Homotopic deviation theory for regular matrix pencils

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Abstract

We generalize the theory of homotopic deviation of square complex matrices to regular matrix pencils. To this end, we study the existence and the analyticity of the resolvent of the matrix pencils whose matrices are under homotopic deviation with the deviation parameter $t \in \mathbb{C}$. Moreover, we investigate and identify the limits of both the resolvent and the spectrum of the deviated matrix pencils, as $|t| \to \infty$. We also study the special cases where t tends to the eigenvalues of the related matrix pairs. We use the notions and the results of the generalized homotopic deviation theory to analyze the Weierstrass structure of the deviated matrix pencils under two different assumptions, in particular, either the eigenvalues of the deviated matrix pencils are independent parameters, or the deviation parameter t is an independent parameter. Numerical examples illustrate and support the theoretical results.

Key words: Homotopic deviation, matrix pencil, resolvent, frontier point, critical point, limit point, Weierstrass structure.

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1 Introduction

Consider a matrix pencil $P(\lambda) = A - \lambda B$, where A and B are fixed $n \times n$ complex matrices and λ is a variable which varies over $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The *spectrum* of $P(\lambda)$ is defined and denoted by $\sigma(A, B) = \Big\{\lambda \in \hat{\mathbb{C}} : \det(P(\lambda)) = 0\Big\}$, where $\infty \in \sigma(A, B)$ if and only if $\det(B) = 0$. The elements of $\sigma(A, B)$ are known as the *eigenvalues* of $P(\lambda) = A - \lambda B$. Suppose that the coefficient matrices A and B are perturbed by $t\Delta A$ and $t\Delta B$, respectively, where ΔA and ΔB are fixed $n \times n$ complex matrices and $t \in \hat{\mathbb{C}}$ is a parameter. The resulting deviated matrix pencil is

$$P(\lambda, t) = (A + t\Delta A) - \lambda(B + t\Delta B). \tag{1}$$

In this paper, we study the conditions for the existence and the analyticity of the *resolvent* of the deviated matrix pencil $P(\lambda, t)$, that is, $R(\lambda, t) = P(\lambda, t)^{-1}$. We characterize singularities of $P(\lambda, t)$ which are far from $t = 0$, and use them to introduce the analyticity disks for $R(\lambda, t)$ that depend on t (including the one around 0 and the one around ∞). We also investigate the limit of both the resolvent $R(\lambda, t)$ and the *spectrum*

$$\sigma(P(\lambda, t)) = \Big\{\lambda(t) : \det(P(\lambda(t), t)) = 0 \ \text{ for every } t \in \hat{\mathbb{C}}\Big\}$$

of the deviated matrix pencil $P(\lambda, t)$, as $|t| \to \infty$. The proposed results yield a generalization of the homotopic deviation theory of square complex matrices (which, for expositional convenience, we call the basic homotopic deviation theory) to regular matrix pencils. This generalization requires some simple but important modifications of some notions including frontier points, critical points and limit points. Our study provides information regarding the relation between the parameter t and the eigenvalues of $P(\lambda, t)$, and the necessity of considering

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t in $\hat{\mathbb{C}}$. As we shall see, addressing the cases where t tends to finite eigenvalues of the matrix pair $(B, -\Delta B)$, or the matrix pair $(A, -\Delta A)$, provides useful information.

For a complete presentation of the basic homotopic deviation theory, we refer to [2, 11, 13], where, with the same assumption on t, the existence and the analyticity of the resolvent of the deviated matrix $A(t) = A + t\Delta A$ together with the limits of both the resolvent and the spectrum of $A(t)$ (as $|t| \to \infty$) were studied. The results in [2] and some of its main references indicate a tight spectral coupling between A and ΔA, when ΔA is singular. This coupling depends heavily on the Jordan structure of $0 \in \sigma(\Delta A)$, when 0 is defective. The behavior of the eigenvalues of matrices under homotopic deviation [2, 13], $A(t) = A + t\Delta A$, $t \in \mathbb{C}$, combined with the Lidskii theory and the Puiseux expansion of the eigenvalues of matrices under analytic perturbation theory [5, 9, 19, 23, 24] were revisited in [3]. The algebraic results of [3] are helpful in better understanding the limit of analyticity around infinity for the resolvent $(A + t\Delta A - \lambda I)^{-1}$, when $\lambda \in \mathbb{C}\backslash\sigma(A)$ is not a frontier point. Moreover, the structure of the matrix pencil family $\lambda \mapsto P(\lambda, t) = (A - \lambda I) + t\Delta A$, where λ is a complex parameter, has been studied in [1, 2, 12]. Clearly, this is a special case of (1), where $B = I$ and $\Delta B = 0$.

Our discussion is also an extension of the perturbation theory (and more directly, the linear perturbation theory) of matrix pencils. Some works in the literature under the title of perturbation (or linear perturbation) with different assumptions are the following. For pencils whose coefficient matrices are perturbed via matrices that do not preserve the structure, we refer to [14, 25]. Perturbations of matrix pencils with real spectrum are studied in [22] and the references therein. Minimal de-regularizing perturbations of square matrix pencils were investigated in [8] to find normwise distance of a regular pencil to the nearest non-regular (singular) square pencils. To deal with uncertainties in the data of a linear dynamical system with constant coefficients, we refer to [7]. In this kind of problems, one is interested in conditions which guarantee that all the eigenvalues of the perturbed matrix pencil, for some real $t \neq 0$, remain within a particular open subset of the complex plane [7]. It is a simple task to provide examples, see Example 3.2 in [7], in which we cannot guarantee the existence of such a particular open subset when t is restricted to be a nonzero real number. For such cases, all or part of the eigenvalues of the perturbed matrix pencil may remain in a particular desired open subset of the complex plane when t is complex or when $t \in \hat{\mathbb{C}}$. Therefore, studying the case where $t \in \hat{\mathbb{C}}$ may increase our insight to these problems.

One more motivation for this work is investigating the linearizations of parametric polynomial eigenvalue problems. In particular, results on matrix pencils, relying on (strong) linearization of matrix polynomials [21], can be used to analyze parametric polynomial eigenvalue problems. Some collections of practical parametric quadratic eigenvalue problems are available in [6, 28]. A family of parameterized quadratic eigenvalue problems from acoustics, with one parametric matrix, has been analyzed in [11] using the framework of basic homotopic deviation theory for a complex matrix A. The application studied in [11] is the acoustic wave equation (in 1D and 2D) whose boundary conditions are partly pressure release (homogeneous Dirichlet) and partly impedance, with a complex impedance parameter $\zeta = 1/t$. For more details on this application and the necessity of considering the parameter t in $\hat{\mathbb{C}}$, see [29] and [11]. For a more recent work on quadratic matrix polynomials with parametric coefficient matrix, we refer to [26]. The investigation in [26] includes the limit of spectrum of the parametric quadratic eigenvalue problem to compare the behavior of undamped and strongly damped structures. Its analysis is based on the study of a linearization of the associated parametric quadratic matrix polynomial, which is a parametric linear pencil (of double size) whose eigenproblem is strongly related to the original eigenproblem of second degree.

The generic change of the Weierstrass canonical form of a regular matrix pencil $A - \lambda B$ under a low rank perturbation $\Delta A - \lambda\Delta B$ was studied in [27]. Some classes of regular structured matrix pencils, under a certain type of structure-preserving rank-one perturbations, were studied in [4]. We use the notions and the theoretical results of homotopic deviation to analyze the Weierstrass structure of deviated matrix pencils under two different assumptions. In the first assumption, we suppose that λ is a parameter that varies in some subsets of $\mathbb{C}$ (or $\hat{\mathbb{C}}$). In the second assumption, we let the parameter t vary in some subsets of $\mathbb{C}$ (or $\hat{\mathbb{C}}$). This latter case means that we study the structure change of deviated matrix pencils at infinity, i.e., when $|t| \to \infty$.

The rest of the paper is organized as follows. In Section 2, we introduce the general observations concerning the possible values for t and λ in $P(\lambda, t)$. In Section 3, we investigate the existence and the analyticity of the resolvent $R(\lambda, t)$, and we also discuss the limit of the resolvent $R(\lambda, t)$, as $|t| \to \infty$. In Section 4, we study $\lim_{|t|\to\infty} \sigma(P(\lambda, t))$, and in Section 5, we analyze the structure of regular deviated matrix pencils $P(\lambda, t)$ when λ or t is the independent parameter. Numerical verifications of the theoretical results are given whenever needed.

The numerical examples were performed in MATLAB 9.6, where the roundoff is $u = 2^{-53} \approx 1.1 \times 10^{-16}$.

2 General observations

In this section, we study the relations between the possible values of λ in $\sigma(P(\lambda,t))$ and those of the deviation parameter t. The following definition is needed in what follows.

Definition 2.1. A matrix pencil $A-\lambda B$ is called *regular* if A and B are $n\times n$ matrices and the determinant $\det(A-\lambda B)$ does not vanish identically. Otherwise, i.e., when A and B are $m\times n$ with $m\neq n$, or when $\det(A-\lambda B)\equiv 0$, the pencil is called *singular*.

Throughout this work, we suppose that $A, B, \Delta A, \Delta B \in \mathbb{C}^{n\times n}$ and all the considered matrix pencils and deviated matrix pencils, for all values of t and for all values of λ (in the case where we assume that λ is the independent parameter), remain regular. As a direct result of this assumption, for all values of t and for all ΔA and ΔB, we have $\operatorname{rank}(B+t\Delta B) > 1$ and $\operatorname{rank}(A+t\Delta A) > 1$. We also assume that t and λ belong to $\hat{\mathbb{C}}$, since for some specific problems $|t|$ and/or $|\lambda|$ go to $+\infty$, and that both limits $\lim\limits_{|t|\to+\infty} \operatorname{rank}(A+t\Delta A)$ and $\lim\limits_{|t|\to+\infty} \operatorname{rank}(B+t\Delta B)$ are greater than 1.

We recall the notion of the *reverse matrix pencil* of $P(\lambda) = A-\lambda B$, that is, $revP(\lambda) = B-\lambda A$ [21]. It is known that $P(\lambda)$ has an infinite eigenvalue if and only if $revP(\lambda)$ has a zero eigenvalue [21]. More precisely, the sum of the algebraic multiplicities of the infinite eigenvalues of $P(\lambda)$ is equal to the algebraic multiplicity of the zero eigenvalue of $revP(\lambda)$. Of course, the sum of the algebraic multiplicities of the infinite eigenvalues of $revP(\lambda)$ is equal to the algebraic multiplicity of the zero eigenvalue of $P(\lambda) = rev(revP(\lambda))$. Based on the notion of reverse matrix pencil, we have

$$revP(\lambda,t) = (B+t\Delta B) - \lambda(A+t\Delta A) \tag{2}$$

for the deviated matrix pencil in (1).

Before investigating some general cases, we give some introductory examples.

Example 2.1. Consider the pencil (A,B) with

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

and let

$$\Delta A = \begin{bmatrix} 0 & 0 & -10^{-10} \\ 10^{-8} & 0 & 0 \\ 0 & 10^{-6} & 0 \end{bmatrix} \quad \text{and} \quad \Delta B = \begin{bmatrix} 0 & 1 & 0 \\ 10^{-4} & 0 & 0 \\ 0 & 10^{-4} & 0 \end{bmatrix}.$$

It can be easily verified that $\operatorname{rank}(A) = \operatorname{rank}(B) = \operatorname{rank}(\Delta A) = 3 = n$, but $\operatorname{rank}(\Delta B) = 2 < n$. It also holds that $\sigma(B,-\Delta B) = \{-\infty, -141, 141\} = \{t_1, t_2, t_3\}$, $\sigma(A+t_2\Delta A, B+t_2\Delta B) = \{-0.0485, 0.0486, \infty\}$ and $\sigma(A+t_3\Delta A, B+t_3\Delta B) = \big\{3.33\times 10^{-5} + 0.0485\,\mathrm{i},\ 3.33\times 10^{-5} - 0.0485\,\mathrm{i}, \infty\big\}$. Closer investigation of the spectrum $\sigma(A+t\Delta A, B+t\Delta B)$ for $|t|$ large enough shows that it contains one infinite eigenvalue.

