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# FINITE TYPE INVARIANTS FOR KNOTOIDS

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## ABSTRACT

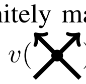
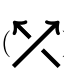
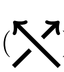
We extend the theory of Vassiliev (or finite type) invariants for knots to knotoids using two different approaches. Firstly, we take closures on knotoids to obtain knots and we use the Vassiliev invariants for knots, proving that these are knotoid isotopy invariant. Secondly, we define finite type invariants directly on knotoids, by extending knotoid invariants to singular knotoid invariants via the Vassiliev skein relation. Then, for spherical knotoids we show that there are non-trivial type-1 invariants, in contrast with classical knot theory where type-1 invariants vanish. We give a complete theory of type-1 invariants for spherical knotoids, by classifying linear chord diagrams of order one, and we present examples arising from the affine index polynomial and the extended bracket polynomial.

***Keywords*** singular knotoids · finite type invariants · Vassiliev invariants · linear chord diagrams · regular diagrams · extended bracket polynomial · affine index polynomial

## Introduction

This paper initiates the study of Vassiliev or finite type invariants for knotoids.

Knotoids are a generalization of classical knot theory introduced by V. Turaev [33]. He showed that the set of classical knots injects into the set of spherical knotoids. Knotoids have caught the attention of the mathematical community in the last years with many different results and applications to other aspects of science, such as the topological study of proteins,[5, 14, 9, 11, 8, 13, 16, 26, 4, 17, 1].

The Vassiliev invariants are extensions of knot invariants to knots which fail to be embeddings in finitely many transversal double points. The extension is possible using the relation called the Vassiliev skein relation: $v($$) =$ $v($$) - v($$)$. These invariants behave like building blocks for knot invariants like the classical Jones polynomial

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whose coefficients in the Taylor expansion are Vassiliev invariants. It is indeed a very strong family of knot invariants, as shown by the proof of the Vassiliev-Kontsevich Theorem in [28]. Nevertheless, it is not yet known whether Vassiliev invariants can classify knots or even if they detect the unknot.

Trying to extend the theory of finite type invariants for knots to knotoids one can use two different approaches. The immediate one is to take various types of closures on singular knotoids, such as the under/over closure and the virtual closure, to obtain singular knots or singular virtual knots, and to use the theory of Vassiliev invariants for knots (or virtual knots), proving that these are (virtual) knotoid isotopy invariants too. The second approach is to define finite type invariants directly on knotoids, by extending knotoid invariants to singular knotoid invariants via the Vassiliev skein relation. This approach is our main focus.

In this work we consider mainly knotoid diagrams on the plane and in the 2-sphere and we show that there are non-trivial Vassiliev invariants of type-1 for such knotoids. This is in contrast with classical knot theory where type-1 invariants vanish.

The first tool in our theory is the introduction of linear chord diagrams, which extend the combinatorial approach of chord diagrams for singular knots. A linear chord diagram of order $n$ is an oriented closed interval, equipped with a distinguished set of $n$ disjoint pairs of distinct points being connected with immersed unknotted arcs in the sphere or in the plane in general position, the 'chords'. See Fig. 1 for examples.

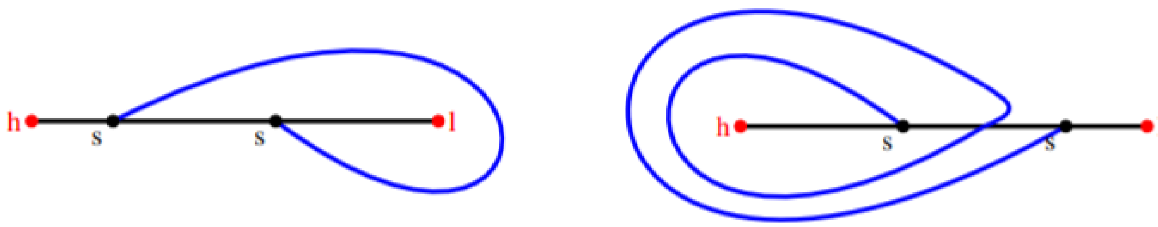



Figure 1: Linear chord diagrams

We give a complete theory of Vassiliev invariants of type-1 for spherical knotoids, by classifying linear chord diagrams of order one. As we show, a spherical linear chord diagram of order one can be abstracted by its winding number, as illustrated in Fig. 2.

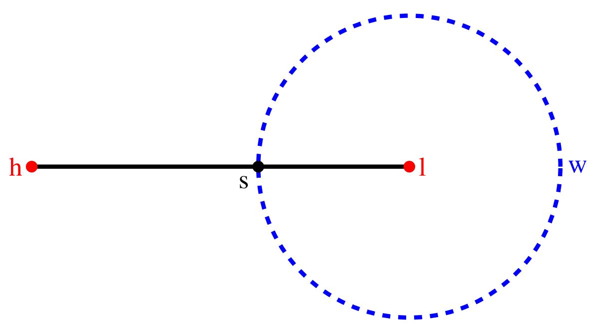



Figure 2: The abstraction of a chord diagram with winding number $w \in \mathbb{Z}$.

We then classify spherical knotoid diagrams with 1 singular crossing, up to singular equivalence (that is, rigid isotopy and crossing switches) by means of Theorem 5.4, which is our first main result and states that:

*Every spherical knotoid diagram with* 1 *singularity is singular equivalent to one and only one regular diagram.*

A regular diagram is a descending spherical knotoid diagram with 1 singular crossing and with the minimal crossing number of real crossings, which number is odd, and such that the singularity is located 'next to' the leg of the knotoid. See Fig. 3 for examples.

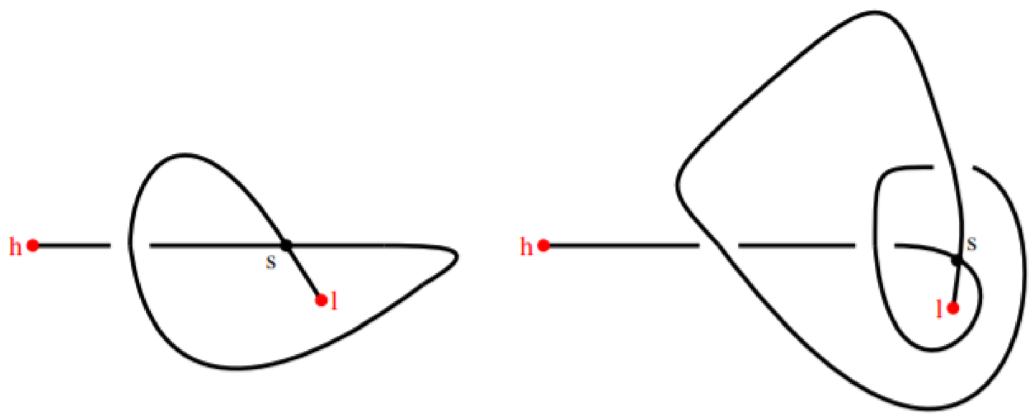



Figure 3: The regular diagrams $R_1$,$R_2$

We proceed with proving Theorem 5.10, which is our second main result, stating that:

*Every function on linear chord diagrams which respects the one-term relation gives rise to a type-1 knotoid invariant.*

As a corollary to Theorems 5.4 and 5.10 we have that there are non-trivial type-1 invariants for spherical knotoids. Moreover, there are non-trivial type-1 invariants also for planar knotoids, since a spherical knotoid corresponds uniquely to a planar knotoid whose head lies in the outer region of $\mathbb{R}^2$ (minus the knotoid), see Lemma 5.6.

We conclude the paper by presenting examples of type-1 invariants for knotoids. We show that the affine index polynomial, originally defined for virtual knots in [21] and reformulated for knotoids in [14], is a type-1 invariant. This appearance of type-1 Vassiliev invariants is related to the work of Henrich [18] who found type-1 Vassiliev invariants for virtual knots. We also consider higher order Vassiliev invariants for spherical knotoids via the Jones polynomial, the Kauffman bracket polynomial and the Turaev extended bracket polynomial for spherical knotoids.

The outline of the paper is as follows. In Section 1 we recall the basic definitions and results from the theory of knotoids and we give examples of some strong knotoid invariants. Then, we summarize the theory of finite type invariants and chord diagrams and recall the specific formulas for the Vassiliev invariants arising from the Jones polynomial. In Section 2 we give the definition of a singular knotoid and present different types of closures for knotoids. We continue with Section 3 in which we define finite type invariants for knotoids in two ways: using the closures to induce invariants of knotoids from invariants of knots, and by defining finite type invariants directly on knotoids. In Section 4 we extend the classical chord diagrams to linear chord diagrams, which correspond to singular equivalence classes of singular knotoids. The main results are in Section 5 in which we describe the notion of a regular knotoid diagram and with this we classify singular knotoid diagrams with one singularity in Theorem 5.4. Then, with Theorem 5.10 we demonstrate a way to produce various type-1 knotoid invariants using functions on linear chord diagrams. Finally, Section 7 includes various examples of type-1 invariants, while in Section 6 we discuss the singular height and in Section 8 we show that the Turaev extended bracket gives rise to higher type invariants.

# 1 Background

## 1.1 Knotoids

In this subsection we shall recall some basic notions from the theory of knotoids. The theory was introduced by V. Turaev in 2011 [33], where he showed that the theory of spherical knotoids extends faithfully classical knot theory. He also introduced several invariants for knotoids, such as the bracket and the extended bracket polymonial, while in [14] Gügümcü and Kauffman defined some more. The theory of knotoids has caught the attention of the mathematical community in the last years , enriching the theory with many more interesting results and finding applications to other aspects of science, [5, 9, 11, 8, 13, 16, 26, 4, 17, 1].

Let $\Sigma$ be a path connected, oriented 2-manifold without boundary smoothly embedded in $\mathbb{R}^3$. A *knotoid diagram* is a generic immersion $\gamma$ of the interval in a surface $\Sigma$ whose singularities are only finitely many transversal double points, which are endowed with over- or undercrossing data. These double points are called crossings. $\gamma(0)$ is called the leg and $\gamma(1)$ is called the head of the knotoid, while both comprise the endpoints of the knotoid. We will actually call a knotoid diagram its image in $\Sigma$ rather than the immersion itself. Similarly, a *multi-knotoid diagram* is a generic immersion of a disjoint union of the interval and some copies of the circle, and an isotopy class of a multi-knotoid diagram is a *multi-knotoid*. For our purposes in this paper the oriented manifold will be mostly the 2-sphere or the plane.

In classical knot theory, ambient isotopy is generated combinatorially by three local moves on diagrams, the *Reidemeister moves*, together with planar or disc isotopy. Note that in the knotoid theory an isotopy may displace the endpoints, but may not pull a strand adjacent to an endpoint over or under a transversal strand. This justifies the notion of the *forbidden moves* $\Omega_+, \Omega_-$, illustrated in Fig. 5.

Two knotoid diagrams $K_1, K_2$ are called *isotopic* if they differ by disc isotopies of $\Sigma$ and the Reidemeister moves. An equivalence class of a knotoid diagram is called a *knotoid*. See Fig. 4.

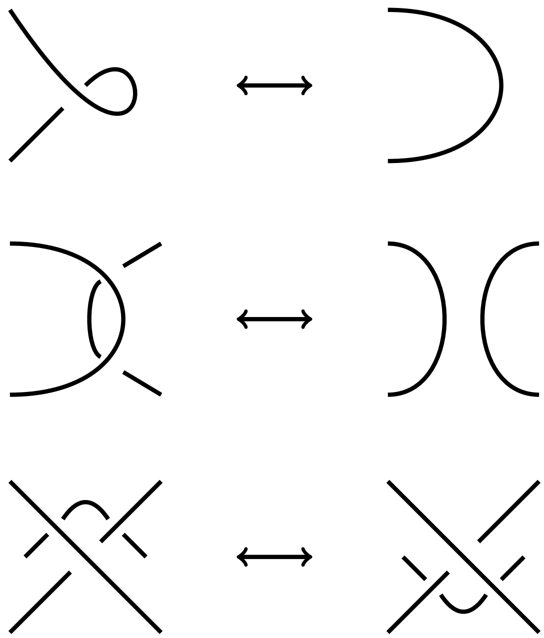


Figure 4: The Reidemeister moves

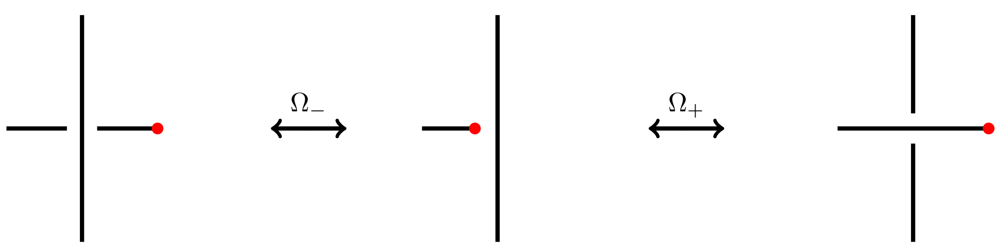



Figure 5: The forbidden moves for knotoids

Note that isotopy of spherical knotoids corresponds to isotopy of an $(1, 1)$-tangle in the annulus with the two ends attached to the two boundary components and allowed to slide along them (where the annulus is formed by removing the two points $l, h$ in $S^2$).

The *height* (or complexity) of a knotoid $K$ is the minimum number of crossings (over all diagrams of $K$) that are created when joining the leg and the head of $K$ by a simple unknotted arc. Such an arc is a *shortcut* for the knotoid. The spherical knotoids of zero height are the *knot-type knotoids* and their isotopy classes correspond bijectively to the classical knot types. The knotoids of height greater than zero are called *proper knotoids*.

Given two spherical knotoid diagrams $K_1, K_2$ of knotoids $k_1, k_2 \in K(S^2)$ one can form the *product* $k_1k_2$ by deforming both $K_1, K_2$ such that their heads are in a small neighbourhood of the point at infinity. This is achievable using isotopy in the sphere. Then we attach a copy of $K_2$ to the head of $K_1$ in a small neighborhood of the latter.

Furthermore, given a knot $\kappa$ and a knotoid $k$ one can define the *product* $k\kappa$ in the following manner. Present $\kappa$ by an oriented knot diagram $D$ in $S^2$ and pick a small open arc $\alpha \subset D$ disjoint from the crossings. Then $K = D - \alpha$ is a knotoid diagram in $S^2$. This knotoid diagram is unique up to isotopy. Then take a knotoid diagram $K'$ of $k$, as described above and multiply it with $K$. That is, $k \cdot \kappa := K' \cdot K$.

