The Formal Indeterminacy of Natural Numbers and Artificial Intelligence

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Introductory Note:

This article is directed to those who are interested in the possibilities and weaknesses of the efforts for a symbolic-mechanistic reproduction of thinking, but it does not require from the reader a deeper occupation with Logic or Mathematics. More specifically, we consider here where we are led by a reduction of Arithmetic into a formalism based on Symbolic Logic. In some sense, this is a revision of the corresponding chapter (Appendix 4.7) in my book THE MECHANISM OF THINKING.: http://www.math.ntua.gr/~jqstel/book1.pdf

For the understanding of the main text the reader will only need to know what are algebraic polynomials. Beyond that only careful reading is needed. The text is simplified on purpose by omitting many symbolisms and replacement of them by verbal descriptions. However, for the understanding of Appendix 2 about the weaknesses of Second Order Logic we need some results from Set Theory which are presented in Appendix 1.

NOTE: Since mathematic notation is not always available we will use here where it is necessary appropriate substitutes of these symbols.

KEYWORDS: Natural Numbers, Models of Natural Numbers, Ambiguousness (non-uniqueness) of Natural Numbers, the concept "infinity", Artificial Intelligence, Gödel, Skolem, intrinsic, latent, unconventional, non-declarative properties of objects and actions.

The non-uniqueness of the concept "infinite"

At first sight, a claim that we don't actually know what Natural Numbers (in short NN) are seems strange, since every child learns to count on his/her fingers 1, 2, 3, 4, 5, … already from a very early age. However, this indeterminacy refers to the formal description of the system of NN and not to their empirical perception. I.e., it is a consequence of the possibilities and weaknesses offered by a formal logical founding of them in the form of a system based on elementary initial principles (axioms).

The problem lies partially not at the beginning of the sequence of NN but at its fictitious "end", the infinite. As Nietzsche [1] correctly observes, many times the fact that we give a specific name to a concept deceives us into believing that it describes something uniquely determined.

Thus, as it will be seen, the concept "infinite" is not uniquely determined[1].

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[1] Already Georg Cantor has shown that there are not denumerable sets, i.e., many kinds of infinite (see Appendix 1). However, here we will see an infinite, A, where A+1>A.

Surely, if it is indeed hazy and ambiguous we might wish to avoid it by all means /totally. However, this is not possible because then not arithmetic proposition would hold in general. All mathematical proofs of general properties of NN extend necessarily to the infinite. When we say that a relation is valid for ALL NN we mean that it is valid for every NN from 1 to the infinite, whatever that may be.

Another side of the impossibility of unique determination of NN by means of a formalism is due to the fact that this formalism does NOT preclude the parallel existence of sets of magnitudes, which do not belong to the usual "chain" of positive integers which is produced by the system of axioms by means of the process of succeeding in NN: 1, 2, 3, …

The first theorems which have upset the hope of unique determination of the natural numbers by means of a formalism are theorems of Formal Logic due to Leopold Löwenheim (1915) and Thoralf Skolem (from 1920 on). Later, in 1931 the much wider known theorems of Kurt Gödel have followed.

However, let us see a very simple example due to Skolem [2] (1929) which shows that the formal determination of NN is unattainable.

How we can extend the system of natural numbers

Skolem constructs here a system of some kind of arithmetic magnitudes which are different from the natural numbers but fulfill exactly the same axioms as they do.

Axioms are called unproven fundamental assumptions, which determine the nature of the magnitudes we study. All other propositions which refer to these magnitudes must be logically provable by means of these axioms.

Skolem has used a variation of the usual axioms of Number Theory, which are due to Peano and Dedekind. He considers 1 as the first natural number (while others start with 0). Here we simplify them somewhat by omitting their formal formulation /expression:

(1) The natural numbers are linearly ordered through the relation '<' (the distinction of smaller – larger).

(2) There is a number, 1, who has no previous number.

