Lecture notes

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Chapter 1

Finite-type arithmetic

Higher types and higher-type entities (functionals) constitute a natural, and constructive, way of extending the expressive power of arithmetic without increasing its proof-theoretic strength. They also provide the syntactic means to express the effective information contained in proofs of arithmetical statements.

As a foundation for our subsequent proof-theoretic considerations, we present a basic theory $\mathbf{HA}^{\boldsymbol{\omega}}$ of intuitionistic finite-type arithmetic, together with an extensional variant $\mathbf{E}\text{-}\mathbf{HA}^{\boldsymbol{\omega}}$ and an intensional one $\mathbf{I}\text{-}\mathbf{HA}^{\boldsymbol{\omega}}$.

1.1 Syntax

The language of finite-type arithmetic The following types are present:

1. An atomic type $\mathcal{N}$ (the type of natural numbers),
2. a type $\sigma \times \tau$ for any two types σ and τ (product types),
3. a type τ^σ for any two types σ and τ (function types).

Notation. *$(\tau^\sigma)^\rho$ is simplified to $\tau^{\rho\sigma}$, and $\tau^{\vec{\sigma}}$ is governed by a similar convention; hence, τ^{σ^ρ} denotes the other alternative.*

Terms, and their types, are generated by

0. There is an inexhaustible supply (infinite set) of variables of each type.
1. 0 is a term of type $\mathcal{N}$; for any term t of type $\mathcal{N}$, $\boldsymbol{S}t$ is a term of type $\mathcal{N}$.
2. For any terms t of type τ, u of type $\tau^{\mathcal{N}\tau}$ and v of type $\mathcal{N}$, $\boldsymbol{R}tuv$ is a term of type τ.
3. For any terms t_l and t_r of types τ_l and τ_r respectively, $\langle t_l, t_r\rangle$ is a term of type $\tau_l \times \tau_r$.
4. For any term t of type $\tau_l \times \tau_r$, $\boldsymbol{p}_s t$ is a term of type τ_s, for $s \in \{l, r\}$.

5. For any variable x of type σ and term t of type τ, $\lambda x\, t$ is a term of type τ^σ.

6. For any terms t of type τ^σ and u of type σ, tu is a term of type τ.

Prime (or atomic) formulae are equations $t = u$ between terms of the same type. *Formulae* are formed from prime formulae by means of &, $\to$, $\forall$, and $\exists$.

Axioms and rules of inference Besides the usual (natural deduction or other) rules for the logical constants present in the system, we have rules for *equality*

$$\frac{}{t = t} \qquad\qquad \frac{t = u \qquad \phi(t)}{\phi(u)}\ ,$$

β-conversion

$$\frac{}{\boldsymbol{R}tu0 = t} \qquad\qquad \frac{}{\boldsymbol{R}tu(Sv) = uvRtuv}$$

$$\frac{}{\boldsymbol{p}_i\langle t_l, t_r\rangle = t_i}\ , \qquad i \in \{l, r\}$$

$$\frac{}{(\lambda x\ t)u = t[x := u]}\ ,$$

and *induction*

$$\frac{\phi(0) \qquad \forall x\, \big[\phi(x) \to \phi(\boldsymbol{S}x)\big]}{\phi(v)}\ .$$

The above axioms and rules constitute $\mathbf{HA}^{\boldsymbol{\omega}}$. We will also be interested in a number of extensions of this theory. *Extensional finite-type arithmetic*, $\mathbf{E\text{-}HA}^{\boldsymbol{\omega}}$, is obtained from $\mathbf{HA}^{\boldsymbol{\omega}}$ by the addition of the *extensionality rules*

$$\frac{\boldsymbol{p}_l t = \boldsymbol{p}_l u \qquad \boldsymbol{p}_r t = \boldsymbol{p}_r u}{t = u}$$

for t, u of product type, and

$$\frac{\forall x\, (tx = ux)}{t = u}\ , \qquad x \notin \mathrm{FV}(t, u)$$

for t, u of function type. *Intensional finite-type arithmetic*, $\mathbf{I\text{-}HA}^{\boldsymbol{\omega}}$, augments the language of $\mathbf{HA}^{\boldsymbol{\omega}}$ with *equality functionals* $\boldsymbol{E}_\tau$, one for each type τ, subject to

$$\frac{}{\boldsymbol{E}_\tau tu = 0 \leftrightarrow t = u} \qquad\qquad \frac{}{\boldsymbol{E}_\tau tu = 1 \leftrightarrow t \neq u}\ .$$

