Lecture notes

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Chapter 1

Finite-type arithmetic

Higher types and higher-type entities (functionals) constitute a natural, and constructive, way of extending the expressive power of arithmetic without increasing its proof-theoretic strength. They also provide the syntactic means to express the effective information contained in proofs of arithmetical statements.

As a foundation for our subsequent proof-theoretic considerations, we present a basic theory $\mathbf{HA}^{\boldsymbol{\omega}}$ of intuitionistic finite-type arithmetic, together with an extensional variant $\mathbf{E\text{-}HA}^{\boldsymbol{\omega}}$ and an intensional one $\mathbf{I\text{-}HA}^{\boldsymbol{\omega}}$.

Based in part on Troelstra and van Dalen (1988).

1.1 Syntax

1.1.1 The language of finite-type arithmetic

The following types are present:

1. An atomic type $\mathcal{N}$ (the type of natural numbers),
2. a type $\sigma \times \tau$ for any two types σ and τ (product types),
3. a type τ^{σ} for any two types σ and τ (function types).

Notation. $(\tau^{\sigma})^{\rho}$ is simplified to $\tau^{\rho\sigma}$, and $\tau^{\vec{\sigma}}$ is governed by a similar convention; hence, $\tau^{\sigma^{\rho}}$ denotes the other alternative.

Terms, and their types, are generated by

0. There is an inexhaustible supply (infinite set) of variables of each type.
1. $\mathbf{0}$ is a term of type $\mathcal{N}$; for any term t of type $\mathcal{N}$, $\boldsymbol{S}t$ is a term of type $\mathcal{N}$.
2. For any terms t of type τ, u of type $\tau^{\mathcal{N}\tau}$ and v of type $\mathcal{N}$, $\boldsymbol{R}tuv$ is a term of type τ.
3. For any terms t_l and t_r of types τ_l and τ_r, $\langle t_l, t_r\rangle$ is a term of type $\tau_l \times \tau_r$.
4. For any term t of type $\tau_l \times \tau_r$, $\boldsymbol{p}_s t$ is a term of type τ_s, for $s \in \{l, r\}$.
5. For any variable x of type σ and term t of type τ, $\lambda x\, t$ is a term of type τ^{σ}.

6. For any terms t of type τ^σ and u of type σ, tu is a term of type τ.

Notation. Lists of variables/terms are conveniently abbreviated $\vec{x}$, $\vec{t}$ etc., with the aid of the following conventions (where $\vec{t} \equiv t_1, \ldots, t_n$ and $\vec{u} \equiv u_1, \ldots, u_m$):

$$\begin{aligned} t\vec{u} &\equiv tu_1 \cdots u_m \equiv (\cdots(tu_1)\cdots)u_m, \\ \vec{t}\vec{u} &\equiv t_1\vec{u}, \ldots, t_n\vec{u}, \\ \lambda x_1, \ldots, x_m\ t &\equiv \lambda x_1 \cdots \lambda x_m\ t, \\ \lambda\vec{x}\ \vec{t} &\equiv \lambda\vec{x}\ t_1, \ldots, \lambda\vec{x}\ t_n. \end{aligned}$$

Prime (or atomic) formulae are equations $t = u$ between terms of the same type. *Formulae* are formed from prime formulae by means of &, $\to$, $\forall$, and $\exists$. We will use the following abbreviations:

$$\begin{aligned} \mathbf{1} &\equiv \boldsymbol{S}\mathbf{0}, \\ \bot &\equiv \mathbf{0} = \mathbf{1}, \\ \neg\phi &\equiv \phi \to \bot, \\ t \neq u &\equiv \neg(t = u), \\ \phi \vee \psi &\equiv \exists z\,((z = \mathbf{0} \to \phi)\ \&\ (z \neq \mathbf{0} \to \psi)). \end{aligned}$$

1.1.2 Axioms and rules of inference

Besides the usual (natural deduction or other) axioms and rules of intuitionistic first-order logic for the logical constants present in the system, we have rules for *equality*

$$\frac{}{t = t} \qquad \frac{t = u \qquad \phi(t)}{\phi(ut)},$$

β-conversion

$$\frac{}{\boldsymbol{R}tu\mathbf{0} = t} \qquad \frac{}{\boldsymbol{R}tu\boldsymbol{S}v = uvRtuv},$$

$$\frac{}{\boldsymbol{p}_i\langle t_l, t_r\rangle = t_i}, \quad i \in \{l, r\},$$

$$\frac{}{(\lambda x\ t)u = t[x := u]},$$

and *induction*

$$\frac{\phi(\mathbf{0}) \qquad \forall x\,[\phi(x) \to \phi(\boldsymbol{S}x)]}{\phi(v)}.$$

The above axioms and rules constitute $\mathbf{HA}^{\boldsymbol{\omega}}$. Occasionally, we will be interested in the following variants of this theory. *Extensional finite-type arithmetic*, $\mathbf{E\text{-}HA}^{\boldsymbol{\omega}}$, is obtained from $\mathbf{HA}^{\boldsymbol{\omega}}$ by the addition of the *extensionality rules*

$$\frac{\boldsymbol{p}_l t = \boldsymbol{p}_l u \qquad \boldsymbol{p}_r t = \boldsymbol{p}_r u}{t = u}$$

for t, u of product type, and

$$\frac{\forall x\,(tx = ux)}{t = u}\ , \quad x \notin \mathrm{FV}(t, u)$$

for t, u of function type. *Intensional finite-type arithmetic*, $\mathbf{I\text{-}HA}^{\boldsymbol{\omega}}$, augments the language of $\mathbf{HA}^{\boldsymbol{\omega}}$ with *equality functionals* $\boldsymbol{E}_\tau$, one for each type τ, subject to

$$\frac{}{\boldsymbol{E}_\tau tu = \mathbf{0} \leftrightarrow t = u} \qquad\qquad \frac{}{\boldsymbol{E}_\tau tu = \mathbf{1} \leftrightarrow t \neq u}\ .$$