(3) For every number x there is a successor of it, Sx (Successor of x), which is usually written as x+1 when we define addition.

(4) There is an operation '+', the addition, which is defined by the recursive relation x+Sy=S(x+y), i.e., x+(y+1)=(x+y)+1.

(5) There is an operation '*', the multiplication, which is determined by the recursive relation x*(y+1)=x*y+x.

(6) Principle of Induction: For formal propositions, P(x), which refer to natural numbers x, the following is valid: If P(x) is valid for x = 1 and from its validity for x=k it follows that it is also valid for x=Sk=k+1, then the proposition is valid for all NN x.

Fundamental idea of Skolem was to consider sets of functions of NN, i.e., functions like x^2 (x square, i.e., x*x) which for every NN x provide us with another NN. Such functions fulfill by necessity whatever relations are fulfilled by NN, because ultimately the values such functions have are only NN.

Thus, he defines the symbol '<' saying that for any functions f(x) and g(x) of the natural numbers we will formally write f(x)<g(x) if the value of f(x) is smaller than that of g(x) for an infinite set of natural numbers x *but not necessarily for all of them.*

For instance, we will write, x+100<x^2 because this is true for every NN from 11 on.

Thus we may extend the sequence of NN in the following way:

1<2<3<…<x<x+1<x+2<…<x+100<…<2x<2x+1<…<x^2<x^2+1<…
<x^2+x+1<…<x^3< x^3+1<…< 2x^3+3x^2+4x+5<…<x^4<
x^4+1<…

Where 1, 2, 3, … are now considered simply as functions with constant value.

The symbolism x^n means x*x*x*…*x with x repeated n times. I.e. x is multiplied n-1 times by itself. Thus, x^2=x*x, x^3=x*x*x and so on.

More descriptively we may write:

[1]<[2]<[3]<…<[x]<[x+1]<[x+2]<…<[x+100]<…<[2x]<[2x+1]<…<
[x^2]< [x^2+1]<… <[x^2+x+1]<…<[x^3]< [x^3+1]<…<
[2x^3+3x^2+4x+5]<…<[x^4]< [x^4+1]<…

Where we put the functions in brackets in order to declare that they are ordered through '<' as entities and not pointwise. Of course, this symbolism is interpretative informal, or extra-formal! I.e., it serves only to show the structure of the model we have created! The formalism knows nothing about this interpretation.

Obviously, when we compare functions of NN in this way, the symbol "<" has a meaning different from the established one. *However, this does not change anything in the formalism, because the formalism only uses this symbol and is not interested in interpreting it*! The axioms, which contain this symbol are invariably valid also for the new system of magnitudes.

The above ordered set fulfills all the established axioms for NN but does not have the same structure as they (it is not isomorphic), because here there are terms like x and 2x, between of which there are infinitely many intermediate terms: x+1, x+2, x+3, …, while between two NN there is always a finite number of intermediate integers. The members of this system of numerical magnitudes were later called *superintegers*.

Based on this interpretation we could say that in this arrangement the expressions x, x+1, x+2, … constitute an extension of the set of NN. They are distinct from each other, but they are "hyper-finite numbers", since they lie in this arrangement beyond the finite NN. Thus, they are distinvt from eavh other static kinds of infinite. Thus, the term "infinite" characterizes different kinds of mathematical magnitudes.

Here we must note that these numbers do not result directly from the given axioms, but they are also not opposed to them. Based on the extra-formalist (beyond formalism) interpretation of this symbolism we can also <u>define</u> the sum and the product of such magnitudes in the following way:

[f(x)]+[g(x)]:=[f(x)+g(x)] , [f(x)]*[g(x)]:=[f(x)*g(x)]

As said before, the brackets declare that the functions are meant as entities, while the operations on the right side are the usual algebraic operations.[2]

For instance, in this way it will be: [2x]+[x]=[3x] και [x]*[x]=[x*x]=[x^2]

However, we must note that these rules are our own choices, because there is no way to go [2x] steps further than [x] adding each time 1 and reach in this way [3x]. This is possible only for the classical NN. For instance, in order to add 2 to 3 we write:

$$3+2=3+(1+1)=(3+1)+1=4+1=5$$

In this way we step two places beyond 3 and reach 5.