Classical (or Peano) finite-type arithmetic $\mathbf{PA}^{\boldsymbol{\omega}}$ is the extension of $\mathbf{HA}^{\boldsymbol{\omega}}$ by the *principle of the excluded middle*

$$\frac{}{\neg\neg\phi \to \phi}\ (\mathbf{PEM})\ .$$

1.2 Semantics

(to be written)

1.3 Exercises

1. Define addition and multiplication with the aid of primitive recursion.

2. Using your preferred logical formalism, show that if

$$\vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \phi,$$

then

$$\vdash_{\mathbf{HA}^{\boldsymbol{\omega}}} \phi[x := t].$$

3. Prove that extensionality is equivalent to the set of equations

$$(\eta) \qquad \begin{aligned} \langle \boldsymbol{p}_l t, \boldsymbol{p}_r t \rangle &= t, \ t \text{ of product type}, \\ \lambda x\ (tx) &= t, \ t \text{ of function type}, x \notin \mathrm{FV}(t). \end{aligned}$$

4. Extensional equality $t =_e u$ between terms t, u of the same type is inductively defined by

$$t =_e u \equiv \begin{cases} t = u & t,\, u \text{ of atomic type}, \\ \boldsymbol{p}_l t =_e \boldsymbol{p}_l u \ \&\ \boldsymbol{p}_r t =_e \boldsymbol{p}_r u & t,\, u \text{ of product type}, \\ \forall x\, (tx =_e ux) & t,\, u \text{ of function type}. \end{cases}$$

Show that extensionality is equivalent to the schema

$$t =_e u \leftrightarrow t = u,$$

and conclude that, in $\mathbf{E}\text{-}\mathbf{HA}^{\boldsymbol{\omega}}$, equality at higher types is reducible to equality between terms of type $\mathcal{N}$.

5. (Closure of $\mathbf{HA}^{\boldsymbol{\omega}}$ under *mutual primitive recursion.*) Let $\vec{\tau} \equiv \tau_1, \ldots, \tau_n$ be a list of types, $\vec{t}$ a list of terms of types $\vec{\tau}$ (i.e., each t_i has type τ_i) and $\vec{u}$ a list of terms of types $\vec{\tau}^{\mathcal{N}\vec{\tau}}$ (i.e., each u_i has type ${\tau_i}^{\mathcal{N}\tau_1\cdots\tau_n}$). Construct terms $\vec{r} \equiv \vec{r}(z)$, z fresh, with the properties

$$\begin{aligned} \vec{r}(0) &= \vec{t}, \\ \vec{r}(Sv) &= \vec{u}v\vec{r}(v). \end{aligned}$$

Chapter 2

Modified realizability

The term *realizability* refers to any one of a family of translations that may be seen as formalizations of the BHK interpretation of the logical constants; for a more complete description of the BHK interpretation, the reader may consult Troelstra and van Dalen (1988).

Modified realizability is a variant of realizability where the realizing objects are functionals. This notion of realizability is well adapted to the study of typed theories; it will be our first, and simplest, example of term extraction.

2.1 Definition

To each formula ϕ in the language of finite-type arithmetic we associate its *modified realizability interpretation* ϕ^{mr}, which is a formula of the form

$$\exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})$$

with the same free variables as ϕ, where $\phi_{\mathrm{mr}}(\vec{x})$ ($\vec{x}$ *modified realizes* ϕ, alternative notation: $\vec{x}\mathbf{mr}\phi$) is an $\exists$-free formula and $\vec{x}$ a possibly empty list of variables. The associations $(\ )_{\mathrm{mr}}$ and $(\ )^{\mathrm{mr}}$ are defined by the following induction:

> For ϕ atomic, $\phi^{\mathrm{mr}} \equiv \phi$,
>
> $(\phi \mathbin{\&} \psi)^{\mathrm{mr}} \equiv \exists \vec{x}, \vec{y} \left[\phi_{\mathrm{mr}}(\vec{x}) \mathbin{\&} \psi_{\mathrm{mr}}(\vec{y})\right]$,
>
> $(\phi \to \psi)^{\mathrm{mr}} \equiv \exists \vec{Y} \left[\forall \vec{x}\, (\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{Y}\vec{x}))\right]$,
>
> $(\forall z\, \phi(z))^{\mathrm{mr}} \equiv \exists \vec{X} \left[\forall z\, (\phi(z)_{\mathrm{mr}}(\vec{X}z))\right]$,
>
> $(\exists z\, \phi(z))^{\mathrm{mr}} \equiv \exists z, \vec{x} \left[\phi(z)_{\mathrm{mr}}(\vec{x})\right]$,

where, in each case, the $\exists$-free kernel is delimited by brackets.