Moreover, our numerical experiments confirm that the pair $(A+t\Delta A, B+t\Delta B)$ for real $0\le t\le 10^{308}$ has no indeterminate eigenvalue (i.e., 0/0). For several real numbers t between 0 and 10^{308}, we checked the null spaces of $A+t\Delta A$ and $B+t\Delta B$. For $0\le t\le 10^{308}$ (resp., $0\le t\le 10^{15}$) matrix $A+t\Delta A$ (resp., $B+t\Delta B$) remains nonsingular, but for $10^{16}\le t$ matrix $B+t\Delta B$ becomes singular with the nonzero vector e_3 in its null space, where e_k stands for the k-th column of the identity matrix. It is necessary to note that, in (finite) double precision arithmetic, any number larger than $\approx 1.79\times 10^{308}$ is considered as infinite. So, we may conclude that the deviated matrix pencil in this example (at least in finite precision arithmetic) remains regular.

Example 2.2. For the same matrix pencil (A,B) as in Example 2.1, let

$$\Delta A = \begin{bmatrix} 0 & 10^{-6} & 0 \\ 10^{-8} & 0 & 0 \\ 0 & 10^{-6} & 0 \end{bmatrix} \quad \text{and} \quad \Delta B = \begin{bmatrix} 0 & 1 & 0 \\ 10^{-4} & 0 & 0 \\ 0 & 10^{-4} & 1 \end{bmatrix}.$$

Then it is easy to see that $\mathrm{rank}(\Delta A) = 2 < n$ and $\mathrm{rank}(\Delta B) = 3 = n$. Moreover, we have $\sigma(B, -\Delta B) = \{-141.4214, 141.4214, -3\}$ and $\sigma(A, -\Delta A) = \{-10^8, -10^6, \infty\} = \{t_1, t_2, t_3\}$. For $t = t_1$ and $t = t_2$, we also get $\sigma(A + t_1\Delta A, B + t_1\Delta B) = \{0, 1.01 \times 10^{-6}, -9.80 \times 10^{-9}\}$, and $\sigma(A + t_2\Delta A, B + t_2\Delta B) = \{2.71 \times 10^{-26}, 10^{-7}, -0.0099\}$. When $|t| \to \infty$, $\sigma(A + t\Delta A, B + t\Delta B)$ contains one zero eigenvalue.

Our numerical experiments confirm that the pair $(A + t\Delta A, B + t\Delta B)$ for real $0 \leq t \leq 10^{308}$ has no indeterminate eigenvalue (i.e., 0/0). For several real numbers t between 0 and 10^{308}, we checked the null spaces of $A + t\Delta A$ and $B + t\Delta B$. For $0 \leq t \leq 10^{20}$ (resp., $0 \leq t \leq 10^{307}$), matrix $A + t\Delta A$ (resp., $B + t\Delta B$) remains nonsingular. For any $10^{21} \leq t \leq 10^{308}$, matrix $A + t\Delta A$ is singular with the nonzero vector e_3 in its null space. For $t = 10^{308}$, matrix $B + t\Delta B$ becomes singular with the nonzero vectors

$$\begin{bmatrix} 0 \\ \approx 0.71 \\ \approx 0.71 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \approx -0.71 \\ \approx 0.71 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

in its null space. So, we may conclude that the deviated matrix pencil in this example (at least in finite precision arithmetic) also remains regular.

Lemma 2.1. *For the matrix pencils $P(\lambda, t)$ and $revP(\lambda, t)$, defined in* (1) *and* (2)*, respectively, we have the following:*

(a) *For any $t_0 \in \sigma(B, -\Delta B)$, there exist k $(1 \leq k < n)$ eigenvalues of $P(\lambda, t)$ (resp., $revP(\lambda, t)$) which go to (resp., tend to) infinity (resp., zero), as $t \to t_0$. When* $\mathrm{rank}(\Delta B) < n$*, there is at least one $\lambda(t) \in \sigma(P(\lambda, t))$ (resp., $\mu(t) \in \sigma(revP(\lambda, t))$) such that $\lim_{|t| \to +\infty} |\lambda(t)| = +\infty$ (resp., $\lim_{|t| \to +\infty} |\mu(t)| = 0$).*

(b) *For any $t_0 \in \sigma(A, -\Delta A)$, there exist k $(1 \leq k < n)$ eigenvalues of $P(\lambda, t)$ (resp., $revP(\lambda, t)$) which tend to (resp., go to) zero (resp., infinity), as $t \to t_0$. When* $\mathrm{rank}(\Delta A) < n$*, there is at least one $\lambda(t) \in \sigma(P(\lambda, t))$ (resp., $\mu(t) \in \sigma(revP(\lambda, t))$) such that $\lim_{|t| \to +\infty} |\lambda(t)| = 0$ (resp., $\lim_{|t| \to +\infty} |\mu(t)| = +\infty$).*

Proof. The proof follows from the theory of matrix pencils and some simple remarks (see below). The statements about the eigenvalues of $revP(\lambda, t)$ follow readily from the relation explained above between $P(\lambda, t)$ and $revP(\lambda, t)$. Thus, their proof is omitted.

For the first claim in (a), we know that for $t \in \sigma(B, -\Delta B)$, $\mathrm{rank}(B + t\Delta B) < n$. Thus, the matrix pencil $P(\lambda, t)$ has some infinite eigenvalues when $t \in \sigma(B, -\Delta B)$. Hence, it follows that k ($1 \leq k < n$, according to the algebraic multiplicity of t_0 and Jordan structure of ΔB or $B + t_0\Delta B$) eigenvalues of $P(\lambda, t)$ go to infinity, as $t \to t_0$. The second claim in (a) is a result of the fact that, for a singular matrix ΔB, there is at least one t with infinite $|t|$ which makes $B + t\Delta B$ singular. Therefore, when $\mathrm{rank}(\Delta B) < n$, there is at least one $\lambda(t) \in \sigma(P(\lambda, t))$ with the property that $\lim_{|t| \to +\infty} |\lambda(t)| = +\infty$.

Part (b) can be proved in a similar way. □

Remark 2.1. If we prefer to avoid working with the case $|t| \to \infty$ (i.e., working in $\mathbb{C}$ instead of $\hat{\mathbb{C}}$), we may think of the known relation between $A + t\Delta A$ (resp., $B + t\Delta B$) and $\Delta A + sA$ (resp., $\Delta B + sB$) for $s = 1/t$. See Chapters 2 and 3 in [2] for the cases where the deviation matrix of the standard eigenvalue problem is regular or singular with semi-simple (or defective) zero eigenvalue. By this means, it is possible to bypass the reference to $\hat{\mathbb{C}}$ and even benefit from well-known theories and results such as Puiseux expansion [19, 24] and Lidskii's theory [23] for a deeper analysis of the problem in terms of the structure of existing matrices. Two objections to this idea which lead us not to use it in this work are the following:

a) The problems we study here are deviated matrix pencils where the absolute value of some of their eigenvalues may go to infinity, as $|t| \to \infty$.

b) As we shall see in Section 5, we analyze the Weierstrass structure of deviated matrix pencils under two different cases, where either t or λ is a parameter in $\hat{\mathbb{C}}$.

Because of these two facts, we naturally have to work in $\hat{\mathbb{C}}$. Beside, for brevity reasons, we leave the idea of considering Puiseux expansion for a future work.

Proposition 2.2. *Suppose that both perturbations ΔA and ΔB are nonzero and singular. Then some eigenvalues of $P(\lambda,t)$ converge to zero and some diverge to infinity, as $|t| \to +\infty$.*

Proof. When ΔB is singular, by Lemma 2.1 (a), some eigenvalues of $P(\lambda,t)$ (that is, some $\lambda(t) \in \sigma(P(\lambda,t))$) go to infinity, as $|t| \to +\infty$. By Lemma 2.1 (b), for the problems with singular ΔA, some eigenvalues of $P(\lambda,t)$ converge to zero, as $|t| \to +\infty$. Therefore, when both ΔA and ΔB are singular, it follows that some eigenvalues of $P(\lambda,t)$ converge to zero and some diverge to infinity, as $|t| \to +\infty$. □

Example 2.3. This example supports Proposition 2.2. For the same matrix pencil (A,B) and the same deviation matrix ΔB as in Example 2.1, let $\Delta A = \begin{bmatrix} 0 & 0 & -10^{-10} \\ 0 & 0 & 0 \\ 0 & 10^{-6} & 0 \end{bmatrix}$. Then $\text{rank}(\Delta A) = \text{rank}(\Delta B) = 2 < n$, and $\sigma(B,-\Delta B) = \{-\infty,-141,141\}$ and $\sigma(A,-\Delta A) = \{-\infty,-10^{11},10^{7}\}$. When $|t| \to \infty$, the spectrum of $(A+t\Delta A, B+t\Delta B)$ contains one zero eigenvalue and one infinite eigenvalue.

Our experiments confirm that the pair $(A+t\Delta A, B+t\Delta B)$ for real $0 \le t \le 10^{308}$ has no indeterminate eigenvalue (i.e., 0/0). Also $P(\lambda,t)$ for real $10^{31} \le t \le 10^{308}$ has one infinite eigenvalue. For several real numbers t between 0 and 10^{307}, we checked the null spaces of $A+t\Delta A$ and $B+t\Delta B$. For $0 \le t \le 10^{21}$ (resp., $0 \le t \le 10^{15}$), matrix $A+t\Delta A$ (resp., $B+t\Delta B$) remains nonsingular, but for $10^{21} < t \le 10^{307}$ (resp., $10^{15} < t \le 10^{307}$), matrix $A+t\Delta A$ (resp., $B+t\Delta B$) becomes singular with the nonzero vector e_1 (resp., the nonzero vector e_3) in its null space. However, for $t = 10^{308}$, matrices $A+t\Delta A$ (resp., $B+t\Delta B$) are singular with one common nonzero vector e_1 in their null spaces. This means that the singularity of the deviated matrix pencil $P(\lambda,t)$ appears at $t = 10^{308}$.

The following arrangement for the deviated matrix pencil (1) will help us in the remainder. Let us rewrite $P(\lambda,t)$ as

$$P(\lambda,t) = P(\lambda) + t\Delta P(\lambda) \tag{3}$$

for $P(\lambda) = A - \lambda B$ and $\Delta P(\lambda) = \Delta A - \lambda \Delta B$.

At this point, let us provide the following remark on Example 2.3, when we use the arrangement (3).

Remark 2.2. We consider the matrices in Example 2.3 and look at what happens for $P(\lambda,t) = P(\lambda)+t\Delta P(\lambda)$, $A - \lambda B$ and $\Delta A - \lambda \Delta B$ when λ varies in $0 \le \lambda \le 10^{308}$; here is where we consider λ as parameter and t as eigenvalue of $P(\lambda,t)$. Based on the same arguments as those in Example 2.3, we expect that one eigenvalue, t, of $P(\lambda,t)$ goes to infinity as $|\lambda| \to \infty$. Our experiments confirm that, for real $0 \le \lambda \le 10^{307}$, $P(\lambda,t)$ has no indeterminate eigenvalue (i.e., 0/0). Also $P(\lambda,t)$ has one infinite eigenvalue for real $10^2 \le \lambda \le 10^{307}$. For $\lambda = 10^{308}$, the eigenvalues, t's, of $P(\lambda,t)$ are not computable using MATLAB. For $0 \le \lambda \le 10^{307}$ (resp., $0.02 \le \lambda \le 15$), matrix $A - \lambda B$ (resp., $\Delta A - \lambda \Delta B$) remains nonsingular. For $\lambda = 10^{308}$ (resp., $\lambda = 10^{309}$), the null space of matrix $A - \lambda B$ (resp., $\Delta A - \lambda \Delta B$) is not computable using MATLAB. So, we may conclude that the deviated matrix pencil in Example 2.3 (at least in finite precision arithmetic) stays regular for $0 \le \lambda \le 10^{307}$.

Let us also denote the matrix

$$E_\lambda = -P(\lambda)\Delta P(\lambda)^{-1} \tag{4}$$

for (fixed) $\lambda \in \mathbb{C}\backslash\sigma(\Delta A, \Delta B)$.

Proposition 2.3. *A scalar $\lambda \in \mathbb{C}\backslash\sigma(\Delta A, \Delta B)$ is an eigenvalue of $P(\lambda,t)$ if and only if $t = \nu(\lambda)$ for some $\nu(\lambda) \in \sigma(E_\lambda)$, where E_λ is defined by* (4).

