A Birkhoff-James cosine function for normed linear spaces

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Abstract

The cosine function is a classical tool for measuring angles in inner product spaces, and it has various extensions to normed linear spaces. In this paper, we investigate a cosine function for the convex angle formed by two nonzero elements of a complex normed linear space, in connection with recent results on the Birkhoff-James approximate orthogonality sets.

Key words: norm, Birkhoff-James orthogonality, Birkhoff-James ε-orthogonality, linear functional, cosine, sine.

AMS Subject Classifications: 46B99, 47A12.

1 Introduction

The study of angle functions and their use to designate the measure of angles have a long history, especially in the context of inner product spaces. In the last decades, the problem of extending angle functions to normed linear spaces in connection with the notion of orthogonality has attracted the attention of researchers; see [2] and the references therein. To this direction, and motivated by the Birkhoff-James orthogonality, Szostok [26] introduced the sine function $s(\chi,\psi)=\inf\limits_{\lambda\in\mathbb{R}}\dfrac{\|\chi+\lambda\psi\|}{\|\chi\|}$ for two nonzero elements χ and ψ of a real normed linear space $(\mathcal{V},\|\cdot\|)$. He also obtained that this function is continuous, and if the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $s(\chi,\psi)$ coincides with the standard sine function $\sqrt{1-\left(\dfrac{\langle\chi,\psi\rangle}{\|\chi\|\,\|\psi\|}\right)^2}$. Furthermore, Chmieliński [7] observed that for any $\varepsilon\in[0,1)$, $\sqrt{1-s(\chi,\psi)^2}\leq\varepsilon$ if and only if $s(\chi,\psi)\geq\sqrt{1-\varepsilon^2}$, or equivalently, if and only if $\|\chi+\lambda\psi\|\geq\sqrt{1-\varepsilon^2}\,\|\chi\|$ for all $\lambda\in\mathbb{R}$, without discussing further this concept. In this article, we consider complex normed linear spaces, and describe and study the function $\sqrt{1-s(\chi,\psi)^2}$ in terms of the Birkhoff-James approximate orthogonality sets, taking advantage of the geometrical properties and the rich structure of these sets.

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Recall that, if $(\mathcal{A}, \|\cdot\|)$ (for simplicity, $\mathcal{A}$) is a unital normed algebra over $\mathbb{C}$, with identity element $\mathbf{1}$, and $\mathcal{A}^*$ is the dual space of $\mathcal{A}$ (i.e., the Banach space of all continuous linear functionals of $\mathcal{A}$), then the *numerical range* (also known as the *field of values*) of an element $\alpha \in \mathcal{A}$ is defined and denoted by $F(\alpha) = \{f(\alpha) : f \in \mathcal{A}^*, f(\mathbf{1}) = 1, \|f\| = 1\}$. This set has been studied extensively for decades, and it is useful in understanding matrices and operators; see [5, 6, 15, 25] and the references therein. Stampfli and Williams [25, Theorem 4], and later Bonsall and Duncan [6, Lemma 6.22.1], observed that the numerical range $F(\alpha)$ coincides with an infinite intersection of closed (circular) disks, namely,

$$F(\alpha) = \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \|\alpha - \lambda \mathbf{1}\|\right), \tag{1}$$

where, for any scalar $\lambda \in \mathbb{C}$, $\mathcal{D}\left(\lambda, \|\alpha - \lambda \mathbf{1}\|\right) = \{\mu \in \mathbb{C} : |\mu - \lambda| \leq \|\alpha - \lambda \mathbf{1}\|\}$.

Let $(\mathcal{X}, \|\cdot\|)$ (for simplicity, $\mathcal{X}$) be a complex normed linear space. For two elements χ and ψ of $\mathcal{X}$, χ is said to be *Birkhoff-James orthogonal* to ψ, denoted by $\chi \perp_{BJ} \psi$, if $\|\chi + \lambda\psi\| \geq \|\chi\|$ for all $\lambda \in \mathbb{C}$ [4, 16]. This orthogonality is homogeneous, but it is neither symmetric nor additive [16]. Furthermore, for any $\varepsilon \in [0, 1)$, χ is said to be *Birkhoff-James ε-orthogonal* to ψ, denoted by $\chi \perp_{BJ}^{\varepsilon} \psi$, if [7, 12]

$$\|\chi + \lambda\psi\| \geq \sqrt{1 - \varepsilon^2}\,\|\chi\|, \quad \forall\, \lambda \in \mathbb{C}. \tag{2}$$

The Birkhoff-James ε-orthogonality is also homogeneous. If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then a $\chi \in \mathcal{X}$ is said to be *ε-orthogonal* to a $\psi \in \mathcal{X}$, denoted by $\chi \perp^{\varepsilon} \psi$, if $|\langle\chi, \psi\rangle| \leq \varepsilon\,\|\chi\|\,\|\psi\|$; apparently, for $\varepsilon = 0$, we get the standard orthogonality. Moreover, for any $\varepsilon \in [0, 1)$, $\chi \perp^{\varepsilon} \psi$ if and only if $\chi \perp_{BJ}^{\varepsilon} \psi$ [7, 12].

Motivated by (1) and (2), Chorianopoulos and Psarrakos [9] (for matrices) and Karamanlis and Psarrakos [18] (for elements of a normed linear space) introduced and studied the following set: For any $\chi, \psi \in \mathcal{X}$, with $\chi \neq 0$, and any $\varepsilon \in [0, 1)$, the *Birkhoff-James ε-orthogonality set of ψ with respect to χ* is defined and denoted by[1] $F_{\varepsilon}(\psi; \chi) = \{\mu \in \mathbb{C} : \chi \perp_{BJ}^{\varepsilon} (\psi - \mu\chi)\}$.

The Birkhoff-James ε-orthogonality set $F_{\varepsilon}(\psi; \chi)$ is a *compact* and *convex* subset of the complex plane, and it contains the origin if and only if $\chi \perp_{BJ}^{\varepsilon} \psi$. It is a direct generalization of the numerical range, and appears to have interesting properties [9, 18, 23] (see Section 2). Moreover, the set valued function $\varepsilon \mapsto F_{\varepsilon}(\psi; \chi)$ is continuous with respect to the Hausdorff metric (see Proposition 2.1 below) and increasing for χ and ψ not co-linear (see Remark 2.1 in [23]). This behaviour of $F_{\varepsilon}(\psi; \chi)$ leads in a natural way to a definition of a cosine for the convex angle formed by span$\{\chi\}$ and span$\{\psi\}$ (see Sections 3 and 4); in particular, one can define a cosine as the minimum value of parameter $\varepsilon \in [0, 1)$ such that $0 \in F_{\varepsilon}(\psi; \chi)$. Remarkably, this cosine function coincides with the function $\sqrt{1 - s(\chi, \psi)^2}$ [7, 26].

Basic properties of the proposed cosine function are derived in Section 5, and the case of semi-inner product spaces is considered in Section 6. The relation of the new cosine with the Phythagorean cosine (obtained by the Phythagorean orthogonality and the law of cosines) and the isosceles cosine (obtained by the isosceles orthogonality) is investigated in Section 7. Finally, in Section 8, the proposed cosine function is discussed for bounded linear operators, and two characterizations of the Birkhoff-James orthogonality of bounded linear operators are presented (extending Theorems 2.1 and 2.8 of [24] to the complex case).

[1]For notational convenience, in this article, we consider the set $F_{\varepsilon}(\psi; \chi)$ instead of $F_{\varepsilon}(\chi; \psi)$ which was introduced and studied in [18, 23].

2 Definition and basic properties of the Birkhoff-James ε-orthogonality set

Consider a complex normed linear space $(\mathcal{X}, \|\cdot\|)$ (for simplicity, $\mathcal{X}$), and let $\chi, \psi \in \mathcal{X}$ with $\chi \neq 0$. For any $\varepsilon \in [0,1)$, it is straightforward to see that [9, 18, 23]:

$$\begin{aligned}
F_\varepsilon(\psi;\chi) &= \{\mu \in \mathbb{C} : \chi \perp_{BJ}^{\varepsilon} (\psi - \mu\chi)\} \\
&= \left\{\mu \in \mathbb{C} : \|\chi + \lambda(\psi - \mu\chi)\| \geq \sqrt{1-\varepsilon^2}\,\|\chi\|, \ \forall\, \lambda \in \mathbb{C}\right\} \\
&= \left\{\mu \in \mathbb{C} : \left\|\chi - \frac{1}{\lambda}(\psi - \mu\chi)\right\| \geq \sqrt{1-\varepsilon^2}\,\|\chi\|, \ \forall\, \lambda \in \mathbb{C}\setminus\{0\}\right\} \\
&= \left\{\mu \in \mathbb{C} : \frac{1}{|\lambda|}\,\|\lambda\chi - (\psi - \mu\chi)\| \geq \sqrt{1-\varepsilon^2}\,\|\chi\|, \ \forall\, \lambda \in \mathbb{C}\setminus\{0\}\right\} \\
&= \left\{\mu \in \mathbb{C} : \|\psi - (\mu - \lambda)\chi\| \geq \sqrt{1-\varepsilon^2}\,\|\chi\|\,|\lambda|, \ \forall\, \lambda \in \mathbb{C}\right\} \\
&= \left\{\mu \in \mathbb{C} : \|\psi - \lambda\chi\| \geq \sqrt{1-\varepsilon^2}\,\|\chi\|\,|\mu - \lambda|, \ \forall\, \lambda \in \mathbb{C}\right\} \\
&= \bigcap_{\lambda\in\mathbb{C}} \mathcal{D}\left(\lambda, \frac{\|\psi - \lambda\chi\|}{\sqrt{1-\varepsilon^2}\,\|\chi\|}\right). \qquad (3)
\end{aligned}$$

Corollary 2.2 of [16] implies that $F_\varepsilon(\psi;\chi)$ is always *non-empty* (see also Proposition 2.1 of [18]), and the defining formula (3) implies that $F_\varepsilon(\psi;\chi)$ is a *compact* and *convex* subset of $\mathbb{C}$ that lies in the closed disk $\mathcal{D}\left(0, \frac{\|\psi\|}{\sqrt{1-\varepsilon^2}\,\|\chi\|}\right)$. Moreover, for any $0 \leq \varepsilon_1 < \varepsilon_2 < 1$, $F_{\varepsilon_1}(\psi;\chi) \subseteq F_{\varepsilon_2}(\psi;\chi)$. The Birkhoff-James ε-orthogonality set is a direct generalization of the standard numerical range. In particular, for $\mathcal{X} = \mathcal{A}$, $\chi = \mathbf{1}$, $\psi = \alpha$ and $\varepsilon = 0$, we have $F_0(\alpha;\mathbf{1}) = F(\alpha)$; see (1) and (3).