Classical (or Peano) finite-type arithmetic $\mathbf{PA}^{\boldsymbol{\omega}}$ is the extension of $\mathbf{HA}^{\boldsymbol{\omega}}$ by the *principle of the excluded middle*

$$\frac{}{\neg\neg\phi \to \phi}\ (\mathbf{PEM})\ .$$

1.2 Exercises

1. Show that equality at type $\mathcal{N}$ is decidable, i.e., $x = y \vee x \neq y$.
2. Using your preferred logical formalism, show that if

   $$\vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \phi,$$

   then

   $$\vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \phi[x := t].$$

3. Prove that extensionality is equivalent to the set of equations

   $$(\eta) \qquad \begin{aligned} \langle \boldsymbol{p}_l t, \boldsymbol{p}_r t\rangle &= t, && t \text{ of product type,} \\ \lambda x\ (tx) &= t, && t \text{ of function type, } x \notin \mathrm{FV}(t). \end{aligned}$$

4. Extensional equality $t =_e u$ between terms t, u of the same type is inductively defined by

   $$t =_e u \equiv \begin{cases} t = u & t,\ u \text{ of atomic type,} \\ \boldsymbol{p}_l t =_e \boldsymbol{p}_l u \ \&\ \boldsymbol{p}_r t =_e \boldsymbol{p}_r u & t,\ u \text{ of product type,} \\ \forall x\,(tx =_e ux) & t,\ u \text{ of function type.} \end{cases}$$

   Show that extensionality is equivalent to the schema

   $$t =_e u \leftrightarrow t = u,$$

   and conclude that, in $\mathbf{E\text{-}HA}^{\boldsymbol{\omega}}$, atomic formulae are equivalent to purely universal formulae involving equality at type $\mathcal{N}$ only.

5. (Closure of $\mathbf{HA}^{\boldsymbol{\omega}}$ under *mutual primitive recursion*.) Let $\vec{\tau} \equiv \tau_1, \ldots, \tau_n$ be a list of types, $\vec{t}$ a list of terms of types $\vec{\tau}$ (i.e., each t_i has type τ_i) and $\vec{u}$ a list of terms of types $\vec{\tau}^{\mathcal{N}\vec{\tau}}$ (i.e., each u_i has type ${\tau_i}^{\mathcal{N}\tau_1\cdots\tau_n}$). Construct terms $\vec{r} \equiv \vec{r}(z)$, z fresh, with the properties

   $$\begin{aligned} \vec{r}(\mathbf{0}) &= \vec{t}, \\ \vec{r}(\boldsymbol{S}v) &= \vec{u}v\vec{r}(v). \end{aligned}$$

Chapter 2

Modified realizability

The name *realizability* refers to any one of a family of translations that may be seen as formalizations of the BHK interpretation of the logical constants; for a more complete description of the BHK interpretation, the reader may consult Troelstra and van Dalen (1988).

Modified realizability is a variant of realizability introduced in Kreisel (1959) for the purpose of showing that Markov's principle is not derivable in intuitionistic logic. It could as well be called *typed realizability* because it uses functionals instead of numbers as realizing objects. This notion of realizability is well adapted to the study of typed theories; it will be our first, and simplest, example of term extraction.

2.1 Definition

To each formula ϕ in the language of finite-type arithmetic we associate its *modified realizability interpretation* ϕ^{mr}, which is a formula of the form

$$\exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})$$

with the same free variables as ϕ, where $\phi_{\mathrm{mr}}(\vec{x})$ ($\vec{x}$ *modified realizes* ϕ, alternative notation: $\vec{x}\mathbf{mr}\phi$) is an $\exists$-free formula and $\vec{x}$ a possibly empty list of variables. The associations $(\ )_{\mathrm{mr}}$ and $(\ )^{\mathrm{mr}}$ are defined by the following induction:

For ϕ atomic, $\phi^{\mathrm{mr}} \equiv \phi$,

$(\phi \;\&\; \psi)^{\mathrm{mr}} \equiv \exists \vec{x}, \vec{y} \left[\phi_{\mathrm{mr}}(\vec{x}) \;\&\; \psi_{\mathrm{mr}}(\vec{y})\right]$,

$(\phi \to \psi)^{\mathrm{mr}} \equiv \exists \vec{Y} \left[\forall \vec{x}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{Y}\vec{x}))\right]$,

$(\forall z\, \phi(z))^{\mathrm{mr}} \equiv \exists \vec{X} \left[\forall z\, (\phi(z)_{\mathrm{mr}}(\vec{X}z))\right]$,

$(\exists z\, \phi(z))^{\mathrm{mr}} \equiv \exists z, \vec{x} \left[\phi(z)_{\mathrm{mr}}(\vec{x})\right]$,

where, in each case, the $\exists$-free kernel is delimited by brackets.

Remark on notation. Expressions like $\phi(v)_{\mathrm{mr}}(\vec{x})$ are unambiguous, since the interpretation commutes with substitution:

$$\phi[z := v]_{\mathrm{mr}}(\vec{x}) \equiv \phi_{\mathrm{mr}}(\vec{x})[z := v].$$

Proposition 2.1. *Let* $\phi^{\mathrm{mr}} \equiv \exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})$.

1. $\phi_{\mathrm{mr}}(\vec{x})$ *is* $\exists$*-free, and if* ϕ *is* $\exists$*-free, then* $\vec{x}$ *is empty and* $\phi^{\mathrm{mr}} \equiv \phi_{\mathrm{mr}} \equiv \phi$.
2. *If* ψ *is* $\exists$*-free, then* $(\exists \vec{y}\, \psi)^{\mathrm{mr}} \equiv \exists \vec{y}\, \psi$*; in particular,* $(\phi^{\mathrm{mr}})^{\mathrm{mr}} \equiv \phi^{\mathrm{mr}}$.