One thing that is worth mentioning now for our purpose is a special way to represent planar knotoids, which is due to N. Gügümcü and L.H. Kauffman [14]. Intuitively speaking, one could take a knot in the thickened surface $\Sigma \times F$ out of knotoid diagram that lives in a surface $\Sigma$. For $K(\mathbb{R}^2)$ there is something more specific. Identify the plane of the planar knotoid $K$ with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. $K$ can be embedded into $\mathbb{R}^3$ by pushing the overpasses of the diagram into the upper half-space and the underpasses into the lower half-space in the vertical direction. This creates an embedding of $[0, 1]$ in $\mathbb{R}^3$. The leg and the head of the diagram are attached to the two lines, $\{l\} \times \mathbb{R}$ and $\{h\} \times \mathbb{R}$ that pass through the leg and the head, respectively and are perpendicular to the plane of the diagram. Moving the endpoints of $K$ along these lines gives rise to open oriented curves embedded in $\mathbb{R}^3$ with two endpoints lying on each line. Such a curve is said to be the rail lifting of a knotoid diagram. An example of a knotoid $K_2$ and its rail lifting $K_1$ is illustrated in Fig.6.

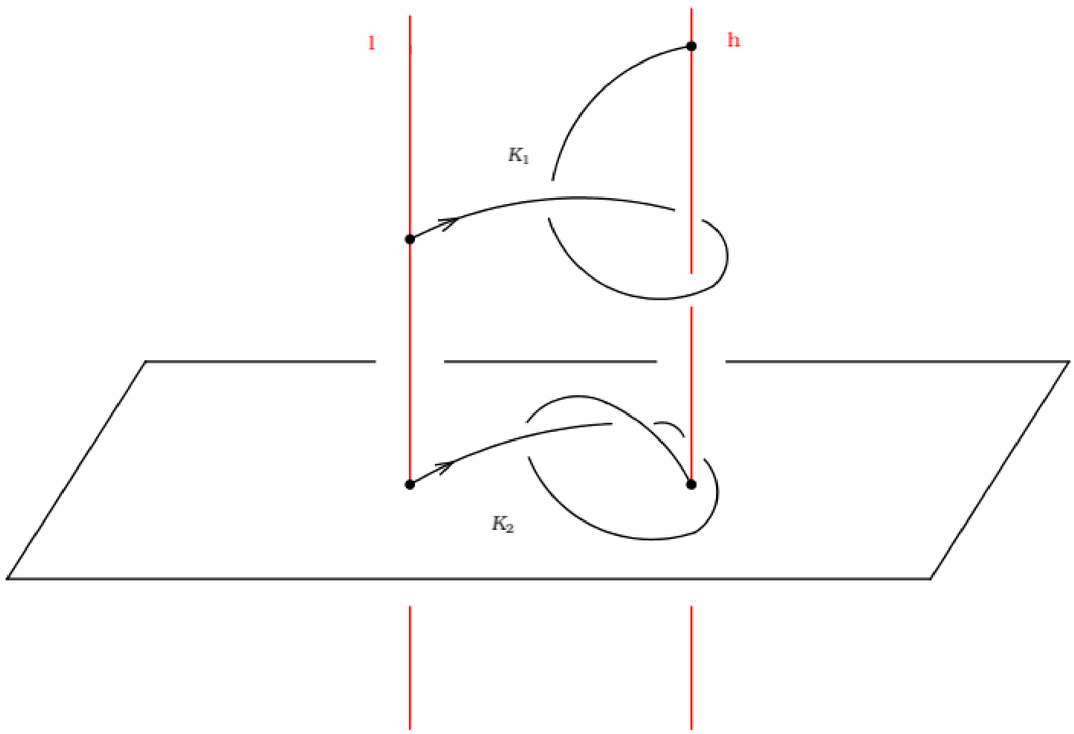



Figure 6: Example of a knotoid and its rail lifting

Two smooth open oriented curves embedded in $\mathbb{R}^3$ with the endpoints that are attached to two special lines, are said to be *line isotopic* (or *rail isotopic* [26]) if there is a smooth ambient isotopy of the pair $(\mathbb{R}^3/\{t \times \mathbb{R}, h \times \mathbb{R}\}, t \times \mathbb{R} \cup h \times \mathbb{R})$, taking one curve to the other curve in the complement of the lines, taking endpoints to endpoints, and lines to lines.

In [14] Gügümcü and Kauffman prove that there is a one-to-one correspondence between the set of knotoids in $\mathbb{R}^2$ and the set of line-isotopy classes of smooth open oriented curves in $\mathbb{R}^3$, which have their two ends attached to the two parallel lines. Furthermore, in [26] D. Kodokostas and S. Lambropoulou develop the theory of *rail knotoids*, which are projections of line curves in the plane of the two parallel lines (the 'rails').

### 1.2 A digression on virtual knots

Virtual knot theory was introduced by L.H. Kauffman in 1998 [21]. We recall that a *virtual knot diagram* is an immersion of $S^1$ in the plane, containing finitely many double points, some of which are classical crossings and some are virtual crossings. The virtual crossings can be viewed as permutation crossings with no 'under' or 'over' specification. Virtual isotopy comprises the planar isotopy, the Reidemeister moves among classical crossings and the extra isotopy moves involving also virtual crossings, as illustrated in Fig. 7. A *virtual knot* is a virtual isotopy class of virtual knot diagrams.

We note that Reidemeister moves of type III, involving two classical crossings and one virtual are not allowed in this theory.

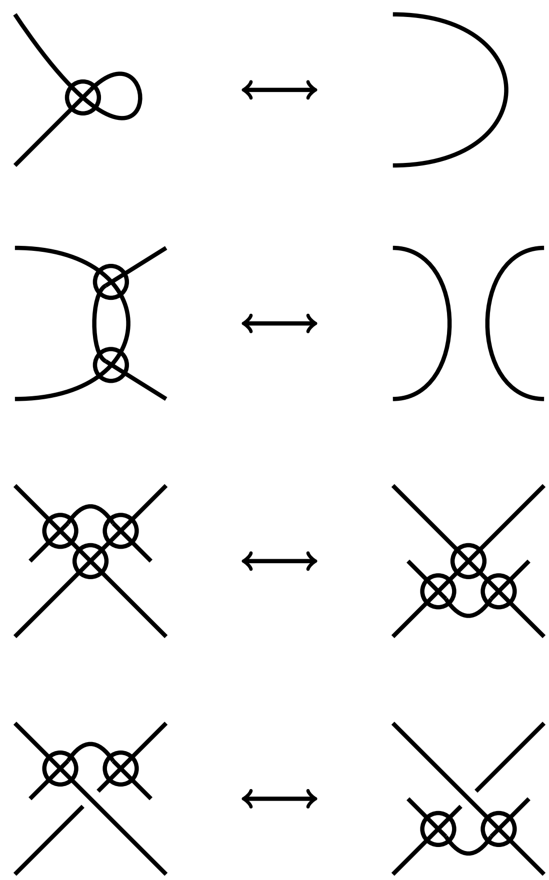


Figure 7: Isotopy moves involving virtual crossings.

### 1.3 Finite type invariants and chord diagrams

A *singular knot* is a mapping of $S^1$ in $\mathbb{R}^3$ that fails to be an embedding only in finitely many points where we have only transversal self-intersections, the *singular crossings*.

Two singular knots are said to be *rigid vertex isotopic* if any two diagrams of theirs differ by a fine sequence of disc isotopies, the Reidemeister moves for classical knots, and the rigid vertex isotopy moves involving singular crossings. See Figs. 4 and 8. See for example [34], [2], [23].

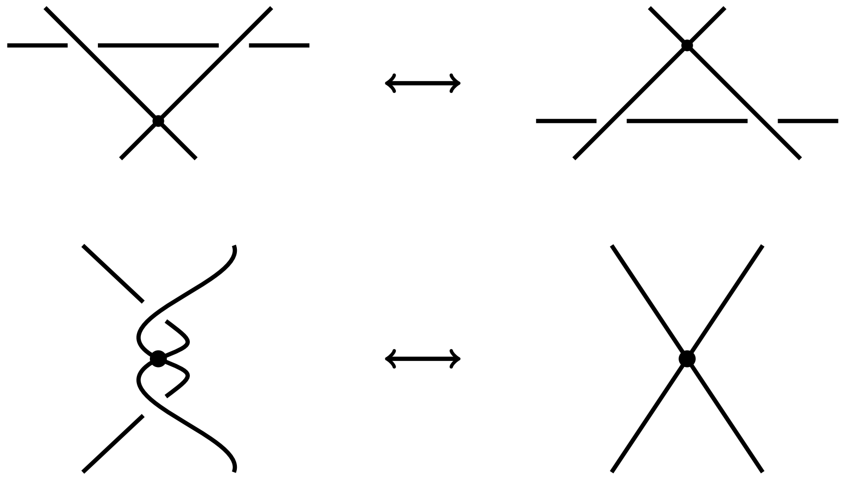
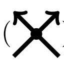
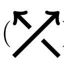
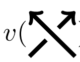


Figure 8: Rigid vertex isotopy moves for singular crossings

Any knot invariant $v$ can be extended to an invariant of singular knots using the *Vassiliev skein relation*:

$v($$) = v($$) - v($$)$ (1)

The small diagrams in the relation denote diagrams that are identical except for the regions in which they differ only as indicated in the small diagrams.

Using this relation successively, one can extend $v$ to singular knots with an arbitrary number of singular crossings. While there are many choices when taking and resolving a sequence of singular crossings, in fact the complete resolution yields the alternating sum

$$\sum_{\varepsilon_1=\pm 1,\ldots,\varepsilon_n=\pm 1} (-1)^{|\varepsilon|} v(K_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n})$$

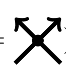
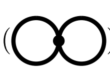
where $|\varepsilon|$ is the number of $-1$'s in the sequence of $\varepsilon_i$ and $K_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}$ the knot obtained by positive/negative resolution of each singular crossing of $K$.

A knot invariant $v$ is said to be of *finite type* $k$ if there exists a $k \in \mathbb{N}$ such that $v$ vanishes on every knot with more than $k$ singular crossings. We say that $v$ is of order (or type) $\leqslant k$. See [34], [7]

Rigid vertex isotopy together with crossing switches generate an equivalence relation, the *singular equivalence*, which is used for the so-called 'top row' diagrams.

**Remark 1.1.** Every knot invariant of type 1 is trivial.

Indeed, let $v$ be an invariant of type 1 and let $K$ be a singular knot with exactly 1 singular crossing ($v$ vanishes on knots with more singularities). The singular crossing divides the knot into two disjoint closed curves. After appropriate classical crossing switches, using the Vassiliev skein relation, the two closed curves can turn into simple unknotted closed curves, with no self-crossings or shared crossings between them. Now, these crossing switches cost nothing, because each use of Eq. 1 will add a singular crossing to the one of the two resulting diagrams and, by definition, $v$ will vanish on this diagram. Hence, we derive the relation:

$v(K :=$$) = v($$)$

which, again, using Eq. 1, is clearly equal to zero, by the first Reidemeister move.

**Remark 1.2.** The technique of (classical) crossing switches can be applied to any finite type invariant of order $k$ when applied to a 'top row' diagram (that is, a singular knot with precisely $k$ singular crossings). Namely, one can use Eq. 1 for switching any real crossing at will. As a result, the value of a finite type invariant of order $k$ depends only on the nodal structure of the singularities (the underlying 4-valent graph) and not on the embedding of the graph in space.

In the original work of V. Vassiliev [34], finite type invariants correspond to the zero-dimensional classes of a special spectral sequence. In fact, the space of knots is understood by the complement of the space of mappings of $S^1$ in $\mathbb{R}^3$ that fail in some way to be an embedding (the so-called discriminant). Knot invariants are just locally constant functions in the space of knots and Vassiliev's formalism was trying to show that there exists such a spectral sequence that converges to the cohomology of the space of knots. These definitions and calculations have been very much simplified by the works of Bar-Natan [2], Birman & Lin [6], Stanford [31] and Polyak & Viro [30], as we are interested only in this zero-class, and as the use of the chord diagrams made a big part of the theory combinatorial.

A *chord diagram* encodes the order of double points along a singular knot. A Vassiliev invariant of order $k$ gives rise to a function on chord diagrams with $k$ chords.The conditions that a function on chord diagrams should satisfy in order to come from a Vassiliev invariant are the so-called one-term and four-term relations. The vector space spanned by chord diagrams modulo these relations has the structure of a Hopf algebra. This Hopf algebra turns out to be dual to the algebra of the Vassiliev invariants and this isomorphism is the famous Vassiliev-Kontsevich theorem [28]. There are some interesting questions concerning this fact including if there exists a similar strucure for knotoids.

## 2 Singular knotoids

In this section we define the notion of singular knotoid in any connected, oriented surface $\Sigma$.

### 2.1 Singular knotoid diagrams and rigid vertex isotopy

**Definition 2.1.** A *singular knotoid diagram* is a knotoid diagram with some (finite) undeclared double points which are transversal (i.e. the two corresponding velocity vectors of the curve are linearly independent). See Fig. 9 for some examples.

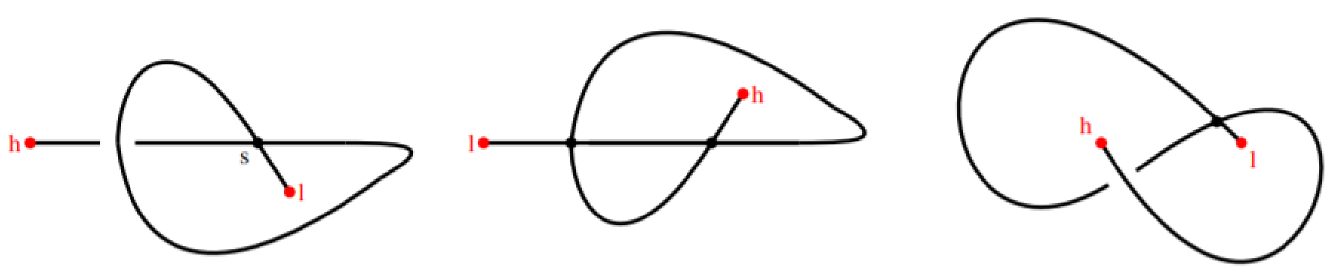



Figure 9: Singular knotoids

**Definition 2.2.** A *rigid vertex isotopy for singular knotoid diagrams* is generated by locally planar isotopy and the Reidemeister moves for classical knotoids (recall Fig. 4), extended by the rigid vertex isotopy moves involving singular crossings (recall Fig. 8). An isotopy class of singular knotoid diagrams is called *singular knotoid*.