Let $\mathcal{X}^*$ denote the dual space of $\mathcal{X}$, i.e., the normed linear space of all continuous linear functionals of $\mathcal{X}$ (using the induced operator norm), and for any $\chi, \psi \in \mathcal{X}$, with $\chi \neq 0$, and $\varepsilon \in [0,1)$, define the set

$$L_\varepsilon(\chi) = \left\{f \in \mathcal{X}^* : f(\chi) = \sqrt{1-\varepsilon^2}\,\|\chi\| \text{ and } \|f\| \leq 1\right\},$$

which is always non-empty, closed and convex [23, Lemma 2.1]. Then, it holds that [23, Theorem 2.2]

$$F_\varepsilon(\psi;\chi) = \left\{\frac{f(\psi)}{\sqrt{1-\varepsilon^2}\,\|\chi\|} : f \in L_\varepsilon(\chi)\right\}. \qquad (4)$$

Let $\chi, \psi \in \mathcal{X}$ with $\chi \neq 0$. For convenience, we summarize some results of [18, 23] (see also [8, 9] for matrices), describing basic properties of the Birkhoff-James ε-orthogonality set. We also remark that, in the remainder of the paper, the zero vector is always considered as a scalar multiple of χ.

(P_1) For any $a, b \in \mathbb{C}$ and any $\varepsilon \in [0,1)$, $F_\varepsilon(a\psi + b\chi;\chi) = a\,F_\varepsilon(\psi;\chi) + b$.

(P_2) For any nonzero $b \in \mathbb{C}$ and any $\varepsilon \in [0,1)$, $F_\varepsilon(\psi;b\chi) = \frac{1}{b}\,F_\varepsilon(\psi;\chi)$.

(P_3) If ψ is a nonzero element of $\mathcal{X}$, then for any $\varepsilon \in [0,1)$,

$$\left\{\mu^{-1} \in \mathbb{C} : \mu \in F_\varepsilon(\psi;\chi),\ |\mu| \geq \frac{\|\psi\|}{\|\chi\|}\right\} \subseteq F_\varepsilon(\chi;\psi).$$

(P_4) Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two equivalent norms acting on $\mathcal{X}$, i.e., there exist two real numbers $C, c > 0$ such that $c\,\|\zeta\|_\alpha \le \|\zeta\|_\beta \le C\,\|\zeta\|_\alpha$ for all $\zeta \in \mathcal{X}$, and denote by $F_\varepsilon^{\|\cdot\|_\alpha}(\psi;\chi)$ and $F_\varepsilon^{\|\cdot\|_\beta}(\psi;\chi)$ the corresponding Birkhoff-James ε-orthogonality sets. Then, for any $\varepsilon \in [0,1)$, it holds that $F_\varepsilon^{\|\cdot\|_\alpha}(\psi;\chi) \subseteq F_{\varepsilon'}^{\|\cdot\|_\beta}(\psi;\chi)$, where $\varepsilon' = \sqrt{1 - \dfrac{c^2(1-\varepsilon^2)}{C^2}}$.

(P_5) $\psi = a\chi$ for some $a \in \mathbb{C}$ if and only if $F_\varepsilon(\psi;\chi) = \{a\}$ for every $\varepsilon \in [0,1)$.

(P_6) If ψ is not a scalar multiple of χ, then for any $0 \le \varepsilon_1 < \varepsilon_2 < 1$, $F_{\varepsilon_1}(\psi;\chi)$ lies in the interior of $F_{\varepsilon_2}(\psi;\chi)$.

(P_7) If ψ is not a scalar multiple of χ, then for any $\varepsilon \in (0,1)$, $F_\varepsilon(\psi;\chi)$ has a non-empty interior.

(P_8) If ψ is not a scalar multiple of χ, then for any bounded region $\Omega \subset \mathbb{C}$, there is an $\varepsilon_\Omega \in [0,1)$ such that $\Omega \subseteq F_{\varepsilon_\Omega}(\psi;\chi)$. (This means that if ψ is not a scalar multiple of χ, then $F_\varepsilon(\psi;\chi)$ can be arbitrarily large for ε sufficiently close to 1.)

(P_9) For any $\varepsilon \in [0,1)$, a scalar $\mu_0 \in F_\varepsilon(\psi;\chi)$ lies on the boundary $\partial F_\varepsilon(\psi;\chi)$ if and only if $\inf_{\lambda\in\mathbb{C}} \left\{ \|\psi - \lambda\chi\| - \sqrt{1-\varepsilon^2}\,\|\chi\|\,|\mu_0 - \lambda| \right\} = 0$. For $\varepsilon \in (0,1)$, infimum can be replaced by minimum, and in this case, $\mu_0 \in \partial F_\varepsilon(\psi;\chi)$ if and only if $\|\psi - \lambda_0\chi\| = \sqrt{1-\varepsilon^2}\,\|\chi\|\,|\mu_0 - \lambda_0|$ for some $\lambda_0 \in \mathbb{C}$.

(P_{10}) For any $\varepsilon \in (0,1)$,

$$\mathrm{Int}\,[F_\varepsilon(\psi;\chi)] = \left\{ \mu \in \mathbb{C} : \|\psi - \lambda\chi\| > \sqrt{1-\varepsilon^2}\,\|\chi\|\,|\mu - \lambda|,\ \forall\, \lambda \in \mathbb{C} \right\}.$$

(P_{11}) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then for any $\varepsilon \in [0,1)$,

$$F_\varepsilon(\psi;\chi) = \mathcal{D}\left(\frac{\langle\psi,\chi\rangle}{\|\chi\|^2},\ \left\| \psi - \frac{\langle\psi,\chi\rangle}{\|\chi\|^2}\,\chi \right\| \frac{\varepsilon}{\sqrt{1-\varepsilon^2}\,\|\chi\|} \right).$$

(P_{12}) For any $\chi, \psi_1, \psi_2, \in \mathcal{X}$ and $\varepsilon \in [0,1)$, it holds that $F_\varepsilon(\psi_1 + \psi_2;\chi) \subseteq F_\varepsilon(\psi_1;\chi) + F_\varepsilon(\psi_2;\chi)$.

The next result was obtained for matrices in [10], and its proof can be easily applied to arbitrary normed linear spaces. Moreover, part (a) of this proposition also follows from Property (P_{12}) (yielding a second direct proof of it).

Proposition 2.1. [10, Theorems 6 and 7] *Let $\chi, \psi \in \mathcal{X}$ with $\chi \neq 0$, and $\varepsilon \in [0,1)$.*

(a) *The map $\psi \mapsto F_\varepsilon(\psi;\chi)$ is continuous (with respect to the Hausdorff metric).*

(b) *The map $\varepsilon \mapsto F_\varepsilon(\psi;\chi)$ is continuous (with respect to the Hausdorff metric).*

3 Definition of the Birkhoff-James cosine function

Consider two nonzero vectors $\chi, \psi \in \mathcal{X}$. If they are not co-linear, then we can define the (positive) *Birkhoff-James cosine* of the convex angle formed by $\mathrm{span}\{\chi\}$ and $\mathrm{span}\{\psi\}$:

$$\begin{aligned}
\cos_{BJ}(\chi,\psi) &= \min\{\varepsilon \in [0,1) : 0 \in F_\varepsilon(\psi;\chi)\} \\
&= \min\{\varepsilon \in [0,1) : \chi \perp_{BJ}^{\varepsilon} \psi\} \\
&= \min\left\{\varepsilon \in [0,1) : \|\chi - \lambda\psi\| \ge \sqrt{1-\varepsilon^2}\,\|\chi\|,\ \forall\, \lambda \in \mathbb{C}\right\}.
\end{aligned}$$

If χ and ψ are co-linear, then we assume that $\cos_{BJ}(\chi,\psi)=1$.

By the continuity and the monotonicity of the set $F_\varepsilon(\psi;\chi)$ with respect to ε (see Proposition 2.1 (b) and [23, Remark 2.1]), it follows that the smallest value of the parameter $\varepsilon\in[0,1)$, say ε_0, that satisfies $0\in F_{\varepsilon_0}(\psi;\chi)$ is unique. As a consequence, the above cosine is well defined. Moreover, if $\cos_{BJ}(\chi,\psi)=\varepsilon_0$, then we can define the associate *Birkhoff-James sine* by

$$\sin_{BJ}(\chi,\psi)=\sqrt{1-\cos_{BJ}(\chi,\psi)^2}=\sqrt{1-{\varepsilon_0}^2}.$$

It is clear that, in general, the Birkhoff-James cosine and sine are not symmetric functions of χ and ψ.

Proposition 3.1. *Let $(\mathcal{X},\|\cdot\|)$ be a complex normed linear space, and let $\chi,\psi\in\mathcal{X}$ be nonzero.*

(i) $\chi\perp_{BJ}\psi$ if and only if $\cos_{BJ}(\chi,\psi)=0$.

(ii) For any scalars $a,b\in\mathbb{C}\backslash\{0\}$, $\cos_{BJ}(a\chi,b\psi)=\cos_{BJ}(\chi,\psi)$. In particular, for $a,b=\pm1$, $\cos_{BJ}(\chi,\psi)=\cos_{BJ}(-\chi,\psi)=\cos_{BJ}(-\chi,-\psi)=\cos_{BJ}(\chi,-\psi)$.

Proof. Suppose $\chi,\psi\in\mathcal{X}$ with $\chi\neq0$.

(i) Apparently, $\chi\perp_{BJ}\psi$ if and only if $0\in F_0(\psi;\chi)$, or equivalently, if and only if $\cos_{BJ}(\chi,\psi)=0$.

(ii) By Properties (P_1) and (P_2), it follows that $F_\varepsilon(b\psi;\alpha\chi)=\dfrac{b}{a}F_\varepsilon(\psi;\chi)$. Hence, $0\in F_\varepsilon(b\psi;a\chi)$ if and only if $0\in\dfrac{b}{a}F_\varepsilon(\psi;\chi)$, or equivalently, if and only if $0\in F_\varepsilon(\psi;\chi)$. As a consequence, $\cos_{BJ}(a\chi,b\psi)=\cos_{BJ}(\chi,\psi)$. □

The definition of the Birkhoff-James cosine is compatible with the standard cosine in inner product spaces. This follows directly from Remark 2.4 of [26]. Here, we give an independent simple proof which is based on Property (P_{11}) and the special shape of the Birkhoff-James ε-orthogonality set $F_\varepsilon(\psi;\chi)$.