Proof. Exercise. □

2.2 Soundness

In the following, we are going to employ the *modified realizability schema*

(MR) $$\phi^{\mathrm{mr}} \leftrightarrow \phi.$$

This is not among the axioms usually considered for arithmetic; we will shortly prove its equivalence to something more familiar (??).

Theorem 2.2 (soundness)**.** *Let* $\boldsymbol{H}$ *be any one of* $\boldsymbol{HA^\omega}$, $\boldsymbol{E\text{-}HA^\omega}$, $\boldsymbol{I\text{-}HA^\omega}$*, and let* $\boldsymbol{H}-\exists$ *be the* $\exists$*-free part of* $\boldsymbol{H}$*. If* $\vdash_{\boldsymbol{H}+MR} \phi$*, then* $\vdash_{\boldsymbol{H}-\exists} \phi_{\mathrm{mr}}(\vec{t})$ *for a suitable list* $\vec{t}$ *of terms satisfying* $\mathrm{FV}(\vec{t}) \subseteq \mathrm{FV}(\phi)$.

Proof. We are going to apply induction on the proofs of **H**+MR, for the purpose of which we will need the (superficially) stronger statement

> If $\Phi \vdash_{\mathbf{H}+\mathrm{MR}} \phi$, then $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H}-\exists} \phi_{\mathrm{mr}}(\vec{t})$, where all free variables of $\vec{t}$ are among those free in ϕ and those free in Φ_{mr}.

where Φ is an arbitrary (finite) set of formulae and $\Phi_{\mathrm{mr}} = \{\phi_{\mathrm{mr}} \mid \phi \in \Phi\}$. Of the axioms and rules of $\mathbf{H}-\exists$, those that are $\exists$-free are *self-realizing* and don't need any further examination; this includes the "extras" of $\mathbf{E\text{-}HA^\omega}$ and $\mathbf{I\text{-}HA^\omega}$. For most of the others, a deduction will be furnished that may be combined with the induction hypotheses in an obvious way to yield the required conclusion. Exception: $\exists$-rules.

Natural deduction

$$\frac{\phi \qquad \psi}{\phi \,\&\, \psi} : \qquad \frac{\phi_{\mathrm{mr}}(\vec{t}) \qquad \psi_{\mathrm{mr}}(\vec{u})}{\phi_{\mathrm{mr}}(\vec{t}) \,\&\, \psi_{\mathrm{mr}}(\vec{u}) \equiv (\phi \,\&\, \psi)_{\mathrm{mr}}(\vec{t},\vec{u})}$$

$$\frac{\phi \,\&\, \psi}{\phi} : \qquad (\phi \,\&\, \psi)_{\mathrm{mr}}(\vec{t},\vec{u}) \equiv \frac{\phi_{\mathrm{mr}}(\vec{t}) \,\&\, \psi_{\mathrm{mr}}(\vec{u})}{\phi_{\mathrm{mr}}(\vec{t})}$$

$$\frac{\begin{matrix}[\phi]\\ \psi\end{matrix}}{\phi \to \psi} : \qquad \cfrac{\cfrac{\begin{matrix}[\phi_{\mathrm{mr}}(\vec{x})]\\ \psi_{\mathrm{mr}}(\vec{u})\end{matrix}}{\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{u})}}{\forall \vec{x}\,(\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{u})) \leftrightarrow (\phi \to \psi)_{\mathrm{mr}}(\lambda \vec{x}\, \vec{u})}$$

$$\frac{\phi \to \psi \qquad \phi}{\psi} : \qquad \cfrac{(\phi \to \psi)_{\mathrm{mr}}(\vec{t}) \equiv \cfrac{\forall \vec{x}\,(\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{t}\vec{x}))}{\phi_{\mathrm{mr}}(\vec{u}) \to \psi_{\mathrm{mr}}(\vec{t}\vec{u})} \qquad \phi_{\mathrm{mr}}(\vec{u})}{\psi_{\mathrm{mr}}(\vec{t}\vec{u})}$$

$$\frac{\phi(z)}{\forall z\, \phi(z)} : \qquad \frac{\phi(z)_{\mathrm{mr}}(\vec{t})}{\forall z\,(\phi(z)_{\mathrm{mr}}(\vec{t})) \leftrightarrow (\forall z\, \phi(z))_{\mathrm{mr}}(\lambda z\, \vec{t})}$$

$\dfrac{\forall z\, \phi(z)}{\phi(v)}$: $\qquad (\forall z\, \phi(z))_{\mathrm{mr}}(\vec{t}) \equiv \dfrac{\forall z\, (\phi(z)_{\mathrm{mr}}(\vec{t}z))}{\phi(v)_{\mathrm{mr}}(\vec{t}v)}$

$\dfrac{\phi(v)}{\exists z\, \phi(z)}$: Nothing to prove; the conclusion coincides with the induction hypothesis (this is because the interpretation of $\exists$ is "trivial", in the sense that it merely converts the existentially quantified variable into a realizing variable).

$\dfrac{\exists z\, \phi(z) \qquad \begin{matrix}[\phi(z)] \\ \psi\end{matrix}}{\psi}$: By hypothesis, there are deductions $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H}-\exists} \phi(v)_{\mathrm{mr}}(\vec{t})$ and $\Phi_{\mathrm{mr}}, \phi(z)_{\mathrm{mr}}(\vec{x}) \vdash_{\mathbf{H}-\exists} \psi_{\mathrm{mr}}(\vec{u})$, whence $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H}-\exists} \psi_{\mathrm{mr}}(\vec{u}[\vec{x} := \vec{t}])$.