Remark on notation. For a (possible) dependence of a formula ϕ on a variable z to be made explicit, it is customary to write $\phi(z)$ in place of ϕ. Then, substitution of a term v for z in ϕ is conveniently denoted $\phi(v)$ instead of $\phi[z := v]$.

This may contribute a lot to readability, but it may also lead to error if sufficient attention is not paid. Fortunately, one possible source of ambiguity is lifted as soon as we know that *modified realizability commutes with substitution*, namely

$$\phi(v)_{\mathrm{mr}}(\vec{x}) \equiv \phi(z)_{\mathrm{mr}}(\vec{x})[z := v],$$

and hence

$$\phi(v)^{\mathrm{mr}} \equiv \phi(z)^{\mathrm{mr}}[z := v].$$

Proposition 2.1.1. *Let* $\phi^{\mathrm{mr}} \equiv \exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})$.

1. $\phi_{\mathrm{mr}}(\vec{x})$ *is* $\exists$*-free, and if* ϕ *is* $\exists$*-free, then* $\vec{x}$ *is empty and* $\phi^{\mathrm{mr}} \equiv \phi_{\mathrm{mr}} \equiv \phi$.
2. *If* ψ *is* $\exists$*-free, then* $(\exists \vec{y}\, \psi)^{\mathrm{mr}} \equiv \exists \vec{y}\, \psi$*; in particular,* $(\phi^{\mathrm{mr}})^{\mathrm{mr}} \equiv \phi^{\mathrm{mr}}$.

Proof. Exercise. □

2.2 Soundness

In the following, we are going to employ the *modified realizability schema*

$$\text{(MR)} \qquad \phi^{\mathrm{mr}} \leftrightarrow \phi.$$

This is not among the axioms usually considered for arithmetic; we will shortly prove its equivalence to something more familiar (??).

Theorem 2.2.1 (soundness)**.** *Let* $\boldsymbol{H}$ *be any one of* $\boldsymbol{HA^\omega}$, $\boldsymbol{E\text{-}HA^\omega}$, $\boldsymbol{I\text{-}HA^\omega}$*, and let* $\boldsymbol{H}-\exists$ *be* $\boldsymbol{H}$ *with the rules for the existential quantifier removed. If* $\vdash_{\boldsymbol{H}+MR} \phi$*, then* $\vdash_{\boldsymbol{H}-\exists} \phi_{\mathrm{mr}}(\vec{t})$ *for a suitable list* $\vec{t}$ *of terms satisfying* $\mathrm{FV}(\vec{t}) \subseteq \mathrm{FV}(\phi)$.

Proof. We are going to apply induction on the proofs of $\mathbf{H}$+MR, for the purpose of which we will need the (superficially) stronger statement

> If $\Phi \vdash_{\mathbf{H}+\mathrm{MR}} \phi$, then $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H}-\exists} \phi_{\mathrm{mr}}(\vec{t})$, where all free variables of $\vec{t}$ are among those free in ϕ and the realizing variables in Φ_{mr}.

where Φ is an arbitrary (finite) set of formulae and $\Phi_{\mathrm{mr}} = \{\phi_{\mathrm{mr}} \mid \phi \in \Phi\}$. Of the axioms and rules of $\mathbf{H} - \exists$, those that are $\exists$-free are *self-realizing* and don't need any further examination; this includes the "extras" of $\mathbf{E\text{-}HA^\omega}$ and $\mathbf{I\text{-}HA^\omega}$. For most of the others, a deduction will be furnished that may be combined with the induction hypotheses in an obvious way to yield the required conclusion. Exception: $\exists$-rules.