Note that in the theory of singular knotoids we still have the restrictions of the forbidden moves (recall Fig. 5).

### 2.2 Types of closures

**Definition 2.3.** We call *classical closure for (classical or singular) knotoids of type 'o' (resp. 'u')* a mapping $c^o$ (resp. $c^u$) from the set of (classical or singular) knotoid diagrams in a surface $\Sigma$ to the set of singular knot diagrams in $\Sigma$, induced by the over- (resp. under-) closure on knotoid diagrams.

$$c^o(K) := K^o \quad \text{and} \quad c^u(K) := K^u$$

where $K^o, K^u$ denote the singular knots in $\Sigma$ obtained via the over- resp. under-closure of a knotoid $K$. See top row of Fig. 10.

As we will see the classical closure is well-defined for planar and spherical knotoids.However, in surfaces of higher genus there is a multiplicity of closures, related to non-equivalent homotopy classes.

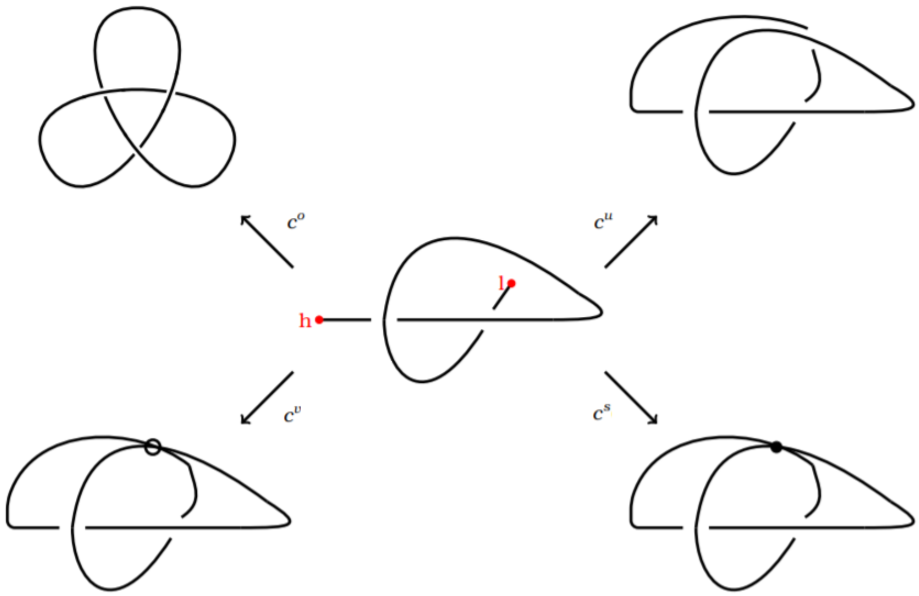



Figure 10: Types of closures: over- and under-closures (top row), virtual closure (bottom left), and singular closure of height 1 (bottom right)

**Lemma 2.4.** *Any classical closure (over or under) for spherical or planar (singular) knotoids is well-defined up to isotopy.*

*Proof.* We want to show that a singular knotoid $K \subset S^2$ determines, via the under-closure, a unique singular knot $K_u \subset S^3$ up to isotopy.

Indeed, pick an embedded arc $\alpha \subset S^2$ connecting the endpoints of $K$ and otherwise meeting $K$ transversely at a finite set of points distinct from the crossings of $K$ (a shortcut for $K$) $K \cup \alpha$ is a knot diagram declaring that $\alpha$ passes everywhere under $K$. The orientation of $K$ from leg to head defines an orientation on $K \cup \alpha$ The knot in $S^3$ represented by $K \cup \alpha$ is denoted by $K^u$. We say that $K$ represents $K^u$.

$K^u$ does not depend on the choice of the shortcut $\alpha$ because, by the Reidemeister moves, any two shortcuts for $K$ are isotopic in the class of embedded arcs in $S^2$ connecting the endpoints of $K$. □

**Remark 2.5.** Suppose that for a knotoid we have a diagram $K$ which realizes its height. One might consider taking all possible closures (using both under and over crossing) by taking a diagram that realizes the height and an arc connecting leg to head and not passing through any other crossing, but creating $c$ new crossings. Then one would declare with $2^c$ options which crossing is under and which over. Given two isotopic knotoid diagrams $K^1$,$K^2$ with the same height, say 2, there is no way to prove that the knot $K^1_{o,u}$ is knot-isotopic with the knot $K^2_{o,u}$ or $K^2_{u,o}$, where $K^n_{o,u}$ is the knot that we obtain if, starting from the leg of $K^n$ and following the arc $\alpha$, we declare the first extra crossing to be over-crossing and the second one to be under-crossing. Respectively, we denote $K^n_{i_1,i_2,\ldots,i_c}$ to be the closure of the knotoid $K^n$ with height $c$, such that $i_1, ..., i_c \in \{o, u\}$ and if $i_k = o$ this means that the $k$th extra crossing is declared to be over, while if $i_k = u$ this means that the $k$th extra crossing is declared to be under, $k = 1, \ldots, c$. The above means that $K^n_{i_1,i_2,\ldots,i_c}$ is not well-defined, as different shortcuts will possibly give non isotopic knots.

For example if we take two distinct shortcuts in the knotoid illustrated in Fig. 11 , we will end up with different normalized bracket polynomials with the $(u, o)$-closure.

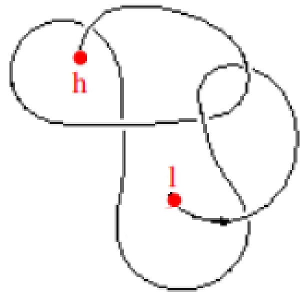



Figure 11: A knotoid that attains two distinct $(u, o)$-closures

We will now introduce the virtual closure. Recall the extra isotopy moves between virtual knot diagrams, illustrated in Fig. 7.

**Definition 2.6.** We call *virtual closure for knotoids* (classical, singular or virtual) a mapping $c^v$ from the set of (classical, singular or virtual) knotoid diagrams in a surface $\Sigma$ to the set of virtual knot diarams in $\Sigma$, induced by the virtual closure on knotoid diagrams, whereby all extra crossings are declared to be virtual when taking a shortcut for the knotoid diagram. Namely,

$$c^v(K) := K^v$$

where $K^v$ denotes the virtual knot in $\Sigma$ obtained via the virtual closure of a knotoid $K$. See bottom left of Fig. 10.

We further have:

**Definition 2.7.** Two singular virtual knotoids differ by *singular virtual isotopy* if they differ by (a finite sequence of) planar isotopy, the Reidemeister moves, extended by the local moves involving virtual crossings, and by rigid vertex isotopy.

**Lemma 2.8.** *Virtual closure is well-defined up to virtual isotopy.*

*Proof.* Clearly, the closing arc may move freely, by the detour move, so the definition does not depend on the choice of shortcut. Moreover, any virtual/rigid vertex isotopy move between knotoid diagrams is also a valid isotopy move for their virtual closures. □

For (singular) knotoids of height one, we can also define a singular closure:

We say that a knotoid is *prime* if it is not a connected sum of a knot with a knotoid or a product of two knotoids.

**Definition 2.9.** We call *singular closure for prime knotoids of height one* (classical, singular or virtual) a mapping $c^s$ from the set of prime knotoid diagrams (classical, singular or virtual) in a surface $\Sigma$ to the set of singular knot diagrams in $\Sigma$, induced by the singular closure on knotoid diagrams which realize the height, whereby the extra crossing is declared to be singular when taking a shortcut for the knotoid diagram. Namely,

$$c^s(K) := K^s$$

where $K^s$ denotes the singular knot diagram in $\Sigma$ obtained via the singular closure of the knotoid $K$. See bottom right of Fig. 10.

**Lemma 2.10.** *Singular closure for prime knotoids of height one is well-defined up to rigid vertex isotopy.*

*Proof.* We will first argue that the definition of singular closure does not depend on the choice of shortcut. Indeed, since our diagram realizes the height one, there is at least one place where the leg is separated by the head by just one boundary arc. If there is only one such place, we are done. Suppose there are more places. We consider two neighbouring. Then the union of the associated shortcuts makes a simple closed curve. This closed curve bounds a disc,and this disc does not contain any knotting, since it would be by definition local so the two shortcuts differ by rigid vertex isotopy.

Moreover, any virtual/rigid vertex isotopy move between knotoid diagrams is also a valid isotopy move for their virtual closures. □

**Remark 2.11.** Singular closure is well-defined only for height-one prime knotoids and, moreover, on diagrams that realize the height of the knotoid they represent. The requirement for prime knotoids becomes apparent in the proof of Lemma 2.10. To see the height-1 requirement, take, for instance, the counterexample of height two, shown in Fig. 12.

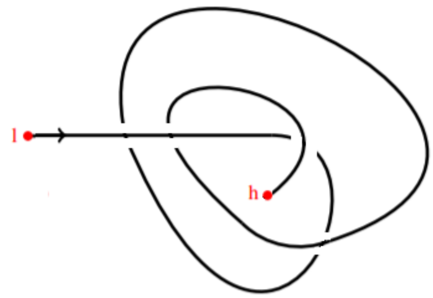



Figure 12: A knotoid diagram that admits two distinct singular closures

Indeed, calculating its affine index polynomial (see Subsection 7.1) we obtain

$$P_K = t^2 + t^1 + t^{-1} + t^{-2} - 4$$

so, by [14] the knotoid is indeed of height 2. Yet, it admits two distinct singular closures, as indicated in Fig. 13: the closure $s_1$ illustrated by the left picture gives rise via an oriented smoothing of the singular crossings to a non-trivial link of 3 components. However, the closure $s_2$ illustrated by the right picture gives rise to the trefoil knot via an oriented smoothing of the singular crossings. So, the two singular closures are not isotopic and, therefore, the singular closure is not well-defined for height two.

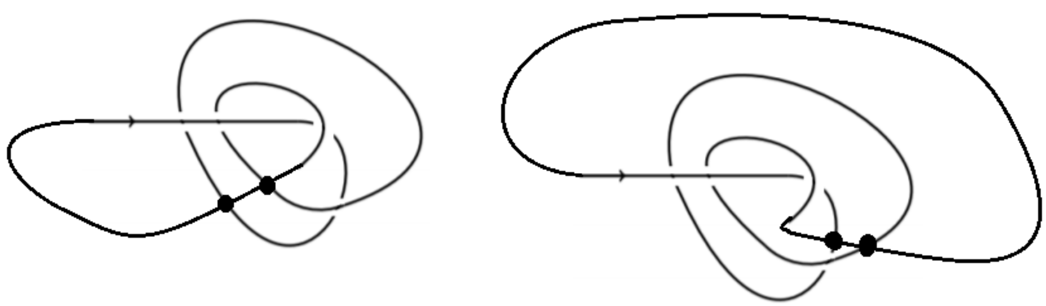


Figure 13: Two distinct singular closures

## 3 Finite type invariants for knotoids

For the rest of the paper we focus on knotoids (classical, singular, virtual, etc) in $S^2$. Occasionally we will be making more general definitions and comments, stating explicitly the generalization.

### 3.1 Finite type invariants obtained by closures

**Definition 3.1.** Let $v$ be any finite type invariant for classical knots. We call *classical closure finite type invariant for (classical) knotoids related to* $v$ a mapping $v^c$ from the set of (classical) knotoid diagrams to $R^2$, where $R$ is a commutative ring, defined in the following way:

$$v^c(K) := (v(K^u), v(K^o))$$

where $K^u, K^o$ are respectively the knot diagrams obtained via under- and over-closure, respectively, of a knotoid diagram $K$.

What we want to prove is that given a finite type invariant for knots, $v$, induces an isotopy invariant for spherical knotoids using the classical closure. For isotopic knotoids $K_1, K_2$, and for any (knot) finite type invariant $v$, we want $v^c(K_1) = v^c(K_2)$, or equivalently we want $v(K_1^u) = v(K_2^u)$ and $v(K_1^o) = v(K_2^o)$ but this is true because $v$ is an isotopy invariant for knots and a knotoid $K \subset S^2$ determines (via the under-closure) a unique knot $K_u \subset S^3$ (resp. $K_o \subset S^3$) up to isotopy from Lemma 2.4.

**Definition 3.2.** Let $v$ be any finite type invariant for virtual knots. We call *virtual closure finite type invariant for knotoids related to* $v$ a mapping $v^v$ from the set of (virtual) knotoid diagrams to a commutative ring $R$, by computing the invariant on the resulting virtual knot. Namely,

$$v^v(K) := v(K^v)$$

where $K$ is a (virtual) knotoid diagram and $K^v$ is the virtual knot diagram obtained by the virtual closure.

For finite type invariants of virtual knots, see for example [21].

**Definition 3.3.** Let $v$ be any finite type invariant for singular knots. We call *singular closure finite type invariant for prime knotoids of height one related to* $v$ a mapping $v^s$ from the set of (singular) prime knotoid diagrams realizing the height to a commutative ring $R$, by computing the invariant on the resulting singular knot. Namely,

$$v^s(K) := v(K^s)$$

where $K$ is a (singular) knotoid diagram realizing the height and $K^s$ is the singular knot diagram obtained by the singular closure. One could then say that this knotoid invariant is of order (or type) $\leqslant n$ if it vanishes on all singular knotoids with more than $n-1$ singular crossings.

### 3.2 Finite type invariants defined directly on knotoids

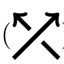
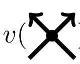
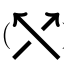
**Definition 3.4.** Two singular knotoid diagrams $K_1, K_2$ in a connected, oriented surface $\Sigma$ are said to be *singular equivalent* if one can be deformed to the other by rigid vertex isotopy moves (recall Definition 2.2, Figs. 4 and 8) and real crossing switches. We will use the symbol $K_1 \sim K_2$ for singular equivalence of the knotoids $K_1, K_2$.

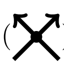
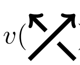
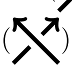
The principal idea of the combinatorial approach to the theory of finite type invariants is to extend a knotoid invariant $v$ to singular knotoids, by making use of the Vassiliev skein relation, recall Eq. 1.

So we define:

**Definition 3.5.** A knotoid invariant, $v$, is said to be a *finite type invariant of order (or type)* $\leqslant n$ if its extension on singular knotoids vanishes on all singular knotoids with more than $n$ singular crossings. Furthermore, $v$ is said to be of order (or type) $n$ if it is of order $\leqslant n$ and not of order $\leqslant n-1$.