Proof. For any $\lambda \in \mathbb{C}\backslash\sigma(\Delta A, \Delta B)$, we have

$$P(\lambda,t) = P(\lambda) + t\Delta P(\lambda) = \left[P(\lambda)\Delta P(\lambda)^{-1} + tI\right]\Delta P(\lambda).$$

Therefore, such a λ belongs to $\sigma(P(\lambda,t))$ if and only if $t = \nu(\lambda) \in \sigma(E_\lambda)$. □

Let us now denote the matrix

$$F_\lambda = -\Delta P(\lambda)P(\lambda)^{-1} \tag{5}$$

for (fixed) $\lambda \in \mathbb{C}\backslash\sigma(A,B)$.

Proposition 2.4. *A scalar $\lambda \in \mathbb{C}\backslash\sigma(A,B)$ is an eigenvalue of $P(\lambda,t)$ if and only if there exists an eigenvalue $\mu(\lambda) \in \sigma(F_\lambda)\backslash\{0\}$ of the matrix F_λ defined by* (5), *such that $t\mu(\lambda) = 1$.*

Proof. For any scalar $\lambda \in \mathbb{C}\backslash\sigma(A,B)$,

$$P(\lambda,t) = P(\lambda) + t\Delta P(\lambda) = \left[I + t\Delta P(\lambda)P(\lambda)^{-1}\right] P(\lambda) = (I - tF_\lambda)\, P(\lambda).$$

Hence, since $\lambda \notin \sigma(A,B)$ and $t \neq 0$, $\lambda \in \sigma(P(\lambda,t))$ if and only if $t = \mu(\lambda)^{-1}$ for some $\mu(\lambda) \in \sigma(F_\lambda)\backslash\{0\}$. □

We call any $\lambda \in \mathbb{C}\backslash\sigma(\Delta A, \Delta B)$ (resp., $\lambda \in \mathbb{C}\backslash\sigma(A,B)$) that satisfies the sufficient condition of Proposition 2.3 (resp., Proposition 2.4) an *inexact eigenvalue* of the matrix pencil $A - \lambda B$ at homotopic distance $|t| = |\nu_\lambda|$ (resp., $|t| = 1/|\mu(\lambda)|$). The latter distance suggests an extension of Proposition 2.4 for $\mu(\lambda) \in \sigma(F_\lambda)$, where the distance can be unbounded when $\mu(\lambda) = 0$.

For the matrices E_λ and F_λ defined by (4) and (5), respectively, we remark the following:

- When $\text{rank}(E_\lambda) = r_E \leq n$, λ is an *exact* eigenvalue of r_E matrix pairs $(A + t_i\Delta A, B + t_i\Delta B)$ with finite $t_i = \nu_{i\lambda} \in \mathbb{C}$, for $\nu_{i\lambda} \in \sigma(E_\lambda)$, $i = 1, 2, \ldots, r_E$. There are at most r_E different homotopic distances $|t_i| = |\nu_{i\lambda}|$.
- When $\text{rank}(F_\lambda) = r_F \leq n$, λ is the *exact* eigenvalue of r_F matrix pairs $(A + t_i\Delta A, B + t_i\Delta B)$ with finite $t_i = \mu_{i\lambda}^{-1} \in \mathbb{C}$, for $\mu_{i\lambda} \in \sigma(F_\lambda)\backslash\{0\}$, $i = 1, 2, \ldots, r_F$. There are at most r_F different homotopic distances $|t_i| = |\mu_{i\lambda}|^{-1}$, $\mu_{i\lambda} \neq 0$.

3 Resolvent of $P(\lambda, t)$

One can use the first companion matrix [20, pp. 490–494] of the matrix pencil $P(\lambda,t)$ to obtain its resolvent, $R(\lambda,t) = P(\lambda,t)^{-1}$, but here, we consider the closer notation to the basic homotopic deviation theory [2]. For this purpose, we investigate the existence and the analyticity of $R(\lambda,t)$, where $P(\lambda,t)$ is given in the form (3). For $\lambda \notin \sigma(A,B)$, let

$$R(\lambda,t) = [P(\lambda) + t\Delta P(\lambda)]^{-1} = \left[(I + t\Delta P(\lambda)P(\lambda)^{-1})P(\lambda)\right]^{-1}.$$

Then, we get

$$R(\lambda,t) = P(\lambda)^{-1}\,(I - tF_\lambda)^{-1} \tag{6}$$

for F_λ as in (5). According to (6), $R(\lambda,t)$ with $\lambda \notin \sigma(A,B)$ exists for $t \neq \mu(\lambda)^{-1}$, where $\mu(\lambda)$ belongs to $\sigma(F_\lambda)\backslash\{0\}$. The Neumann series of $(I - tF_\lambda)^{-1} = \sum\limits_{k=0}^{\infty}(tF_\lambda)^k$ is convergent for $\rho\,(tF_\lambda) < 1$, where $\rho(X)$ stands for the spectral radius of X. Therefore, $R(\lambda,t)$, with $\lambda \notin \sigma(A,B)$ and for $|t| < \rho\left(\Delta P(\lambda)P(\lambda)^{-1}\right)^{-1}$, is computable (i.e., $\sum\limits_{k=0}^{\infty}(tF_\lambda)^k$ is convergent) by $P(\lambda)^{-1}\sum\limits_{k=0}^{\infty}(tF_\lambda)^k$.

If $0 \notin \sigma(F_\lambda)$, then $R(\lambda,t)$ is defined for $\lambda \notin \sigma(A,B)$ and $t \in \mathbb{C}$ with $t \neq t_i$, where $t_i = \mu_{i\lambda}^{-1}$ for $\mu_{i\lambda} \in \sigma(F_\lambda)$, $i = 1, 2, \ldots, n$. The reason is that $R(\lambda,t)$ is not defined when λ is an eigenvalue of the matrix pencil $P(\lambda,t)$.

It is well known that any $n \times n$ nonzero matrix $X \in \mathbb{C}^{n\times n}$ with $\text{rank}(X) = r \leq n$, can be written in the form

$$X = UV^H, \tag{7}$$

where the matrices $U, V \in \mathbb{C}^{n\times r}$ are of rank r and represent a basis of $\text{Im}\,X$ and a basis of $\text{Im}\,X^H$, respectively. Moreover, for computing U and V, we can use the SVD of matrix X. To this end, let $X = U_1S_1V_1^H$ be the SVD of X. Then, using the MATLAB colon notation, the matrices U and V are given by $U = U_1(:, 1:r)S_1(1:r, 1:r)$ and $V = V_1(:, 1:r)$.

For any $n \times n$ nonzero matrix $\Delta P(\lambda) = \Delta A - \lambda\Delta B$, the value of $r_\lambda = \text{rank}(\Delta P(\lambda)) \leq n$ depends on the value of λ. Therefore, for any fixed λ, as above (see (7)), we can write $\Delta P(\lambda)$ in the form

$$\Delta P(\lambda) = U_\lambda V_\lambda^H, \tag{8}$$

where $U_\lambda, V_\lambda \in \mathbb{C}^{n\times r_\lambda}$ are of rank r_λ and represent a basis of $\text{Im}\,\Delta P(\lambda)$ and a basis of $\text{Im}\,\Delta P(\lambda)^H$, respectively.

Proposition 3.1. *Let $\lambda \in \mathbb{C}\backslash\sigma(A,B)$ and $1 \le \text{rank}(\Delta P(\lambda)) = r_\lambda \le n$. Then the nonzero eigenvalues of the $n \times n$ matrix $F_\lambda = -\Delta P(\lambda)P(\lambda)^{-1}$ are exactly the nonzero eigenvalues of the $r_\lambda \times r_\lambda$ matrix*

$$M_\lambda = -V_\lambda^H P(\lambda)^{-1} U_\lambda, \tag{9}$$

where U_λ and V_λ are given by (8).

Proof. This can be proved using the following well-known fact (for example, see [18, Theorem 1.3.20]): For the matrices $X \in \mathbb{C}^{n\times r}$ and $Y \in \mathbb{C}^{r\times n}$, the nonzero eigenvalues of the product $XY \in \mathbb{C}^{n\times n}$ are the same as the nonzero eigenvalues of the product $YX \in \mathbb{C}^{r\times r}$. In particular, if for any arbitrary $\lambda \in \mathbb{C}\backslash\sigma(A,B)$, we use (8) to replace $\Delta P(\lambda)$ by $U_\lambda V_\lambda^H$, then denoting $X = -U_\lambda \in \mathbb{C}^{n\times r_\lambda}$ and $Y = V_\lambda^H P(\lambda)^{-1} \in \mathbb{C}^{r_\lambda\times n}$, we conclude that the nonzero eigenvalues of the $n \times n$ matrix $F_\lambda = XY$ are exactly the nonzero eigenvalues of the $r_\lambda \times r_\lambda$ matrix $M_\lambda = YX$. □

Remark 3.1. It is well known that matrix inversion is not advisable both in terms of the number of arithmetic operations and in terms of maintaining stability in the floating point arithmetic [17, Section 14]. Therefore, the matrix inversion in (9) should be treated as solving a linear system with multiple right-hand sides $P(\lambda)Y = U_\lambda$ in Y, where $U_\lambda \in \mathbb{C}^{n\times r_\lambda}$.

Based on Proposition 3.1, we give the following remark.

Remark 3.2. For any $\lambda \in \mathbb{C}\backslash\sigma(A,B)$, the order of M_λ is equal to $r_\lambda = \text{rank}(\Delta P(\lambda))$ and the value of $r_\lambda \le n$ depends on the value of λ. In particular, if $\sigma(\Delta P(\lambda)) = \{\mu_1, \mu_2, \ldots, \mu_n\}$, then the order of M_λ changes when λ varies over $\mathbb{C}\backslash\sigma(A,B)$, and for $\lambda = \mu_i$, $i = 1, 2, \ldots, n$, the order of M_{μ_i} is less than n.

The following is a direct consequence of Proposition 2.4 and Proposition 3.1.

Corollary 3.2. *Let $\lambda \in \mathbb{C}\backslash\sigma(A,B)$ and $1 \le \text{rank}(\Delta P(\lambda)) = r_\lambda \le n$. Then $\lambda \in \mathbb{C}\backslash\sigma(A,B)$ is an eigenvalue of $P(\lambda,t)$ if and only if there exists an eigenvalue $\mu(\lambda) \in \sigma(M_\lambda)\backslash\{0\}$ such that $t\mu(\lambda) = 1$.*

The Sherman-Morrison-Woodbury formula [16, p. 65] for matrices $A \in \mathbb{C}^{n\times n}$ and $S, T \in \mathbb{C}^{n\times r}$ gives

$$(A + ST^H)^{-1} = A^{-1} - A^{-1}S(I_r + T^H A^{-1} S)^{-1} T^H A^{-1}, \tag{10}$$

provided that A is of order n and $(I_r + T^H A^{-1} S)$ is of order r, and they are both invertible. This means that adding a rank r deviation matrix to an invertible matrix results in a rank r deviation to the inverse of the deviated matrix.

Proposition 3.3. *The resolvent $R(\lambda,t)$ exists for any $\lambda \in \mathbb{C}\backslash\sigma(A,B)$ and $t \ne \mu(\lambda)^{-1}$, where $\mu(\lambda) \in \sigma(M_\lambda)\backslash\{0\}$ for M_λ in* (9).

Proof. Let $\lambda \in \mathbb{C}\backslash\sigma(A,B)$. Using (5) and (8), we obtain

$$(I_n - tF_\lambda)^{-1} = \left(I_n + t\Delta P(\lambda)P(\lambda)^{-1}\right)^{-1} = \left(I_n + tU_\lambda V_\lambda^H P(\lambda)^{-1}\right)^{-1}.$$

This, together with (6) and (10), can be used to write $R(t,\lambda)$ in the form

$$R(\lambda,t) = P(\lambda)^{-1}\left[I_n - tU_\lambda\left(I_{r_\lambda} + tV_\lambda^H P(\lambda)^{-1}U_\lambda\right)^{-1} V_\lambda^H P(\lambda)^{-1}\right].$$

The definition of M_λ implies

$$R(\lambda,t) = P(\lambda)^{-1}\left[I_n - tU_\lambda\left(I_{r_\lambda} - tM_\lambda\right)^{-1} V_\lambda^H P(\lambda)^{-1}\right], \tag{11}$$

which exists for $\lambda \in \mathbb{C}\backslash\sigma(A,B)$ and $t \ne \mu(\lambda)^{-1}$, where $\mu(\lambda) \in \sigma(M_\lambda)\backslash\{0\}$. □

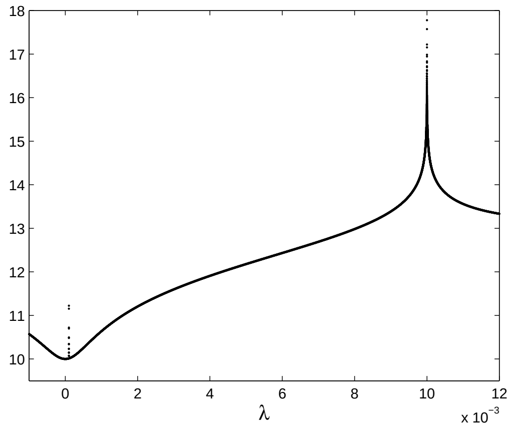

Figure 1: $Log_{10}\left(\max_{1\le i\le r_\lambda}(1/|\mu_i|)\right)$ *for* $\mu_i \in \sigma(F_\lambda)$.