Theorem 3.2. *Let $(\mathcal{X},\|\cdot\|)$ be a complex normed linear space. If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then for any nonzero $\chi,\psi\in\mathcal{X}$,*

$$\cos_{BJ}(\chi,\psi)=\frac{|\langle\psi,\chi\rangle|}{\|\chi\|\,\|\psi\|}.$$

Proof. Without loss of generality, we assume that the vectors χ and ψ are not co-linear. Then, by Property (P_{11}), for any $\varepsilon\in[0,1)$,

$$F_\varepsilon(\psi;\chi)=\mathcal{D}\left(\frac{\langle\psi,\chi\rangle}{\|\chi\|^2}\,,\,\left\|\psi-\frac{\langle\psi,\chi\rangle}{\|\chi\|^2}\,\chi\right\|\frac{\varepsilon}{\sqrt{1-\varepsilon^2}\,\|\chi\|}\right).$$

As a consequence, $\varepsilon_0=\min\{\varepsilon\,:\,0\in F_\varepsilon(\psi;\chi)\}$ if and only if

$$\frac{|\langle\psi,\chi\rangle|}{\|\chi\|^2}=\left\|\psi-\frac{\langle\psi,\chi\rangle}{\|\chi\|^2}\chi\right\|\frac{\varepsilon_0}{\sqrt{1-\varepsilon_0^2}\,\|\chi\|},$$

or equivalently, if and only if

$$|\langle\psi,\chi\rangle|=\big\|\|\chi\|^2\psi-\langle\psi,\chi\rangle\chi\big\|\frac{\varepsilon_0}{\sqrt{1-\varepsilon_0^2}\,\|\chi\|},$$

or equivalently, if and only if

$$|\langle\psi,\chi\rangle|^2 = \big\|\|\chi\|^2\psi - \langle\psi,\chi\rangle\chi\big\|^2 \,\frac{\varepsilon_0^2}{(1-\varepsilon_0^2)\|\chi\|^2},$$

or equivalently, if and only if

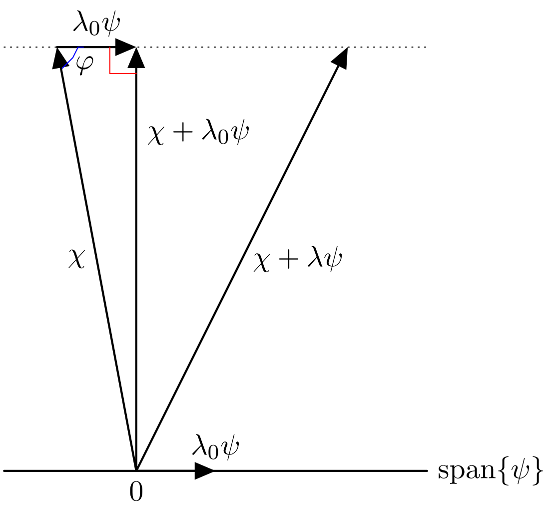
$$(1-\varepsilon_0^2)\|\chi\|^2|\langle\psi,\chi\rangle|^2 = \big\langle \|\chi\|^2\psi - \langle\psi,\ \chi\rangle\chi, \|\chi\|^2\psi - \langle\psi,\chi\rangle\chi\big\rangle\,\varepsilon_0^2,$$

or equivalently, if and only if

$$(1-\varepsilon_0^2)|\langle\psi,\chi\rangle|^2 = \big(\|\chi\|^2\|\psi\|^2 - |\langle\psi,\chi\rangle|^2\big)\,\varepsilon_0^2,$$

or equivalently, if and only if

$$\varepsilon_0 = \frac{|\langle\psi,\chi\rangle|}{\|\chi\|\,\|\psi\|}. \qquad \square$$



Suppose $\chi,\psi\in\mathcal{X}$ are not co-linear, and χ is not Birkhoff-James orthogonal to ψ, i.e., $0\notin F_0(\psi;\chi)$. If $\cos_{BJ}(\chi,\psi)=\varepsilon_0\in(0,1)$, then $0\in F_{\varepsilon_0}(\psi;\chi)$, or equivalently, $\|\chi+\lambda\psi\|\geq\sqrt{1-\varepsilon_0{}^2}\,\|\chi\|$ for all $\lambda\in\mathbb{C}$. Furthermore, since $\varepsilon_0>0$ and $0\in\partial F_{\varepsilon_0}(\psi;\chi)$, by Property (P_9), there exists a nonzero scalar $\lambda_0\in\mathbb{C}$ such that (see the above figure)

$$\|\chi+\lambda\psi\|\geq\|\chi+\lambda_0\psi\|=\sqrt{1-\varepsilon_0^2}\,\|\chi\|,\quad\forall\,\lambda\in\mathbb{C},$$

or equivalently,

$$\|\chi+\lambda_0\psi+(\lambda-\lambda_0)\psi\|\geq\|\chi+\lambda_0\psi\|=\sqrt{1-\varepsilon_0^2}\,\|\chi\|,\quad\forall\,\lambda\in\mathbb{C},$$

or equivalently,

$$(\chi+\lambda_0\psi)\perp_{BJ}\psi.$$

As a consequence,

$$\sqrt{1-\varepsilon_0^2}=\frac{\|\chi+\lambda_0\psi\|}{\|\chi\|}=\min_{\lambda\in\mathbb{C}}\frac{\|\chi+\lambda\psi\|}{\|\chi\|}=\sin_{BJ}(\chi,\psi),$$

that is, the sine function which was introduced by Szostok [26].

Moreover, it is clear that χ and ψ are co-linear if and only if $\min_{\lambda\in\mathbb{C}} \|\chi+\lambda\psi\| = 0,$ or equivalently, if and only if $\sin_{BJ}(\chi,\psi)=0,$ or equivalently, if and only if $\cos_{BJ}(\chi,\psi)=1.$

Remark 3.1. Based on the definition (4) of the Birkhoff-James ε-orthogonality set $F_\varepsilon(\psi;\chi)$, we can use continuous linear functionals to define the Birkhoff-James cosine:

$$\begin{aligned}
\cos_{BJ}(\chi,\psi) &= \min\{\varepsilon\in[0,1) :\ 0\in F_\varepsilon(\psi;\chi)\} \\
&= \min\{\varepsilon\in[0,1) :\ f(\psi)=0 \text{ for some } f\in L_\varepsilon(\chi)\} \\
&= \min\Big\{\varepsilon\in[0,1) :\ f(\psi)=0 \text{ for some } f\in\mathcal{X}^* \text{ with } f(\chi)=\sqrt{1-\varepsilon^2}\,\|\chi\| \text{ and } \|f\|\le 1\}.
\end{aligned}$$

If the (nonzero) vectors χ and ψ are not co-linear, and χ is not Birkhoff-James orthogonal to ψ, then (keeping Proposition 2.4 in [23] in mind) for any $\varepsilon\in(0,1)$,

$$\begin{aligned}
\cos_{BJ}(\chi,\psi) &= \min\{\varepsilon\in(0,1) :\ 0\in \partial F_\varepsilon(\psi;\chi)\} \\
&= \min\Big\{\varepsilon\in(0,1) :\ f(\psi)=0 \text{ for some } f\in\mathcal{X}^* \text{ with } f(\chi)=\sqrt{1-\varepsilon^2}\,\|\chi\| \text{ and } \|f\|=1\Big\} \\
&= \min\left\{\sqrt{1-\left(\frac{f(\chi)}{\|\chi\|}\right)^2}\in(0,1) :\ f\in\mathcal{X}^* \text{ with } f(\psi)=0,\ f(\chi)>0 \text{ and } \|f\|=1\right\}.
\end{aligned}$$

An optimal continuous linear functional $f_{\chi,\psi}\in\mathcal{X}^*$ such that

$$\cos_{BJ}(\chi,\psi)=\sqrt{1-\left(\frac{f_{\chi,\psi}(\chi)}{\|\chi\|}\right)^2},\quad f_{\chi,\psi}(\psi)=0,\quad f_{\chi,\psi}(\chi)>0 \quad\text{and}\quad \|f_{\chi,\psi}\|=1,$$

also satisfies

$$\sin_{BJ}(\chi,\psi)=\frac{f_{\chi,\psi}(\chi)}{\|\chi\|} = \max_{\substack{f\in\mathcal{X}^*\\ \|f\|=1\\ f(\psi)=0\\ f(\chi)>0}} \frac{f(\chi)}{\|\chi\|} = \max_{\substack{f\in\mathcal{X}^*\\ \|f\|=1\\ f(\psi)=0}} \frac{|f(\chi)|}{\|\chi\|} = \min_{\lambda\in\mathbb{C}} \frac{\|\chi+\lambda\psi\|}{\|\chi\|},$$

or equivalently,

$$f_{\chi,\psi}(\chi) = \max_{\substack{f\in\mathcal{X}^*\\ \|f\|=1\\ f(\psi)=0\\ f(\chi)>0}} f(\chi) = \max_{\substack{f\in\mathcal{X}^*\\ \|f\|=1\\ f(\psi)=0}} |f(\chi)| = \min_{\lambda\in\mathbb{C}} \|\chi+\lambda\psi\|.$$

4 Two illustrative examples

In this section, we present two examples to give insight into the definition of the Birkhoff-James ε-orthogonality set and the Birkhoff-James cosine. In the second example, we also describe a mechanism that yields non-smooth points on the boundary of the Birkhoff-James ε-orthogonality set.

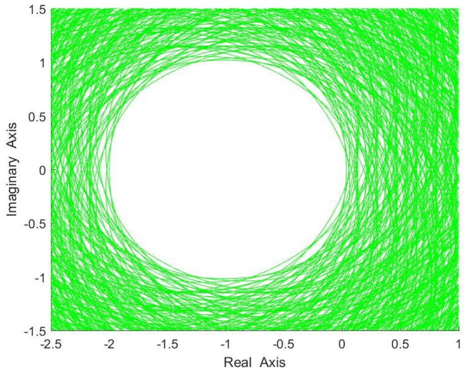


Figure 1: The origin is a smooth point of the boundary $\partial F_{\frac{\sqrt{3}}{2}}(\psi;\chi)$.

Example 4.1. Consider the normed linear space $(\mathcal{X},\|\cdot\|) = (\mathbb{C}^3,\|\cdot\|_\infty)$, and the unit vectors $\chi = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ and $\psi = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. It is easy to see that

$$\|\chi+\lambda\psi\|_\infty = \left\| \begin{bmatrix} \frac{1}{2} \\ \lambda \\ 1-\lambda \end{bmatrix} \right\|_\infty = \max\left\{\frac{1}{2}, |\lambda|, |1-\lambda|\right\} \geq \frac{1}{2} = \|\chi\|_\infty \sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}, \quad \forall\, \lambda\in\mathbb{C},$$

or equivalently, $0\in F_{\frac{\sqrt{3}}{2}}(\psi;\chi)$; see Figure 1, where $F_{\frac{\sqrt{3}}{2}}(\psi;\chi)$ is estimated by the unshaded region that results from having drawn one thousand circles of the defining formula (3). For $\lambda=\frac{1}{2}$, the equality follows, and hence, $0\in\partial F_{\frac{\sqrt{3}}{2}}(\psi;\chi)$. Moreover, for every $\varepsilon<\frac{\sqrt{3}}{2}$ and for $\lambda=\frac{1}{2}$, it holds

$$\frac{1}{2} = \left\| \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \frac{1}{2}+\frac{1}{2}\cdot 0 \\ 0+\frac{1}{2}\cdot 1 \\ 1+\frac{1}{2}\cdot(-1) \end{bmatrix} \right\|_\infty = \left\|\chi+\frac{1}{2}\psi\right\|_\infty < \|\chi\|_\infty\sqrt{1-\varepsilon^2}.$$

Thus, we conclude that

$$\cos_{BJ}(\chi,\psi) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin_{BJ}(\chi,\psi) = \frac{1}{2} = \frac{\min_{\lambda\in\mathbb{C}}\|\chi+\lambda\psi\|_\infty}{\|\chi\|_\infty}.$$

We also observe that $\|\chi+\lambda\psi\|_\infty = \frac{1}{2}$ if and only if $\max\left\{\frac{1}{2},|\lambda|,|1-\lambda|\right\} = \frac{1}{2}$, or equivalently, if and only if $|\lambda|\leq\frac{1}{2}$ and $|1-\lambda|\leq\frac{1}{2}$. Hence, $\lambda_0=\frac{1}{2}$ is the only scalar that satisfies $\|\chi+\lambda_0\psi\|_\infty = \min_{\lambda\in\mathbb{C}}\|\chi+\lambda\psi\|_\infty$. Despite the uniqueness of λ_0, $\cos_{BJ}(\chi,\psi) = \frac{\sqrt{3}}{2} \neq \frac{1}{2} = \frac{\|\lambda_0\psi\|_\infty}{\|\chi\|_\infty}$. This means that essentially we cannot have a direct geometrical implementation for the Birkhoff-James cosine, as the ratio of the adjacent side to the hypotenuse.

