\forall \vec{y}\, \phi_{\wedge}(\vec{x}, \vec{y}) \to \phi'_{\wedge}(\vec{x}', \vec{y}')) \\ &\leftrightarrow \forall \vec{x}\, \exists \vec{x}'\, \forall \vec{y}'\, \exists Z\, \exists \vec{Y}\, (\forall z{<}Z\, \phi_{\wedge}(\vec{x}, \vec{Y}z) \to \phi'_{\wedge}(\vec{x}', \vec{y}')) && [\mathrm{M}_{\wedge}] \\ &\leftrightarrow (\phi \to \phi')^{\wedge}. && [\mathrm{AC}] \end{aligned}$$

$$\begin{aligned} \forall z\, \phi(z) &\leftrightarrow \forall z\, \exists \vec{x}\, \forall \vec{y}\, \phi(z)_{\wedge}(\vec{x}, \vec{y}) && \text{[induction hypothesis]} \\ &\leftrightarrow (\forall z\, \phi(z))^{\wedge}. && [\mathrm{AC}] \end{aligned}$$

$$\begin{aligned} \exists z\, \phi(z) &\leftrightarrow \exists z\, \exists \vec{x}\, \forall \vec{y}\, \phi(z)_{\wedge}(\vec{x}, \vec{y}) && \text{[induction hypothesis]} \\ &\leftrightarrow (\exists z\, \phi(z))^{\wedge}. \end{aligned}$$

□

Corollary 3.8. $\boldsymbol{HA^{\omega}} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\forall} + \mathrm{M}_{\wedge}$ *has the existence property.*

Proof. Assume $\exists z\, \phi(z) \in \mathbf{HA^{\omega}} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\forall} + \mathrm{M}_{\wedge} = \{\phi \mid \vdash_{\mathbf{T}_{\wedge}} \phi_{\wedge}(\vec{t}, \vec{y})\}$. Then, $\phi(v)_{\wedge}(\vec{t}, \vec{y}) \in \mathbf{T}_{\wedge}$ for suitable terms $v, \vec{t}$, whence $\phi(v)^{\wedge} \in \mathbf{HA^{\omega}}$, and finally $\phi(v) \in \mathbf{HA^{\omega}} + \{\phi^{\wedge} \to \phi\} = \mathbf{HA^{\omega}} + \mathrm{AC} + \mathrm{IP}^{\omega}_{\forall} + \mathrm{M}_{\wedge}$. □

3.4 Dialectica

Dialectica was introduced in Gödel (1958) (english translation and extensive comments in Troelstra (1990)) for the purpose of elaborating on the constructive meaning of the intuitionistic logical constants and of providing a reduction of Heyting arithmetic to a finitistic system $\mathbf{T}$. Combined with the negative translation, this reduction yields a consistency proof for Peano arithmetic.

Dialectica does not interact well with higher-type equality, so, in order to extend it to theories involving higher types, such as $\mathbf{HA}^{\boldsymbol{\omega}}$ and its relatives, it seems necessary to treat equality at types $\tau \not\equiv \mathcal{N}$ as a defined relation. This may be accomplished by letting $t =_\tau u$ stand for $v(t) =_{\mathcal{N}} v(u)$ for arbitrary terms $v(x)$ of type $\mathcal{N}$. Dialectica may thus be extended to higher types as a translation of a theory $\mathbf{HA}^{\boldsymbol{\omega}}_{\mathbf{0}}$ with equality only at type $\mathcal{N}$ into its quantifier-free fragment $\mathbf{T_0}$, or of $\mathbf{I\text{-}HA}^{\boldsymbol{\omega}}$ into $\mathbf{I\text{-}T}$.

For the purpose of showing that Dialectica is equivalent to Diller-Nahm, in the following we will assume an arbitrary extension $\mathbf{T_*}$ of $\mathbf{T_0}$ by term constants, and its quantified version $\mathbf{HA}^{\boldsymbol{\omega}}_{\boldsymbol{*}}$.

Free from primitive equality at higher types, $\mathbf{T_*}$ is decidable. What's more,

Lemma 3.9. *$\boldsymbol{T_*}$ possesses* characteristic functions*, i.e., for each formula ϕ there is a term $[\phi]$ of type $\mathcal{N}$ such that*

$$\vdash_{\boldsymbol{T_*}} \phi \leftrightarrow [\phi] = \mathbf{0}.$$

Proof. $[\phi]$ is defined by the structural recursion

$$\begin{aligned} [t = u] &\equiv (t \dot{-} u) + (u \dot{-} t), \\ [\phi \,\&\, \psi] &\equiv [\phi] + [\psi], \\ [\phi \to \psi] &\equiv (\mathbf{1} \dot{-} [\phi]) \cdot [\psi]. \end{aligned}$$ □