Any matrix inversion in (11), whenever needed, should be replaced by an appropriate linear system solving as in Remark 3.1. Proposition 3.3 shows that $R(t,\lambda)$ is not defined for $\lambda \in \mathbb{C}\backslash\sigma(A,B)$ when $t \in \mathbb{C}$ satisfies $t\mu(\lambda) = 1$ for some $\mu(\lambda) \in \sigma(M_\lambda)\backslash\{0\}$. As a consequence, if M_λ is regular for $\lambda \in \mathbb{C}\backslash\sigma(A,B)$, then $R(t,\lambda)$ is not defined for $t \in \sigma\left(M_\lambda^{-1}\right)$.

Next we introduce some useful definitions and notions to be used for the classification of the problems.

Definition 3.1. We say that a $\lambda \in \mathbb{C}\backslash\sigma(A,B)$ is a *frontier point* if $\text{rank}(\Delta P(\lambda)) = r_\lambda < n$. We denote the set of all frontier points by

$$F(P,\Delta P) = \{\lambda \in \mathbb{C}\backslash\sigma(A,B) : \ \text{rank}(\Delta P(\lambda)) < n\} = \{\lambda \in \mathbb{C} : \lambda \in \sigma(\Delta A,\Delta B)\backslash\sigma(A,B)\}\,.$$

At any frontier point, the matrix M_λ in (9) is $r_\lambda \times r_\lambda$ with $r_\lambda < n$. Any scalar in $F(P,\Delta P)$ is called a frontier point, because for $\lambda \notin F(P,\Delta P)$, the matrix pencil $P(\lambda,t)$ has exactly n finite eigenvalues. When $\lambda \in F(P,\Delta P)$, the pencil $P(\lambda,t)$ has r_λ (with $r_\lambda < n$) finite eigenvalues and $n - r_\lambda = a_\lambda$ infinite eigenvalues. The value of r_λ depends on $\lambda \in F(P,\Delta P) \subseteq \sigma(\Delta P(\lambda))$.

The above definition of frontier points is different than the corresponding definition in [2]. The reason is that in [2], we have $P(\lambda,t) = (A - \lambda I) + t\Delta A$ with $\text{rank}(\Delta A) \le n$, whereas here, we have $P(\lambda,t) = P(\lambda) + t\Delta P(\lambda)$, where $P(\lambda) = A - \lambda B$ and $\Delta P(\lambda) = \Delta A - \lambda\Delta B$ are $n \times n$ matrix pencils and $r_\lambda = \text{rank}(\Delta P(\lambda))$ $(1 \le r_\lambda \le n)$ is dependent on λ.

Let us denote $R = \sigma(A,B) \cap \sigma(\Delta A,\Delta B)$. Based on Remark 3.2, we may have only one of the following cases:

Case 1. $F(P,\Delta P)$ is empty. This happens when $\text{rank}(\Delta P(\lambda)) = n$ for all $\lambda \in \mathbb{C}\backslash\sigma(A,B)$. For instance, this is the case when $R \ne \emptyset$ and the eigenvalues in $\sigma(\Delta A,\Delta B)\backslash R$ are infinite.

Case 2. $F(P,\Delta P)$ is discrete with finite cardinality. This is the case when $\Delta P(\lambda)$ is a regular matrix pencil with some finite eigenvalues which do not belong to the set R.

Case 3. $F(P,\Delta P)$ coincides with $\mathbb{C}\backslash\sigma(A,B)$ when $\Delta P(\lambda)$ is a singular matrix pencil, i.e., $\det\Delta P(\lambda) \equiv 0$ for all $\lambda \in \mathbb{C}$.

Example 3.1. This example illustrates the quality of the frontier set associated with the problem of Example 2.1, where we have $\sigma(\Delta P(\lambda)) = \left\{10^{-4}, 10^{-2}, -\infty\right\}$ and $\sigma(P(\lambda)) = \{-0.55, 0.275 + 0.477\,\mathrm{i}, 0.275 - 0.477\,\mathrm{i}\}$. Therefore $\sigma(A,B) \cap \sigma(\Delta A,\Delta B) = \emptyset$ and both finite eigenvalues of $\Delta P(\lambda)$ are frontier points, so $F(P,\Delta P)$ is a discrete set with finite cardinality. Figure 1 illustrates a scaled plot of $\max_{1\le i\le r_\lambda}(1/|\mu_i|)$ for $\mu_i \in \sigma(F_\lambda)$. We can see the abrupt changes happened at the frontier points.

Definition 3.2. A scalar λ in $F(P,\Delta P)$ is called *critical* when for the $r_\lambda \times r_\lambda$ matrix M_λ defined in (9), it holds $\rho(M_\lambda) = 0$. The set of critical points is denoted by $C(P,\Delta P)$.

When $\lambda \in C(P, \Delta P)$, then M_λ is nilpotent, that is, $M_\lambda^\delta = 0$ with $M_\lambda^{\delta-1} \neq 0$, where δ (with $1 \leq \delta \leq r_\lambda$) is the size of the largest Jordan block of $0 \in \sigma(M_\lambda)$ [10]. Therefore, at such a point, we have

$$(I_{r_\lambda} - tM_\lambda)^{-1} = \sum_{k=0}^{\delta-1} (tM_\lambda)^k,$$

and the map $t \mapsto R(t, \lambda)$ in (11) has a finite series expansion.

Let M_λ be regular for some $\lambda \in F(P, \Delta P)$. We can order the eigenvalues $\mu_1, \mu_2, \ldots, \mu_{r_\lambda}$ of M_λ such that

$$|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_{r_\lambda}| > 0,$$

or equivalently, such that (for $t_i = \mu_i^{-1}$)

$$|t_1| \leq |t_2| \leq \cdots \leq |t_{r_\lambda}| < \infty.$$

When M_λ is singular for some $\lambda \in F(P, \Delta P)$, then M_λ has less than r_λ nonzero eigenvalues.

Proposition 3.4. *Suppose M_λ, defined by (9), is regular for some $\lambda \in F(P, \Delta P)$. Then $\lim_{|t| \to \infty} R(\lambda, t)$, denoted by $R(\lambda, \infty)$, exists, and its representation in closed form is*

$$R(\lambda, \infty) = P(\lambda)^{-1} \left(I_n + U_\lambda M_\lambda^{-1} V_\lambda^H P(\lambda)^{-1} \right).$$

Proof. As M_λ^{-1} exists, for $s = t^{-1}$, we have $I_{r_\lambda} - tM_\lambda = (sM_\lambda^{-1} - I_{r_\lambda}) tM_\lambda$. Therefore,

$$(I_{r_\lambda} - tM_\lambda)^{-1} = -sM_\lambda^{-1} \left(I_{r_\lambda} - sM_\lambda^{-1} \right)^{-1}. \tag{12}$$

Multiplying (12), from the left, by $-tU_\lambda$ gives

$$-tU_\lambda \left(I_{r_\lambda} - tM_\lambda \right)^{-1} = U_\lambda M_\lambda^{-1} \left(I_{r_\lambda} - sM_\lambda^{-1} \right)^{-1},$$

which converges to $U_\lambda M_\lambda^{-1}$, as $|s| \to 0$ (or, as $|t| \to \infty$). Using this and (11), we can conclude that $R(\lambda, t)$ converges to $R(\lambda, \infty) = P(\lambda)^{-1} \left(I_n + U_\lambda M_\lambda^{-1} V_\lambda^H P(\lambda)^{-1} \right)$, as $|t| \to \infty$. □

When $\lambda \in (\mathbb{C} \backslash \sigma(A, B)) \backslash F(P, \Delta P)$, the order and the rank of M_λ are equal to n, and we have the following two analytic expansions for $R(\lambda, t)$.

(i) $|t_1|$ defines the largest analyticity disk for $R(\lambda, t)$. It rules the convergence of the initial analytic expansion for t around 0, where

$$R(\lambda, t) = R(\lambda, 0) \left[I_n - tU_\lambda \sum_{k=0}^{\infty} (tM_\lambda)^k V_\lambda^H R(\lambda, 0) \right]$$

based on M_λ and valid for $|t| < |t_1|$.

(ii) $|t_n|$ defines the smallest value for $|t|$ beyond which $R(\lambda, t)$ is analytic in $s = 1/t$. This is analyticity in t around ∞. It rules the convergence of the asymptotic analytic expansion in $s = 1/t$, where

$$\begin{aligned} R(\lambda, t) &= R(\lambda, 0) \left[I_n + U_\lambda M_\lambda^{-1} \sum_{k=0}^{\infty} \left(sM_\lambda^{-1} \right)^k V_\lambda^H R(\lambda, 0) \right] \\ &= R(\lambda, 0) + R(\lambda, 0) U_\lambda M_\lambda^{-1} \sum_{k=0}^{\infty} (tM_\lambda)^{-k} V_\lambda^H R(\lambda, 0), \end{aligned}$$

based on M_λ^{-1} and valid for $|t| > |t_n|$ (around $|t| = \infty$, that is $s = 0$).

4 Spectrum of $P(\lambda, t)$ when $|t| \to \infty$

In this section, we characterize the limit $\lim_{|t|\to\infty} \sigma(P(\lambda,t))$. We need to make precise the notion of limit point of $\sigma(P(\lambda,t))$ (or simply, limit point) and the set of limit points of $\sigma(P(\lambda,t))$.

Let $\lambda(t)$ be an eigenvalue of $P(\lambda,t)$. Then, for the limit $L_\lambda = \lim_{|t|\to\infty} \lambda(t)$, except for some rare cases like oscillating eigenvalues, one of the following holds:

(i) $L_\lambda \notin \sigma(A,B)$ is a finite number.

(ii) $L_\lambda \notin \sigma(A,B)$ and $|L_\lambda| = +\infty$.

(iii) $L_\lambda = \gamma \in \sigma(A,B)$, where $\gamma \neq \lambda(0)$.

(iv) $L_\lambda = \lambda(0) \in \sigma(A,B)$ is an invariant finite or infinite eigenvalue of $A - \lambda B$. This is the case when $\lambda(t)$ is a constant function of t.

Definition 4.1. Any finite number in $\mathbb{C}\backslash\sigma(A,B)$, among $L_\lambda = \lim_{|t|\to\infty} \lambda(t)$, $\lambda(t) \in \sigma(P(\lambda,t))$, is called a *limit point* of $\sigma(P(\lambda,t))$, and the set of all such numbers is denoted by Lim, and it is called the *limit set* of $\sigma(P(\lambda,t))$.

At the first part of this section, we characterize the possible cases for the limit of $\lambda(t) \in \sigma(P(\lambda,t))$ as $|t| \to \infty$. At the second part of the section, we discuss a close relation between frontier points and limit points, using the singularities of F_λ.

4.1 The possible cases

When $\lambda \in \sigma(\Delta A, \Delta B)$, we have the following general result.

Proposition 4.1. *Any finite $\lambda \in \sigma(\Delta A, \Delta B)\backslash\sigma(A,B)$ is an eigenvalue of $P(\lambda,t)$ for at most $n-1$ finite values of t.*

Proof. For any $\lambda \in \sigma(\Delta A, \Delta B)\backslash\sigma(A,B)$, we have $r = \operatorname{rank}(\Delta P(\lambda)) < n$. Therefore, such a λ is an eigenvalue of the matrix pencil (3) for at least one infinite value of t. So, $\lambda \in \sigma(\Delta A, \Delta B)\backslash\sigma(A,B)$ is an eigenvalue of $P(\lambda,t)$ for at most r ($r \leq n-1$) finite values of t. □

We need the following notation. We remark that some of them are related to the cases and the problems where t goes to infinity ($+\infty$ or $-\infty$) on real axis.