Natural deduction

$$\frac{\phi \qquad \psi}{\phi \,\&\, \psi} : \qquad\qquad \frac{\phi_{\mathrm{mr}}(\vec{t}) \qquad \psi_{\mathrm{mr}}(\vec{u})}{\phi_{\mathrm{mr}}(\vec{t}) \,\&\, \psi_{\mathrm{mr}}(\vec{u}) \equiv (\phi \,\&\, \psi)_{\mathrm{mr}}(\vec{t}, \vec{u})}$$

$$\frac{\phi \ \&\ \psi}{\phi} : \qquad (\phi \ \&\ \psi)_{\mathrm{mr}}(\vec{t},\vec{u}) \equiv \frac{\phi_{\mathrm{mr}}(\vec{t}) \ \&\ \psi_{\mathrm{mr}}(\vec{u})}{\phi_{\mathrm{mr}}(\vec{t})}$$

$$\frac{\begin{array}{c}[\phi]\\ \psi\end{array}}{\phi \to \psi} : \qquad \cfrac{\cfrac{\begin{array}{c}[\phi_{\mathrm{mr}}(\vec{x})]\\ \psi_{\mathrm{mr}}(\vec{u})\end{array}}{\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{u})}}{\forall \vec{x}\,(\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{u})) \leftrightarrow (\phi \to \psi)_{\mathrm{mr}}(\lambda \vec{x}\ \vec{u})}$$

$$\frac{\phi \to \psi \qquad \phi}{\psi} : \qquad \cfrac{(\phi \to \psi)_{\mathrm{mr}}(\vec{t}) \equiv \cfrac{\forall \vec{x}\,(\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{t}\vec{x}))}{\phi_{\mathrm{mr}}(\vec{u}) \to \psi_{\mathrm{mr}}(\vec{t}\vec{u})} \qquad \phi_{\mathrm{mr}}(\vec{u})}{\psi_{\mathrm{mr}}(\vec{t}\vec{u})}$$

$$\frac{\phi(z)}{\forall z\,\phi(z)} : \qquad \frac{\phi(z)_{\mathrm{mr}}(\vec{t})}{\forall z\,(\phi(z)_{\mathrm{mr}}(\vec{t})) \leftrightarrow (\forall z\,\phi(z))_{\mathrm{mr}}(\lambda z\ \vec{t})}$$

$$\frac{\forall z\,\phi(z)}{\phi(v)} : \qquad (\forall z\,\phi(z))_{\mathrm{mr}}(\vec{t}) \equiv \frac{\forall z\,(\phi(z)_{\mathrm{mr}}(\vec{t}z))}{\phi(v)_{\mathrm{mr}}(\vec{t}v)}$$

$\dfrac{\phi(v)}{\exists z\,\phi(z)}$: Nothing to prove; the conclusion coincides with the induction hypothesis (this is because the interpretation of $\exists$ is "trivial", in the sense that it merely turns the existentially quantified variable into a realizing variable).

$\dfrac{\exists z\,\phi(z) \qquad \begin{array}{c}[\phi(z)]\\ \psi\end{array}}{\psi}$: By hypothesis, there are deductions $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H}-\exists} \phi(v)_{\mathrm{mr}}(\vec{t})$ and $\Phi_{\mathrm{mr}}, \phi(z)_{\mathrm{mr}}(\vec{x}) \vdash_{\mathbf{H}-\exists} \psi_{\mathrm{mr}}(\vec{u})$, whence $\Phi_{\mathrm{mr}} \vdash_{\mathbf{H}-\exists} \psi_{\mathrm{mr}}(\vec{u}[\vec{x} := \vec{t}])$.

Equality

$$\frac{t = u \qquad \phi(t)}{\phi(u)} : \qquad \frac{t = u \qquad \phi(t)_{\mathrm{mr}}(\vec{v})}{\phi(u)_{\mathrm{mr}}(\vec{v})}$$

(using the fact that modified realizability commutes with substitution).

Induction

$$\frac{\phi(0) \qquad \forall z\,(\phi(z) \to \phi(\boldsymbol{S}z))}{\phi(v)} :$$

$$\cfrac{\phi(0)_{\mathrm{mr}}(\vec{t}) \qquad \cfrac{\forall z, \vec{x}\,(\phi(z)_{\mathrm{mr}}(\vec{x}) \to \phi(\boldsymbol{S}z)_{\mathrm{mr}}(\vec{u}z\vec{x}))}{\forall z\,(\phi(z)_{\mathrm{mr}}(\vec{w}(z)) \to \phi(\boldsymbol{S}z)_{\mathrm{mr}}(\vec{u}z\vec{w}(z)))}}{\phi(v)_{\mathrm{mr}}(\vec{w}(v))}$$