For any finite type invariant $v$ we consider the top row singular knotoid diagrams up to singular equivalence. Now, as in the classical case, any finite type invariant remains unchanged under crossing switches on top row diagrams, due to the Vassiliev skein relation. Indeed, if $K$ is a top row diagram and has a positive (resp. negative) crossing we can switch this crossing by invoking the Vassiliev relation

$v($$) = v($$) + v($$).$

But $v($$) = 0$ because this diagram has $n+1$ singular crossings, since $K$ is a top row diagram. Hence $v($$) =$ $v($$)$.

From the above, a top row singular knotoid may be represented by an appropriately chosen 'regular' representative of its singular equivalence class.

For examples of finite type invariants defined directly on knotoids we prompt the reader to Sections 7 and 8.

## 4 Linear chord diagrams

In the theory of finite type invariants for classical knots there is a natural way to encode the order of the singularities in a circular diagram and then join each pair of points of the circle with a simple interior arc (obtaining the chord diagram).

Since a knotoid diagram is an immersion of $[0, 1]$ in some surface, we must think of chord diagrams as 'chords' joining points of an open-ended smooth curve in the surface, that is, a connected, oriented 1-manifold with boundary two endpoints, which correspond to the two endpoints of the singular knotoid diagram. Each chord corresponds to precisely one singular crossing of a singular knotoid diagram, and the chords respect the forbidden moves. With these restrictions there are many non-isotopic ways to join two points of the interval with a simple arc. This is not a technicality but a first attempt to understand the complexity of the problem since chord diagrams are closely connected with singular equivalence classes of knotoids.

### 4.1 Linear chord diagrams in $S^2$

**Definition 4.1.** A *linear chord diagram* of order (or degree) $n$ is an oriented closed interval, with its endpoints marked with '$h$' and '$l$' (for 'head' and 'leg'), equipped with a distinguished set of $n$ disjoint pairs of distinct points being connected with immersed unknotted arcs in the sphere in general position, the 'chords', considered up to orientation preserving diffeomorphisms of the interval, and up to isotopy of the immersed arcs which fixes the ends. All the intersections between the chords themselves and between the chords and the interval are declared to be flat. The set of all linear chord diagrams of order $n$ will be denoted by $A_n^l$.

Fig. 1 illustrates examples of linear chord diagrams. The symbol '$s$' stands for 'singular crossing'. It is clear that there are many ways that two points can be connected with an immersed arc even though being considered up to arc isotopy and diffeomorphisms of the interval. For example, consider two points connected by two different arcs, the first one wrapping around the head with the counterclockwise orientation and the second wrapping around the leg with the counterclockwise orientation. See Fig.14. These two arcs belong to different homotopy classes of the annulus (that is, the sphere with the points $h$ and $l$ removed). So one thinks that even the set $A_1^l$ is not trivial.

For the time we will restrict ourselves to the case $n = 1$. We will need some lemmas in succession to classify all different linear chord diagrams of order 1. First we consider two generators $a, b$: Generator $a$ represents a wrapping of the chord around the leg with the counterclockwise orientation, meaning that, if we have a disjoint pair of points in the interval starting with the point closer to the leg with destination the point closer to the head and following the immersed arc, we go exactly once around the leg. Generator $b$ represents a wrapping of the chord around the head respectively. For the clockwise orientations we will use the symbols $a^{-1}$ and $b^{-1}$ respectively.

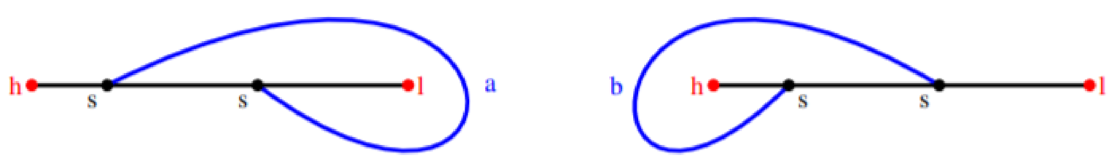



Figure 14: The generators $a, b$

We now define a multiplication operation on the set of linear chord diagrams $A_1^l$ . Multiplication $x \cdot y$ means that we place all four ends in the same interval such that the two involving $x$ are both before the ones involving $y$, with respect to the natural orientation of the interval, and then we do the following procedure. All the intersections between $x$ and $y$ are declared to be flat. We then take a neighbourhood of the sub-interval starting from the destination of $x$ and ending at the start of $y$. Inside there we connect the destination of $x$ with the start of $y$ in the unique way that the resulting curve is transversal to the interval precisely at both points. For an illustration see Fig. 16. The result is a pair of points of the interval together with an immersed curve joining them, whose interior meets the interval at every point transversely.

The operation multiplication is well-defined up to flat isotopy and diffeomerphisms of the interval. It is also clearly associative. Note, further, that there are two trivial chord diagrams, in which the chord does not wind around the leg or the head: the one that the chord lies below the interval and the one that the chord lies above the interval, denoted by $1_u, 1_o$ respectively. Of course they are flat isotopic, hence $1_u = 1_o := 1$. They both correspond to knot-type singular knotoids and so they can be identified with the classical chord diagram with one chord.

The following result justifies the choice of the notation for $a^{-1}$ and $b^{-1}$.

**Lemma 4.2.** $a \cdot a^{-1} = 1 \quad b \cdot b^{-1} = 1$.

*Proof.* The statement is an immediate consequence of the definition of the multiplication with $x = a$ and $y = a^{-1}$, as illustrated in Fig. 15. It is clear that $a \cdot a^{-1}$ is flat isotopic to the trivial chord diagram.

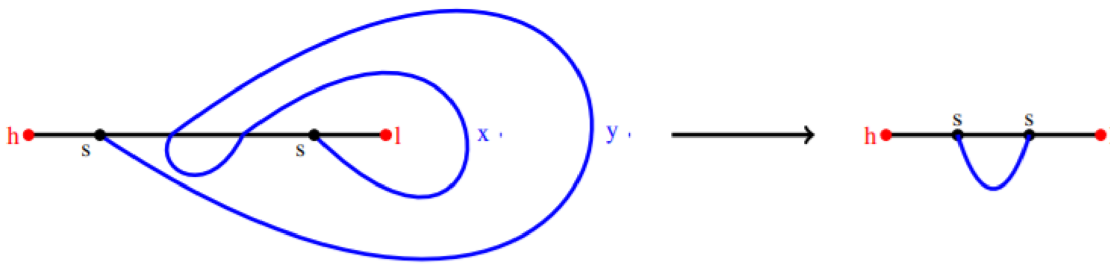



Figure 15: $a \cdot a^{-1} = 1$

□

**Lemma 4.3.** *$a \cdot b = 1 \quad b \cdot a = 1$ in $S^2$.*

*Proof.* Now the idea is this. Take $x = a$ and $y = b$. Connect the endpoints as the definition suggests such that both arcs cross the interval transversely. Then Fig. 16 shows that in $S^2$ $a \cdot b$ is trivial.

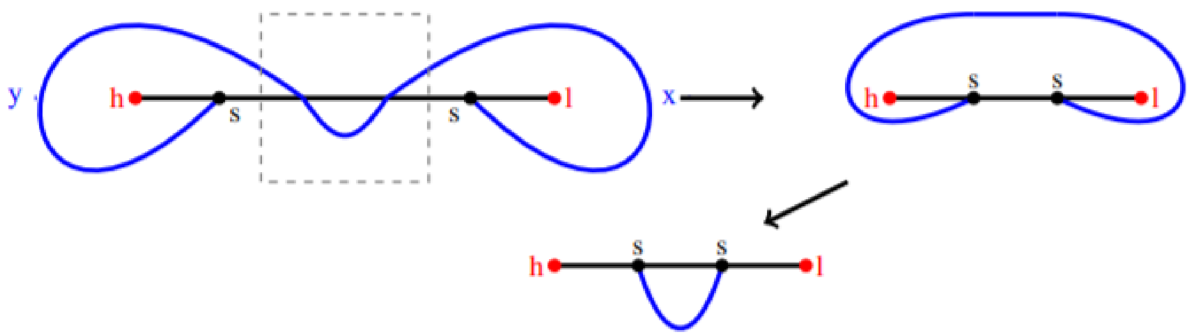



Figure 16: $a \cdot b = 1$

□

**Corollary 4.4.** *From the above it follows that the set $A_1^l$ of spherical linear chord diagrams equipped with the multiplication forms a group. Lemma 4.3 leaves us with only one generator in our construction, say $a$. So we can assume that every spherical chord diagram has its head in the outer region.*

From the above, it is clear that we may rethink of a linear chord diagram with one chord, representing a singular knotoid with exactly one singularity, as a closed interval with a loop winding (only) around the leg $w$ times. The integer $w$ is the *winding number* of the loop. So, the two points $s$ can collapse into a single point and the loop can be abstracted to a circle with the indication $w$. See Fig. 2.

Furthermore, there is clearly no torsion in the cyclic group $A_1^l$. So, from the above we have proven the following:

**Proposition 4.5.** *In $S^2$ for any linear chord diagram $C \in A_1^l$ there exists some $w \in \mathbb{Z}$ such that $C = a^w$. As a group, $A_1^l$ is infinite cyclic.*

Note that in $\mathbb{R}^2$ this technique wouldn't work, since, for example, the second diagram in Fig. 16 is locked and non-isotopic to the third one. So, if one tries to extend this theory to $\mathbb{R}^2$ or a surface $\Sigma$, one would have to deal with more complexities.

### 4.2 From a singular knotoid in $S^2$ to its chord diagram

Given a singular spherical knotoid diagram with one or more singular crossings we will correspond to it unambiguously a linear chord diagram.

If we allowed to join two points without winding around $l$ or $h$ we would have the same chord diagram for non singular equivalent knotoids. For example, the knotoids shown in Fig. 17 are clearly non singular equivalent, yet, with the above convention they would both correspond to the trivial chord diagram (illustrated in the second row of Fig. 16).

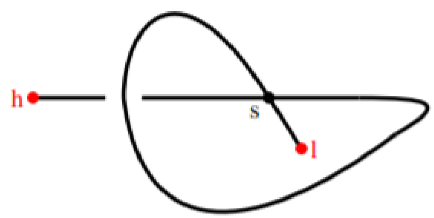



Figure 17: Two non singular equivalent knotoid diagrams

The first observation to address this problem is that a knotoid diagram which winds once around the leg counter-clockwise from the first time we traverse the singular crossing until the second time, should correspond to a chord diagram winding once around the leg in the counter-clockwise sense, as in Fig. 18

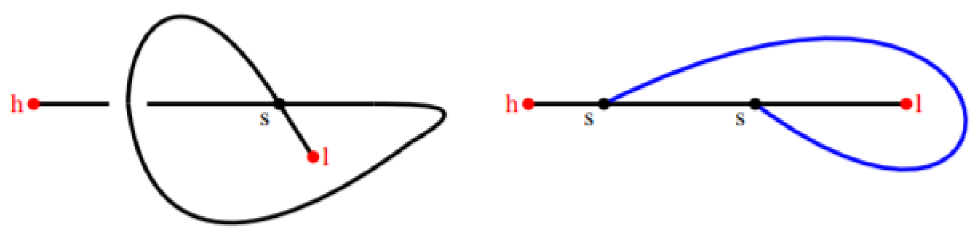



Figure 18: A knotoid diagram with its corresponding chord diagram

We also note that there is no point in considering paths that wind around the head since we know that this would correspond to a path winding around the leg the same amount of times but with reversed orientations.

With these observations in mind, we propose the following algorithm for corresponding a linear chord diagram to a singular spherical knotoid diagram $K$ of order 1.

**Algorithm for order 1 singular spherical knotoid diagrams**

- Split $K$ into 3 paths: The path joining leg to singular point ($l \to s$), the *singular loop* ($s \to s$), and the path joining the singular loop with the head ($s \to h$). See Fig. 19 top left.
- Erase the paths $l \to s$ and $s \to h$, but keep track of the points $l, h$. See Fig. 19 top right.
- What remains is a loop that winds a number of times around $l$. This integer is by convention positive if the winding is counter-clockwise and negative otherwise.
- We now construct a linear chord diagram of class $a^w$ with $w$ being the winding number we just obtained. One can compute directly $w$ by the usual isomorphism $\pi_1(D^2 \setminus \{l, s\}) \cong \mathbb{Z}$ and so $w \in \mathbb{Z}$ corresponds to the class of $\gamma(s) \in \pi_1(D^2 \setminus \{l, s\})$. See Fig. 19 bottom.

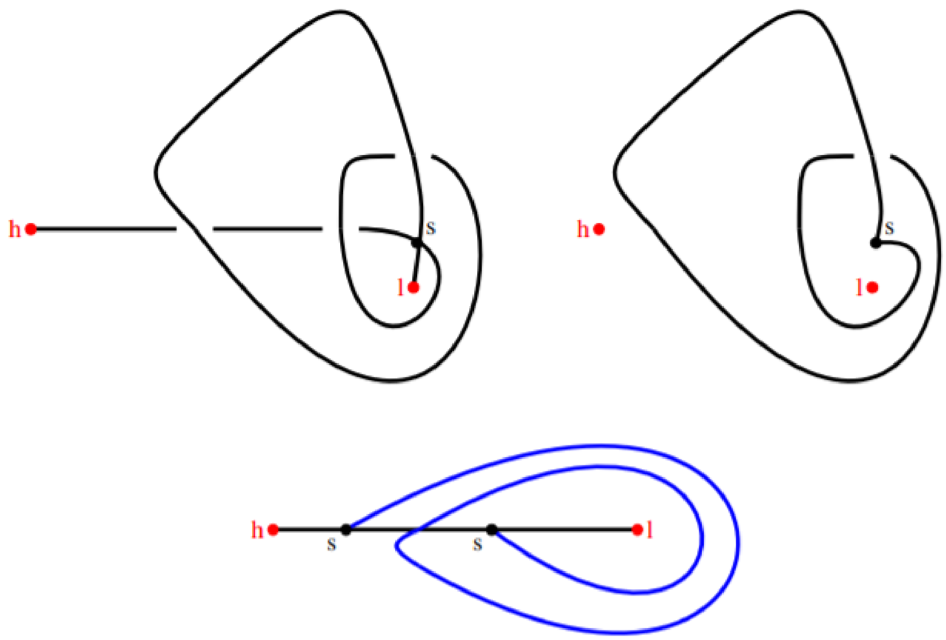



Figure 19: The singular loop and the corresponding chord diagram

The above generalize easily to an algorithm for corresponding a linear chord diagram to a singular spherical knotoid diagram $K$ of higher order:

**Algorithm for order $> 1$ singular spherical knotoid diagrams**

- Let $K$ be a singular knotoid diagram with $n$ singular crossings, $s_1, s_2, \dots s_n$, given in the order that one meets them for the first time as one traverses $K$ from leg to head.
- Split $K$ into 3 paths: The path joining the leg to $s_1$ ($l \to s_1$), the *singular path* starting from the first passage of $s_1$ and ending at the last encounter of a singular crossing, say $s_j$, ($s_1 \to s_j$), and the path joining $s_j$ with the head ($s_j \to h$).
- As before, erase the paths $l \to s_1$ and $s_j \to h$, but keep track of the points $l, h$.
- We now construct a linear chord diagram by first indicating on the interval from $l$ to $h$ all encounters of singular crossings in the sequence one meets them while traversing $K$.
- The singular path contains $n$ singular loops, one for each singular crossing, and each one winding a number of times around $l$. This information is carried on the interval by pairing the singular points with same index, joining them with a corresponding winding arc that starts from the rightmost encounter of the singular indication on the interval.