- The number of finite eigenvalues of $\Delta P(\lambda)$ are denoted by n_f.
- σ_f stands for the set of finite eigenvalues of $\Delta P(\lambda)$.
- Φ_m (resp., Φ_p) is the set of finite eigenvalues in $\sigma(\Delta P(\lambda))$ whose elements are the limit of $\sigma(P(\lambda,t))$ at negative infinity, i.e., $\Phi_m = \sigma_f \cap (\lim_{t\to-\infty} \sigma(P(\lambda,t)))$ (resp., at positive infinity, i.e., $\Phi_p = \sigma_f \cap (\lim_{t\to+\infty} \sigma(P(\lambda,t)))$).
- We denote by n_i, the number of infinite eigenvalues of $\Delta P(\lambda)$.
- σ_i stands for the set of infinite eigenvalues of $\Delta P(\lambda)$.
- Ψ_m (resp., Ψ_p) is the set of infinite eigenvalues in $\sigma(\Delta P(\lambda))$ whose elements are the limit of $\sigma(P(\lambda,t))$ at negative infinity, i.e., $\Psi_m = \sigma_i \cap (\lim_{t\to-\infty} \sigma(P(\lambda,t)))$ (resp., at positive infinity, i.e., $\Psi_p = \sigma_i \cap (\lim_{t\to+\infty} \sigma(P(\lambda,t)))$).
- $\operatorname{card} X$ denotes the cardinality of the set X.

This notation is used in the following two propositions.

Proposition 4.2. *Suppose t goes to infinity ($-\infty$ or $+\infty$) on real axis. Then we have the following:*

(*a*) card Φ_m+ card $\Phi_p \geq n_f$ *and* $\Phi_m \cup \Phi_p = \sigma_f$.

(*b*) card Ψ_m+ card $\Psi_p \geq n_i$ *and* $\Psi_m \cup \Psi_p = \sigma_i$.

(*c*) *The limit set of* $\sigma(P(\lambda,t))$ *is a subset of* $\sigma_f \backslash \sigma(A,B)$.

Proof. There is a possibility that some finite $\lambda \in \sigma(\Delta P(\lambda))$ might be the limit for both cases $t \to -\infty$ and $t \to +\infty$. This means that, in some cases, we may have $\Phi_m \cap \Phi_p \neq \emptyset$. Therefore, the case (a) is proved. Almost the same argument is valid for the case (b). The statement (c) is directly achieved via the definitions of σ_f and the limit set of $\sigma(P(\lambda,t))$. □

Proposition 4.3. *If* Φ_m *and* Ψ_m *(resp.,* Φ_p *and* Ψ_p*) are not empty, then* $\lim_{t\to-\infty} \lambda_i(t) = \{\lambda \in \Phi_m \cup \infty\}$ *(resp.,* $\lim_{t\to+\infty} \lambda_i(t) = \{\lambda \in \Phi_p \cup \infty\}$*), where* ∞ *stands for some appropriate* $-\infty$ *or* $+\infty$ *in* $\sigma_i \subset \sigma(\Delta P(\lambda))$.

Proof. For any $\lambda \in \sigma(\Delta P(\lambda))$, rank$(\Delta P(\lambda)) < n$. Based on the theory of matrix pencils, for any finite or infinite $\lambda \in \sigma(\Delta P(\lambda))$, we can use (3) to conclude that there is at least one infinite value, i.e., $t = -\infty$ or $t = +\infty$, in $\sigma(P(\lambda), -\Delta P(\lambda))$. Therefore, when $t \to -\infty$ (resp., $t \to +\infty$), $n_{\Phi_m} = \text{card}\Phi_m$ (resp., $n_{\Phi_p} = \text{card}\Phi_p$) eigenvalues of $\lambda_i(t) \in \sigma(P(\lambda,t))$ converge to the eigenvalues in $\Phi_m \subseteq \sigma(\Delta P(\lambda))$ (resp., $\Phi_p \subseteq \sigma(\Delta P(\lambda))$), and the rest, i.e., $n_i = n - n_f$ eigenvalues of $\lambda_i(t) \in \sigma(P(\lambda,t))$, go to some associated $-\infty$ or $+\infty$ in $\sigma_i \subset \sigma(\Delta P(\lambda))$. □

Table 1: *Correspondence of* $\lambda \in \sigma(\Delta P(\lambda))$ *and* $t \in \sigma(P(\lambda,t))$*, for* $\lambda = \lambda_i \in \sigma(\Delta P(\lambda))$*,* $i = 1,2$*, and* $t_j \in \sigma(P(\lambda_i,t))$*,* $j = 1,2,3$.

Eigenvalues of $\Delta P(\lambda)$	$t_1 \in \sigma(P(\lambda_i,t))$	$t_2 \in \sigma(P(\lambda_i,t))$	Infinite t_3 in $\sigma(P(\lambda_i,t))$
$\lambda_1 = 10^{-4}$	$t_1 = 9.80 \times 10^5$	$t_2 = 1.03 \times 10^{10}$	$+\infty$
$\lambda_2 = 10^{-2}$	$t_1 = -1.01 \times 10^6$	$t_2 = 3.33 \times 10^3$	$+\infty$

Example 4.1. This example illustrates the classification made in Proposition 4.3 for the matrix pair (A,B) in Example 2.1. As reported in Example 3.1, $\sigma(\Delta A, \Delta B) \cap \sigma(A,B) = \emptyset$. Also for any finite $\lambda_i \in \sigma(\Delta P(\lambda))$, $i = 1,2$, we have $r_{\lambda_i} = \text{rank}(\Delta P(\lambda_i)) = 2 < 3 = n$. Table 1 shows that corresponding to each finite $\lambda_i \in \sigma(\Delta P(\lambda))$, $i = 1,2$, there are two finite t (t_j, $j = 1,2$) which result in $\det P(\lambda_i,t) = 0$, i.e., $t_j \in \sigma(P(\lambda_i), -\Delta P(\lambda_i))$, for $i = 1,2$, and $j = 1,2$. For each finite $\lambda_i \in \sigma(\Delta P(\lambda))$, $i = 1,2$, there is one (i.e., $n-2 = 3-2$) infinite eigenvalue, t, in $\sigma(P(\lambda_i), -\Delta P(\lambda_i))$ such that Φ_m is empty and $\Phi_p = \{\lambda_1, \lambda_2\}$. Two eigenvalues of $\lambda_i(t) \in \sigma(P(\lambda,t))$, $i = 1,2$, converge to the elements of the set $\Phi_p = \{\lambda_1, \lambda_2\}$ as $t \to +\infty$. In addition, there is one eigenvalue of $\sigma(P(\lambda,t))$ that goes to $+\infty$ as $t \to +\infty$.

In order to verify the case where t is a complex parameter, let us write the homotopic parameter $t = he^{i\theta}$, with $h = |t| \in \mathbb{R}^+$ and $\theta = \text{Arg}\, t \in [0, 2\pi)$. Obviously, the case where t is positive (resp., negative) real and goes to $+\infty$ (resp., to $-\infty$) on the real axis, is a special case of $t = he^{i\theta}$ where $\theta = 0$ (resp., $\theta = \pi$). Let θ be a fixed number in $[0, 2\pi)$. Then the map $h \in \mathbb{R}^+ \mapsto \lambda_j(t)$, $j = 1, 2, \ldots, n$, defines a set of n *spectral rays.* The spectral rays either diverge to infinity or converge to a limit point of $\sigma(P(\lambda,t))$ (or possibly, an eigenvalue in $\sigma(A,B)$). Any spectral ray provides a trajectory in the complex plane starting from any element of $\sigma(A,B)$. The value of the parameter θ in $[0, 2\pi)$ does not affect the limit set of $\sigma(P(\lambda,t))$ as $h \to \infty$. In fact, when $h \to \infty$, the parameter θ adjusts the one-to-one correspondence

$$\sigma(A,B) \mapsto \sigma(\Delta A, \Delta B). \tag{13}$$

Examples 4.2 and 4.3 provide some sets of spectral rays and illustrate the role of θ mentioned in (13). In the figures of both examples, the blue stars show the elements of $\sigma(A,B)$ and the red circles indicate the elements of $\sigma(\Delta A, \Delta B)$.

Example 4.2. In this example, we consider

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 40 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 50 & 0 \\ 0 & 8 & 19 \end{bmatrix}.$$

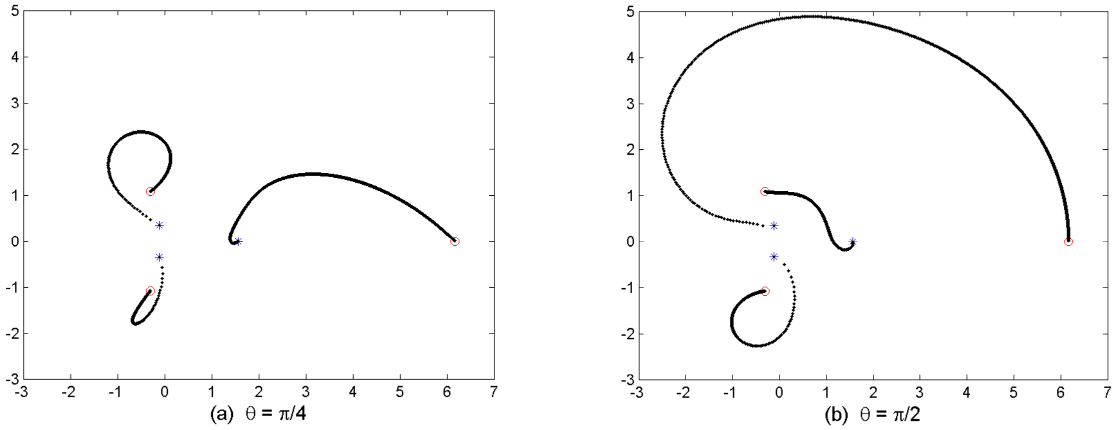

Figure 2: *Two different sets of 3 rays for* $t \mapsto \sigma(P(\lambda, t))$, *where* $0 \le h \le 10^4$.

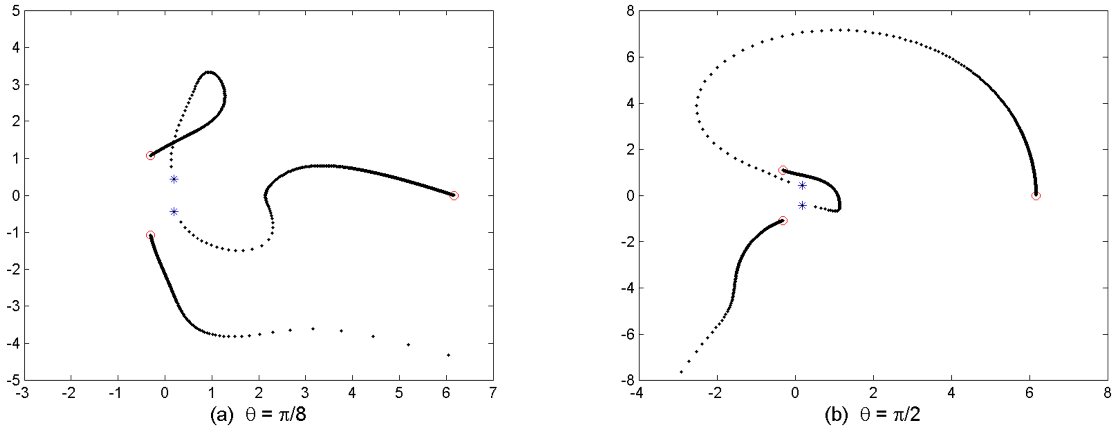

Figure 3: *Two different sets of 3 rays for* $t \mapsto \sigma(P(\lambda, t))$, *where* $0 \le h \le 10^4$.