The axiomatic definition of addition presupposes the continuation of the sequence of NN by adding each time one unit. However, the new magnitudes which we insert in the sequence of NN by re-interpreting the symbol "<" are hyper-finite. We cannot reach them by means of successive additions of 1. However, they do also not oppose the above axioms.

We should note that the interpretation which allows us the above extension is performed extra-formally! The formalism is a blind mechanism which simply follows the rules set by the axioms without interpreting them. I.e., it operates like the interior of a vehicle with closed windows; we may know everything about what happens in the vehicle but, if we do not look out, we do not know where we are and what is outside this vehicle.

Similarly, the formalism allows us to produce as many formal propositions as we like, but it is unable to distinguish from each other the magnitudes it uses. Only when we climb to a higher level and consider the objects of the formalism as sets we are able to establish, like Skolem did, that there are totally different sets which fulfill the same axioms and validate all the propositions produced by the formalism.

It is also obvious that nothing changes even if we add more axioms to the original set of axioms (i.e. fundamental assumptions), because the magnitudes that have been constructed have only NN as values and will verify any proposition which holds for NN. Whatever happens for NN happens also for the Ordered set of natural number Polynomials (OP), because ultimately their values are only natural numbers. However, it is not certain that all properties of Ordered Polynomials are also properties of the natural numbers, because Ordered Polynomials have a different structure. As we have seen, there are properties of OP which are not shared by NN.

What is more, Skolem has shown that we can construct much wider sets of magnitudes which fulfill the same axioms as the NN. According to a theorem of Skolem published in 1935, even a (countably) infinite system of axioms cannot define uniquely the natural numbers.

However, here we must also note a property which distinguishes the traditional set of natural numbers from other sets that fulfill Peano's axioms: According to a theorem due to Stanley Tennenbaum (1959) in nonstandard models of Arithmetic it is not always possible to compute the sums and products of magnitudes belonging to such a set only by means of the given axioms [3].

[2] If we don't use brackets then the expression f(x)+g(x):=f(x)+g(x) does not clarify how these operations are performed.

Thus, the traditional NN are distinguished as the only set which allows the performance of these operations always. In the above definitions of sums and products of polynomials the arithmetic operations are performed only on the coefficients. We cannot add x to x+1 by going x numbers further, as we do for the usual NN.

The establishment of the formal indeterminacy of natural numbers is a problem for many theorists, because it opposes the intention of knowing what we are speaking about. Inwardly, we wish to believe that the concepts are unique, so that the communication between us is clear. Ultimately, we seek something, which we call "absolute truth".

In order to ensure uniqueness of the traditional /classical system of Natural Numbers some researchers of formal logic were lead to adopting a more general than the usual formal logic which is called "Second Order Predicate Logic" (SOL). This formalism guarantees indeed the formal uniqueness of the set of NN and restores thus the feeling of a safe determination of the meanings at least in the field of NN. However, as we shall show in Appendix 2, this is an illogical /absurd logic, because it fails to fulfill the basic requirement from any logic, the possibility of logical control /checking of the validity of propositions subjected to its judgment.

The proof of such results is, of course, conducted formally and with great care. However, here we will adopt a more intuitive and insightful justification based on the Theory of Sets. We will, therefore, describe in Appendix 1 some elementary results of this theory which refer to infinite sets. Based on these results we will then attempt to explain in Appendix 2 why the extension of formal logic ends up in an interpretative impasse (why it becomes absurd).

How are these results related to the, so called, Artificial Intelligence?