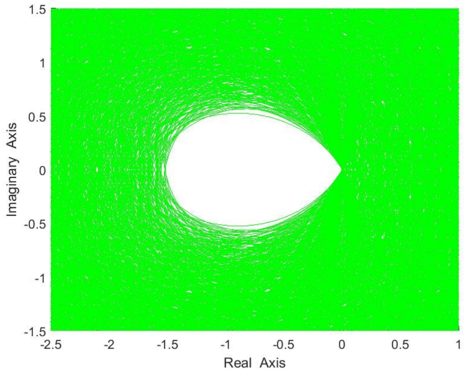


Figure 2: The origin is a non-smooth point of the boundary $\partial F_{\frac{\sqrt{3}}{2}}(\psi;\chi)$.

Example 4.2. Consider the normed linear space $(\mathcal{X}, \|\cdot\|) = (\mathbb{C}^3, \|\cdot\|_\infty)$, and the unit vectors $\chi = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ and $\psi = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -1 \end{bmatrix}$. Then,

$$\|\chi + \lambda\psi\|_\infty = \left\| \begin{bmatrix} \frac{1}{2} \\ \frac{\lambda}{2} \\ 1-\lambda \end{bmatrix} \right\|_\infty = \max\left\{ \frac{1}{2}, \frac{|\lambda|}{2}, |1-\lambda| \right\} \geq \frac{1}{2} = \|\chi\|_\infty \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2}, \quad \forall\, \lambda \in \mathbb{C},$$

or equivalently, $0 \in F_{\frac{\sqrt{3}}{2}}(\psi;\chi)$; see Figure 2, where $F_{\frac{\sqrt{3}}{2}}(\psi;\chi)$ is again estimated by the unshaded region. For $\lambda = 1$, we get the equality $\|\chi + \psi\|_\infty = \frac{1}{2}$, and thus, $0 \in \partial F_{\frac{\sqrt{3}}{2}}(\psi;\chi)$. Moreover, for every $\varepsilon < \frac{\sqrt{3}}{2}$ and for $\lambda = 1$, we have

$$\frac{1}{2} = \left\| \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \frac{1}{2} + 1\cdot 0 \\ 0 + 1\cdot \frac{1}{2} \\ 1 + 1\cdot(-1) \end{bmatrix} \right\|_\infty = \|\chi + \psi\|_\infty < \|\chi\|_\infty \sqrt{1-\varepsilon^2}.$$

As a consequence,

$$\cos_{BJ}(\chi,\psi) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin_{BJ}(\chi,\psi) = \frac{1}{2} = \frac{\min_{\lambda\in\mathbb{C}} \|\chi + \lambda\psi\|_\infty}{\|\chi\|_\infty}.$$

We also observe that $\|\chi + \lambda\psi\|_\infty = \frac{1}{2}$ if and only if $\max\left\{ \frac{1}{2}, \left|\frac{\lambda}{2}\right|, |1-\lambda| \right\} = \frac{1}{2}$, or equivalently, if and only if $|\lambda| \leq 1$ and $|1-\lambda| \leq \frac{1}{2}$. Hence, for every $\lambda_0 \in \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \cap \left\{ \lambda \in \mathbb{C} : |1-\lambda| \leq \frac{1}{2} \right\}$ (i.e., for all scalars in the intersection of the two circular disks), we get $\|\chi + \lambda_0\psi\|_\infty = \min_{\lambda\in\mathbb{C}} \|\chi + \lambda\psi\|_\infty$. For the same scalars, the ratio $\frac{\|\lambda_0\psi\|_\infty}{\|\chi\|_\infty}$ takes all the values in the interval $\left[\frac{1}{2}, 1\right]$. Thus, it is no use comparing

$\cos_{BJ}(\chi, \psi) = \dfrac{\sqrt{3}}{2}$ and $\dfrac{\|\lambda_0 \psi\|_\infty}{\|\chi\|_\infty}$, and it is verified that we cannot have a geometric implementation for the cosine $\cos_{BJ}(\chi, \psi)$ as we have for the sine $\sin_{BJ}(\chi, \psi)$.

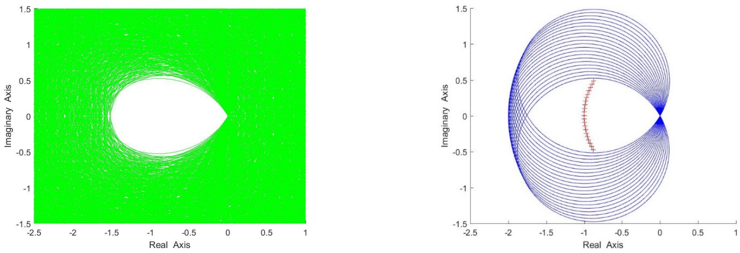


Figure 3: The set $F_{\frac{\sqrt{3}}{2}}(\psi; \chi)$ (left), and circles with centers on the arc $\Theta\left(0, 1, \frac{1}{2}\right) - 1$ and radii equal to 1, which illustrate $F_{\frac{\sqrt{3}}{2}}(\psi; \chi)$ (right).

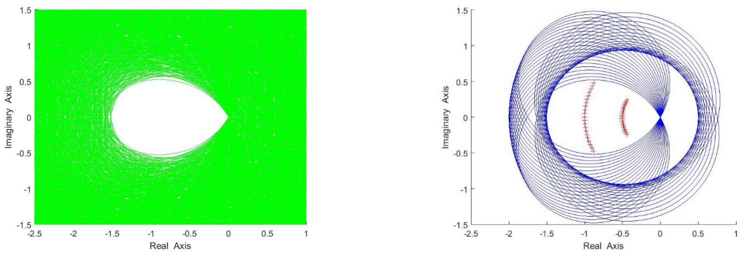


Figure 4: The set $F_{\frac{\sqrt{3}}{2}}(\psi; \chi)$ (left), and circles with centers on the arcs $\Theta\left(0, 1, \frac{1}{2}\right) - 1$ and $\dfrac{1}{2}\left[\Theta\left(0, 1, \frac{1}{2}\right) - 1\right]$, which illustrate $F_{\frac{\sqrt{3}}{2}}(\psi; \chi)$ (right).

In this example, it is worth mentioning that

$$0 \in F_{\frac{\sqrt{3}}{2}}(\psi; \chi) = \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \frac{\|\psi - \lambda \chi\|_\infty}{\frac{1}{2}\|\chi\|_\infty}\right) = \bigcap_{\lambda \in \mathbb{C}} \mathcal{D}\left(\lambda, \max\left\{|\lambda|, 1, 2|1 + \lambda|\right\}\right).$$

So, if $\Theta\left(0, 1, \frac{1}{2}\right)$ denotes the arc of the circle $\mathcal{C}(1, 1)$ with center at 1 and radius equal to 1 that lies in the disk $\mathcal{D}\left(0, \frac{1}{2}\right)$ with center at the origin and radius equal to $\dfrac{1}{2}$, then for every $\lambda \in \Theta\left(0, 1, \frac{1}{2}\right) - 1$, it follows $\max\{|\lambda|, 1, 2|\lambda + 1|\} = |\lambda| = 1$. As a consequence, there is an infinite number of circles with centers on $\Theta\left(0, 1, \frac{1}{2}\right) - 1$ and radii equal to 1, which illustrate the region $F_{\frac{\sqrt{3}}{2}}(\psi; \chi)$ and pass through the origin (see Figures 3 and 4). Thus, the origin is a corner of $F_{\frac{\sqrt{3}}{2}}(\psi; \chi)$.

5 Properties of the Birkhoff-James cosine function

In this section, we obtain some basic properties of the Birkhoff-James cosine function. As above, we consider a complex normed linear space $\mathcal{X}$.

Proposition 5.1. *The map* $(\chi,\psi)\mapsto \cos_{BJ}(\chi,\psi)$, $(\chi,\psi)\in(\mathcal{X}\setminus\{0\})\times(\mathcal{X}\setminus\{0\})$, *is continuous.*

Proof. By Theorem 2.5 of [26] (which holds for complex normed linear spaces), the sine function $\sin_{BJ}(\chi,\psi)$ is continuous. Hence, the cosine function $\cos_{BJ}(\chi,\psi)=\sqrt{1-\sin_{BJ}(\chi,\psi)^2}$ is also continuous. □

Remark 5.1. Suppose that the dimension of the normed linear space $\mathcal{X}$ is greater than or equal to three. Then, the cosine is symmetric, i.e., $\cos_{BJ}(\chi,\psi)=\cos_{BJ}(\psi,\chi)$ for all nonzero $\chi,\psi\in\mathcal{X}$, if and only if the norm $\|\cdot\|$ is induced by an inner product. Indeed, if $\cos_{BJ}(\chi,\psi)=\cos_{BJ}(\psi,\chi)$ for all $\chi,\psi\in\mathcal{X}\setminus\{0\}$, then $\cos_{BJ}(\chi,\psi)=0$ if and only if $\cos_{BJ}(\psi,\chi)=0$. This means that $\chi\perp_{BJ}\psi$ if and only if $\psi\perp_{BJ}\chi$, or equivalently, the norm $\|\cdot\|$ is induced by an inner product [11, 17]. The converse is obvious.

Proposition 5.2. *The norm* $\|\cdot\|$ *is induced by an inner product if and only if for every nonzero vectors* $\chi,\psi\in\mathcal{X}$, *it holds that*

$$\cos_{BJ}(\chi,\psi)=\frac{\left|\|\chi\|^2+\|\psi\|^2-\|\chi-\psi\|^2\right|}{2\,\|\chi\|\,\|\psi\|}.$$

Proof. Let χ and ψ be any two nonzero vectors of $\mathcal{X}$, and suppose

$$\cos_{BJ}(\chi,\psi)=\frac{\left|\|\chi\|^2+\|\psi\|^2-\|\chi-\psi\|^2\right|}{2\,\|\chi\|\,\|\psi\|}.$$

If χ is Birkhoff-James orthogonal to ψ, then

$$\cos_{BJ}(\chi,\psi)=\frac{\left|\|\chi\|^2+\|\psi\|^2-\|\chi-\psi\|^2\right|}{2\,\|\chi\|\,\|\psi\|}=0,$$

or equivalently, $\|\chi\|^2+\|\psi\|^2=\|\chi-\psi\|^2$. This means that the Birkhoff-James orthogonality yields the Pythagorean orthogonality, and hence, the norm $\|\cdot\|$ is induced by an inner product [1, 11, 22].

Conversely, if the norm $\|\cdot\|$ is induced by an inner product, then it is apparent that the law of cosines holds. □

Proposition 5.3. *If* $\chi,\psi\in\mathcal{X}\setminus\{0\}$ *are not co-linear, then, for any* $\varepsilon_0\in(0,1)$, *there are infinitely many scalars* $\mu_0\in\mathbb{C}$ *such that* $\cos_{BJ}(\chi,\psi-\mu_0\chi)=\varepsilon_0$. *In particular, these scalars* μ_0 *are exactly the boundary points of the set* $F_{\varepsilon_0}(\psi;\chi)$.