**Remark 4.6.** Clearly, the above algorithm does not change under rigid vertex isotopy moves. So, the linear chord diagram obtained by a singular knotoid diagram represents its whole rigid vertex isotopy class.

### 4.3 From a linear chord diagram to a singular knotoid in $S^2$

It is a natural question to ask whether a linear chord diagram corresponds bijectively to a singular equivalence class. This question will only be answered in full (and in positive) in the next section for linear chord diagrams of order 1.

Here we will describe an algorithm for obtaining a singular spherical knotoid diagram from a linear chord diagram. We will follow the classical construction of surgery along the two ends of a chord. The idea behind the technique is that a chord diagram illustrates the nodal structure of the singularities. Furthermore, especially in pure knotoids the chords illustrate the winding numbers of the singular loops around the leg.

**Algorithm for order 1 linear chord diagrams**

- We start with a linear chord diagram with one chord, say $c$, which has ends $a, b$ with $a$ being closer to $l$ and $b$ closer to $h$. See top left illustration in Fig. 20.
- Take now a flat thickening of $c$: $c \times [-\varepsilon, \varepsilon]$
- Cut out from the initial interval the two small open sub-intervals $(a-\varepsilon, a+\varepsilon), (b-\varepsilon, b+\varepsilon)$. See top right illustration in Fig. 20.
- The chord $c$ has now been duplicated into two new chords $c_-$ and $c_+$ comprising the remaining parts of the band's boundary. See top right illustration in Fig. 20.
- Introduce between the arcs $c_-$ and $c_+$ a singular crossing and at the same time a positive classical crossing, for retaining the connectivity pattern. See bottom left illustration in Fig. 20.
- The result is an order one singular knotoid diagram, say $K$, which can be simplified by isotopy to the bottom right illustration in Fig. 20.

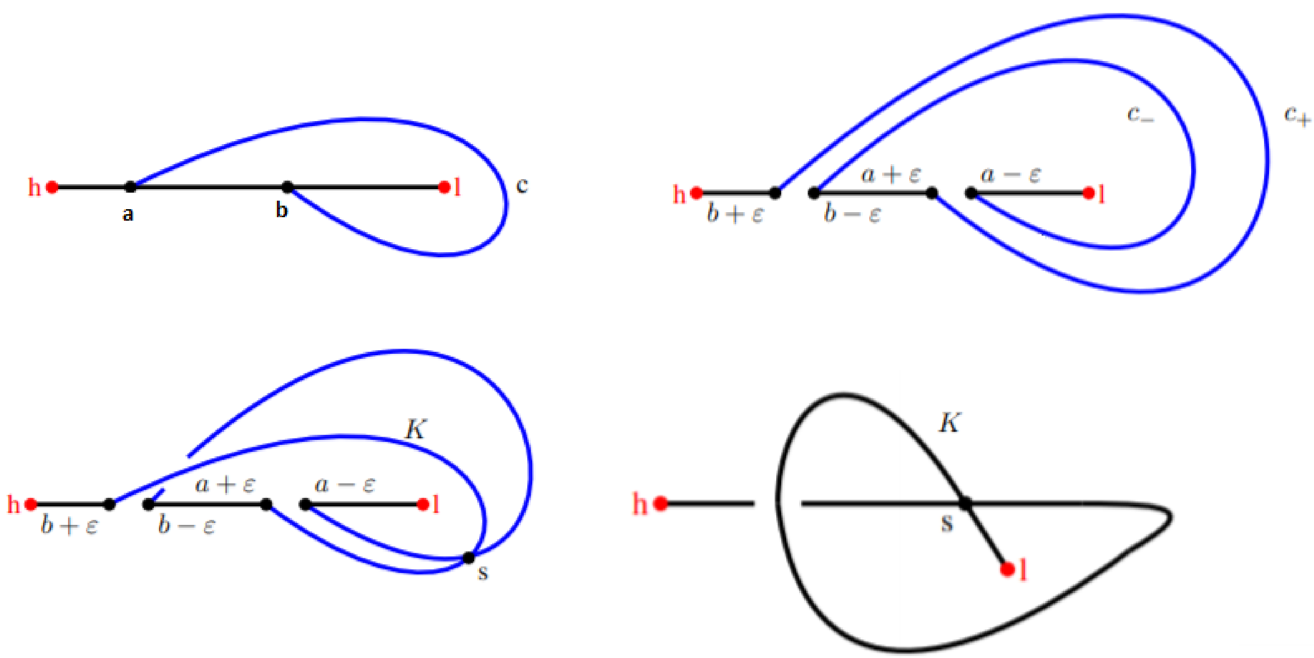



Figure 20: Recovering a singular knotoid from a chord diagram

**Remark 4.7.** Clearly, the resulting singular knotoid does not change under chord diagram equivalence (orientation preserving diffeomorphisms of the interval and isotopy of the immersed curve which fixes the endpoints). Moreover, as it will become clear in the next section, the singular knotoid obtained by a linear chord diagram is unique up to singular equivalence.

**Algorithm for order $> 1$ linear chord diagrams**

- Consider now a linear chord diagram with $n$ chords, say $c_1, c_2, \ldots c_n$, numbered from leg to head.
- Apply the above algorithm for the first chord $c_1$, declaring at the same time positive all crossings created between the resulting knotoid arc, say $K_1$, with the remaining chords.
- Continue with each one of the rest of the chords, respecting the crossings with the already created knotoid arcs and declaring positive all crossings created between the new resulting knotoid arc, with the remaining chords.
- At the end we obtain a singular knotoid diagram with $n$ singular crossings.

# 5 Type-1 invariants, regular diagrams and weight systems for spherical knotoids

Thinking of classical finite type invariants, the one-term relation applies on trivial chord diagrams, yielding zero evaluation of any type-1 invariant. In the theory of knotoids, however, as we already discussed, linear chord diagrams are not trivial.

## 5.1 The regular spherical knotoid diagrams for $v_1$

Of crucial importance here is the notion of regular diagram, which gives rise to our main result for any type-1 invariant, denoted generically $v_1$.

**Definition 5.1.** A knotoid diagram in a surface $\Sigma$ is *descending* if when walking from leg to head every classical crossing that we meet for the first time is an overcrossing. Similarly, a singular knotoid diagram in a surface $\Sigma$ is *descending* if when walking from leg to head every segment that we meet containing classical crossings is descending with respect to the classical crossings that we meet for the first time.

**Lemma 5.2.** *Every singular knotoid diagram in a surface $\Sigma$ is singular equivalent to a descending singular knotoid diagram.*

*Proof.* It it clear, since singular equivalence allows crossing switches. So, applying appropriate crossing switches to the initial singular knotoid diagram, in can be deformed into a descending one. □

**Definition 5.3.** A *regular diagram* is a descending spherical knotoid diagram with 1 singular crossing and with the minimal number of real crossings, which number is odd, among all diagrams in its rigid vertex isotopy class, and such that the singularity is located 'next to' the leg of the knotoid, in the sense that there exist no other crossings between the leg and the singular crossing. See Fig. 3.

Recall now Definition 3.4 of singular equivalence. Our goal is to prove the following theorem.

**Theorem 5.4.** *Every spherical singular knotoid diagram with 1 singularity is singular equivalent to one and only one regular diagram.*

With this in hand we will be ready to show that we can handle the complexity occurring by the forbidden moves and the rigidity of the singularity just by the winding number of the singular loop.

Following Turaev [33], the spherical knotoid diagrams which have the head in the region of the point at infinity and the leg in the region of the point 0 are called *special*. Clearly we have the lemma:

**Lemma 5.5.** *Every knotoid $K \in \mathcal{K}(S^2)$ is isotopic to a $\tilde{K} \in \mathcal{K}(S^2)$, such that there is a path joining the head of $\tilde{K}$ with the point at infinity $\infty$ and does not intersect any other part of the knotoid, that is, $\tilde{K}$ is special.*

*Proof.* Take a point $p \neq h$ in the connected component of $h$. By the fact that $S^2$ is rotationally symmetric, $p$ can be thought of as the point at infinity via an isometry. Alternatively, one could also think that, given $h$ and $\infty$, we are not allowed by the forbidden moves to pass branches of the knotoid $K$ through $h$, but we can pass them through $\infty$ and, so, if in the initial position we had $k$ intersection points when joining $h$, $\infty$, by passing $k$ branches to the other side of $\infty$ we have the desired special diagram only by using locally-planar isotopy. The same reasoning applies for $l$ and $0$. □

**Lemma 5.6.** *The set of spherical knotoids $\mathcal{K}(S^2)$ is in bijection with the set of planar knotoids whose head lies in the non-bounded connected component of $\mathbb{R}^2$ minus the knotoid, which is viewed as planar knotoid:*

$$\mathcal{K}(S^2) \hookrightarrow K(\mathbb{R}^2)$$

We could equivalently think of these special planar knotoids as "long knots in one direction", in the sense that: even though the leg is trapped, yet following the knotoid diagram we can approach the head via a straight line outside a compact set. The same effect is achieved by identifying the head of a spherical knotoid with the point at infinity.

*Proof.* Let $K \in \mathcal{K}(S^2)$. From Lemma 5.5 we can assume, without loss of generality, that $K$ is special. By the decompactification of $S^2$, using the usual stereographic projection:

$$\pi : S^2 \setminus \{\infty\} \longrightarrow \mathbb{R}^2$$

the initial diagram $K$ corresponds uniquely to a diagram $\tilde{K} \in K(\mathbb{R}^2)$. The diagram $K$ except a neighbourhood of $h$, is in a compact subset $B$ which does not contain $\infty$. Then we have that $\pi(B)$ is compact in $\mathbb{R}^2$ and $\pi(h) \in \pi(S^2 \setminus B)$

which is non-bounded and has no common point with the knotoid other than the small path near $\pi(h)$. Moreover, we can assume that outside the compact subset $\pi(B)$ the knotoid $\tilde{K}$ reaches the head with a straight line since, from our hypothesis, $h$ is in the same region as $\infty$. The converse is analogous, since we begin with a knotoid $\tilde{K} \in K(\mathbb{R}^2)$, which lies inside a compact set $F$ except a simple straight arc from the boundary of $F$ to $h$. Applying $\pi^{-1}$ yields a special knotoid which is what we want.

□

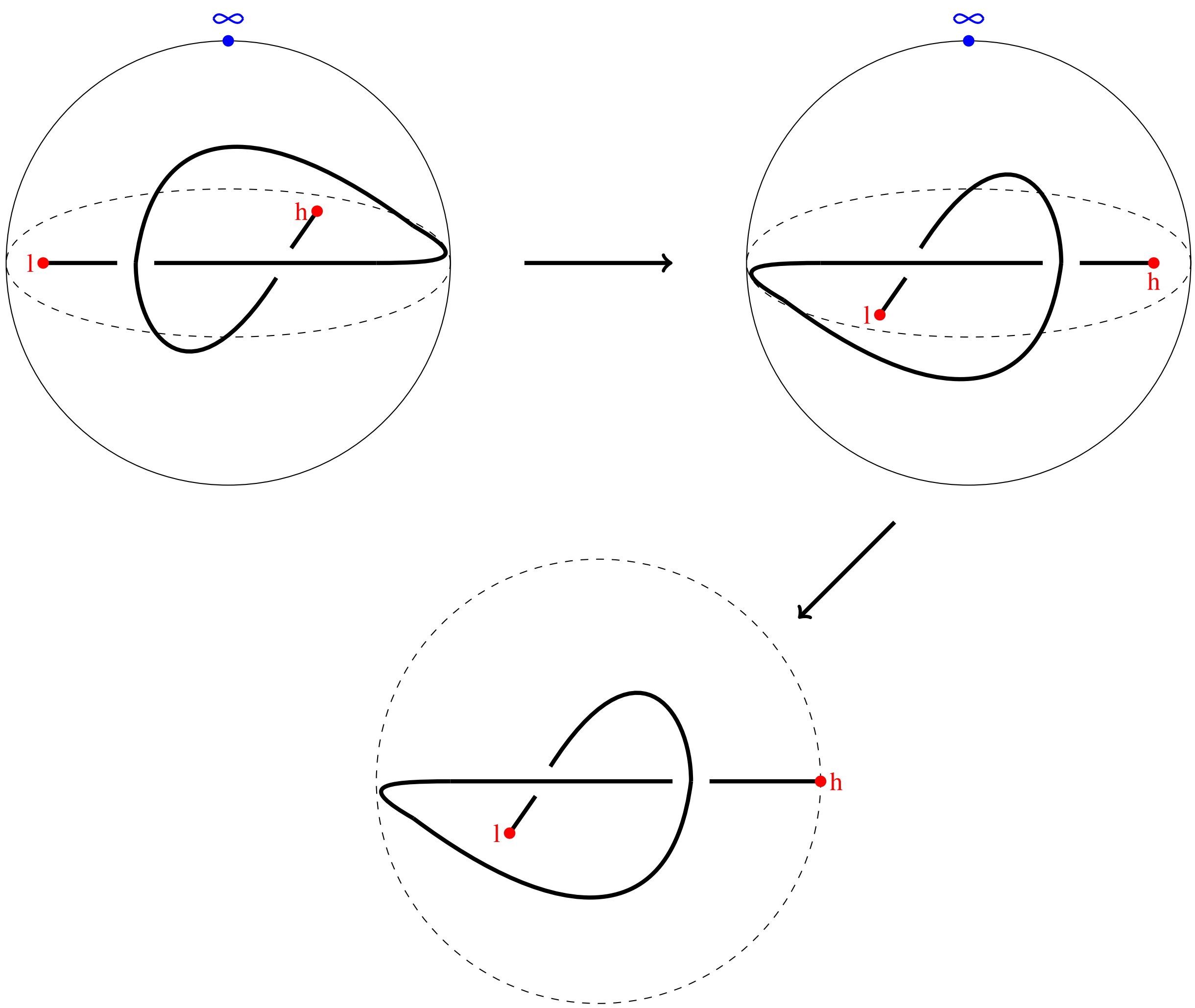



Figure 21: Transformation to a special knotoid diagram and then to a planar

**Lemma 5.7.** *Every descending spherical knotoid diagram is isotopic to the trivial.*

*Proof.* Since in the isotopy class of any spherical knotoids there is at least one special representative, we choose, without loss of generality, $K$ to be a descending special spherical knotoid diagram. Take a disc $D$ in $S^2$ centered at $\infty$ which intersects $K$ only at a single straight segment $a$ starting from $h$. Then, the disc $S^2 \setminus D$ centered at $l$, after possible application of some planar isotopy. This disc encloses the rest of $K$ except for the segment $a$.