Equality

$\dfrac{t = u \qquad \phi(t)}{\phi(u)}$: $\qquad \dfrac{t = u \qquad \phi(t)_{\mathrm{mr}}(\vec{v})}{\phi(u)_{\mathrm{mr}}(\vec{v})}$

(using the fact that modified realizability commutes with substitution).

Induction

$\dfrac{\phi(\mathbf{0}) \qquad \forall z\, (\phi(z) \to \phi(\boldsymbol{S}z))}{\phi(v)}$:

$$\dfrac{\phi(\mathbf{0})_{\mathrm{mr}}(\vec{t}) \qquad \dfrac{\forall z, \vec{x}\, (\phi(z)_{\mathrm{mr}}(\vec{x}) \to \phi(\boldsymbol{S}z)_{\mathrm{mr}}(\vec{u}z\vec{x}))}{\forall z\, (\phi(z)_{\mathrm{mr}}(\vec{w}(z)) \to \phi(\boldsymbol{S}z)_{\mathrm{mr}}(\vec{u}z\vec{w}(z)))}}{\phi(v)_{\mathrm{mr}}(\vec{w}(v))}$$

where $\vec{w} \equiv \vec{w}(z)$ is a list of terms such that

$$\vec{w}(\mathbf{0}) = \vec{t},$$
$$\vec{w}(\boldsymbol{S}z) = \vec{u}z\vec{w}(z).$$

One way to guarantee the existence of $\vec{w}(z)$ is by formulating the system with *mutual primitive recursion.* In the presence of product types, however, mutual primitive recursion is reducible to ordinary primitive recursion; e.g., $\vec{w}(z)$ may be constructed as follows: Assuming $\vec{t} \equiv t_1, \ldots, t_n$ and $\vec{u} \equiv u_1, \ldots, u_n$, define

$$\boldsymbol{t} \equiv \langle \vec{t} \rangle \equiv \langle t_1, \ldots, t_n \rangle,$$
$$\boldsymbol{u} \equiv \lambda z, y\, \langle \vec{u}z(\vec{q}y) \rangle,$$

where $\langle\ \rangle$ denotes an arbitrary representation of n-tuples (using pairing), with corresponding projections $\vec{q}y \equiv q_1 y, \ldots, q_n y$. Then, the terms

$$\vec{w}(z) \equiv \vec{q}\boldsymbol{Rtu}z$$

have the required properties.

MR

Since $(\phi^{\mathrm{mr}})_{\mathrm{mr}}(\vec{x}) \equiv \phi_{\mathrm{mr}}(\vec{x})$, a simple calculation yields

$$(\phi^{\mathrm{mr}} \leftrightarrow \phi)^{\mathrm{mr}} \equiv \exists \vec{X}, \vec{Y}\, \left[\forall \vec{x}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \phi_{\mathrm{mr}}(\vec{X}\vec{x}))\ \&\ \forall \vec{y}\, (\phi_{\mathrm{mr}}(\vec{y}) \to \phi_{\mathrm{mr}}(\vec{Y}\vec{y}))\right]$$

which has the trivial realizers $\lambda \vec{x}\, \vec{x}, \lambda \vec{y}\, \vec{y}$. □

2.3 Axiomatization

Here, we are going to show that $\mathbf{HA}^{\boldsymbol{\omega}} + \mathrm{MR}$ may be axiomatized by familiar principles.

Theorem 2.3. *Over $\boldsymbol{HA}^{\boldsymbol{\omega}}$, the following schemata are equivalent:*

1. *MR:* $\phi^{\mathrm{mr}} \leftrightarrow \phi$,
2. $\phi^{\mathrm{mr}} \to \phi$,
3. $AC + IP^{\omega}_{ef}$, *where*

$$\text{(AC)} \qquad \forall \vec{x}\, \exists \vec{y}\, \phi(\vec{x}, \vec{y}) \to \exists \vec{Y}\, \forall \vec{x}\, \phi(\vec{x}, \vec{Y}\vec{x}),$$
$$(\mathrm{IP}^{\omega}_{\mathrm{ef}}) \qquad (\phi \to \exists x\, \psi) \to \exists x\, (\phi \to \psi), \qquad \phi\ \exists\text{-free.}$$

Proof. *1. → 2.* Obvious.

2. → 3. It suffices to show that each instance θ of one of AC and $\mathrm{IP}^{\omega}_{\mathrm{ef}}$ is modified realizable, $\vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \theta^{\mathrm{mr}}$. In each case, this holds trivially, and is left as an exercise.

3. → 1. We proceed by structural induction, where $\phi^{\mathrm{mr}} \equiv \exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})$ and $\psi^{\mathrm{mr}} \equiv \exists \vec{y}\, \psi_{\mathrm{mr}}(\vec{y})$:

(a) Atomic formulae are self-realizing.

(b)

$$\begin{aligned}(\phi\ \&\ \psi)^{\mathrm{mr}} &\equiv \exists \vec{x}, \vec{y}\, (\phi_{\mathrm{mr}}(\vec{x})\ \&\ \psi_{\mathrm{mr}}(\vec{y}))\\ &\leftrightarrow (\exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x}))\ \&\ (\exists \vec{y}\, \psi_{\mathrm{mr}}(\vec{y}))\\ &\equiv \phi^{\mathrm{mr}}\ \&\ \psi^{\mathrm{mr}}\\ &\leftrightarrow \phi\ \&\ \psi.\end{aligned}$$