Proof. Consider an $\varepsilon_0\in(0,1)$. For any $\mu_0\in\partial F_{\varepsilon_0}(\psi;\chi)$, $0\in\partial F_{\varepsilon_0}(\psi-\mu_0\chi;\chi)$ by Property (P_1), and thus, $\cos_{BJ}(\chi,\psi-\mu_0\chi)=\varepsilon_0$. Moreover, the set $F_{\varepsilon_0}(\psi;\chi)$ is compact and convex. So, since the vectors χ and ψ are not co-linear, $F_{\varepsilon_0}(\psi;\chi)$ is not a singleton and has an infinite number of boundary points. □

Proposition 5.4. *If* $\chi,\psi\in\mathcal{X}\setminus\{0\}$ *are not co-linear, then*

$$\sin_{BJ}(\chi,\chi\pm\psi)\leq\frac{\|\psi\|}{\|\chi\|}\quad \textit{and}\quad \cos_{BJ}(\chi,\chi\pm\psi)\geq\sqrt{\left|1-\frac{\|\psi\|^2}{\|\chi\|^2}\right|}.$$

Proof. Suppose $\cos_{BJ}(\chi,\chi\pm\psi)=\varepsilon_0<1$. Then $0\in F_{\varepsilon_0}(\chi\pm\psi;\chi)$, or equivalently, $\chi\perp_{BJ}^{\varepsilon_0}(\chi\pm\psi)$. Thus, $\|\chi+\lambda(\chi\pm\psi)\|\geq\sqrt{1-{\varepsilon_0}^2}\,\|\chi\|$ for all $\lambda\in\mathbb{C}$. For $\lambda=-1$, it follows $\|\psi\|\geq\sqrt{1-{\varepsilon_0}^2}\,\|\chi\|$, or equivalently, $\sin_{BJ}(\chi,\chi\pm\psi)\leq\dfrac{\|\psi\|}{\|\chi\|}$. The second inequality is obvious. □

Next we obtain that, if a triangle has two sides of the same length, then the Birkhoff-James cosines of the corresponding angles (keeping the vectors in a specific order due to the lack of symmetry) are equal.

Proposition 5.5. *If $\chi,\psi\in\mathcal{X}\setminus\{0\}$ are not co-linear and satisfy $\|\psi-\chi\|=\|\psi\|$, then $\cos_{BJ}(\psi,\chi)=\cos_{BJ}(\psi-\chi,\chi)$.*

Proof. Assume that $\cos_{BJ}(\psi,\chi)=\varepsilon_1$ and $\cos_{BJ}(\psi-\chi,\chi)=\varepsilon_2$. Then, by definition, $0\in F_{\varepsilon_1}(\chi,\psi)$, and hence, $\psi\perp_{BJ}^{\varepsilon_1}\chi$. As a consequence,

$$\|\psi-\lambda\chi\|\geq\sqrt{1-{\varepsilon_1}^2}\,\|\psi\|,\quad\forall\,\lambda\in\mathbb{C},$$

or equivalently,

$$\|\psi-\chi-(\lambda-1)\chi\|\geq\sqrt{1-{\varepsilon_1}^2}\,\|\psi\|,\quad\forall\,\lambda\in\mathbb{C},$$

or equivalently,

$$\|(\psi-\chi)-\lambda\chi\|\geq\sqrt{1-{\varepsilon_1}^2}\,\|\psi-\chi\|,\quad\forall\,\lambda\in\mathbb{C}.$$

Thus, $\psi-\chi\perp_{BJ}^{\varepsilon_1}\chi$, i.e., $0\in F_{\varepsilon_1}(\chi;\psi-\chi)$, and hence,

$$\cos_{BJ}(\psi-\chi,\chi)=\varepsilon_2=\min\{\varepsilon\in[0,1):\,0\in F_\varepsilon(\chi;\psi-\chi)\}\leq\varepsilon_1=\cos_{BJ}(\psi,\chi).$$

Considering $\psi-\chi$ and $-\chi$ instead of ψ and χ, respectively, and keeping Proposition 3.1 (ii) in mind, one can verify that $\cos_{BJ}(\psi,\chi)\leq\cos_{BJ}(\psi-\chi,\chi)$. □

Remark 5.2. If $\cos_{BJ}(\chi+\psi,\chi-\psi)=0$ for every $\chi,\psi\in\mathcal{X}$ with $\|\chi\|=\|\psi\|=1$, then by [1, p. 33] and [11], it follows that the norm $\|\cdot\|$ is induced by an inner product; i.e., $(\mathcal{X},\|\cdot\|)$ is an inner product space if and only if the two diagonals of any rhombus are Birkhoff-James orthogonal.

Remark 5.3. Consider two nonzero vectors $\chi,\psi\in\mathcal{X}$ such that $\chi\perp_{BJ}\psi$, or equivalently, $0\in F^0_{\|\cdot\|}(\psi;\chi)$. Suppose also that there is a real $\rho_0>0$ such that $\mathcal{D}(0,\rho_0)\subseteq F^0_{\|\cdot\|}(\psi;\chi)$ (i.e., the origin lies in the interior of $F^0_{\|\cdot\|}(\psi;\chi)$). Then, by the relation $\mathcal{D}(0,\rho_0)\subseteq F^0_{\|\cdot\|}(\psi;\chi)=\bigcap_{\lambda\in\mathbb{C}}\mathcal{D}\left(\lambda,\dfrac{\|\psi-\lambda\chi\|}{\|\chi\|}\right)\subseteq\mathcal{D}\left(0,\dfrac{\|\psi-0\chi\|}{\|\chi\|}\right)$, it follows that $\rho_0\leq\dfrac{\|\psi\|}{\|\chi\|}$, and for every $\lambda\in\mathbb{C}$, $\dfrac{\|\psi-\lambda\chi\|}{\|\chi\|}\geq|\lambda|+\rho_0$. As a consequence, $\rho_0\leq\inf_{\lambda\in\mathbb{C}}\left\{\dfrac{\|\psi-\lambda\chi\|}{\|\chi\|}-|\lambda|\right\}\leq\min_{\lambda\in\mathbb{C}}\dfrac{\|\psi-\lambda\chi\|}{\|\chi\|}=\dfrac{\|\psi\|}{\|\chi\|}\sin_{BJ}(\psi;\chi)$. So, if $\chi\perp_{BJ}\psi$ and $\mathcal{D}(0,\rho_0)\subseteq F^0_{\|\cdot\|}(\psi;\chi)$, then $\dfrac{\|\chi\|}{\|\psi\|}\rho_0\leq\sin_{BJ}(\psi;\chi)\leq 1$, or equivalently, $0\leq\cos_{BJ}(\psi;\chi)\leq\sqrt{1-\left(\dfrac{\|\chi\|}{\|\psi\|}\rho_0\right)^2}$.

More generally, suppose that $0\in F_\varepsilon(\psi;\chi)$ for some $\varepsilon\in[0.1)$, and that there is a real $\rho_\varepsilon>0$ such that $\mathcal{D}(0,\rho_\varepsilon)\subseteq F_\varepsilon(\psi;\chi)$. Then one can similarly see that $\rho_\varepsilon\leq\dfrac{\|\psi\|}{\sqrt{1-\varepsilon^2}\,\|\chi\|}$ and $\cos_{BJ}(\psi,\chi)\leq\sqrt{1-\left(\dfrac{\|\chi\|}{\|\psi\|}\rho_\varepsilon\right)^2(1-\varepsilon^2)}$.

Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two equivalent norms on a linear space $\mathcal{X}$, and let C and c be two positive numbers such that $c\|\zeta\|_\alpha \le \|\zeta\|_\beta \le C\|\zeta\|_\alpha$ for all $\zeta \in \mathcal{X}$. By Property (P_4), for any $\varepsilon \in [0,1)$, $F_\varepsilon^{\|\cdot\|_\alpha}(\psi;\chi) \subseteq F_{\varepsilon'}^{\|\cdot\|_\beta}(\psi;\chi)$, where $\varepsilon' = \sqrt{1-\frac{c^2(1-\varepsilon^2)}{C^2}}$. Moreover, denoting by $\cos_{BJ}^{\|\cdot\|_\alpha}(\chi,\psi)$ and $\cos_{BJ}^{\|\cdot\|_\beta}(\chi,\psi)$ the corresponding cosine functions, and by $\sin_{BJ}^{\|\cdot\|_\alpha}(\chi,\psi)$ and $\sin_{BJ}^{\|\cdot\|_\beta}(\chi,\psi)$ the corresponding sine functions, the following holds:

Proposition 5.6. *For any nonzero vectors $\chi, \psi \in \mathcal{X}$,*

$$\sin_{BJ}^{\|\cdot\|_\alpha}(\chi,\psi) \le \frac{c}{C} \sin_{BJ}^{\|\cdot\|_\beta}(\chi,\psi).$$

Proof. Let $\varepsilon_1 = \cos_{BJ}^{\|\cdot\|_\alpha}(\chi,\psi)$ and $\varepsilon_2 = \sqrt{1-\frac{c^2(1-\varepsilon^2)}{C^2}}$. Then, $0 \in F_{\varepsilon_1}^{\|\cdot\|_\alpha}(\psi;\chi) \subseteq F_{\varepsilon_2}^{\|\cdot\|_\beta}(\psi;\chi)$, and thus, $\cos_{BJ}^{\|\cdot\|_\beta}(\chi,\psi) \le \varepsilon_2$. As a consequence,

$$\cos_{BJ}^{\|\cdot\|_\beta}(\chi,\psi)^2 \le 1 - \frac{c^2(1-{\varepsilon_1}^2)}{C^2},$$

or

$$C^2(\cos_{BJ}^{\|\cdot\|_\beta}(\chi,\psi)^2 - 1) \le -c^2\left(1-\cos_{BJ}^{\|\cdot\|_\alpha}(\chi,\psi)^2\right),$$

or

$$C^2(1-\cos_{BJ}^{\|\cdot\|_\beta}(\chi,\psi)^2) \ge c^2\left(1-\cos_{BJ}^{\|\cdot\|_\alpha}(\chi,\psi)^2\right),$$

or

$$\frac{C}{c} \ge \frac{\sin_{BJ}^{\|\cdot\|_\alpha}(\chi,\psi)}{\sin_{BJ}^{\|\cdot\|_\beta}(\chi,\psi)}. \qquad \square$$

6 Semi-inner product

Recall that a map $[\cdot,\cdot] : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$ is said to be a *semi-inner product* if it satisfies the following properties [13, 14, 19]:

1. $[\psi,\psi] \ge 0$ for every $\psi \in \mathcal{X}$, and $[\psi,\psi] = 0$ if and only if $\psi = 0$;
2. $[a\psi,\chi] = a[\psi,\chi]$ for every $\psi,\chi \in \mathcal{X}$ and $a \in \mathbb{C}$;
3. $[\psi,a\chi] = \overline{a}[\psi,\chi]$ for every $\psi,\chi \in \mathcal{X}$ and $a \in \mathbb{C}$;
4. $[\psi+\zeta,\chi] = [\psi,\chi] + [\zeta,\chi]$ for every $\psi,\zeta,\chi \in \mathcal{X}$;
5. $|[\psi,\chi]|^2 \le [\psi,\psi]\,[\chi,\chi]$ for every $\psi,\chi \in \mathcal{X}$.