We also use

$$\Delta A = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 5 & 6 \\ 17 & 80 & 9 \end{bmatrix} \quad \text{and} \quad \Delta B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 8 & 9 \end{bmatrix}.$$

Then $\sigma(A, B) = \{1.56, -0.11 + 0.34\,\mathrm{i}, -0.11 - 0.34\,\mathrm{i}\}$ and $\sigma(\Delta A, \Delta B) = \{6.17, -0.30 + 1.08\,\mathrm{i}, -0.30 - 1.08\,\mathrm{i}\}$. Figure 2 (a) (resp., (b)) shows a set of spectral rays corresponding to $\theta = \pi/4$ (resp., $\theta = \pi/2$). As we can see, for $\theta = \pi/4$, the ray originated from 1.56 (resp., $-0.11 + 0.34\,\mathrm{i}$) converges to 6.17 (resp., $-0.30 + 1.08\,\mathrm{i}$). On the other hand, for $\theta = \pi/2$, the ray originated from 1.56 (resp., $-0.11 + 0.34\,\mathrm{i}$) converges to $-0.30 + 1.08\,\mathrm{i}$ (resp., 6.17).

Example 4.3. For the matrices A, ΔA, and ΔB in Example 4.2, let $B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 3 \\ 0 & 8 & 19 \end{bmatrix}$ be a rank 2 matrix. Then $\sigma(A, B) = \{+\infty, -0.19 + 0.43\,\mathrm{i}, -0.19 - 0.43\,\mathrm{i}\}$ and $\sigma(\Delta A, \Delta B) = \{6.17, -0.30 + 1.08\,\mathrm{i}, -0.30 - 1.08\,\mathrm{i}\}$. Figure 3 (a) (resp., (b)) shows a set of spectral rays corresponding to $\theta = \pi/8$ (resp., $\theta = \pi/2$). For $\theta = \pi/8$, the ray originated from $-0.19 + 0.43\,\mathrm{i}$ (resp., $-0.19 - 0.43\,\mathrm{i}$) converges to $-0.30 + 1.08\,\mathrm{i}$ (resp., 6.17). On the other hand, for $\theta = \pi/2$, the ray originated from $-0.19 + 0.43\,\mathrm{i}$ (resp., $-0.19 - 0.43\,\mathrm{i}$) converges to 6.17 (resp., $-0.30 + 1.08\,\mathrm{i}$). In both Figure 3 (a) and Figure 3 (b), $-0.30 - 1.08\,\mathrm{i}$ is a limit point of some spectral ray that comes from infinity, i.e., from $+\infty \in \sigma(A, B)$.

4.2 Close relation between frontier points and limit points

Now we can clarify the close relation between frontier points and limit points of $\sigma(P(\lambda,t))$ as $|t|\to\infty$. To this end, we provide a method that uses the singularities of F_λ. Proposition 2.4 states that a scalar $\lambda\in\mathbb{C}\backslash\sigma(A,B)$ is an eigenvalue of $P(\lambda,t)$ if and only if there exists at least one eigenvalue $\mu(\lambda)\in\sigma(F_\lambda)\backslash\{0\}$ such that $t\mu(\lambda)=1$, with F_λ as in (5). This fact can be used to state that any singularity of F_λ corresponds to some infinite t. Singularity of F_λ means any complex (or real) λ such that

$$\det F_\lambda=\mu_1(\lambda)\mu_2(\lambda)\cdots\mu_n(\lambda)=0,$$

that is, any complex (or real) λ such that $\mu_i(\lambda)=0$ for at least one i in $\{1,2,\dots,n\}$. This means that any singularity of F_λ in $\mathbb{C}\backslash\sigma(A,B)$ is a limit point, and as a consequence, $F(P,\Delta P)=\text{Lim}$.

4.3 Limit points as subset of finite eigenvalues of a block matrix pencil

Definition 3.1 says that when $0\in\sigma(F_\lambda)$ (or $\text{rank}(\Delta P(\lambda))<n$), then λ is a frontier point. Let us denote

$$\hat{A}_1(\lambda)=\left[\begin{array}{c|c} P(\lambda) & I \\ \hline \Delta P(\lambda) & 0\end{array}\right]=\begin{bmatrix} A & I \\ \Delta A & 0\end{bmatrix}-\lambda\begin{bmatrix} B & 0 \\ \Delta B & 0\end{bmatrix}.$$

For $\lambda\notin\sigma(A,B)$, the Schur complement [16, p. 119] of $A-\lambda B$ in matrix $\hat{A}_1(\lambda)$ is

$$0-\Delta P(\lambda)P(\lambda)^{-1}=0-(\Delta A-\lambda\Delta B)(A-\lambda B)^{-1}=F_\lambda,$$

i.e., it coincides with F_λ in (5). Using this, we have

$$\det\hat{A}_1(\lambda)=\det(A-\lambda B)\det F_\lambda. \tag{14}$$

The relation (14) reconfirms that the finite numbers in $\sigma(\hat{A}_1(\lambda))\backslash\sigma(A,B)$ are singularities of F_λ (i.e., the limit points of $\sigma(P(\lambda,t))$).

5 Structure analysis of regular matrix pencils under homotopic deviation

In this section, we analyze the structure of the deviated matrix pencil $P(\lambda,t)$ under the following two different assumptions:

- λ is a parameter and varies in $\mathbb{C}\backslash\sigma(A,B)$. In this case, we write $P(\lambda,t)$ as in (3).
- t is a parameter which varies in $\mathbb{C}\backslash\sigma(A,-\Delta A)$ (or in $\hat{\mathbb{C}}\backslash\sigma(A,-\Delta A)$). In such cases, we consider the form (1) of $P(\lambda,t)$.

Before analyzing the structure of the deviated matrix pencil $P(\lambda,t)$, we need to review some necessary definitions and notions. Our analysis follows from Gantmacher [15] and is a development of the discussion in [1, 2].

In 1867, Weierstrass established a criterion for strict equivalence of regular pencils of matrices and also a canonical form for such pencils. In 1890, Kronecker solved the same problems for singular matrix pencils [15]. For a review on finite and infinite elementary divisors, we refer to [15, pp. 24–28].

We need the following definition.

Definition 5.1. Two $m\times n$ matrix pencils $S+\alpha T$ and $S_1+\alpha T_1$ satisfying $P(S+\alpha T)Q=S_1+\alpha T_1$, where P and Q are constant square nonsingular matrices (i.e., matrices independent of α) of orders m and n, respectively, are called *strictly equivalent*.

The next theorem is essential for the remainder of this section.

Theorem 5.1. ([15]) *Every regular pencil $S + \alpha T$ can be reduced to a (strictly equivalent) canonical quasi-diagonal form*

$$\mathrm{diag}\{N^{(u_1)}, N^{(u_2)}, \ldots, N^{(u_s)},\ J + \alpha I\} \tag{15}$$

for $N^{(u)} = I_u + \alpha H^{(u)}$, where $H^{(u)}$ is a matrix of order u whose elements in the first superdiagonal are 1, while the remaining elements are 0. In (15), the first s diagonal blocks correspond to infinite elementary divisors $\lambda^{u_1}, \lambda^{u_2}, \ldots, \lambda^{u_s}$ of the pencil $S + \alpha T$ and the normal form of the last diagonal block $J + \alpha I$ is uniquely determined by the finite elementary divisors of the given pencil.

5.1 When λ is a parameter

We consider the pencil $P(\lambda, t) = P(\lambda) + t\Delta P(\lambda)$, where $P(\lambda)$ and $\Delta P(\lambda)$ are defined by (3) for $A, B, \Delta A, \Delta B \in \mathbb{C}^{n\times n}$, and $\lambda \in \mathbb{C}\backslash\sigma(A, B)$. We also assume that both ΔA and ΔB are nonzero matrices. This pencil is a regular pencil, because for the considered set of λ, i.e., $\lambda \in \mathbb{C}\backslash\sigma(A, B)$, we have $\det P(\lambda, t) \neq 0$ when $t = 0$. Now the question is: *How does the structure of the pencil $P(\lambda, t)$ changes as λ varies in $\mathbb{C}\backslash\sigma(A, B)$?* We shall answer this question using the notion of frontier points.

In some study about the effect of extended linear perturbations on the structured matrix pencils, stemmed from problems in control theory [7], one assumes that λ lies in some specific subset of $\mathbb{C}$, and looks for some real nonzero t which implies $\det P(\lambda, t) = 0$. This is another example where λ is considered as a parameter.

We know that at any frontier point $\lambda \in F(P, \Delta P)$, $r_\lambda = \mathrm{rank}(\Delta P(\lambda)) < n$, and $\mathrm{rank}(F_\lambda) = r_\lambda < n$, for $F_\lambda = -\Delta P(\lambda)P(\lambda)^{-1}$. Now we distinguish between Weierstrass structure of the matrix pencil $P(\lambda, t)$ at frontier points and at non-frontier points.

Theorem 5.2. *Let a_λ, with $0 \leq a_\lambda = n - r_\lambda$, be the algebraic multiplicity of $0 \in \sigma(F_\lambda)$ for $\lambda \in \mathbb{C}\backslash\sigma(A, B)$.*

(*a*) *For $\lambda \in (\mathbb{C}\backslash\sigma(A, B))\backslash F(P, \Delta P)$, the pencil $P(\lambda, t)$ is strictly equivalent to the matrix*

$$\begin{bmatrix} \frac{1}{\mu_1} & \zeta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \zeta_{n-1} \\ & & & \frac{1}{\mu_n} \end{bmatrix},$$

where ζ_j for $j = 1, 2, \ldots, n-1$ represents 0 or 1, and $\mu_i \in \sigma(F_\lambda)$, $i = 1, 2, \ldots, n$, are nonzero.

(*b*) *For $\lambda \in F(P, \Delta P)$, the pencil $P(\lambda, t)$ is strictly equivalent to the 2×2 block matrix*

$$\left[\begin{array}{cccc|cccc} 1 & t\eta_1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & t\eta_{a_\lambda-1} & & & & \\ & & & 1 & & & & \\ \hline & & & & \frac{1}{\mu_1} & \zeta_1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & \zeta_{r_\lambda-1} \\ & & & & & & & \frac{1}{\mu_{r_\lambda}} \end{array}\right] \tag{16}$$

corresponding to the partition $n = a_\lambda + (n - a_\lambda) = a_\lambda + r_\lambda$, where η_i (for $i = 1, 2, \ldots, a_\lambda - 1$) and ζ_j (for $j = 1, 2, \ldots, r_\lambda - 1$) represent 0 or 1, $t \in \mathbb{C}$, and $\mu_i \in \sigma(F_\lambda)$ are nonzero for $i = 1, 2, \ldots, r_\lambda$.

Proof. For the case (a), we know from Proposition 2.4, that F_λ is nonsingular for any $\lambda \in (\mathbb{C}\backslash\sigma(A, B))\backslash F(P, \Delta P)$. Beside for such λ, we have $\det P(\lambda, t) = 0$ if and only if λ is an eigenvalue of $P(\lambda, t)$, or equivalently, if and only if $t = 1/\mu_i$ for $\mu_i \in \sigma(F_\lambda)$ and $i = 1, 2, \ldots, n$. Therefore, the matrix pencil $P(\lambda, t)$ is strictly equivalent to the normal form of the last diagonal block of (15) which is determined by substituting $t = 1/\mu_i$ for $i = 1, 2, \ldots, n$. This means that the n eigenvalues of the pencil $P(\lambda, t)$ are finite.

For the case (b), i.e., when $\lambda \in F(P, \Delta P)$, we have $r_\lambda = \operatorname{rank}(\Delta P(\lambda)) < n$ and $P(\lambda, t)$ has $a_\lambda = n - r_\lambda$ infinite eigenvalues. Thus, we can apply Theorem 5.1 to verify that the structure of $P(\lambda, t)$ is strictly equivalent to that in (16) with the partition $n = a_\lambda + (n - a_\lambda) = a_\lambda + r_\lambda$. □

Theorem 5.2 shows how the structure of the pencil $P(\lambda, t)$ depends on λ in $F(P, \Delta P)$ and the value of a_λ. When λ is a critical point, i.e., when $\rho(M_\lambda) = \rho(F_\lambda) = 0$, then $a_\lambda = n$. Hence, the pencil $P(\lambda, t)$ has no finite eigenvalue.