In spite of objections of important thinkers like Friedrich Nietzsche and Ludwig Wittgenstein (in his later period) [3] in many people a view still persists that our conceptual system can be fully determined formally and thus become understood by a computer. Novels or movies which assume that a man's memory can be digitally stored in a computer's memory are essentially based on this delusion. However, the only way to store and process information in a computer is symbolic, i.e. formal. The impression that the concepts are

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[3] **Nietzsche**: 'On Truth and Falsity in their Extra-Moral Sense', 1873 p.4/9: "What, then, is truth? A mobile army of Metaphors, metonyms, and anthropomorphisms—in short, a sum of human relations which have been enhanced, transposed, and embellished poetically and rhetorically, and which after long use seem firm, canonical, and obligatory to a people: truths are illusions about which one has forgotten that this is what they are; Metaphors which are worn out and without sensuous power; coins which have lost their pictures and now matter only as metal, no longer as coins".
Wikipedia: **Wittgenstein**: In his *Philosophical Investigations,* published posthumously in 1953 in two parts, Wittgenstein urges the reader to think of language as a multude of language games within which parts of the language develop and operate. He argues that philosophical problems are illusions resulting from misguided attempts of the philosophers to consider the meaning of concepts independent from the context, the usage and the grammar, something which he calls "language going for vacations".
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determined by appropriate definitions and thus can be stored formally is, obviously, created by the existence of Interpretative and Encyclopedic Lexicons. However, the empirical knowledge, which we store in our memory is usually unwritten and not verbally translatable, because it is based on direct experience. For instance, try to describe all the movements of a bicyclist as well as the body's position by means of which he/she manages to balance each time on two wheels. Such a thing is impossible. This is why we learn the use of a bicycle by direct experience and not by reading appropriate treatises of bicycling. We are also able to recognize faces and other shapes automatically, without being able to describe them geometrically. Even more elementarily we unconsciously know that a potato or a plate can be used as projectiles, while a broken plate can be used in order to cut or scratch. These properties are nowhere mentioned in the definition of the potato and the plate. Beyond the lexicographic there is also an associational content of each object's conception. This is related to all possible interactions which an object can have with all other objects of our environment and it is learned only empirically.

But let us discuss the range and the possibilities of the formal systems which refer to natural numbers (NN), i.e., the non-negative integers.

The study of these systems has shown that the set of natural numbers CAN NOT be formally determined in a unique way unless it is done by means of a requirement which is beyond the axiom system: that the operations of addition and multiplication are always performable, i.e., that they have a definite outcome (Tennenbaum).

If the computer's knowledge is restricted to the usual axioms then it does not know what magnitudes it handles. The realization that its arithmetic operations are always possible does not result from these axioms but from the extra-arithmetic study of the structure of the models, the sets of magnitudes which satisfy these axioms. This is how Tennenbaum proved his theorem. I.e., a computer does not have a deeper understanding of what it does, unless it is given by means of programming extra-arithmetic rules for evaluating structures.

Here there may possibly be an objection: The computer produces anyway only classical NN! It does not care whatever other number systems are formally possible![4] This is true! The interpretation of a formal system is a concern of people and not of computers. We are the seekers of sense.

What properties and what shortcomings does Second Order Arithmetic (i.e., the Arithmetic which is based on Second Order Logic) have?

The theorems of Löwenheim - Skolem, and Gödel

Skolem's theorems, which we have already mentioned assume that Arithmetic is built up by means of the more restricted Predicate Logic of the

[4] In rder to have a computer that is able to consider the structure of such system, we must equip it with an appropriate formal tool for studying properties of the formal system which produces the NN and not only the NN themselves. This tool is called Second Order Propositional Logic and it studies properties of properties, i.e. categories of properties. This logic should be combined with a system that formalizes the study of algebraic structures so that it may reach similar conclusions as those to which we came above. This endeavor would be a formal analogon of what we call "Meta-mathematics" or Model Theory and it would be burdened with all shortcomings of a Second Order System.

First Order. However, as we have already said, Second Order Logic is not always accepted as a tool, because it does not have the basic property that we require from any Logic: the ability of verifying propositions we subject to its judgment.