By Lummer [19] and Giles [14], in any normed linear space $\mathcal{X}$, one can find a (not necessarily unique) semi-inner product $[\cdot,\cdot]$ which generates the given norm $\|\cdot\|$, i.e., $[\zeta,\zeta] = \|\zeta\|^2$ for all $\zeta \in \mathcal{X}$. This semi-inner product is uniquely defined exactly when all the sets $F_0(\psi;\chi)$ $(\chi \neq 0)$ are singletons (see the next theorem and remark).

Theorem 6.1. *Let $\mathcal{X}$ be a complex normed linear space, and suppose that for every $\chi, \psi \in \mathcal{X}$ with $\chi \neq 0$, the set $F_0(\psi;\chi)$ is a singleton, say $F_0(\psi;\chi) = \{\mu(\psi,\chi)\}$. Then the map*

$$[\psi,\chi] = \begin{cases} \mu(\psi,\chi)\|\chi\|^2, & \text{if } \chi \neq 0, \\ 0, & \text{if } \chi = 0 \end{cases}$$

is a semi-inner product.

Proof. Using basic properties of $F_0(\psi;\chi)$, one can verify the conditions of the above definition:

1. Since by (4), $F_0(\psi;\psi) = \left\{ \dfrac{f(\psi)}{\|\psi\|} : f \in L_0(\psi) \right\} = \{1\}$, it follows that $[\psi,\psi] = \|\psi\|^2 \geq 0$, and $[\psi,\psi] = 0$ if and only if $\psi = 0$.

2. Consider a scalar $a \in \mathbb{C}$, and observe that $a[\psi,\chi] = a\mu(\psi,\chi)\|\chi\|^2$ and $[a\psi,\chi] = \mu(a\psi,\chi)\|\chi\|^2$, where $F_0(a\psi;\chi) = \{\mu(a\psi,\chi)\}$. By Property (P_1), the proof of the second condition follows readily.

3. For $a = 0$, the third condition holds trivially. Consider a nonzero scalar $a \in \mathbb{C}$, and observe that $\overline{a}[\psi,\chi] = \overline{a}\mu(\psi,\chi)\|\chi\|^2$ and $[\psi,a\chi] = \mu(\psi,a\chi)\|a\chi\|^2 = |a|^2\mu(\psi,a\chi)\|\chi\|^2$, where $F_0(\psi;a\chi) = \{\mu(\psi,a\chi)\}$. It is enough to see that $\overline{a}\mu(\psi,\chi) = |a|^2\mu(\psi,a\chi)$, or equivalently, $\mu(\psi,a\chi) = \dfrac{\mu(\psi,\chi)}{a}$. The latter equality follows directly from Property (P_2).

4. For any $\psi,\zeta,\chi \in \mathcal{X}$, with $\chi \neq 0$, it holds $\mu(\psi+\zeta,\chi) = \dfrac{f(\psi+\zeta)}{\|\chi\|} = \dfrac{f(\psi)}{\|\chi\|} + \dfrac{f(\zeta)}{\|\chi\|} = \mu(\psi,\chi)+\mu(\zeta,\chi)$.

5. Clearly, $|[\psi,\chi]| = |\mu(\psi,\chi)|\,\|\chi\|^2 \leq \dfrac{\|\psi\|}{\|\chi\|}\|\chi\|^2 = \|\chi\|\|\psi\|$. □

Apparently, the semi-inner product defined in the above theorem is an inner product if and only if $\mu(\psi,\chi)\|\chi\|^2 = \overline{\mu(\chi,\psi)}\|\psi\|^2$ for every nonzero $\chi,\psi \in \mathcal{X}$.

Moreover, it is worth mentioning that Theorem 6.1 also follows by Theorem 2 in [14].

Remark 6.1. By Theorem 48 in [13] and Theorem 4.2 in [16], the Birkhoff-James orthogonality is right additive (i.e., $\chi \perp_{BJ} \psi$ and $\chi \perp_{BJ} \zeta$ yield $\chi \perp_{BJ} (\psi+\zeta)$) if and only if $F_0(\psi;\chi)$ is a singleton for any $\chi,\psi \in \mathcal{X}$ $(\chi \neq 0)$, or equivalently, if and only if the normed linear space $\mathcal{X}$ is smooth.

Example 6.1. Consider the linear space $\mathbb{C}^3$ with the norm $\|\cdot\|_3$, and for any vectors $\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix}$ and $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$, define

$$[\psi,\chi] = \begin{cases} \dfrac{1}{\|\chi\|_3} \sum\limits_{1\leq i\leq 3} \overline{\chi}_i|\chi_i|\psi_i, & \text{if } \chi \neq 0, \\ 0, & \text{if } \chi = 0. \end{cases}$$

Then, $[\psi,\chi]$ is a semi-inner product. Indeed, for any $\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix}$, $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$, $\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \in \mathbb{C}^3 \setminus \{0\}$ and $a \in \mathbb{C}$, the first four conditions of the definition are obtained by straightforward computations, and for the fifth condition, we have:

$$
\begin{aligned}
\frac{|[\psi,\chi]|}{\|\chi\|_3} &= \frac{1}{\|\chi\|_3^2}\left|\sum_{1\le i\le 3}\overline{\chi}_i\,|\chi_i|\,\psi_i\right| \\
&\le \frac{1}{\|\chi\|_3^2}\sum_{1\le i\le 3}|\overline{\chi}_i|\;|\chi_i|\;|\psi_i| \\
&\le \frac{1}{\|\chi\|_3^2}\left(\sum_{1\le i\le 3}\left(|\chi_i|^2\right)^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(\sum_{1\le i\le 3}|\psi_i|^3\right)^{\frac{1}{3}} \;=\; \|\psi\|_3,
\end{aligned}
$$

where the last inequality follows by the Hölder inequality for $p=\dfrac{3}{2}$ and $q=3$.

The normed linear space $(\mathbb{C}^3,\|\cdot\|_3)$ is smooth [20, Corollary 5.5.17], and thus, the above semi-inner product is the only semi-inner product induced by the norm $\|\cdot\|_3$. Hence, it coincides with the semi-inner product of Theorem 6.1, that is,

$$\mu(\psi,\chi)\|\chi\|_3^2=[\psi,\chi]=\frac{1}{\|\chi\|_3}\sum_{1\le i\le 3}\overline{\chi}_i|\chi_i|\psi_i\quad(\chi\neq 0).$$

As a consequence,

$$\mu(\psi,\chi)=\frac{1}{\|\chi\|_3^3}\sum_{1\le i\le 3}\overline{\chi}_i|\chi_i|\psi_i\quad(\chi\neq 0).$$

For the vectors $\chi=\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\psi=\begin{bmatrix}1\\0\\1\end{bmatrix}$, we have $\chi\perp_{BJ}^0(\psi-\chi)$ and $\mu(\psi,\chi)=1$; indeed, $\|\chi+\lambda(\psi-\chi)\|_3=(1+|\lambda|^3)^{\frac{1}{3}}\ge 1=\|\chi\|_3$ for all $\lambda\in\mathbb{C}$. It is also straightforward to see that

$$f_\chi(\zeta)=\frac{1}{\|\chi\|_3^2}\sum_{1\le i\le 3}\overline{\chi}_i|\chi_i|\zeta_i=\zeta_1$$

is a continuous linear functional in $L_0(\chi)=\{f\in\mathcal{X}^*:\ f(\chi)=\|\chi\|_3=1\ \text{ and }\ \|f\|_3=1\}$ and satisfies $f_\chi(\psi-\mu(\psi,\chi)\chi)=f_\chi(\psi-\chi)=0$. Hence, $1=\mu(\psi,\chi)=\dfrac{f_\chi(\psi)}{f_\chi(\chi)}=\dfrac{f_\chi(\psi)}{\|\chi\|_3}$. Moreover,

$$[\psi,\chi]=\frac{1}{\|\chi\|_3}\sum_{1\le i\le 3}\overline{\chi}_i|\chi_i|\psi_i=1=\|\chi\|_3^2,$$

verifying Theorem 6.1.

Finally, we observe that for every scalar $\lambda\in\mathbb{C}$,

$$
\begin{aligned}
\|\chi+\lambda\psi\|_3 &= \left\|\begin{bmatrix}1+\lambda\\0\\\lambda\end{bmatrix}\right\|_3 = \left(|1+\lambda|^3+|\lambda|^3\right)^{\frac{1}{3}} \\
&\ge \frac{\sqrt[3]{2}}{2} = \frac{\sqrt[3]{2}}{2}\|\chi\|_3 = \sqrt{1-\left(1-\left(\frac{\sqrt[3]{2}}{2}\right)^2\right)}\,\|\chi\|_3.
\end{aligned}
$$

For $\lambda = -\frac{1}{2}$, $\left\|\chi - \frac{1}{2}\psi\right\|_3 = \left\|\begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}\right\|_3 = \left(\frac{1}{8} + \frac{1}{8}\right)^{\frac{1}{3}} = \frac{\sqrt[3]{2}}{2} = \frac{\sqrt[3]{2}}{2}\|\chi\|_3$. By Property (P_9), for $\varepsilon = \sqrt{1 - \left(\frac{\sqrt[3]{2}}{2}\right)^2}$, $0 \in \partial F_\varepsilon(\psi;\chi)$, and thus, $\cos_{BJ}(\chi,\psi) = \sqrt{1 - \left(\frac{\sqrt[3]{2}}{2}\right)^2} = 0.776$. This cosine is different than $\frac{|[\psi,\chi]|}{\|\psi\|_3\,\|\chi\|_3} = \frac{1}{\sqrt[3]{2}} = 0.7936$, as expected by the discussion in [7].

7 Relation with other cosine functions

Inspired by the Phythagorean orthogonality and the law of cosines, in [27], Wilson introduced the P-cosine between two non-zero vectors $\chi, \psi \in \mathcal{X}$ to be

$$\cos_P(\chi,\psi) = \frac{\|\chi\|^2 + \|\psi\|^2 - \|\chi - \psi\|^2}{2\|\chi\|\,\|\psi\|};$$

see also [2]. Clearly, this cosine is symmetric but not homogeneous. Moreover, by Proposition 5.2, its absolute value coincides with the Birkhoff-James cosine if and only if the norm is induced by an inner product.