5.2 When t is a parameter

Here, we assume that the independent parameter is t, and we study the changes that may occur in the structure of the regular matrix pencil $P(\lambda, t)$ when t varies in $\mathbb{C}\backslash\sigma(A, -\Delta A)$. This is a natural extension of linear perturbation for regular matrix pencils, where we are interested in the probable singularities far from $|t| = 0$. We suppose that $P(\lambda, t) = (A + t\Delta A) - \lambda(B + t\Delta B)$ in (1) is obtained from $P(\lambda) = A - \lambda B$ when A and B are affected by $t\Delta A$ and $t\Delta B$, respectively. This pencil, for $t \in \mathbb{C}\backslash\sigma(A, -\Delta A)$ and $\lambda = 0$, satisfies $\det P(\lambda, t) \neq 0$. Therefore, it is a regular matrix pencil. As a complement of our study, we also investigate the possible changes in the structure of the regular matrix pencil $P(\lambda, t)$ when t varies in $\hat{\mathbb{C}}\backslash\sigma(A, -\Delta A)$.

To proceed, we first develop some notions analogous to those in Section 3, but for a different independent parameter, namely, t instead of λ. We denote

$$F_t = (B + t\Delta B)(A + t\Delta A)^{-1} \tag{17}$$

for $t \in \mathbb{C}\backslash\sigma(A, -\Delta A)$.

For the primal form of $P(\lambda, t) = (A+t\Delta A)-\lambda(B+t\Delta B)$ in (1), Proposition 2.4 has the following counterpart.

Proposition 5.3. *A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of $P(\lambda, t)$ if and only if there exists at least one eigenvalue $\nu_t \in \sigma(F_t)\backslash\{0\}$ such that $\lambda\nu_t = 1$.*

Proof. For $t \in \mathbb{C}\backslash\sigma(A, -\Delta A)$, we use (1) to write

$$P(\lambda, t) = \left[I - \lambda(B + t\Delta B)(A + t\Delta A)^{-1}\right](A + t\Delta A) = (I - \lambda F_t)(A + t\Delta A).$$

Hence, $\lambda \in \sigma(P(\lambda, t))$ if and only if $\lambda = 1/\nu_t$ for some $\nu_t \in \sigma(F_t)\backslash\{0\}$. □

We use the factorization (7) to conclude that for any fixed t, $B + t\Delta B$ can be written under the form

$$B + t\Delta B = U_t V_t^H, \tag{18}$$

where the matrices $U_t, V_t \in \mathbb{C}^{n\times r_t}$ are of rank r_t. Based on our notation, we can provide the following counterpart of Proposition 3.1.

Proposition 5.4. *Let $t \in \mathbb{C}\backslash\sigma(A, -\Delta A)$ and $1 \leq \operatorname{rank}(B + t\Delta B) = r_t \leq n$. Then the nonzero eigenvalues of the $n \times n$ matrix F_t defined in (17) are the nonzero eigenvalues of the $r_t \times r_t$ matrix*

$$N_t = -V_t^H (A + t\Delta A)^{-1} U_t \in \mathbb{C}^{r_t\times r_t} \tag{19}$$

for U_t and V_t in (18).

The idea of the proof of Proposition 5.4, with different characters, is similar to the one in proof of Proposition 3.1, so we do not include it here.

Whenever we need some counterparts for Proposition 3.3 and Proposition 3.4, based on the notation of this section, we can use the matrix N_t defined in (19).

Definition 5.2. We call $t \in \mathbb{C}\backslash\sigma(A, -\Delta A)$ a t-frontier point if $t \in \sigma(B, -\Delta B)$. We denote the set of all t-frontier points by

$$TF(P, \Delta P) = \{t \in \mathbb{C}\backslash\sigma(A, -\Delta A) : \ r_t = \operatorname{rank}(B + t\Delta B) < n\} = \{t \in \mathbb{C} : t \in \sigma(B, -\Delta B)\backslash\sigma(A, -\Delta A)\}.$$

Any number in $TF(P,\Delta P)$ is called t-frontier point because for $t \notin TF(P,\Delta P)$ the matrix pencil $P(\lambda,t)$ has exactly n finite eigenvalues. When $t \in TF(P,\Delta P)$, there are r_t ($r_t < n$) finite eigenvalues and $n-r_t = a_t$ infinite eigenvalues for the pencil $P(\lambda,t)$. The exact number r_t depends on the location of t in $TF(P,\Delta P) \subseteq \sigma(B,-\Delta B)$.

Definition 5.3. Any $t \in TF(P,\Delta P)$ with $\rho(F_t) = \rho(N_t) = 0$ is called t-critical point. The set of t-critical points is denoted by $TC(P,\Delta P)$.

The next theorem concerns the possible changes in the structure of $P(\lambda,t)$ when t varies in $\mathbb{C}\backslash\sigma(A,-\Delta A)$. We shall show that t-frontier points have an essential role in this regard.

Theorem 5.5. *Let a_t, with $0 \le a_t = n - r_t$, be the algebraic multiplicity of $0 \in \sigma(F_t)$ for F_t defined in (17) and $t \in \mathbb{C}\backslash\sigma(A,-\Delta A)$.*

(*a*) *For $t \in (\mathbb{C}\backslash\sigma(A,-\Delta A))\backslash TF(P,\Delta P)$, the pencil $P(\lambda,t)$ is strictly equivalent to the matrix*

$$\begin{bmatrix} \frac{1}{\nu_1} & \zeta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \zeta_{n-1} \\ & & & \frac{1}{\nu_n} \end{bmatrix},$$

where ζ_j (for $j = 1,2,\dots,n-1$) represents 0 or 1 and $\nu_i \in \sigma(F_t)$, $i = 1,2,\dots,n$, are nonzero.

(*b*) *For $t \in TF(P,\Delta P)$, the pencil $P(\lambda,t)$ is strictly equivalent to the 2×2 block matrix*

$$\left[\begin{array}{cccc|cccc} 1 & \lambda\eta_1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & \lambda\eta_{a_t-1} & & & & \\ & & & 1 & & & & \\ \hline & & & & \frac{1}{\nu_1} & \zeta_1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & \zeta_{r_t-1} \\ & & & & & & & \frac{1}{\nu_{r_t}} \end{array}\right] \tag{20}$$

with the partition $n = a_t + (n-a_t) = a_t + r_t$, where η_i (for $i = 1,2,\dots,a_t-1$) and ζ_j (for $j = 1,2,\dots,r_t-1$) represent 0 or 1, $\lambda \in \mathbb{C}$, and $\nu_i \in \sigma(F_t)$ for $i = 1,2,\dots,r_t$ are nonzero.

Proof. For the case (a), we know from Proposition 5.3 that F_t is nonsingular for any $t \in (\mathbb{C}\backslash\sigma(A,-\Delta A))\backslash TF(P,\Delta P)$. For such a t, we have $\det P(\lambda,t) = 0$ if and only if t is an eigenvalue of $P(\lambda,t)$, or equivalently, if and only if $\lambda = 1/\nu_i$ for $0 \ne \nu_i \in \sigma(F_t)$ and $i = 1,2,\dots,n$. Therefore, the matrix pencil $P(\lambda,t)$ is strictly equivalent to a modification of the last diagonal block in the normal form (15) (i.e., $(J-\lambda I)$), where $\lambda = 1/\nu_i$ for $i = 1,2,\dots,n$. This means that the n eigenvalues of the pencil $P(\lambda,t)$ are finite.

For the case (b), i.e., when $t \in TF(P,\Delta P)$, we have $r_t = \operatorname{rank}(B + t\Delta B) < n$ and $P(\lambda,t)$ has $a_t = n - r_t$ infinite eigenvalues. Thus, we can use Theorem 5.1 to state that the structure of $P(\lambda,t)$ is strictly equivalent to that in (20) with the partition $n = a_t + (n - a_t)$. □

The following example verifies the structure changing discussed in Theorem 5.5 when t is an independent parameter.

Example 5.1. For the problem in Example 2.1, we have $\sigma(B,-\Delta B) = \{-141, 141, -\infty\}$ and $\sigma(A,-\Delta A) = \{-10^{10}, -10^{8}, -10^{6}\}$. Therefore, $\sigma(B,-\Delta B) \cap \sigma(A,-\Delta A) = \emptyset$, and both finite eigenvalues of $B + t\Delta B$ are t-frontier points and $TF(P,\Delta P)$ is a discrete set with finite cardinality. Since the two t-frontier points are real, in Figure 4, we plot $\max_{1\le i\le 3}(1/|\nu_i|)$, as t is maintained real and varies in $[-225\ 225]$, to indicate how the corresponding values $\lambda = (1/\nu_i)$ for the pencil $P(\lambda,t)$ escapes to infinity at $t = t_k \in TF(P,\Delta P)$, $k = 1,2$.

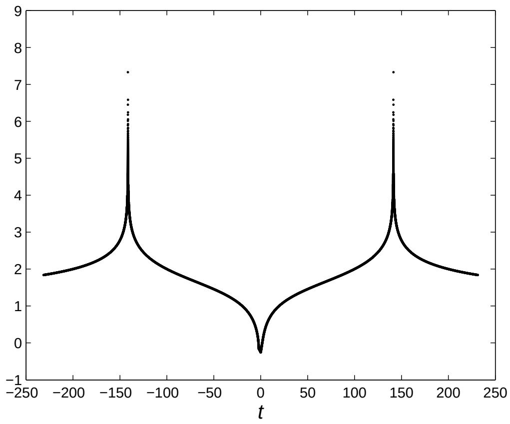

Figure 4: $Log_{10}\left(\max_{1\leq i\leq 3}(1/|\nu_i|)\right)$ *for* $\nu_i \in \sigma(F_t)$, $i = 1, 2, 3$.

Figure 4 illustrates a scaled plot of $\max_{1\leq i\leq 3}(1/|\nu_i|)$ for $\nu_i \in \sigma(F_{t_k})$, $i = 1, 2, 3$. We can see the abrupt changes happened at t-frontier points.

The algebraic multiplicity of $0 \in \sigma(F_{t_k})$, for each t-frontier point t_k, is $a_{t_k} = 1$, $k = 1, 2$. Therefore, according to Theorem 5.5, the structure of $P(\lambda, t_k)$ for $t_k \in TF(P, \Delta P)$ is determined by the partition $n = 3 = a_{t_k} + (n - a_{t_k}) = 1 + 2$. This means that the pencil $P(\lambda, t_k)$, for $t_k \in TF(P, \Delta P)$, $k = 1, 2$, is strictly equivalent to the 2×2 block matrix $V^{(k)} = \left[\begin{array}{c|cc} 1 & & \\ \hline & \frac{1}{\nu_1^{(k)}} & \zeta_1 \\ & & \frac{1}{\nu_2^{(k)}} \end{array}\right]$. corresponding to the partition $3 = 1 + 2$, where ζ_1 represents 0 or 1 and $\nu_i^{(k)} \in \sigma(F_{t_k})$ for $i = 1, 2$. Table 2 illustrates the computed nonzero values $\nu_i^{(k)}$, $i = 1, 2$, in matrix $V^{(k)}$ above and the very small values of $\nu_3^{(k)}$ corresponding to infinite eigenvalue, λ, for each $t_k \in TF(P, \Delta P)$, $k = 1, 2$.

Table 2: The computed values of $\nu_i^{(k)}$, $i = 1, 2, 3$, for each $t_k \in TF(P, \Delta P)$, $k = 1, 2$.

$t_k \in TF(P, \Delta P)$	$\nu_1^{(k)}$	$\nu_2^{(k)}$	$\nu_3^{(k)}$
$t_1 = -141$	-20.6	25.8	10^{-17}
$t_2 = 141$	$1.41 \times 10^{-2} + 20.6\,\mathrm{i}$	$1.41 \times 10^{-2} - 20.6\,\mathrm{i}$	-4.04×10^{-17}

For $t \notin TF(P, \Delta P)$, the pencil $P(\lambda, t)$ is strictly equivalent to the 3×3 matrix $W = \begin{bmatrix} \frac{1}{\nu_1} & \zeta_1 & 0 \\ & \frac{1}{\nu_2} & \zeta_2 \\ & & \frac{1}{\nu_3} \end{bmatrix}$, where ζ_j (for $j = 1, 2$) represents 0 or 1 and $\nu_i \in \sigma(F_t)$, $i = 1, 2, 3$.

Example 5.1 suggests some cases beyond those considered in Theorem 5.5 because $\sigma(B, -\Delta B)$ includes $-\infty$, i.e., there is a $t \in \sigma(B, -\Delta B)$ with infinite $|t|$. These cases are discussed in Lemma 2.1 and Proposition 2.2. The following proposition extends the assumptions of Theorem 5.5 to the cases where t is not limited to finite numbers and t does not necessarily belong to $TF(P, \Delta P)$.