However, there is another theorem of mathematical logic, which upsets our feeling that we know what the natural numbers are; Kurt Gödel's Incompleteness theorem, which is valid for all kinds of formal logic.

This theorem says that, no matter how many fundamental assumptions (axioms) for natural numbers we make, there are always propositions of Arithmetic, which we can neither prove, nor disprove. It even tells us how we can construct such formal propositions, which are called "undecidable".

Of what kind are these propositions?

For instance, such an undecidable proposition can be of the form: A certain polynomial of many variables p(x,y,z,…) with integer coefficients does not vanish for any combination of integer values for x,y,z, …[4].

However, this theorem of Gödel leaves open the possibility that the natural numbers are a unique set even with an infinity of unprovable properties. May be, the natural numbers exist somehow in a Platonic world of ideas.

Thus, we don't know finally what the natural numbers are or we cannot determine it by means of a finite number of fundamental properties (axioms) of them.

How does man understand the natural numbers?

You may, of course note:

Man has is then in the same position as a computer; he does not understand what he speaks about.

This is partially true! For people there are also no logical properties of the concepts available, which determine them uniquely. However, people determine concepts not logically, but functionally (based on sensorimotor procedures), i.e. by means of the usually imperceptible, unconscious associative knowledge of how he can use the various objects. For instance, the first ten natural numbers are essentially determined by their correspondence with the fingers in our hands. This is how we learn to count and not by memorizing certain properties of natural numbers.

Usually people claim that they understand these concepts by means of a strange property of the mind, which they call "intuition". Thus, even many logicians believe that the natural numbers 1, 2, 3, … are "intuitively" determined although it has been proved that they cannot be uniquely determined by means of formal systems (This seems to have been also what Gödel himself believed).

People do not have better logical definitions of the concepts and particularly of the natural numbers, but they memorize and handle, i.e. understand concepts empirically as procedures. Very few people know exactly Peano's axioms (stated in 1899). However, this does not hinder us to make any kind of calculations using automatically and unconsciously empirical thought schemata compatible with those axioms, but also with many other properties of natural numbers, which we consider as obvious.

We do not need the uniqueness of natural numbers in order to prove any theorems of Arithmetic. Indeed, all known theorems are valid also for the new systems of magnitudes which we may invent by adding new axioms to the already established. There is no hindrance in the performance of formal proofs.

However, formal proofs are secondarily important for us. The way in which we "discover" new properties of NN before we prove them is not formalistic. It is rather "intuitive" (whatever that may be). For instance, there is no formalism that inspires us with Goldbach's conjecture (made in 1742), a seemingly simple eventual property of NN which is still unproven after 275 years. This conjecture says that each even number: 2, 4, 6, 8, … is a sum of two prime numbers, i.e. of numbers like 1, 3, 5, 7, 11, 13, 17, 19, 23, …, which have no other divisors except themselves and 1.

How do we discover new properties of the natural numbers?

Very often through the observation of geometric or symbolic-algebraic forms. For instance, the observation of triangular arrangements of points has shown already to the Pythagoreans that the sum of the arithmetic sequence of 'odd integers' is always a perfect square. Odd integers are called those that are not divisible by 2, i.e., have the form 2k+1, where k is a natural number. Thus we have:

$$1+3=4=2^2,\ 1+3+5=9=3^2,\ 1+3+5+7=16=4^2,$$
$$1+3+5+7+9=25=5^2,\ \dots$$

If we depict in a triangular arrangement rows of equally spaced points whose numbers are the successive odd integers we see immediately that we can move the right part of this arrangement to complete the left part so that we obtain a square arrangement of points. This property can then be strictly proved by means of the axiom of induction and the relation:

$$k^2+(2k+1)=(k+1)^2$$

Even a proof of Pythagoras's theorem which is supposedly due to him is purely imaginal (visual – kinetic). See [5].