(c)

$$\begin{aligned}(\phi \to \psi)^{\mathrm{mr}} &\equiv \exists \vec{Y}\, \forall \vec{x}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{Y}\vec{x}))\\ &\leftrightarrow \forall \vec{x}\, \exists \vec{y}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{y}))\\ &\leftrightarrow \forall \vec{x}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \exists \vec{y}\, \psi_{\mathrm{mr}}(\vec{y}))\\ &\leftrightarrow (\exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})) \to (\exists \vec{y}\, \psi_{\mathrm{mr}}(\vec{y}))\\ &\equiv \phi^{\mathrm{mr}} \to \psi^{\mathrm{mr}}\\ &\leftrightarrow \phi \to \psi.\end{aligned}$$

(d)

$$\begin{aligned}(\forall z\, \phi(z))^{\mathrm{mr}} &\equiv \exists \vec{X}\, \forall z\, (\phi(z)_{\mathrm{mr}}(\vec{X}z))\\ &\leftrightarrow \forall z\, \exists \vec{x}\, (\phi(z)_{\mathrm{mr}}(\vec{x}))\\ &\equiv \forall z\, \phi(z)^{\mathrm{mr}}\\ &\leftrightarrow \forall z\, \phi(z).\end{aligned}$$

(e)

$$\begin{aligned}(\exists z\,\phi(z))^{\mathrm{mr}} &\equiv \exists z, \vec{x}\,(\phi(z)_{\mathrm{mr}}(\vec{x}))\\ &\equiv \exists z\,(\phi(z)^{\mathrm{mr}})\\ &\leftrightarrow \exists z\,\phi(z).\end{aligned}$$

□

2.4 Exercises

1. Prove the following:
   (a) $(\forall \vec{z}\,\phi(\vec{z}))^{\mathrm{mr}} \equiv \exists \vec{X}\,\forall \vec{z}\,(\phi(\vec{z})_{\mathrm{mr}}(\vec{X}\vec{z}))$.
   (b) $(\exists \vec{z}\,\phi(\vec{z}))^{\mathrm{mr}} \equiv \exists \vec{z}, \vec{x}\,(\phi(\vec{z})_{\mathrm{mr}}(\vec{x}))$.

2. Show that modified realizability commutes with substitution,

$$\phi(v)_{\mathrm{mr}}(\vec{x}) \equiv \phi(z)_{\mathrm{mr}}(\vec{x})[z := v], \qquad (\vec{x} \text{ not free in } v).$$

3. Prove whatever has been left as an exercise in the text.

4. Expand $(\neg\neg\phi \to \phi)^{\mathrm{mr}}$ and show that it is provable in $\mathbf{PA}^{\boldsymbol{\omega}}$. Conclude that if $\vdash_{\mathbf{PA}^{\boldsymbol{\omega}}} \phi$, then $\vdash_{\mathbf{PA}^{\boldsymbol{\omega}}} \phi^{\mathrm{mr}}$ (soundness for $\mathbf{PA}^{\boldsymbol{\omega}}$).

5. *Harrop* formulae are defined by the induction
   (a) atomic formulae are Harrop,
   (b) if ϕ and ψ are Harrop, then $\phi \,\&\, \psi$ is Harrop,
   (c) if ψ is Harrop, then $\phi \to \psi$ is Harrop (ϕ any formula),
   (d) if ϕ is Harrop, then $\forall x\,\phi$ is Harrop.

   Prove that a formula ϕ is Harrop if and only if ϕ^{mr} is $\exists$-free, i.e., if in $\phi^{\mathrm{mr}} \equiv \exists \vec{x}\,\phi_{\mathrm{mr}}(\vec{x})$, $\vec{x}$ is the empty list.

6. Show that for any instance θ of schema

   $(\mathrm{IP}^{\omega}_{\mathrm{Harrop}})$ $\qquad (\phi \to \exists x\,\psi) \to \exists x\,(\phi \to \psi), \qquad \phi$ Harrop

   there are terms $\vec{t}$ such that $\vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \theta_{\mathrm{mr}}(\vec{t})$.

Chapter 3

Functional interpretation

This chapter is loosely based on Diller and Nahm (1974).

3.1 Definition and elementary properties

3.1.1 Bounded universal quantification

Bounded universal quantification is generally a finitistic operation on formulae, in contrast to its usual definition,

$$\forall x{<}t\, \phi(x) \equiv \forall x\, (x < t \to \phi(x)), \tag{3.1}$$

which employs unrestricted quantification. For the purpose of making sense of bounded universal quantification in quantifier-free settings below, we will treat the bounded universal quantifier as a primitive logical constant, with introduction rules

$$\frac{}{\forall z{<}\mathbf{0}\, \phi(z)} \qquad\qquad \frac{\forall z{<}v\, \phi(z) \qquad \phi(v)}{\forall z{<}\boldsymbol{S}v\, \phi(z)} \tag{3.2}$$

and elimination rule

$$\frac{\forall z{<}v\, \phi(z) \qquad \psi(\mathbf{0}) \qquad \begin{matrix}[\phi(z)] \quad [\psi(z)] \\ \psi(\boldsymbol{S}z)\end{matrix}}{\psi(v)} \quad , \tag{3.3}$$

where, in the last rule, z may not occur in any open assumptions.

3.1.2 The interpretation

We let $\mathbf{T}$ be the quantifier-free fragment of $\mathbf{HA}^{\boldsymbol{\omega}}$ (with the induction rule adapted as appropriate), and we define $\mathbf{T}_\wedge$ to be $\mathbf{T}$ augmented with bounded universal quantifiers.