Applying the stereographic projection $\pi$ yields in the plane the knotoid $\tilde{K}$ of Lemma 5.6, in which $h$ lies outside a compact region enclosing all the rest of $\tilde{K}$ except for the small straight segment $\pi(a)$ ending at $h$.

We can now take the rail lifting of $\tilde{K}$ in $\mathbb{R}^3$. This gives rise to a solid cylinder with central axis the rail of $l$, denoted by $L$. See Fig. 22. Denote also by $H$ the rail of $h$. $H$ is allowed to move rigidly in the boundary of the solid cylinder.

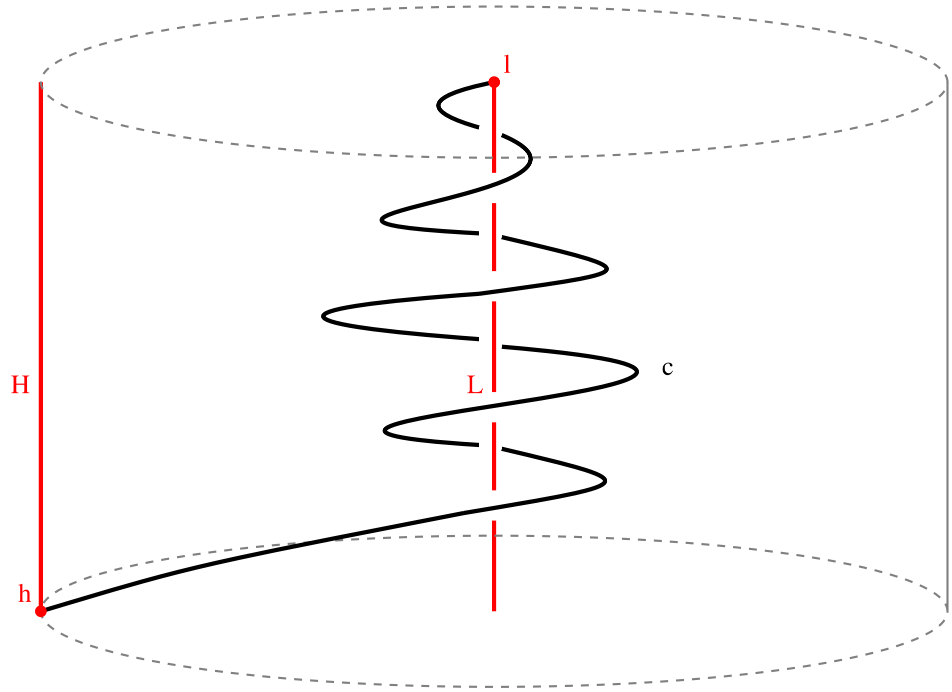



Figure 22: Rail lifting of a descending knotoid diagram

Now, our knotoid $\tilde{K}$ lifts to a curve $c$ in $\mathbb{R}^3$, parametrized by the unit interval, such that $c(0) = l$ and $c(1) = h$. The fact that $\tilde{K}$ is descending means that if two points $p_1 = c(t_1), p_2 = c(t_2)$ with $t_1 < t_2$ and with $p_1 = (x, y, z_1), p_2 = (x, y, z_2)$, then $z_1 > z_2$. Here we use the convention that $L, H$ are perpendicular to the $(x, y)$-plane, with $l$ fixed at the point $(0, 0, 1)$ on $L$ and $h$ lying in the $(x, y)$-plane at the lower end of the rail $H$. Hence, the curve $c$ is a helical curve that winds around the rail $L$ but does not wind around $H$, since $K$ is a special knotoid diagram.

Now, the curve $c$ together with $L$, both oriented downward, form a 2-braid. The parts that correspond to a braid word $\sigma_1\sigma_1^{-1}$ are cancelled by isotopy, so we have a braid word of the form $\sigma_1^n$, where $n \in \mathbb{Z}$. Now we can unwind $c$ from the bottom to the top by a continuous rotation of $H$ around $L$ in the direction opposite to the direction of the winding for $|n|$ half twists. Hence, the curve $c$ is line isotopic to a trivial arc and, so, the projection of $c$ on the $(x, y)$-plane is the trivial knotoid diagram. □

**Remark 5.8.** Lemma 5.7 does not hold for planar knotoids. For example, the knotoid of Fig. 23 is not trivial.

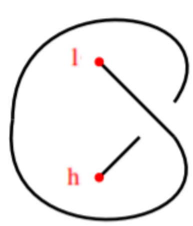



Figure 23: A non-trivial descending planar knotoid diagram

***Proof of Theorem 5.4***

Let $K$ be a singular spherical knotoid diagram with one singular crossing. By Lemma 5.5, $K$ can be considered to be special and, by Lemma 5.2, $K$ can be considered to be descending.

Using, now, the injection of Lemma 5.6, we consider the rail lifting $c$ of $\tilde{K}$ in the solid cylinder $C_R$ of radius $R$ centered at $L$. The curve $c$ lives in the interior of $C_R \setminus L$, except $l$ and $h$, and the head $h$ can be considered to be situated in the line $H$, in the boundary of $C_R$. $c$ comprises three segments: $(l \to s)$ from $l$ to $s$, the singular loop $(s \to s)$, and $(s \to h)$ from $s$ to $h$.

Since $K$ is descending, up to space isotopy, the segment $(l \to s)$ can be viewed as a classical descending spherical knotoid and, hence, by Lemma 5.7, it is an unknotted arc that lies above the rest of $c$. This arc can be straightened by isotopy, employing the unwinding technique that we described earlier, that is, by rotating around the rail $L$. We also take $s$ to lie 'close' to the rail $L$. See Fig. 24. Further on, again by the fact that $K$ is descending and using space isotopy, the singular loop $(s \to s)$ can be viewed to wind around $L$ with decreasing radii. See Fig. 24. Finally, by the same arguments and employing the unwinding technique, the segment $(s \to h)$ is also an unknotted straight arc, descending from $s$ to $h$. See Fig. 24.

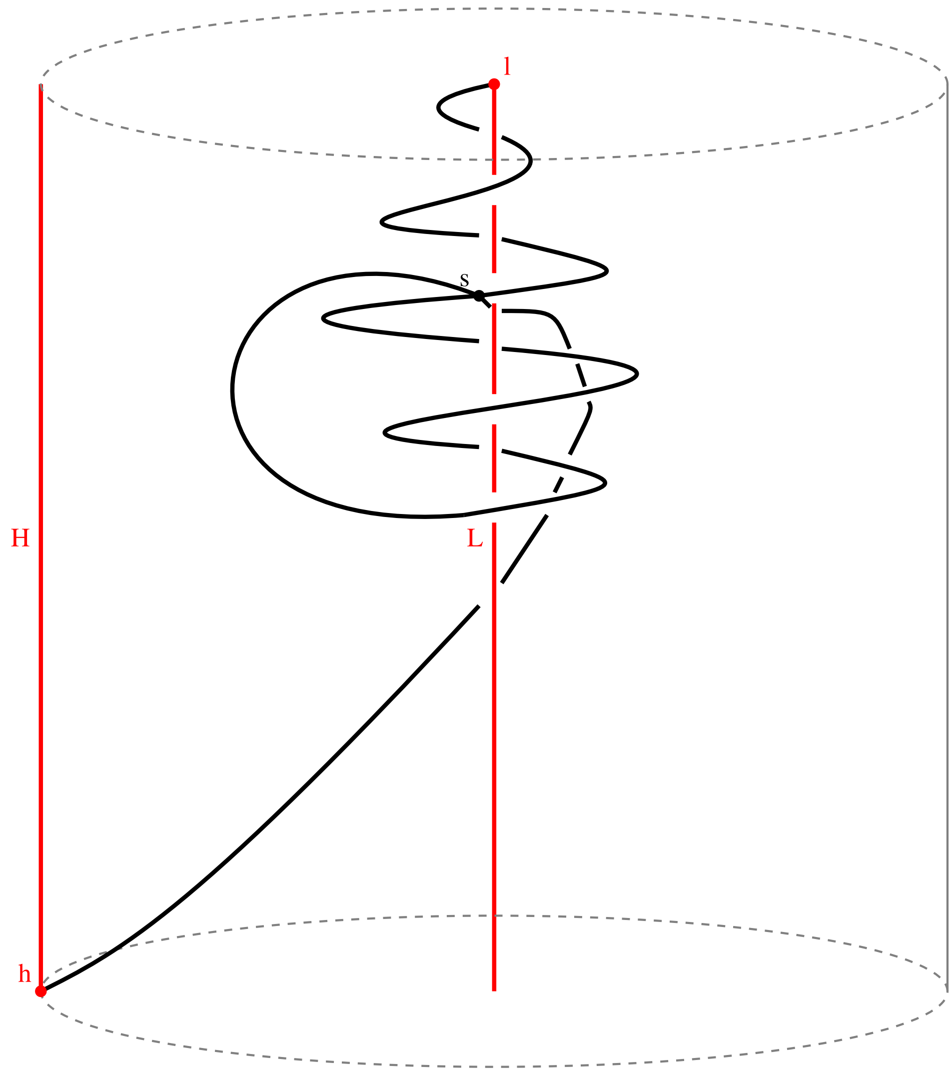



Figure 24: Rail lifting of a singular descending knotoid

We now pass to the projection of the thickened annulus $C_R \setminus L$ projects in the $(x, y)$-plane to the annulus $D^2 \setminus \{l\}$. Then the curve $c$ projects to a singular knotoid $\bar{K}$ comprising three segments analogous to the ones of $c$ and for which we shall use the same notation: $(l \to s)$ from $l$ to $s$, the singular loop $(s \to s)$, and $(s \to h)$ from $s$ to $h$. From the above, the subarc $(l \to s)$ contains no crossings. The segment $(s \to s)$ contains $w - 1$ decreasing self-intersections, where $w$ is its winding number around $l$. Finally, the segment $(s \to h)$ lies entirely under the rest of the diagram and creates $w$ crossings with the singular loop. Therefore, by Definition 5.3 $\bar{K}$ is a regular diagram. Recall, for example, the top left diagram in Fig. 19.

We finally observe that two regular diagrams correspond to two singular loops. So, they are classified by the fundamental group $\pi_1(D^2 \setminus \{l\}, s)$. Therefore, they are distinct from each other.

From the above, each singular equivalence class has one and only one regular diagram as a representative. $\square$

**Corollary 5.9.** *Combining the results of Section 4 and Theorem 5.4 we have that the set $A_1^l$ of all spherical linear chord diagrams of order 1 corresponds bijectively to the set of regular knotoid diagrams.*

### 5.2 The integration theorem

With the proof of Theorem 5.4 and by Corollary 5.9, a natural question arises. Namely, given a *weight system* $W$, which is a function $W : A_1^l \to R$ where $R$ is a commutative ring, does $W$ determine a type-1 invariant of spherical knotoids? The first, obvious thing to observe is that $W$ must give the value zero to the trivial linear diagram, since this diagram corresponds to the unknot with one singular crossing.

We start exploring the question by the following idea. Given a spherical knotoid diagram $K$ we bring $K$ to a descending form, by appropriate crossing switches and creating at the same time a number of singular knotoid diagrams with one singular crossing. More precisely, we apply the following algorithm.

- Start walking from the leg of $K$ toward its head.
- Let $c_1$ be the first crossing that we traverse from underneath as we arrive at it for the first time. We switch $c_1$ and we obtain a new knotoid diagram $K_1$ and a singular knotoid diagram $S_1$.
- We repeat the above step for the diagram $K_1$ and its first crossing $c_2$ that fails $K_1$ to be descending, obtaining a new knotoid diagram $K_2$ and a singular knotoid diagram $S_2$.
- After an appropriate number of repetitions, say $r$, we arrive at a descending diagram $K_r$, which is trivial, and a singular knotoid diagram $S_r$, and so the algorithm terminates.
- Next, correspond each $S_i$ to the linear chord diagram $L_i$ representing the regular diagram in its singular equivalence class.
- Then evaluate each $L_i$ according to the weight system $W$.
- Finally, define

$$v_1(K) := \sum_{c \in Cr(K)} \delta_c sgn(c) W(K_c) \tag{2}$$

  where

$$\delta_c = \begin{cases} 0, & \text{c is an over crossing the first time} \\ 1, & \text{otherwise} \end{cases}$$

  $sgn(c)$ is the sign of the crossing $c$ and $W(K_c)$ is the value of $W$ on the linear chord diagram that corresponds to the singular knotoid diagram $K_c$, which is $K$ with $c$ nodified.

**Theorem 5.10.** *Every weight system $W : A_1^l \to R$ that has zero evaluation on the trivial linear chord diagram, gives rise to a type-1 spherical knotoid invariant, by means of formula (2). Equivalently, the sum (2) is constant on the isotopy class of any spherical knotoid diagram.*

*Proof.* We shall prove that the sum (2) remains invariant under the application of Reidemeister moves.

Indeed, consider first a Reidemeister I move, creating a crossing, say $c$. Then the diagram $K_c$ with the crossing nodified corresponds to the trivial linear diagram, on which the weight system is 0. So Reidemeister I moves contribute nothing to our sum.

A Reidemeister III move does not create any crossings and it does not hide any of the crossings involved. Furthermore, in an oriented Reidemeister III move there are some choices involved, because of the orientations. In any such choice, we will see the contribution of the change to be zero. Consider the case where all three arcs illustrated in Fig. 25 are oriented from left to right. The central crossing contributes to the sum the same before and after the move, since the two corresponding singular diagrams are rigid vertex isotopic (recall Fig. 8). Let now $A$ be the left crossing and $B$ the right one before the move is performed, and let $A', B'$ be their corresponding crossings after the move. See Fig. 25.