Proposition 7.1. *Let χ and ψ be two nonzero vectors of a complex normed linear space $\mathcal{X}$. Then,*

$$\cos_P(\chi,\psi) \le \frac{\|\psi\|}{2\|\chi\|} + \cos_{BJ}(\chi,\psi)^2\frac{\|\chi\|}{2\|\psi\|}.$$

Proof. Without loss of generality, assume that $\|\chi\|^2 + \|\psi\|^2 \ge \|\chi - \psi\|^2$. Let $\cos_{BJ}(\chi,\psi) = \varepsilon_0 < 1$. Then, $0 \in F_{\varepsilon_0}(\psi;\chi)$ and $\chi \perp_{BJ}^{\varepsilon_0} \psi$. Hence, $\|\chi - \lambda\psi\| \ge \sqrt{1-{\varepsilon_0}^2}\,\|\chi\|$ for all $\lambda \in \mathbb{C}$, and thus (for $\lambda = 1$), $-\|\chi - \psi\|^2 \le -(1 - {\varepsilon_0}^2)\|\chi\|^2$. As a consequence,

$$\begin{aligned} 0 \le \cos_P(\chi,\psi) &= \frac{\|\chi\|^2 + \|\psi\|^2 - \|\chi - \psi\|^2}{2\|\chi\|\,\|\psi\|} \\ &\le \frac{\|\chi\|^2 + \|\psi\|^2 - (1 - {\varepsilon_0}^2)\|\chi\|^2}{2\|\chi\|\,\|\psi\|} \\ &= \frac{\|\psi\|}{2\|\chi\|} + \frac{\cos_{BJ}(\chi,\psi)^2\|\chi\|}{2\|\psi\|}. \qquad \square \end{aligned}$$

By the above proposition, it follows that if $\cos_{BJ}(\chi,\psi) = 0$, then $\cos_P(\chi,\psi) \le \frac{\|\psi\|}{2\|\chi\|}$, or equivalently, $\|\chi\| \le \|\chi - \psi\|$.

Consider now the I-cosine

$$\cos_I(\chi,\psi) = \frac{\|\chi + \psi\|^2 - \|\chi - \psi\|^2}{4\|\chi\|\,\|\psi\|},$$

which follows from the isosceles orthogonality [2]. The I-cosine is also symmetric, and its absolute value coincides with the Birkhoff-James cosine if and only if the norm is induced by an inner product [22].

Proposition 7.2. *Let χ and ψ be two nonzero vectors of a complex normed linear space $\mathcal{X}$. Then,*

$$\cos_I(\chi,\psi) \leq \frac{\|\chi+\psi\|^2 - \|\chi\|^2}{4\|\chi\|\,\|\psi\|} + \cos_{BJ}(\chi,\psi)^2\frac{\|\chi\|}{4\|\psi\|}$$

and

$$\cos_I(\chi,\psi) \geq \frac{\|\chi\|^2 - \|\chi-\psi\|^2}{4\|\chi\|\,\|\psi\|} - \cos_{BJ}(\chi,\psi)^2\frac{\|\chi\|}{4\|\psi\|}.$$

Proof. Let $\cos_{BJ}(\chi,\psi) = \varepsilon_0$. Then, $0 \in F_{\varepsilon_0}(\psi;\chi)$ and $\chi \perp_{BJ}^{\varepsilon_0} \psi$. As a consequence, $\|\chi - \lambda\psi\| \geq \sqrt{1-\varepsilon_0{}^2}\,\|\chi\|$ for all $\lambda \in \mathbb{C}$. For $\lambda = 1$, it follows $\|\chi-\psi\|^2 \geq (1-\varepsilon_0{}^2)\,\|\chi\|^2$, and thus,

$$\begin{aligned}\cos_I(\chi,\psi) &= \frac{\|\chi+\psi\|^2 - \|\chi-\psi\|^2}{4\|\chi\|\,\|\psi\|}\\ &\leq \frac{\|\chi+\psi\|^2 - (1-\varepsilon_0{}^2)\|\chi\|^2}{4\|\chi\|\,\|\psi\|}\\ &= \frac{\|\chi+\psi\|^2 - \|\chi\|^2}{4\|\chi\|\,\|\psi\|} + \cos_{BJ}(\chi,\psi)^2\frac{\|\chi\|}{4\|\psi\|}.\end{aligned}$$

For $\lambda = -1$, it follows $\|\chi+\psi\|^2 \geq (1-\varepsilon_0{}^2)\,\|\chi\|^2$, and hence,

$$\begin{aligned}\cos_I(\chi,\psi) &= \frac{\|\chi+\psi\|^2 - \|\chi-\psi\|^2}{4\|\chi\|\,\|\psi\|}\\ &\geq \frac{(1-\varepsilon_0{}^2)\|\chi\|^2 - \|\chi-\psi\|^2}{4\|\chi\|\,\|\psi\|}\\ &= \frac{\|\chi\|^2 - \|\chi-\psi\|^2}{4\|\chi\|\,\|\psi\|} - \cos_{BJ}(\chi,\psi)^2\frac{\|\chi\|}{4\|\psi\|}. \qquad \square\end{aligned}$$

8 Cosines of operators

Consider two complex normed linear spaces $\mathcal{X}$ and $\mathcal{Y}$, and let $T, A : \mathcal{X} \longrightarrow \mathcal{Y}$ be two (nonzero) bounded linear operators. Let also $\chi, \psi \in \mathcal{X}$ be nonzero, and recall the following cosines:

$$\begin{aligned}\cos_{BJ}(T\chi, T\psi) &= \min\{\varepsilon \in [0,1) : T\chi \perp_{BJ}^{\varepsilon} T\psi\}\\ &= \min\left\{\varepsilon \in [0,1) : \|T\chi - \lambda T\psi\| \geq \sqrt{1-\varepsilon^2}\,\|T\chi\|,\ \forall\,\lambda \in \mathbb{C}\right\},\end{aligned}$$

$$\begin{aligned}\cos_{BJ}(T\chi, A\chi) &= \min\{\varepsilon \in [0,1) : T\chi \perp_{BJ}^{\varepsilon} A\chi\}\\ &= \min\left\{\varepsilon \in [0,1) : \|T\chi - \lambda A\chi\| \geq \sqrt{1-\varepsilon^2}\,\|T\chi\|,\ \forall\,\lambda \in \mathbb{C}\right\},\end{aligned}$$

and

$$\begin{aligned}\cos_{BJ}(T, A) &= \min\{\varepsilon \in [0,1) : T \perp_{BJ}^{\varepsilon} A\}\\ &= \min\left\{\varepsilon \in [0,1) : \|T - \lambda A\| \geq \sqrt{1-\varepsilon^2}\,\|T\|,\ \forall\,\lambda \in \mathbb{C}\right\}.\end{aligned}$$

Moreover, for any $\chi \in \mathcal{X}$ and $r \geq 0$, denote by $\mathcal{S}_{\mathcal{X}}(\chi, r)$ and $\mathcal{B}_{\mathcal{X}}(\chi, r)$ the sphere and the ball in $\mathcal{X}$ with center at χ and radius equal to r, respectively.

Recall that a bounded linear operator $U : \mathcal{X} \longrightarrow \mathcal{Y}$ is called a δ-isometry for some $\delta \in (0,1)$ if for every $\chi \in \mathcal{X}$, $(1-\delta)\|\chi\| \leq \|U\chi\| \leq (1+\delta)\|\chi\|$. Note that if a bounded linear operator $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is a scalar multiple of a 0-isometry, then the definitions imply readily that $\cos_{BJ}(T\chi, T\psi) = \cos_{BJ}(\chi, \psi)$. For general δ-isometries, we have the following result (see also [21]).

Proposition 8.1. *Let $T : \mathcal{X} \longrightarrow \mathcal{Y}$ be a scalar multiple of a δ-isometry. Then, for any nonzero vectors $\chi, \psi \in \mathcal{X}$,*

$$\cos_{BJ}(T\chi, T\psi) \leq \sqrt{1 - \left(\frac{1-\delta}{1+\delta}\right)^2 \sin_{BJ}(\chi, \psi)^2}.$$

Proof. Let $\cos_{BJ}(\chi, \psi) = \varepsilon_0$, and let $T = cU$ for some δ-isometry U and $c \in \mathbb{C}$. Then, $\chi \perp_{BJ}^{\varepsilon_0} \psi$, and hence, for every $\lambda \in \mathbb{C}$,

$$\begin{aligned} \|T\chi - \lambda T\psi\| &= \|T(\chi - \lambda\psi)\| = \|cU(\chi - \lambda\psi)\| \\ &\geq |c|(1-\delta)\|\chi - \lambda\psi\| \geq |c|(1-\delta)\sqrt{1-\varepsilon_0{}^2}\,\|\chi\| \\ &\geq |c|\frac{(1-\delta)}{(1+\delta)}\sqrt{1-\varepsilon_0{}^2}\,\|U\chi\| = \sqrt{1-\rho^2}\,\|T\chi\|, \end{aligned}$$

where $\rho = \sqrt{1 - \left(\frac{1-\delta}{1+\delta}\right)^2 (1-\varepsilon_0^2)} = \sqrt{1 - \left(\frac{1-\delta}{1+\delta}\right)^2 \sin_{BJ}(\chi, \psi)^2}$. □

Following the notation of [24], for a bounded linear operator T, we denote by $\mathcal{M}_T$ the set of all vectors in $\mathcal{S}_{\mathcal{X}}(0,1)$ at which T attains its norm. For a finite dimensional Hilbert space $\mathcal{H}$, Bhatia and Šemrl [3, Theorem 1.1] proved that for any two bounded linear operators $T, A : \mathcal{H} \longrightarrow \mathcal{H}$, $T \perp_{BJ} A$ if and only if there exists a $\chi \in \mathcal{M}_T$ such that $T\chi \perp_{BJ} A\chi$. Next, we extend the sufficiency part of this result.

Proposition 8.2. *Let $T, A : \mathcal{X} \longrightarrow \mathcal{Y}$ be two bounded linear operators, and let $\{\chi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_{\mathcal{X}}(0,1)$ be a sequence of unit vectors such that $\|T\chi_n\| \longrightarrow \|T\|$. Then, $\cos_{BJ}(T, A) \leq \sup_{n\in\mathbb{N}} \cos_{BJ}(T\chi_n, A\chi_n)$.*

Proof. Suppose that $\sup_{n\in\mathbb{N}} \cos_{BJ}(T\chi_n, A\chi_n) < 1$, and there is an $\varepsilon_0 \in [0,1)$ such that $\cos_{BJ}(T\chi_n, A\chi_n) \leq \varepsilon_0$ for all $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, it holds that $T\chi_n \perp_{BJ}^{\varepsilon_0} A\chi_n$, or

$$\|T\chi_n - \lambda A\chi_n\| \geq \sqrt{1-\varepsilon_0{}^2}\,\|T\chi_n\|, \quad \forall\, \lambda \in \mathbb{C},$$

or

$$\|T - \lambda A\| \geq \sqrt{1-\varepsilon_0{}^2}\,\|T\chi_n\|, \quad \forall\, \lambda \in \mathbb{C}.$$

By continuity, since $\|T\chi_n\| \longrightarrow \|T\|$, it follows

$$\|T - \lambda A\| \geq \sqrt{1-\varepsilon_0{}^2}\,\|T\|, \quad \forall\, \lambda \in \mathbb{C}.$$

Hence, $T \perp_{BJ}^{\varepsilon_0} A$ and $\cos_{BJ}(T, A) \leq \varepsilon_0$. □

Corollary 8.3. *Let $T, A : \mathcal{X} \longrightarrow \mathcal{Y}$ be two bounded linear operators.*

(i) *If* $\mathcal{M}_T \neq \emptyset$, *then for any* $\chi \in \mathcal{M}_T$, $\cos_{BJ}(T, A) \leq \cos_{BJ}(T\chi, A\chi)$.