The *Diller-Nahm interpretation* $\phi^\wedge$ of a formula ϕ in the language of $\mathbf{HA}^{\boldsymbol{\omega}}$ is a formula of the form

$$\exists \vec{x}\, \forall \vec{y}\, \phi_\wedge(\vec{x}, \vec{y})$$

with the same free variables as ϕ, where $\phi_\wedge(\vec{x},\vec{y})$ is a formula of $\mathbf{T}_\wedge$ and $\vec{x}$, $\vec{y}$ are possibly empty lists of variables. The associations $(\ )^\wedge$ and $(\ )_\wedge$ are inductively defined by

$$
\begin{aligned}
\phi^\wedge &\equiv \phi \quad \text{for } \phi \text{ an atomic formula,}\\
(\phi \,\&\, \phi')^\wedge &\equiv \exists \vec{x}\vec{x}'\, \forall \vec{y}\vec{y}'\, \big[\phi_\wedge(\vec{x},\vec{y}) \,\&\, \phi'_\wedge(\vec{x}',\vec{y}')\big],\\
(\phi \to \phi')^\wedge &\equiv \begin{cases} \exists Z\vec{X}\vec{Y}\, \forall \vec{x}\vec{y}\, \big[\forall z{<}Z\vec{x}\vec{y}\, \phi_\wedge(\vec{x},\vec{Y}\vec{x}\vec{y}z) \to \phi'_\wedge(\vec{X}\vec{x},\vec{y})\big], & \vec{Y} \text{ non-nil,}\\ \exists \vec{X}\, \forall \vec{x}\vec{y}\, \big[\phi_\wedge(\vec{x},) \to \phi'_\wedge(\vec{X}\vec{x},\vec{y})\big], & \text{otherwise,} \end{cases}\\
(\forall z\, \phi(z))^\wedge &\equiv \exists \vec{X}\, \forall \vec{y}z\, \big[\phi(z)_\wedge(\vec{X}z,\vec{y})\big],\\
(\exists z\, \phi(z))^\wedge &\equiv \exists z\vec{x}\, \forall \vec{y}\, \big[\phi(z)_\wedge(\vec{x},\vec{y})\big].
\end{aligned}
$$

Optionally, one may add

$$(\forall z{<}v\, \phi(z))^\wedge \equiv \exists \vec{X}\, \forall \vec{y}\, \big[\forall z{<}v\, \phi(z)_\wedge(\vec{X}z,\vec{y})\big].$$

The last clause is logically equivalent to the one obtained by expanding the left hand side using (3.1) and then translating into $\mathbf{T}_\wedge$. It sole purpose is to allow bounded universal quantification in $\mathbf{HA}^{\boldsymbol{\omega}}$ as a primitive, which serves to render $\mathbf{T}_\wedge$ a subsystem of $\mathbf{HA}^{\boldsymbol{\omega}}$. Similarly, the two branches in the definition of $(\phi \to \phi')^\wedge$ are equivalent in case $\vec{Y}$ is the empty list, whence the first, more general one suffices for both cases, and we will silently assume this simpler definition. With this case distinction, however, the formulae of $\mathbf{T}_\wedge$ are translated onto themselves:

Proposition 3.1. *Let* $\phi^\wedge \equiv \exists \vec{x}\, \forall \vec{y}\, \phi_\wedge(\vec{x},\vec{y})$.

1. $\phi_\wedge(\vec{x},\vec{y})$ *is q.f., and if* ϕ *is q.f., then* $\vec{x}$, $\vec{y}$ *are empty and* $\phi^\wedge \equiv \phi_\wedge \equiv \phi$.
2. *If* ψ *is q.f., then* $(\exists \vec{x}\, \forall \vec{y}\, \psi)^\wedge \equiv \exists \vec{x}\, \forall \vec{y}\, \psi$*; in particular,* $(\phi^\wedge)^\wedge \equiv \phi^\wedge$.

Proof. Exercise. □

3.2 Soundness & term extraction

Theorem 3.2 (soundness)**.** *If* $\vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \phi$, *then* $\vdash_{\mathbf{T}_\wedge} \phi_\wedge(\vec{t},\vec{y})$ *for suitable terms* $\vec{t}$ *in which* $\vec{y}$ *do not occur.*

Proof. In the following, $\vdash$ will denote provability in $\mathbf{T}_\wedge$. For the purpose of applying induction on $\mathbf{HA}^{\boldsymbol{\omega}}$-derivations, we will prove that if $\{\phi^i\}_{i\in I} \vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \phi$, then $\{\forall w{<}V\, \phi^i_\wedge(\vec{x}^i, \vec{U}^i w)\}_{i\in I} \vdash \phi_\wedge(\vec{t},\vec{y})$ for suitable terms $(V^i)_{i\in I}, (\vec{U}^i)_{i\in I}, \vec{t}$, with $\vec{y}$ not occuring in $\vec{t}$.

Some preparation: In reference to the previous paragraph, let $\mathcal{A}$ be the collection of assumption sets $\{\forall w{<}V\, \phi^i_\wedge(\vec{x}^i, \vec{U}^i w)\}_{i\in I}$ for all possible choices of $(V^i)_{i\in I}, (\vec{U}^i)_{i\in I}$. $\mathcal{A}$ is closed under $\vec{y}$-substitution, i.e., $\Gamma \in \mathcal{A} \Rightarrow \Gamma[\vec{y} := \vec{u}] \in \mathcal{A}$. A slightly less trivial fact, which will be employed in the treatment of rules with several premises, is that assumptions may be merged, i.e., for $\Gamma_1, \Gamma_2 \in \mathcal{A}$ there is $\Gamma \in \mathcal{A}$ satisfying $\Gamma \vdash \Gamma_1$ and $\Gamma \vdash \Gamma_2$. This is done formulawise:

Lemma 3.3. *Given $\vec{x}$, there are terms $v(z_1, z_2), \vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2)$ satisfying*

$$\forall w{<}v(z_1, z_2)\, \psi(\vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2)w) \vdash \forall w{<}z_i\, \psi(\vec{y}_i w), \quad i = 1, 2$$

for any formula $\psi(\vec{x})$.