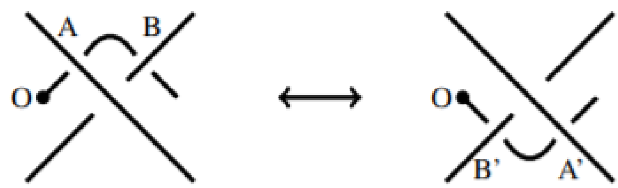



Figure 25: Study of the summation contribution of a Reidemeister III move

Then $K_A$ and $K_{A'}$ correspond to the same regular diagram, since neither the leg nor the head are in the local region of the move, or else we performed a forbidden move, and so the winding number of the singular loop does not change. Hence, $W(K_A) = W(K_{A'})$ and $W(K_B) = W(K_{B'})$. Moreover, $\delta_A = \delta_{A'}, \delta_B = \delta_{B'}$ since the same arcs are involved and of course $sgn(A) = sgn(A'), sgn(B) = sgn(B')$. Therefore, again, there is no contribution to our sum. All other cases of Reidemeister III moves are treated analogously.

Last we check the Reidemeister II move. The idea is again to perform such a move and show that it contributes nothing to the sum. In Fig. 26 we sum over diagrams, for simplicity, meaning the evaluations on these diagrams. In the diagrams in Fig. 26 we choose the orientations from top to bottom and we assume that the understrand is approached first from a local initial point $O$.

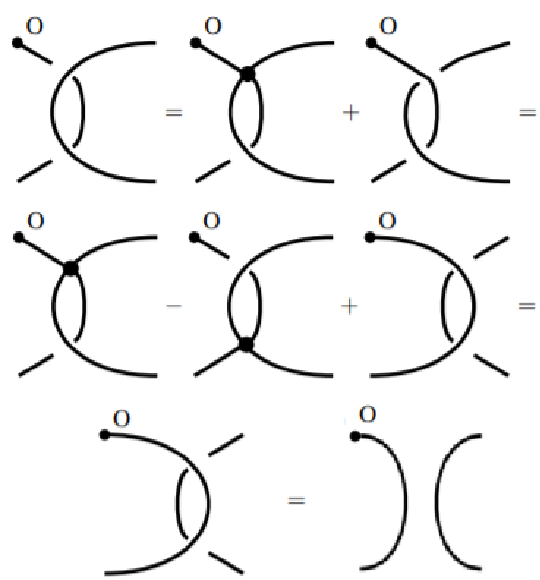



Figure 26: Study of the summation contribution of a Reidemeister II move

Then, invoking twice the Vassiliev skein relation, we end up with the diagram where both crossings in the move are first overcrossings. See Fig. 26. But this is invisible by the algorithm and, consequently, by the summation. So, the Reidemister II move contributes to the sum nothing more than the diagram with two parallel lines in the local region of the move. The proof for all other choices is analogous. $\square$

**Corollary 5.11.** *Theorems 5.4 and 5.10 imply that there are non-trivial type-1 invariants for spherical knotoids. Moreover, there are non-trivial type-1 invariants also for planar knotoids since, by Lemma 5.6, the set of spherical knotoids corresponds bijectively to the set of planar knotoids whose head is in the non-bounded component of* $\mathbb{R}^2$ *minus the knotoid.*

# 6 The singular height

In this section we introduce the singular height, an invariant for singular knotoids with one singularity, analogous to the height (or complexity) of classical knotoids [33]. We conclude the section with a conjecture for planar singular knotoids with one singularity.

**Definition 6.1.** The *singular height* of a knotoid diagram $K$ is the minimal height of its singular equivalence class. $sh(K) = min\{h(K')|K' \sim K\}$, where $\sim$ means singular equivalence. Obviously $sh(K) \leqslant h(K)$.

**Proposition 6.2.** A regular diagram realizes the singular height of its singular equivalence class.

*Proof.* Let $R_n$ be the regular knotoid diagram whose singular loop winds $n$ times around the leg of the singular knotoid. By $n$ consecutive applications of the Jordan curve theorem equivalently we need to intersect at least $n$ times the singular loop to join the leg with the head with a simple arc, which is homotopy invariant. So $sh(R_n) \geqslant n$ Moreover $R_n$ itself has height $n$. Hence, it realizes the height $n$. $\square$

Thus, it is natural to say that since a regular diagram realizes height and a linear chord diagram corresponds to exactly one regular diagram, chord diagrams realize singular height.

The question here is simple: In planar knotoids can we detect singular height, just by seeing the word corresponding to the chord diagram?

The farther we can reach for the moment is conjecture the formula that holds in this occasion and search it further in the future. First of all recall that in planar knotoids $b$ is a free generator as well as $a$. We don't have that $ab = 1$ as in Lemma 4.2 so we should think linear chord diagrams for planar knotoids as words of $F_2[a, b]$.

So let $c \in F_2[a, b]$, then $c = a^{n_1} b^{m_1} \dots a^{n_k} b^{m_k}, k < \infty, n_i, m_i \in \mathbb{Z}$

- $m_i \neq 0 \quad \forall i \leqslant k - 1$
- $n_i \neq 0 \quad \forall i \geqslant 2$
- if $m_1 = 0 \Rightarrow m_i = 0$ for every $i$ and then $c = a^r$ for some $r \in \mathbb{Z}$

We search the first two consecutive exponents which have the same sign.

1. If the first such pair is $n_j > 0, m_j > 0$ for some $j$, then let $d$ be the number of (consecutive) pairs $n_i, m_i$ where $n_i > 0, m_i > 0$ added by the number of (consecutive) pairs $m_i, n_{i+1}$ where $m_i < 0, m_{i+1} < 0$
2. If it is $m_j < 0, m_{j+1} < 0$ let $d$ be the same as in $(1)$.
3. If the first such pair is $n_j < 0, m_j < 0$ for some $j$, then let $d$ be the number of (consecutive) pairs $n_i, m_i$ where $n_i < 0, m_i < 0$ added by the number of (consecutive) pairs $m_i, n_{i+1}$ where $m_i > 0, m_{i+1} > 0$
4. If it is $m_j > 0, m_{j+1} > 0$ let $d$ be the same as in $(3)$

**Conjecture 6.3.** *Given a linear chord diagram D with one chord, for planar knotoids, then the singular height of the corresponding singular planar knotoid diagram is given by the formula*

$$sh(K) = \left[ \sum_{i=1}^{k} (|n_i| + |m_i|) \right] - 2d$$

# 7 Examples of type-1 invariants for knotoids

## 7.1 The affine index polynomial

The affine index polynomial was defined for virtual knots and links by L.H. Kauffman [21] and then for knotoids, virtual or classical, by N. Gügümcü and L.H. Kauffman [14]. It is based on an integer labeling assigned to flat knotoid diagrams (i.e. diagrams with the information 'under' or 'over' on classical crossings omitted) in the following way. A flat knotoid diagram, classical or virtual, is associated with a graph where the flat classical crossings and the endpoints are regarded as the vertices of the graph. An arc of an oriented flat knotoid diagram is an edge of the graph it represents, that extends from one vertex to the next vertex.

We start labeling the edges as exemplified in Fig. 27. At each flat crossing, the labels of the arcs change by one. If the incoming arc labeled by $a \in \mathbb{Z}$ crosses another arc which goes to the right then the next arc is labeled by $a + 1$; if the incoming arc $b \in \mathbb{Z}$ crosses another arc which goes to the left then it is labeled by $b-1$. There is no change of labels at virtual crossings. Note that it is convenient to label the first arc from the leg to the first crossing by $0$.

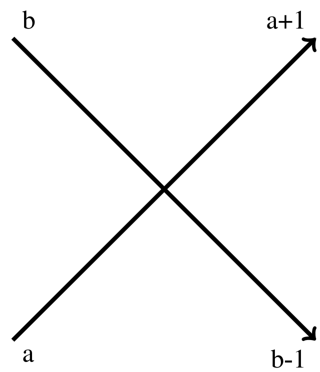



Figure 27: Integer labeling

Let $c$ be a classical crossing of a knotoid diagram $K$ in a surface $\Sigma$. We denote $w_+(c), w_-(c)$ the following integers deriving from the labels at the flat crossing corresponding to $c$:

$$w_+(c) := b - (a+1)$$
$$w_-(c) := a - (b-1)$$

where $a$ and $b$ are the labels for the left and the right incoming arcs, respectively. The numbers $w_+(c)$ and $w_-(c)$ are called positive and negative weights of $c$, respectively. We also denote $sgn(c)$ the sign of the crossing $c$. Define, now, the *weight* of $c$ to be:

$$w_K(c) := w_{sgn(c)}(c)$$

**Definition 7.1.** The *affine index polynomial* of $K$ is defined as

$$P_K(t) := \sum_{c \in Cr(K)} sgn(c)(t^{w_K(c)} - 1)$$

where $Cr(K)$ is the set of all classical crossings of $K$.

The affine index polynomial is an isotopy invariant and has some very interesting properties, as shown in [14], including that its higher degree is smaller than or equal to the height of the knotoid. Furthermore, it is very easy to calculate it by hand, as in the following example which will be very useful in what follows.

**Example 7.2.** We will calculate the affine index polynomial of the knotoid below:

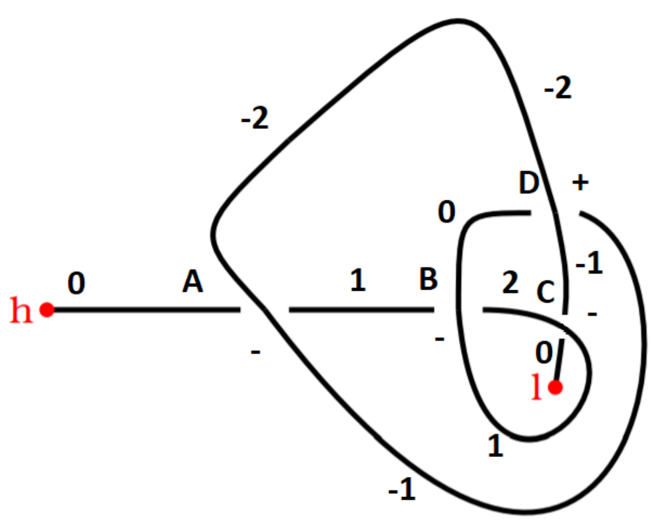



Figure 28: Labeling the knotoid

| | $w_+ = b - (a+1)$ | $w_- = a - (b-1)$ |
|---|---|---|
| A | 2 | $-2$ |
| B | 1 | $-1$ |
| C | $-2$ | 2 |
| D | $-1$ | 1 |

Table 1: Crossing weights for the affine index polynomial

With the labeling of Fig. 28 and of Table 1 we have everything we need for calculating the polynomial. Namely:

$$P_K = \sum_{c\in\{A,B,C,D\}} sgn(c)(t^{\omega_k(c)} - 1) = -(t^{-2}-1) - (t^{-1}-1) - (t^{2}-1) + (t^{-1}-1) =$$

$$= -(t^2 + t^{-2} - 2).$$

Note that, by the discussion before the example, we have a simple proof that $K$, as illustrated in Fig. 28, has height 2.

**Proposition 7.3.** The affine index polynomial is a type-1 invariant for knotoids in $\Sigma$.

*Proof.* We have to prove that if we extend the affine index polynomial by the Vassiliev skein relation to singular knotoids. We have that

$$P_K(t) = \sum_{c\in Cr(K)} sgn(c)(t^{\omega_k(c)} - 1)$$

So for every singularity inserted in a knot diagram we have the following contribution to the sum

$$P\left(\quad\right) = P\left(\quad\right) - P\left(\quad\right) = t^{\omega_+(c)} - 1 + (t^{\omega_-(c)} - 1) = t^{\omega_+} + t^{\omega_-} - 2$$

The $-2$ is also justified by the change of writhe by 2 when one changes a positive crossing to a negative.

Take a knotoid diagram with two singularities $.c, d$ with corresponding $\omega_+$ and $\omega'_+$ This yields the following contributions.

$$P\left[\ _c,\ _d\right] = P_{++} - P_{+-} - P_{-+} + P_{--} =$$

$$(t^{\omega_+} - 1) + (t^{\omega'_+} - 1) - (-t^{\omega_-} + 1) - (t^{\omega'_+} - 1) - (t^{\omega_+} - 1) - (-t^{\omega'_-} + 1) + (-t^{\omega_-} + 1) + (-t^{\omega'_-} + 1) = 0$$

So $P_K$ is a Vassiliev invariant of knotoids of type 1. $\square$

Now recall the Example 7.2 where we computed the affine index polynomial of the knotoid $K$ of Fig. 29. It was $P_K = -(t^2 + t^{-2} - 2)$. We also have (omitting the $P_K(t)$) the computation illustrated in Fig. 29.

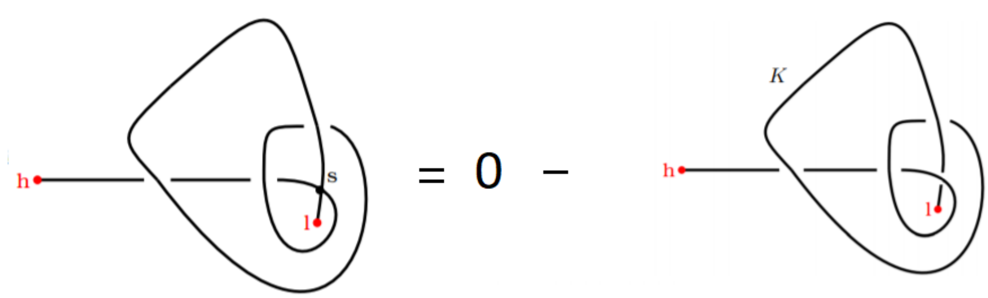



Figure 29: The affine index polynomial of the knotoid $K$ is $P_K = -(t^2 + t^{-2} - 2)$

So the regular diagram with winding number 2 has an affine index polynomial $P_K = t^2 + t^{-2} - 2$, which of course was what we expected, by the calculation of the affine index polynomial on an abstract knotoid diagram with one singularity.

### 7.2 The $\bar{v}$ invariant

As shown by the integration theorem (Theorem 5.10), any weight system $W$ gives rise to a type-1 knotoid invariant iff it respects the one-term relation, via the formula

$$v_1(K) = \sum_{c \in Cr(K)} \delta_c sgn(c) W(K_c)$$

We've also seen that the affine index polynomial is a type-1 invariant and it detects the winding of the singular loop around the leg, but it does not detect the direction. as every singular knotoid has its affine index polynomial evaluation equal to $t^w + t^{-w} - 2$ which has symmetric exponents. So, we want to look for a type-1 invariant which detects every singular equivalence class of singular knotoids.