(ii) *If* $\{\chi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_{\mathcal{X}}(0,1)$ *is a sequence of unit vectors such that* $\|T\chi_n\| \longrightarrow \|T\|$, *and* $T\chi_n \perp_{BJ} A\chi_n$ *for all* $n \in \mathbb{N}$, *then* $T \perp_{BJ} A$.

(iii) *If* $\mathcal{M}_T \neq \emptyset$ *and there exists a* $\chi \in \mathcal{M}_T$ *such that* $T\chi \perp_{BJ} A\chi$, *then* $T \perp_{BJ} A$.

We conclude this section (and paper) by extending two characterizations of the Birkhoff-James orthogonality of bounded linear operators defined on real normed linear spaces [24, Theorems 2.1 and 2.8] (see also the references in [24]) to the complex case. For two vectors $\chi, \psi \in \mathcal{X}$ and two scalars $\theta \in [0, 2\pi]$ and $\varepsilon \in [0, 1)$, we say that $\psi \in \chi^{(\theta,\varepsilon)}$ if $\|\chi + (e^{i\theta}r)\psi\| \geq \sqrt{1-\varepsilon^2}\|\chi\|$ for all $r \geq 0$. Apparently, for any $\varepsilon \in [0,1)$, $\chi \perp_{BJ}^{\varepsilon} \psi$ if and only if $\psi \in \chi^{(\theta,\varepsilon)}$ for all $\theta \in [0, 2\pi]$. If $\psi \in \chi^{(\theta,\varepsilon)}$ for some $\theta \in [0, 2\pi]$ and $\varepsilon \in [0,1)$, then $a\psi \in (b\chi)^{(\theta,\varepsilon)}$ for any $a, b \geq 0$. Moreover, for every $\theta, \phi \in [0, 2\pi]$ with $0 < \phi \leq \theta \leq 2\pi$, $\|\chi + (e^{i\theta}r)\psi\| = \|\chi + (e^{i(\theta-\phi)}r)e^{i\phi}\psi\|$, and thus, $\psi \in \chi^{(\theta,\varepsilon)}$ if and only if $e^{i\phi}\psi \in \chi^{(\theta-\phi,\varepsilon)}$.

Lemma 8.4. *Consider two vectors* $\chi, \psi \in \mathcal{X}$ *and two scalars* $\theta \in [0, 2\pi]$ *and* $r_0 > 0$ *such that* $\|\chi + (e^{i\theta}r_0)\psi\| < \|\chi\|$. *Then, for every* $r \in (0, r_0]$, $\|\chi + (e^{i\theta}r)\psi\| < \|\chi\|$.

Proof. By hypothesis, $\|\chi + (e^{i\theta}r_0)\psi\| < \|\chi\|$. For any $r \in (0, r_0]$, it holds that

$$\begin{aligned}
\|\chi + (e^{i\theta}r)\psi\| &= \left\| \left(\frac{r_0 - r}{r_0}\right)\chi + \left(\frac{r}{r_0}\right)\chi + \left(\frac{r}{r_0}\right)(e^{i\theta}r_0)\psi \right\| \\
&\leq \left(\frac{r_0 - r}{r_0}\right)\|\chi\| + \left(\frac{r}{r_0}\right)\|\chi + (e^{i\theta}r_0)\psi\| \\
&< \left(\frac{r_0 - r}{r_0}\right)\|\chi\| + \left(\frac{r}{r_0}\right)\|\chi\| = \|\chi\|. \qquad \square
\end{aligned}$$

Proposition 8.5. *Consider two vectors* $\chi, \psi \in \mathcal{X}$. *Then, for any* $\theta \in [0, \pi]$, $\psi \in \chi^{(\theta,0)}$ *or* $\psi \in \chi^{(\theta+\pi,0)}$.

Proof. Suppose that $\psi \notin \chi^{(\theta,0)}$ and $\psi \notin \chi^{(\theta+\pi,0)}$. This means that there are two complex numbers $\lambda_\theta = e^{i\theta}r_\theta$ and $\lambda_{\theta+\pi} = e^{i(\theta+\pi)}r_{\theta+\pi}$, with $r_\theta, r_{\theta+\pi} > 0$, such that $\|\chi + \lambda_\theta\psi\| < \|\chi\|$ and $\|\chi + \lambda_{\theta+\pi}\psi\| < \|\chi\|$. By Lemma 8.4, if $\hat{r} = \min\{r_\theta, r_{\theta+\pi}\} > 0$, then for every $r \in [-\hat{r}, 0) \cup (0, \hat{r}]$, $\|\chi + (e^{i\theta}r)\psi\| < \|\chi\|$ (where, for any $r \in [-\hat{r}, 0) \cup (0, \hat{r}]$, $-r$ also lies in $[-\hat{r}, 0) \cup (0, \hat{r}]$). As a consequence,

$$\begin{aligned}
\|\chi\| &= \left\| \frac{1}{2}(\chi + (e^{i\theta}r)\psi) + \frac{1}{2}(\chi + (-e^{i\theta}r)\psi) \right\| \\
&\leq \frac{1}{2}\|\chi + (e^{i\theta}r)\psi\| + \frac{1}{2}\|\chi + (-e^{i\theta}r)\psi\| \\
&< \frac{1}{2}\|\chi\| + \frac{1}{2}\|\chi\| = \|\chi\|,
\end{aligned}$$

which is a contradiction. $\square$

Finally, we extend Theorems 2.1 and 2.8 of [24] to complex normed linear spaces.

Theorem 8.6. *Consider a reflexive complex Banach space* $\mathcal{X}$ *and a complex normed linear space* $\mathcal{Y}$. *Let* $T, A : \mathcal{X} \longrightarrow \mathcal{Y}$ *be two compact linear operators. Then,* $T \perp_{BJ} A$ *if and only if, for any* $\theta \in [0, 2\pi]$, *there is a vector* $\chi_\theta \in \mathcal{M}_T$ *such that* $A\chi_\theta \in (T\chi_\theta)^{(\theta,0)}$.

Proof. Suppose that for any $\theta \in [0, 2\pi]$, there is a vector $\chi_\theta \in \mathcal{M}_T$ such that $A\chi_\theta \in (T\chi_\theta)^{(\theta,0)}$. Then for every $\theta \in [0, 2\pi]$ and $r \geq 0$, $\|T + (re^{i\theta})A\| \geq \|(T + (re^{i\theta})A)\chi_\theta\| \geq \|T\chi_\theta + (re^{i\theta})A\chi_\theta\| \geq \|T\chi_\theta\| = \|T\|$, i.e., $T \perp_{BJ} A$.

For the converse, we follow the arguments of the proof of Theorem 2.1 in [24], replacing the operators $T + \frac{1}{n}A$ and $T + \lambda A$ $(\lambda \geq 0)$ by the operators $T + \frac{1}{n}e^{i\theta}A$ and $T + (e^{i\theta}r)A$ $(r \geq 0)$, respectively. □

Remark 8.1. Let $\mathcal{X}$ be a reflexive complex Banach space and $\mathcal{Y}$ be a complex normed linear space. Let also $T, A : \mathcal{X} \longrightarrow \mathcal{Y}$ be two compact linear operators with $\cos_{BJ}(T, A) = \varepsilon_0 > 0$. Then $0 \in \partial F_{\varepsilon_0}(A, T)$, or equivalently, there is a scalar $\lambda_0 \in \mathbb{C}$ such that

$$\|T + \lambda_0 A\| = \min_{\lambda\in\mathbb{C}} \|T + \lambda A\| = \sqrt{1 - {\varepsilon_0}^2}\|T\| = \sin_{BJ}(T, A)\|T\|.$$

Hence, $(T + \lambda_0 A) \perp_{BJ} A$, and by Theorem 8.6, for any $\theta \in [0, 2\pi]$, there is a vector $\chi_\theta \in \mathcal{M}_{T+\lambda_0 A}$ such that $A\chi_\theta \in [(T + \lambda_0 A)\chi_\theta]^{(\theta,0)}$. Equivalently, for any $\theta \in [0, 2\pi]$, there is a $\chi_\theta \in \mathcal{S}_\mathcal{X}(0, 1)$ such that for every $r > 0$,

$$\|T\chi_\theta + (\lambda_0 + e^{i\theta}r)A\chi_\theta\| \geq \|T\chi_\theta + \lambda_0 A\chi_\theta\| = \|T + \lambda_0 A\| = \sin_{BJ}(T, A)\|T\|.$$

Theorem 8.7. *Consider two complex normed linear spaces $\mathcal{X}$, $\mathcal{Y}$, and let $T, A : \mathcal{X} \longrightarrow \mathcal{Y}$ be two nonzero bounded linear operators. Then, $T \perp_{BJ} A$ if and only if one of the following holds:*

(i) *There exists a sequence $\{\chi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_\mathcal{X}(0, 1)$ such that $\|T\chi_n\| \longrightarrow \|T\|$ and $\|A\chi_n\| \longrightarrow 0$.*

(ii) *For any $\theta \in [0, 2\pi]$, there is a sequence of vectors $\{\chi_{\theta,n}\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_\mathcal{X}(0, 1)$ and a sequence of real numbers $\{\varepsilon_{\theta,n}\}_{n\in\mathbb{N}} \subseteq (0, 1)$ such that*

   (a) $\varepsilon_{\theta,n} \longrightarrow 0$,

   (b) $\|T\chi_{\theta,n}\| \longrightarrow \|T\|$, *and*

   (c) $A\chi_{\theta,n} \in (T\chi_{\theta,n})^{(\theta,\varepsilon_{\theta,n})}$ *for all* $n \in \mathbb{N}$.

Proof. Suppose that (i) holds. Then for every $\lambda \in \mathbb{C}$, $\|T + \lambda A\| \geq \|T\chi_n + \lambda A\chi_n\| \geq \|T\chi_n\| - |\lambda|\,\|A\chi_n\|$. For $n \longrightarrow +\infty$, it follows $\|T + \lambda A\| \geq \|T\|$, and hence, $T \perp_{BJ} A$.

Suppose that (ii) holds, and let $\theta \in [0, 2\pi]$. Then there exist two sequences $\{\chi_{\theta,n}\}_{n\in\mathbb{N}} \subseteq \mathcal{S}_\mathcal{X}(0, 1)$ and $\{\varepsilon_{\theta,n}\}_{n\in\mathbb{N}} \subseteq (0, 1)$ such that (a)–(c) are satisfied. Then, for every $r \geq 0$, $\|T + (e^{i\theta}r)A\| \geq \|T\chi_{\theta,n} + (e^{i\theta}r)A\chi_{\theta,n}\| \geq \sqrt{1 - {\varepsilon_{\theta,n}}^2}\|T\chi_{\theta,n}\|$. For $n \longrightarrow +\infty$, it follows $\|T + (e^{i\theta}r)A\| \geq \|T\|$ for all $r \geq 0$. Since the latter inequality holds for any $\theta \in [0, 2\pi]$, $T \perp_{BJ} A$.

For the converse, we follow the arguments of the proof of Theorem 2.8 in [24], replacing the operators $T + \frac{1}{n}A$ and $T + \lambda A$ $(\lambda \geq 0)$ by the operators $T + \frac{1}{n}e^{i\theta}A$ and $T + (e^{i\theta}r)A$ $(r \geq 0)$, respectively. □

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