where $\vec{w} \equiv \vec{w}(z)$ is a list of terms such that

$$\vec{w}(0) = \vec{t},$$
$$\vec{w}(\boldsymbol{S}z) = \vec{u}z\vec{w}(z).$$

One way to guarantee the existence of $\vec{w}(x)$ is by formulating the system with *mutual primitive recursion*. In the presence of product types, however, mutual primitive recursion is reducible to ordinary primitive recursion; e.g., $\vec{w}(z)$ may be constructed as follows: Assuming $\vec{t} \equiv t_1, \ldots, t_n$ and $\vec{u} \equiv u_1, \ldots, u_n$, define

$$\boldsymbol{t} \equiv \langle \vec{t} \rangle \equiv \langle t_1, \ldots, t_n \rangle,$$
$$\boldsymbol{u} \equiv \lambda z, y \, \langle \vec{u}z(\vec{q}y) \rangle,$$

where $\langle \ \rangle$ denotes an arbitrary representation of n-tuples (using pairing), and $\vec{q}y \equiv q_1 y, \ldots, q_n y$ are the corresponding projections. Then,

$$\vec{w}(z) \equiv \vec{q}(\boldsymbol{Rtu}z)$$

are our desired terms.

MR

Since $(\phi^{\mathrm{mr}})_{\mathrm{mr}}(\vec{x}) \equiv \phi_{\mathrm{mr}}(\vec{x})$, a simple calculation yields

$$(\phi^{\mathrm{mr}} \leftrightarrow \phi)^{\mathrm{mr}} \equiv \exists \vec{X}, \vec{Y} \left[\forall \vec{x} \, (\phi_{\mathrm{mr}}(\vec{x}) \to \phi_{\mathrm{mr}}(\vec{X}\vec{x})) \; \& \; \forall \vec{y} \, (\phi_{\mathrm{mr}}(\vec{y}) \to \phi_{\mathrm{mr}}(\vec{Y}\vec{y})) \right]$$

which has the trivial realizers $\lambda\vec{x}\,\vec{x}, \lambda\vec{y}\,\vec{y}$. □

2.3 Axiomatization

Here, we are going to show that $\mathbf{HA}^{\boldsymbol{\omega}} + \mathrm{MR}$ may be axiomatized by familiar principles.

Theorem 2.3.1. *Over $\boldsymbol{HA}^{\boldsymbol{\omega}}$, the following schemata are equivalent:*

1. *MR: $\phi^{\mathrm{mr}} \leftrightarrow \phi$,*
2. *$\phi^{\mathrm{mr}} \to \phi$,*
3. *$AC + IP^{\omega}_{ef}$, where*

   (AC) $\quad \forall \vec{x} \, \exists \vec{y} \, \phi(\vec{x}, \vec{y}) \to \exists \vec{Y} \, \forall \vec{x} \, \phi(\vec{x}, \vec{Y}\vec{x}),$

   ($\mathrm{IP}^{\omega}_{\mathrm{ef}}$) $\quad (\phi \to \exists x \, \psi) \to \exists x \, (\phi \to \psi), \qquad \phi$ *∃-free.*

Proof. *1. → 2.* Obvious.

2. → 3. It suffices to show that each instance θ of one of AC and $\mathrm{IP}^{\omega}_{\mathrm{ef}}$ is modified realizable, $\vdash_{\mathbf{HA}^{\omega}} \theta^{\mathrm{mr}}$. In each case, this holds trivially, and is left as an exercise.

3. → *1.* We proceed by structural induction, where $\phi^{\mathrm{mr}} \equiv \exists\vec{x}\,\phi_{\mathrm{mr}}(\vec{x})$ and $\psi^{\mathrm{mr}} \equiv \exists\vec{y}\,\psi_{\mathrm{mr}}(\vec{y})$:

(a) Atomic formulae are self-realizing.

(b)

$$\begin{aligned}(\phi \mathbin{\&} \psi)^{\mathrm{mr}} &\equiv \exists\vec{x},\vec{y}\,(\phi_{\mathrm{mr}}(\vec{x}) \mathbin{\&} \psi_{\mathrm{mr}}(\vec{y}))\\ &\leftrightarrow (\exists\vec{x}\,\phi_{\mathrm{mr}}(\vec{x})) \mathbin{\&} (\exists\vec{y}\,\psi_{\mathrm{mr}}(\vec{y}))\\ &\equiv \phi^{\mathrm{mr}} \mathbin{\&} \psi^{\mathrm{mr}}\\ &\leftrightarrow \phi \mathbin{\&} \psi.\end{aligned}$$