Let $K_s$ be a regular diagram (with one singular crossing), such that the singular loop winds around the leg $w$ times. Choose:

$$W(K_s) := t^w - 1.$$

The trivial linear chord diagram, $t$, does not wind around the leg, so $w_t = 0$ and $W(K_t) = t^0 - 1 = 0$. Hence this weight system respects the one-term relation.

**Definition 7.4.** We define:

$$\bar{v}(K) := \sum_{c \in Cr(K)} \delta_c sgn(c) W(K_s) = \sum_{c \in Cr(K)} \delta_c sgn(c)(t^{w_c} - 1)$$

As it follows by the integration theorem, $\bar{v}$ is a type-1 knotoid invariant.

**Corollary 7.5.** *Since every knotoid with one singularity has a $\bar{v}$ evaluation equal to $t^w - 1$, $\bar{v}$ is a winding number detector.*

In the same way we can define a new invariant $\zeta$ by choosing $W(K_s) := t^w + t^{-w} - 2$. The proof that $\zeta$ is an invariant under isotopy is identical as the one for the invariant $\bar{v}$.

**Proposition 7.6.** *The invariant $\zeta$ defined by the weight system $W(K_s) := t^w + t^{-w} - 2$ is the affine index polynomial.*

*Proof.* Both the $\zeta$-invariant and the affine index polynomial are type-1 invariants for knotoids. Moreover, as we already saw, in each regular diagram they take values that depend only on the winding number, namely $\zeta(R_n) = P(R_n) = t^n + t^{-n} - 2$, where $R_n$ is the regular diagram with winding number of its singular loop around the leg equal to $n$.

Since their top row evaluations are identical we get that they differ only by type-0 invariants and, so, they differ by their trivial knotoid evaluations, which vanish in both cases. Hence, the two invariants are identical. $\square$

**Remark 7.7.** The affine index polynomial and the $\bar{v}$ invariant are related through the following equation:

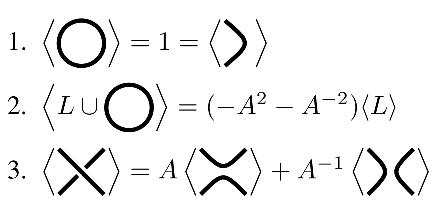
$$P_K(t) = \bar{v}(K)(t) + \bar{v}(K)(t^{-1}).$$

This is a consequence of the conversation above, just by comparing the weight assignments.

# 8 Higher type invariants coming from the Jones polynomial, the Kauffman bracket and the Turaev extended bracket

### 8.1 The Kauffman bracket and the Turaev extended bracket for knotoids

In [33] several invariants for knotoids have been defined, mainly for spherical ones. One of them is the *Kauffman bracket* polynomial. Namely, for $K$ a spherical knotoid or multi-knotoid diagram we define inductively $\langle K \rangle$ to be the element of the ring $\mathbb{Z}[A^{\pm 1}]$ by means of the rules:



Proceeding now with a closed formula: If $G$ is the underlying planar graph for $K$, then a state of $G$ is a choice of splitting marker for every vertex of $U$. We can choose between the $A$-splitting which is  or the $B$-splitting which is . We call the underlying planar graph for a diagram $K$ the *universe* for $K$. This terminology distinguishes the underlying planar graph from the link projection and from other graphs that can arise. Thus we speak of the states of a universe. Then the formula for the bracket is

$$\langle K\rangle = \sum_{S:state} \langle K|S\rangle\,(-A^2 - A^{-2})^{|S|-1} = \sum_{S:state} A^{\sigma_s}(-A^2 - A^{-2})^{|S|-1}$$

where $\sigma_s \in \mathbb{Z}$ is the sum of the values $\pm 1$ of the states over all crossings of $K$.

The bracket polynomial is a Laurent polynomial ( $\langle K\rangle \in \mathbb{Z}[A^{\pm 1}]$ ), and it is a regular isotopy invariant.

Under a Reidemeister I move the bracket polynomial is multiplied by $-A^{\pm 3}$, so $(-A^3)^{-w(K)}\langle K\rangle$ is an isotopy invariant, where $w(K)$ is the writhe of $K$, and (for classical knots) it coincides with the *Jones polynomial* with an appropriate change of variable.

We shall now recall the definition of the *Turaev extended bracket*, which is a two-variable extension of the Kauffman bracket.

- Pick a shortcut $\alpha \in S^2$ for a spherical knotoid diagram $K$.
- given a state $s$ call the smoothed 1-manifold $K_s$ and its segment component $k_s$. $k_s$ coincides with $K$ in a small neighborhood of the endpoints of $K$, and $\partial k_s = \partial\alpha$ consists of the endpoints of $K$.
- orient $K$, $k_s$, and $\alpha$ from the leg to the head of $K$.
- Set $k_s \cdot \alpha$ be the algebraic number of intersections of $k_s$ with $\alpha$ and $K \cdot \alpha$ be the algebraic number of intersections of $K$ with $\alpha$, with the endpoints not being counted.
- Define a Laurent polynomial in variables $A, u$, $T_K(A,u) \in \mathbb{Z}[A^{\pm 1}, u^{\pm 1}]$ as follows

$$T_K(A,u) = (-A^3)^{-w(K)} u^{-K\cdot\alpha} \sum_{s:state} A^{\sigma_s} u^{k_s\cdot\alpha}(-A^2 - A^{-2})^{|s|-1}$$

As it turns out, $T_K(A,u)$ is an isotopy invariant for knotoids.

**Remark 8.1.** Clearly, the Kauffman bracket polynomial can be extracted from the Turaev extended bracket polynomial by specializing $u = 1$, see also [33].

### 8.2 Finite type invariants and the Jones polynomial

Recall the definition of the normalized bracket for a knot $K$ to be

$$f_K(A) = (-A^{-3})^{-w(K)}\langle K\rangle(A)$$

where the writhe $w(K) = n_+ - n_-$ is this sum of crossing sums and $n_\pm$ is the number of positive/negative crossings.

Making the substitution $t^{1/4} = A^{-1}$ we obtain the classical Jones polynomial.

The skein relation for the Jones polynomial together with an initial condition for the unknot is

$$t^{-1}J(\;) - tJ(\;) = (t^{1/2} - t^{-1/2})J(\;)\;; \qquad J(\;) = 1\;.$$

Making a substitution $t := e^x$ and then taking the Taylor expansion into a formal power series in $x$, we can represent the Jones polynomial of a knot $K$ as a power series

$$J(K) = \sum_{n=0}^{\infty} j_n(K)x^n\;.$$

We claim that the coefficient $j_n(K)$ is a Vassiliev invariant of order $\leqslant n$. Indeed, substituting $t = e^x$ into the skein relation gives

$$(1 - x + \dots)\cdot J\left(\;\right) - (1 + x + \dots)\cdot J\left(\;\right) = (x + \frac{x^2}{4} + \dots)\cdot J\left(\;\right)$$

From which we get

$$J\left( \quad \right) = J\left( \quad \right) - J\left( \quad \right) = x\Big(\text{some mess}\Big)$$

The only summands that are not divisible by $x$ are on the left hand side. This means that the value of the Jones polynomial on a knot with a single double point is divisible by $x$. Therefore, the Jones polynomial of a singular knot with $k > n$ double points is divisible by $x^k$, and thus its $n$th coefficient vanishes on such a knot.

We can be more specific here as far as top row evaluations are concerned. Recall the oriented bracket as state expansion for the Jones polynomial with the its basic rules.

$$J\left( \quad \right) = -t^{\frac{1}{2}} J\left( \quad \right) - tJ\left( \quad \right)$$
$$J\left( \quad \right) = -t^{-\frac{1}{2}} J\left( \quad \right) - t^{-1} J\left( \quad \right)$$
$$J\left( \quad \right) = -t^{\frac{1}{2}} - t^{-\frac{1}{2}} =$$

With the substitution $t = e^x$ we get

$$J\left( \quad \right) = -e^{\frac{x}{2}} J\left( \quad \right) - e^{x} J\left( \quad \right)$$
$$J\left( \quad \right) = -e^{-\frac{x}{2}} J\left( \quad \right) - e^{-x} J\left( \quad \right)$$
$$J\left( \quad \right) = -e^{\frac{x}{2}} - e^{-\frac{x}{2}} =$$

And thus,

$$J\left( \quad \right) = J\left( \quad \right) - J\left( \quad \right) = +(-e^{\frac{x}{2}} + e^{-\frac{x}{2}})J\left( \quad \right) + (-e^{x} + e^{-x})J\left( \quad \right)$$

What follows from this computation is that top row evaluations correspond to leading terms of the expansion.

$$J\left( \quad \right) = -e^{\frac{x}{2}} - e^{-\frac{x}{2}} = -2\ldots\text{(higher order terms)}$$
$$-e^{\frac{x}{2}} + e^{-\frac{x}{2}} = -x + \ldots\text{(higher order terms)}$$
$$-e^{x} + e^{-x} = -2x + \ldots\text{(higher order terms)}$$

Hence, we can compute the top row for the Jones polynomial using the formulas.

$$j_*\left( \quad \right) = -j_*\left( \quad \right) - 2j_*\left( \quad \right)$$
$$j_*\left( \quad \right) = -2$$

### 8.3 Higher type invariants for knotoids coming from the Turaev extended bracket

We now proceed with making in the Turaev extended bracket polynomial $T_K(A, u)$ the substitution $A = e^x$.

**Proposition 8.2.** *Substituting $A = e^x$ in the Turaev extended bracket polynomial $T_K(A, u)$ of a knotoid $K$ yields the power series expansion:*

$$T(K) = \sum_{k=0}^{\infty} \sum_{l=-n_1}^{m_1} t_{k,l} u^l x^k$$

*Then $\sum_{l=-n_1}^{m_1} t_{k,l} u^l$ and $t_{k,l}$ for all $l \in \{-n_1, -n_1+1, \ldots, m_1\}$ are type-$k$ invariants of knotoids.*

*Proof.* We have the skein relation

$$-A^4 T\left(\;\right) + A^{-4} T\left(\;\right) = (A^2 - A^{-2}) T\left(\;\right) =$$

$$-\left(1 + 4x + \frac{16x^2}{2} + \dots\right) T\left(\;\right) + \left(1 - 4x + \frac{16x^2}{2} - \dots\right) T\left(\;\right) =$$

$$= ((1 + 2x + \dots) - (1 - 2x + \dots))\, T\left(\;\right)$$

Hence, by the Vassiliev skein relation

$$T\left(\;\right) = T\left(\;\right) - T\left(\;\right) = 4x(\text{some mess}).$$

So, $T\left(\;\right)$ is divisible by $x$. Hence, if a knotoid has $n > k$ singularities, then the coefficient of $x^k$ vanishes.

So we have the following results:

1. $\sum_{l=-n_1}^{m_1} t_{k,l} u^l$ is a type-$k$ invariant.
2. The fact that the coefficient of $x^k$ vanishes, means that for knotoids with $n > k$ singularities we have: $\sum_{l=-n_1}^{m_1} t_{k,l} u^l = 0 \quad \forall u$, which implies that $t_{k,l} = 0 \quad \forall l \in \{-n_1, -n_1 + 1, \dots, m_1\}$. So, every $t_{k,l}$ is a type-$k$ invariant.

□

### 8.4 Some specific computations

We shall now calculate some specific finite type invariants arising from

$$f_K(A) = (-A^3)^{-w(K)} \langle K \rangle = c_{-k} A^{-k} + \dots c_m A^m = \sum_{l=-k}^{m} c_l A^l$$

Expanding we obtain:

$$f_K(e^x) = \sum_{l=-k}^{m} c_l e^{lx} = \sum_{l=-k}^{m} c_l \sum_{n=0}^{\infty} \frac{(xl)^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{1}{n!} \sum_{l=-k}^{m} c_l l^n \right) x^n$$

So,

$$v_n = \frac{1}{n!} \sum_{l=-k}^{m} c_l l^n$$

Now we can calculate the finite type invariants arising from the bracket for the knotoid diagram of Fig. 30.

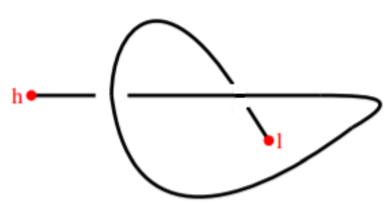



Figure 30: The knotoid diagram $K$

Following the definition of the normalized bracket $f_K$ we get

$$\langle K\rangle = A^2(-A^2 - A^{-2}) + 1 + 1 + A^{-2} = -A^4 + 1 + A^{-2}$$

Hence:

$$f_K(A) = (-A^3)^{-w(K)}\langle K\rangle = A^6(-A^4 + 1 + A^{-2}) = -A^{10} + A^6 + A^4$$

From the previous conversation we have:

$$f_K(x) = \sum_{n=0}^{\infty} v_n x^n = \sum_{n=0}^{\infty} \frac{-10^n + 6^n + 4^n}{n!} x^n$$
$$v_1(K) = -10^1 + 6^1 + 4^1 = 0$$
$$v_2(K) = \frac{-10^2 + 6^2 + 4^2}{2!} = -24$$
$$v_3(K) = \frac{-10^3 + 6^3 + 4^3}{3!} = -120$$

Now if $K^*$ is the mirror image of $K$ then:

$$f_{K^*}(A) = -A^{-10} + A^{-6} + A^{-4}$$

and following in the same way

$$v_1(K^*) = 10^1 - 6^1 - 4^1 = 0$$
$$v_2(K^*) = \frac{-(-1)^2 10^2 + (-1)^2 6^2 + (-1)^2 4^2}{2!} = -24$$
$$v_3(K^*) = \frac{-(-1)^3 10^3 + (-1)^3 6^3 + (-1)^3 4^3}{3!} = 120$$

Hence we obtained:

**Corollary 8.3.** *$v_3$ is the first finite type invariant arising from the bracket that distinguishes the knotoid $K$ from its mirror image.*

The same calculations yield that

$$T_K(A,u) = A^4 + (A^6 - A^{10})u^2$$

$$T_K(x,u) = \sum_{n=0}^{\infty} v'_n x^n = \sum_{n\in\mathbb{N},k=0,2} t_{n,k} x^n u^k = \sum_{n=0}^{\infty}\left(\frac{4^n}{n!} + \frac{6^n - 10^n}{n!}u^2\right)x^n$$

Then

$$v'_1(K) = 4 - 4u^2$$
$$v'_2(K) = \left(\frac{4^2}{2!} + \frac{6^2 - 10^2}{2!}u^2\right) = 8 - 32u^2$$

But also $t_{n,k}$ are finite type invariants, as we saw, and for example $t_{1,0}(K) = 4, \quad t_{1,2}(K) = -4$ and they are non-trivial since the Turaev extended bracket can see the winding around the leg, expressed as intersection number.

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