[Proof hint: Let $v(z_1, z_2) \equiv z_1 + z_2$. $\vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2)$ may be defined as

$$\vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2) \equiv \lambda w \text{ if } z_1 \dot{-} w = \mathbf{0} \text{ then } \vec{y}_2(w \dot{-} z_1) \text{ else } \vec{y}_1 w,$$

or, elementarily, by

$$\begin{aligned} \vec{u}(\vec{y}_1, \vec{y}_2, z_1, \mathbf{0}) &= \vec{y}_1, \\ \vec{u}(\vec{y}_1, \vec{y}_2, z_1, \boldsymbol{S} z_2) &= \lambda w \text{ if } w = z_1 + z_2 \text{ then } \vec{y}_2 z_2 \text{ else } \vec{u}(\vec{y}_1, \vec{y}_2, z_1, z_2)w. \quad] \end{aligned}$$

To the induction. We will examine the more interesting cases, leaving the verification of the other ones as a (relatively trivial) exercise.

Case $\phi \equiv \phi^i$: Take $V^i \equiv \mathbf{1}$, $\vec{U}^i \equiv \vec{y}$ and $\vec{t} \equiv \vec{x}^i$.

Case $\dfrac{\phi \qquad \phi'}{\phi \,\&\, \phi'}$: Use lemma 3.3.

Case $\dfrac{\begin{matrix}[\phi] \\ \phi'\end{matrix}}{\phi \to \phi'}$: The induction hypothesis provides us with terms $V, \vec{U}, \vec{t}$ such that $\forall w{<}V\, \phi_\wedge(\vec{x}, \vec{U} w) \to \phi'_\wedge(\vec{t}, \vec{y})$, or, equivalently,

$$(\phi \to \phi')_\wedge(\lambda\vec{x}\, \lambda\vec{y}\, V, \lambda\vec{x}\, \vec{t}, \lambda\vec{x}\, \lambda\vec{y}\, \vec{U}; \vec{x}, \vec{y}).$$

Case $\dfrac{\phi \to \phi' \qquad \phi}{\phi'}$: The induction hypotheses are

$$\forall w{<}V\vec{x}\vec{y}'\, \phi_\wedge(\vec{x}, \vec{U}\vec{x}\vec{y}'\vec{w}) \to \phi'_\wedge(\vec{t}'\vec{x}, \vec{y}') \tag{3.4}$$

and

$$\phi_\wedge(\vec{t}, \vec{y}). \tag{3.5}$$

Substituting $\vec{t}$ for $\vec{x}$ in (3.4) and $\vec{U}\vec{t}\vec{y}'w$ for $\vec{y}$ in (3.5) we obtain

$$\begin{gathered} (\forall w{<}V\vec{t}\vec{y}'\, \phi_\wedge(\vec{t}, \vec{U}\vec{t}\vec{y}'w)) \to \phi'_\wedge(\vec{t}'\vec{t}, \vec{y}'), \\ \phi_\wedge(\vec{t}, \vec{U}\vec{t}\vec{y}'w). \end{gathered}$$

To complete the argument, we need the following

Lemma 3.4. *The bounded universal quantifier admits the introduction rule of the unbounded one:*

$$\frac{\phi(z)}{\forall z{<}v\, \phi(z)}\ .$$

[Proof hint: Induction on t.]
Using the above lemma, we eventually arrive at

$$\phi'_{\wedge}(\vec{t}'\vec{t}, \vec{y}').$$

Case $\dfrac{\phi(\mathbf{0}) \qquad \forall z\,(\phi(z) \to \phi(\boldsymbol{S}z))}{\phi(v)}$: The induction hypotheses are

$$\Gamma^o \vdash \phi(\mathbf{0})_{\wedge}(\vec{t}^o, \vec{y}) \tag{3.6}$$

and

$$\Gamma^s \vdash (\forall w{<}Vz\vec{x}\vec{y}\,\phi(z)_{\wedge}(\vec{x}, \vec{U}z\vec{x}\vec{y}w)) \to \phi(\boldsymbol{S}z)_{\wedge}(\vec{t}^s z\vec{x}, \vec{y}). \tag{3.7}$$

It is advisable to develop a general intuition regarding the existence, and form, of the witnesses $\vec{t}$ in

$$\phi(v)_{\wedge}(\vec{t}, \vec{y}) \tag{3.8}$$

given (3.6) and (3.7), namely, $\vec{t} \equiv \vec{r}(v)$, where

$$\vec{r}(\mathbf{0}) = \vec{t}^o,$$
$$\vec{r}(\boldsymbol{S}z) = \vec{t}^s z\vec{r}(z).$$

The actual proof that these satisfy (3.8), while important to have, may be skipped at first reading.

By substituting $\vec{r}(z)$ for $\vec{x}$ in (3.7) and replacing equals with equals we obtain

$$\Gamma^o \vdash \phi(\mathbf{0})_{\wedge}(\vec{r}(\mathbf{0}), \vec{y}),$$
$$\Gamma^s[\vec{x} := \vec{r}(z)] \vdash \forall w{<}Vz\vec{r}(z)\vec{y}\,\phi(z)_{\wedge}(\vec{r}(z), \vec{U}z\vec{r}(z)\vec{y}w) \to \phi(\boldsymbol{S}z)_{\wedge}(\vec{r}(\boldsymbol{S}z), \vec{y}).$$

To simplify notation, let $\Gamma' \equiv \Gamma^s[\vec{x} := \vec{r}(z)]$, $\psi(z, \vec{y}) \equiv \phi(z)_{\wedge}(\vec{r}(z), \vec{y})$, $V' \equiv Vz\vec{r}(z)$ and $\vec{U}' \equiv \vec{U}z\vec{r}(z)$. Then,

$$\Gamma^o \vdash \psi(\mathbf{0}, \vec{y}), \tag{3.9}$$

$$\Gamma' \vdash \forall w{<}V'\vec{y}\,\psi(z, \vec{U}'\vec{y}w) \to \psi(\boldsymbol{S}z, \vec{y}). \tag{3.10}$$