(c)

$$\begin{aligned}(\phi \to \psi)^{\mathrm{mr}} &\equiv \exists\vec{Y}\,\forall\vec{x}\,(\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{Y}\vec{x}))\\ &\leftrightarrow \forall\vec{x}\,\exists\vec{y}\,(\phi_{\mathrm{mr}}(\vec{x}) \to \psi_{\mathrm{mr}}(\vec{y}))\\ &\leftrightarrow \forall\vec{x}\,(\phi_{\mathrm{mr}}(\vec{x}) \to \exists\vec{y}\,\psi_{\mathrm{mr}}(\vec{y}))\\ &\leftrightarrow (\exists\vec{x}\,\phi_{\mathrm{mr}}(\vec{x})) \to (\exists\vec{y}\,\psi_{\mathrm{mr}}(\vec{y}))\\ &\equiv \phi^{\mathrm{mr}} \to \psi^{\mathrm{mr}}\\ &\leftrightarrow \phi \to \psi.\end{aligned}$$

(d)

$$\begin{aligned}(\forall z\,\phi(z))^{\mathrm{mr}} &\equiv \exists\vec{X}\,\forall z\,(\phi(z)_{\mathrm{mr}}(\vec{X}z))\\ &\leftrightarrow \forall z\,\exists\vec{x}\,(\phi(z)_{\mathrm{mr}}(\vec{x}))\\ &\equiv \forall z\,\phi(z)^{\mathrm{mr}}\\ &\leftrightarrow \forall z\,\phi(z).\end{aligned}$$

(e)

$$\begin{aligned}(\exists z\,\phi(z))^{\mathrm{mr}} &\equiv \exists z,\vec{x}\,(\phi(z)_{\mathrm{mr}}(\vec{x}))\\ &\equiv \exists z\,(\phi(z)^{\mathrm{mr}})\\ &\leftrightarrow \exists z\,\phi(z).\end{aligned}$$

□

2.4 Exercises

1. Prove the following:
   (a) $(\forall\vec{z}\,\phi(\vec{z}))^{\mathrm{mr}} \equiv \exists\vec{X}\,\forall\vec{z}\,(\phi(\vec{z})_{\mathrm{mr}}(\vec{X}\vec{z}))$.
   (b) $(\exists\vec{z}\,\phi(\vec{z}))^{\mathrm{mr}} \equiv \exists\vec{z},\vec{x}\,(\phi(\vec{z})_{\mathrm{mr}}(\vec{x}))$.
2. Show that modified realizability commutes with substitution,

$$\phi(v)_{\mathrm{mr}}(\vec{x}) \equiv \phi(z)_{\mathrm{mr}}(\vec{x})[z := v], \qquad (\vec{x} \text{ not free in } v).$$

3. Prove whatever has been left as an exercise for you (the reader).

4. Expand $(\neg\neg\phi \to \phi)^{\mathrm{mr}}$ and show that it is provable in $\mathbf{PA}^{\boldsymbol{\omega}}$. Conclude that if $\vdash_{\mathbf{PA}^{\omega}} \phi$, then $\vdash_{\mathbf{PA}^{\omega}} \phi^{\mathrm{mr}}$ (soundness for $\mathbf{PA}^{\boldsymbol{\omega}}$).

5. *Harrop* formulae are defined by the induction

   (a) atomic formulae are Harrop,

   (b) if ϕ and ψ are Harrop, then $\phi \mathbin{\&} \psi$ is Harrop,

   (c) if ψ is Harrop, then $\phi \to \psi$ is Harrop (ϕ any formula),

   (d) if ϕ is Harrop, then $\forall x\, \phi$ is Harrop.

   Prove that a formula ϕ is Harrop if and only if ϕ^{mr} is $\exists$-free, i.e., if in $\phi^{\mathrm{mr}} \equiv \exists \vec{x}\, \phi_{\mathrm{mr}}(\vec{x})$, $\vec{x}$ is the empty list.

6. Show that for any instance θ of schema

   $$(\mathrm{IP}^{\omega}_{\mathrm{Harrop}}) \qquad (\phi \to \exists x\, \psi) \to \exists x\, (\phi \to \psi), \qquad \phi \text{ Harrop}$$

   there are terms $\vec{t}$ such that $\vdash_{\mathbf{HA}^{\omega}} \theta_{\mathrm{mr}}(\vec{t})$.

Bibliography

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