The definition of $[\phi]$ may be extended to bounded universal quantification:

$$[\forall z{<}v\ \phi] \equiv \boldsymbol{R}\mathbf{0}(\lambda z\ \lambda x\ (x + [\phi]))v,$$

which shows that

Corollary 3.10. *$\boldsymbol{T}_\wedge$ is a subsystem of $\boldsymbol{T_*}$.*

In a decidable system, a conjunction is equivalent to the "least true" of the conjuncts. The same holds for bounded universal quantification:

Lemma 3.11. *There is a term t such that*

$$\vdash_{\boldsymbol{T_*}} \phi[z := t] \to \forall z{<}v\ \phi.$$

Proof. The required term is $u(v)$, where

$$\begin{aligned} u(\mathbf{0}) &= \text{anything}, \\ u(\boldsymbol{S}z) &= \text{if } [\phi] = \mathbf{0} \text{ then } u(z) \text{ else } z. \end{aligned}$$ □

Dialectica differs from Diller-Nahm only in its treatment of implication:

$$(\phi \to \phi')^{\mathrm{D}} \equiv \exists \vec{X}\vec{Y}\ \forall \vec{x}\vec{y}\ \Big[\phi_{\mathrm{D}}(\vec{x}, \vec{Y}\vec{x}\vec{y}) \to \phi'_{\mathrm{D}}(\vec{X}\vec{x}, \vec{y})\Big].$$

This is equivalent to letting Z be $\lambda\vec{x}\ \lambda\vec{y}\ \mathbf{1}$ in the definition of $(\phi \to \phi')^\wedge$.

Theorem 3.12. *In the context of* $\boldsymbol{HA}^{\boldsymbol{\omega}}_{\boldsymbol{*}}$*, the two interpretations are equivalent, in the sense that for all* $\boldsymbol{HA}^{\boldsymbol{\omega}}_{\boldsymbol{*}}$*-formulae* ϕ, $\vdash_{\boldsymbol{HA}^{\boldsymbol{\omega}}_{\boldsymbol{*}}} \phi^{\mathrm{D}} \leftrightarrow \phi^{\wedge}$.

Proof. By induction on ϕ: Only the case of implication needs some attention, and this is taken care of by the previous lemma. □

3.5 Exercises

1. Prove that the elimination rule for the bounded universal quantifier is equivalent to the inversion

$$\frac{\forall z{<}\boldsymbol{S}v\ \phi(z)}{\forall z{<}v\ \phi(z)} \qquad\qquad \frac{\forall z{<}\boldsymbol{S}v\ \phi(z)}{\phi(v)}$$

   of its introduction rules.

2. Derive the following generalization of elimination rule (3.3):

$$\frac{\forall z{<}v\ \Phi \qquad \psi(\mathbf{0}) \qquad \begin{matrix}[\Phi]\ [\psi(z)] \\ \psi(\boldsymbol{S}z)\end{matrix}}{\psi(v)}$$

   where Φ is an arbitrary set of formulae and $\forall z{<}v\ \Phi = \{\forall z{<}v\ \phi \mid \phi \in \Phi\}$. [Assume Φ finite and apply rule (3.3) to the conjunction of its elements.]

3. Prove the following:

   (a) $(\forall \vec{z}\ \phi(\vec{z}))^{\wedge} \equiv \exists \vec{X}\ \forall \vec{y}\vec{z}\ \left[\phi(\vec{z})_{\wedge}(\vec{X}\vec{z}, \vec{y})\right]$.

   (b) $(\exists \vec{z}\ \phi(\vec{z}))^{\wedge} \equiv \exists \vec{z}\vec{x}\ \forall \vec{y}\ \left[\phi(\vec{z})_{\wedge}(\vec{x}, \vec{y})\right]$.

4. Prove proposition 3.1.

5. Prove lemma 3.6. [Hint: Your task will be simplified if you first show that if $\phi^{\wedge} \equiv \psi^{\wedge}$, then $\vdash_{\mathbf{T}_{\wedge}} (\phi \to \psi)_{\wedge}(\lambda\vec{x}\ \lambda\vec{y}\ \mathbf{1}, \vec{t}, \vec{u}; \vec{x}, \vec{y})$ for suitable terms $\vec{t}, \vec{u}$.]

6. Show that the restriction of $\mathrm{M}_{\wedge}$ to $\mathbf{HA}^{\boldsymbol{\omega}}_{\boldsymbol{*}}$ is equivalent to the more familiar

$$\neg\forall x\ \phi \to \exists x\ \neg\phi, \ \ \phi \text{ quantifier-free.}$$

   [In one direction, take ψ to be $\perp$; in the other, use the decidability of ψ.]

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