Substituting $\vec{c}(z', \boldsymbol{S}z, w')$ for $\vec{y}$ in (3.10) and applying $\forall w'{<}d(z', \boldsymbol{S}z)$ to both sides ($\vec{c}, d$ to be defined later), we obtain

$$\begin{aligned}\forall w'{<}d(z', \boldsymbol{S}z)\,\Gamma'[\vec{y} := \vec{c}(z', \boldsymbol{S}z, w')] \vdash \\ \forall w'{<}d(z', \boldsymbol{S}z)\,\forall w{<}V'\vec{c}(z', \boldsymbol{S}z, w')\,\psi(z, \vec{U}'\vec{c}(z', \boldsymbol{S}z, w')w) \\ \to \forall w'{<}d(z', \boldsymbol{S}z)\,\psi(\boldsymbol{S}z, \vec{c}(z', \boldsymbol{S}z, w'))\end{aligned} \tag{3.11}$$

(operations on sets of formulae are understood pointwise). Consecutive bounded universal quantifiers may be condensed by means of

Lemma 3.5. *Let j, j_1, j_2 satisty $\vdash j_i(j(x_1, x_2)) = x_i$ for $i = 1, 2$. Given terms t, t' there is a term b such that*

$$\forall w{<}b\,\phi(j_1 w, j_2 w) \vdash_{\boldsymbol{T}_{\wedge}} \forall w{<}t\,\forall w'{<}t'(w)\,\phi(w, w').$$

[As concerns the applicability of the lemma, let us mention that there are well-known primitive recursive pairing functions, e.g. $\frac{1}{2}((x+y)^2+3x+y)$ or 2^x3^y. Proof hint: Define

$$\begin{aligned} a(\mathbf{0}) &= \mathbf{0},\\ a(\boldsymbol{S}w') &= \max\{a(w'), \boldsymbol{S}j(t,w')\} \end{aligned}$$

and

$$\begin{aligned} b(\mathbf{0}) &= \mathbf{0},\\ b(\boldsymbol{S}w) &= \max\{b(w), a(t'(w))\}. \end{aligned}$$

The required term is $b(t)$.]

From the lemma we conclude that

(3.12)
$$\Gamma'' \vdash \forall w{<}b\, \psi(z, \vec{U}\vec{c}(z', \boldsymbol{S}z, j_1w)j_2w) \to \forall w{<}d(z', \boldsymbol{S}z)\, \psi(\boldsymbol{S}z, \vec{c}(z', \boldsymbol{S}z, w))$$

for some $\Gamma'' \in \mathcal{A}$ and some term b. By defining

$$\begin{aligned} \vec{c}(\mathbf{0}, z, w) &= \vec{y},\\ \vec{c}(\boldsymbol{S}z', z, w) &= \vec{U}\vec{c}(z', \boldsymbol{S}z, j_1w)j_2w, \end{aligned}$$

and

$$\begin{aligned} d(\mathbf{0}, z) &= \mathbf{1},\\ d(\boldsymbol{S}z', z) &= b, \end{aligned}$$

(3.12) becomes

$$\Gamma'' \vdash \theta(\boldsymbol{S}z', z) \to \theta(z', \boldsymbol{S}z) \tag{3.13}$$

where

$$\theta(z', z) \equiv \forall w{<}d(z', z)\, \psi(z, \vec{c}(z', z, w)).$$

For $z' := v \dot- \boldsymbol{S}z$, (3.13) implies

$$\Gamma''', \theta(\boldsymbol{S}(v \dot- \boldsymbol{S}z), z) \vdash \theta(v \dot- \boldsymbol{S}z, \boldsymbol{S}z) \tag{3.14}$$

Letting $\Delta \equiv \Gamma''' \cup \{v \dot- z = \boldsymbol{S}(v \dot- \boldsymbol{S}z)\}$ and merging $\forall z{<}v\, \Gamma'''$ and Γ^o into one assumption set $\Gamma \in \mathcal{A}$, everything may be put together into one big deduction:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \forall z{<}v\,\Delta\end{array} \qquad \dfrac{\begin{array}{c}\Gamma\\ \psi(\mathbf{0},\vec{y})\end{array}}{\theta(v \dot- \mathbf{0},\mathbf{0})} \qquad \dfrac{[\Gamma'''] \qquad \dfrac{[v \dot- z = \boldsymbol{S}(v \dot- \boldsymbol{S}z)] \qquad [\theta(v \dot- z, z)]}{\theta(\boldsymbol{S}(v \dot- \boldsymbol{S}z), z)}}{\theta(v \dot- \boldsymbol{S}z, \boldsymbol{S}z)}\,(3.14)}{\theta(v \dot- v, v)}\,(*)}{\psi(v,\vec{y}) \equiv \phi(v)_\wedge(\vec{r}(v),\vec{y})}$$

where horizontal lines may conceal several steps. Rule $(*)$ is a generalization of (3.3) where multiple occurences of the same bounded universal quantifier are eliminated at once; its validity is left to the reader (exercise 1). $\square$

3.3 Exercises

1. Prove the following generalization of elimination rule (3.3):

$$\dfrac{\forall z{<}v\,\Phi \qquad \psi(\mathbf{0}) \qquad \begin{matrix}[\Phi] & [\psi(z)] \\ \psi(\boldsymbol{S}z) & \end{matrix}}{\psi(v)}$$

   where Φ is an arbitrary set of formulae and $\forall z{<}v\,\Phi = \{\forall z{<}v\,\phi \mid \phi \in \Phi\}$.

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