Proposition 5.6. *Let* a_t, *with* $0 \leq a_t = n - r_t$, *be the algebraic multiplicity of* $0 \in \sigma(F_t)$ *for* F_t *defined in (17) and* $t \in \hat{\mathbb{C}} \backslash \sigma(A, -\Delta A)$.

(a) *For any* $t \in (\hat{\mathbb{C}}\backslash\sigma(A,-\Delta A))\backslash\sigma(B,-\Delta B)$, *the pencil* $P(\lambda,t)$ *is strictly equivalent to the matrix*

$$\begin{bmatrix} \frac{1}{\nu_1} & \zeta_1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \zeta_{n-1} & \\ & & & \frac{1}{\nu_n} & \end{bmatrix},$$

where ζ_j *(for* $j = 1,2,\ldots,n-1$*) represents* 0 *or* 1*, and* $\nu_i \in \sigma(F_t)$*,* $i = 1,2,\ldots,n$*, are nonzero.*

(b) *For any* $t \in \sigma(B,-\Delta B)\backslash\sigma(A,-\Delta A)$, *the pencil* $P(\lambda,t)$ *is strictly equivalent to the* 2×2 *block matrix*

$$\left[\begin{array}{cccc|cccc} 1 & \lambda\eta_1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & \lambda\eta_{a_t-1} & & & & \\ & & & 1 & & & & \\ \hline & & & & \frac{1}{\nu_1} & \zeta_1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \ddots & \zeta_{r_t-1} \\ & & & & & & & \frac{1}{\nu_{r_t}} \end{array}\right]$$

with the partition $n = a_t+(n-a_t) = a_t+r_t$*, where* η_i *for* $i = 1,2,\ldots,a_t-1$ *and* ζ_j *(for* $j = 1,2,\ldots,r_t-1$*) represent* 0 *or* 1*,* $\lambda \in \mathbb{C}$*, and* $\nu_i \in \sigma(F_t)$ *for* $i = 1,2,\ldots,r_t$*, are nonzero.*

Proposition 5.6 concerns the structure of deviated matrix pencils at any $t \in \hat{\mathbb{C}} = \mathbb{C}\cup\{-\infty,\infty\}$. Therefore, a new notion and result provided by Proposition 5.6 (b) is the structure of $P(\lambda,t)$ at infinity, i.e., when $|t| \to \infty$. In Example 2.1, we have seen that $\sigma(A+t\Delta A, B+t\Delta B)$, for t with $|t|$ large enough, includes one infinite eigenvalue. This means that 0 is a simple eigenvalue of the matrix $B+t\Delta B$ and the structure of $P(\lambda,t)$ is determined by the partition $n = 3 = a_t + (n-a_t) = 1+2$ when $t \to -\infty$.

The following example investigates a problem which concerns two different types of t in $TF(P,\Delta P)$; in particular, for some t in $TF(P,\Delta P)$, 0 is a simple eigenvalue of $(B,-\Delta B)$, but for others t in $TF(P,\Delta P)$, 0 is a semi-simple or defective eigenvalue of $(B,-\Delta B)$.

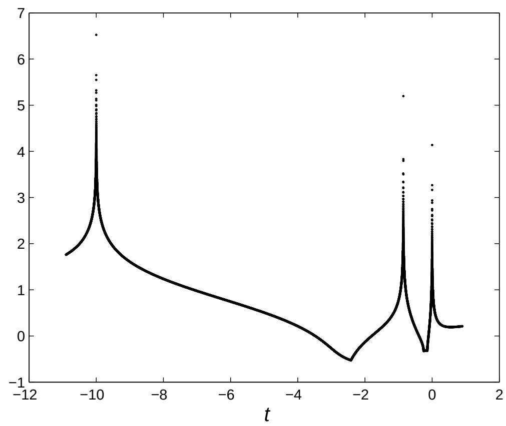

Figure 5: $Log_{10}\left(\max_{1\le i\le 4}(1/|\nu_i|)\right)$ *for* $\nu_i \in \sigma(F_t)$*,* $i = 1,2,3,4$.

Example 5.2. In this example, we consider

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 3 & 2 & 6 & 9 \\ -9 & 2 & 0 & 5 \\ 0 & 0 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 6 & -6 & 6 \\ 6 & 8 & -6 & 4 \\ -6 & -6 & 6 & -6 \\ 6 & 4 & -6 & 8 \end{bmatrix}.$$

We also use

$$\Delta A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \Delta B = \begin{bmatrix} 156 & 44 & 54 & -26 \\ 44 & 16 & 26 & -12 \\ 54 & 26 & 156 & -44 \\ -26 & -12 & -44 & 16 \end{bmatrix}.$$

Here, we have $\sigma(B, -\Delta B) = \{-10, -0.857, 2\times 10^{-17}, 7.17\times 10^{-18}\}$ and $\sigma(A, -\Delta A) = \{-25.8, -1.97, 0.849, 4.98\}$. Therefore, $\sigma(B, -\Delta B) \cap \sigma(A, -\Delta A) = \emptyset$ and all the eigenvalues of $B + t\Delta B$ are t-frontier points, and $TF(P, \Delta P) = \{-10, -0.857, 2\times 10^{-17}, 7.17\times 10^{-18}\} = \{t_1, t_2, t_3, t_4\}$ is a discrete set with finite cardinality. A scaled plot of $\max\limits_{1\le i\le 4}(1/|\nu_i|)$ for $\nu_i \in \sigma(F_t)$, $i = 1, 2, 3, 4$, is illustrated in Figure 5. We can see the abrupt changes happened at the t-frontier points.

Since the numerical rank of F_t equals 3 for each $t = t_k \in TF(P, \Delta P)$, $k = 1, 2$, the pencil $P(\lambda, t_k)$, for each $k = 1, 2$, is strictly equivalent to the 2×2 block matrix $V^{(k)} = \left[\begin{array}{c|ccc} 1 & & & \\ \hline & \frac{1}{\nu_1^{(k)}} & \zeta_1 & \\ & & \frac{1}{\nu_2^{(k)}} & \zeta_2 \\ & & & \frac{1}{\nu_3^{(k)}} \end{array}\right]$ corresponding to the partition $4 = 1 + 3$, where ζ_j (for $j = 1, 2$) represents 0 or 1 and $\nu_i^{(k)} \in \sigma(F_{t_k})$, for $i = 1, 2, 3$, are nonzero.

The numerical rank of F_t is 2 for each $t = t_k \in TF(P, \Delta P)$, when $k = 3, 4$. This means that, for each $k = 3, 4$, the pencil $P(\lambda, t_k)$ is strictly equivalent to the 2×2 block matrix $V^{(k)} = \left[\begin{array}{cc|cc} 1 & \lambda\eta_1 & & \\ & 1 & & \\ \hline & & \frac{1}{\nu_1^{(k)}} & \zeta_1 \\ & & & \frac{1}{\nu_2^{(k)}} \end{array}\right]$ corresponding to the partition $4 = 2 + 2$, where η_1 and ζ_1 represent 0 or 1 and $\nu_i^{(k)} \in \sigma(F_{t_k})$, for $i = 1, 2$, are nonzero. Table 3 illustrates the computed values of $\nu_i^{(k)} \in \sigma(F_t)$, $i = 1, 2, 3, 4$, for each $t_k \in TF(P, \Delta P)$, $k = 1, 2, 3, 4$.

Table 3: The computed values of ν_i, $i = 1, 2, 3, 4$, for each $t_k \in TF(P, \Delta P)$.

$t_k \in TF(P, \Delta P)$	$\nu_1^{(k)}$	$\nu_2^{(k)}$	$\nu_3^{(k)}$	$\nu_4^{(k)}$
$t_1 = -10$	$7.97\times 10^4 - 144\,\mathrm{i}$	$7.97\times 10^4 + 144\,\mathrm{i}$	-4.07	-4.98×10^{-15}
$t_2 = -0.857$	$3.75 - 11.5\,\mathrm{i}$	$3.75 + 11.5\,\mathrm{i}$	-5.72	1.55×10^{-15}
$t_3 = 7.17\times 10^{-18}$	-8.42	-2.65 $\times 10^{-2}$	1.90×10^{-13}	2.57×10^{-16}
$t_4 = 2\times 10^{-17}$	-8.42	-2.65×10^{-2}	3.29×10^{-13}	-4.90×10^{-16}

It is worth mentioning that the two very small t-frontier points t_3 and t_4 show how sensitive could be the structure of a matrix pencil $P(\lambda, t)$ to the value of t. For $t \notin TF(P, \Delta P)$, the pencil $P(\lambda, t)$ is strictly equivalent to the 4×4 matrix $W = \begin{bmatrix} \frac{1}{\nu_1} & \zeta_1 & 0 & 0 \\ & \frac{1}{\nu_2} & \zeta_2 & 0 \\ & & \frac{1}{\nu_3} & \zeta_3 \\ & & & \frac{1}{\nu_4} \end{bmatrix}$, where ζ_j for $j = 1, 2, 3$ represents 0 or 1 and $\nu_i \in \sigma(F_t)$, $i = 1, 2, 3, 4$.

5.3 Interconnection between λ and t via F_λ and F_t

Based on Proposition 2.4, $\lambda \in \mathbb{C}\backslash\sigma(A, B)$ is an eigenvalue of $P(\lambda, t)$ if and only if there exists at least one eigenvalue $\mu(\lambda) \in \sigma(F_\lambda)\backslash\{0\}$ such that $t = 1/\mu(\lambda)$. This means that

$$\det\left(P\left(\lambda, \frac{1}{\mu(\lambda)}\right)\right) = 0 \quad \text{for any } \mu(\lambda) \in \sigma(F_\lambda)\backslash\{0\}. \tag{21}$$

Proposition 5.3 states that a scalar $\lambda \in \mathbb{C}$ is an eigenvalue of $P(\lambda, t)$ if and only if there exists at least an eigenvalue $\nu_t \in \sigma(F_t)\backslash\{0\}$ such that $\lambda\nu_t = 1$. This means that

$$\det\left(P\left(\frac{1}{\nu_t}, t\right)\right) = 0 \quad \text{for any } \nu_t \in \sigma(F_t)\backslash\{0\}. \tag{22}$$

Suppose we are given the values of λ and $0 \neq \mu(\lambda)$ which satisfy (21). Then we may use them to find the corresponding t and ν_t satisfying (22). The reverse is also true, i.e., having the values of $0 \neq \nu_t$ and t satisfying (22), we can retrieve the values of λ and $\mu(\lambda)$ which satisfy (21).

6 Conclusions

We generalized the theory of homotopic deviation to regular matrix pencils. We discussed the existence and analyticity of the resolvent $R(\lambda, t)$ together with the limit of the resolvent and the limit of the spectrum $\sigma(P(\lambda, t))$, as $|t| \to \infty$. Moreover, we characterized the possibilities where the deviation parameter t (resp., the eigenvalues λ) of the deviated matrix pencil can go to infinity. We also studied the cases where t tends to finite eigenvalues of the matrix pair $(B, -\Delta B)$, or the matrix pair $(A, -\Delta A)$.

The connections between the deviation parameter t and the eigenvalues $\lambda(t)$ of the deviated matrix pencil were studied. An immediate and interesting result is that $t, \lambda(t) \in \hat{\mathbb{C}}$. Definitions of frontier points (with a little difference from that in basic homotopic deviation for complex square matrices), critical points and limit points were introduced. The relation between the frontier points and the limit points was also discussed. Some ways for predicting and computing the limit points were proposed.

Under two different assumptions (i.e., either λ or t are considered as an independent parameter), we analyzed the Weierstrass structure of the deviated matrix pencils. Under the first assumption, we generalized the results in [1, 2]. With the second assumption, we provided some new opportunities to look at the structure change of the deviated matrix pencils when t varies in $\mathbb{C}\backslash\sigma(A, -\Delta A)$. We also studied the structure change of $P(\lambda, t)$ when t varies in $\hat{\mathbb{C}}\backslash\sigma(A, -\Delta A)$ which may include the cases when $|t|$ goes to infinity. Interconnection between λ and t via F_λ and F_t was explained.

Numerical examples validated our theoretical analysis and illustrated the properties proved by the theories. The possible generalizations of this theory to regular matrix polynomials will be a future task.

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