Although the formalism allows us the strict formulation of a proof, it does not show us how we can find it. This is something we may achieve primarily by exploiting mainly our sense for the symmetry of various algebraic or geometric forms /figures and our imaginal and kinetic perception of various magnitudes. For instance, in geometric proofs it is almost allways necessary to imagine auxiliary straight lines or circles which reveal to us various initially hidden properties of the figure we study.

In what way does human thought differ from the processing of symbols made by a computer? Why can't a robot think like a man?

There are indeed impressive achievements of Artificial Intelligence in fully controlled environments, i.e. environments built up so that they can me described sufficiently well for our needs in a symbolic language used by the computer. However, no mathematical technique of processing information will ever produce an "intelligent" robot able to realize the intrinsic, but imperceptible latent unconventional and unexpected properties which are inherent in a not controlled environment and the objects existing in it.

They are mostly non-declarative. The reason is simple. We do not store in our memory concepts and attributes /features of the objects as logical properties but as procedures for interacting with the environment. For instance, in lack of a hammer we seek almost instantly and without extensive logical analysis a heavy compact object to use instead, e.g. a stone or an ash tray and in lack of a sweep, we may use a newspaper or a journal in order to remove crumbles from a table top. From what logical definition of a stone or a newspaper are such uses implied? A broken plate can be used, as we have said, in order to cut or scratch. Is this implied from a logical definition of a plate or immediate experience?

Actually, all objects are mentally stored (not only by logical definitions, but also) in terms of hierarchies of procedures for interacting which proceed from an associatively not so specialized schema of interaction to more specialized ones. However, something which is associatively general, i.e. unspecific, unspecialized, is not also by all means logically general. Thus, concepts like "nut shell" or "thimble" and "water glass" have partially a common associative content although they are logically unrelated. The nut shell as well as the water glass are characterized by a sensorimotor (or sensory-motor) procedure which recognizes approximately convex (i.e., hollow) objects, objects in which we can insert our fingers. The differentiation of these objects happens subsequently by additional procedures which distinguish one from the other. Thus, a very young child may use in play without much reflection a nut shell as a substitute for a water glass.

In order to perceive things in the way we do we must gradually and for a long period of time collect even more specialized experiences of interaction with the objects. This is the way in which children learn to handle objects. In particular, as Peter Wasson and Phillip Johnson-Laird have experimentally established, in order to evaluate a situation we must build up in our mind a Mental Model or script /scenario of this situation, and play it out in our mind in order to see where it leads to. Thus, the intrinsic, latent properties of all objects are stored in our mind associatively.

To what extent can this kind of thinking be reproduced by a computer?

If we naively think that all objects of our environment and all our actions correspond to words or verbal descriptions, then for every word we may declare as code-number a natural number. Thus, if we teach a computer or rather a robot (a machine controlled by a computer) to use grammar and syntactic analysis of sentences can we, possibly, teach them also how to evaluate situations as we do?

No! We can teach them how to follow instructions based on an elementary image of the environment but we cannot teach them the intrinsic, latent, and not immediately perceptible properties of the objects surrounding it. Each new and not recognizable object which enters in their perceptual field will cause in them confusion. Even a recognition of known objects from a different angle of view will be extremely difficult for them. No extent of formal coding is able to "grasp" the properties of objects and procedures relative to each other, i.e., each object's properties in relation to all others.

The inability of simulating human thought by a robot is, thus, not a matter of uniqueness or not of the numerical code it uses for the language, but it is due to being unfamiliar with the latent properties of objects especially when they interact with each other and with their environment.

The "intelligent" robots or "thinking robots" will have to learn to handle and examine their environment building up appropriate associations and storing them in their memory as hierarchies of sensorimotor processes. Later on they will have also to be taught names for the sensorimotor schemata they have formed, i.e. a language of memorizing scripts and communicating.

Thus, we will have to educate a robot just like we educate a young child. However, here there is a fundamental difficulty. The exploration /examination of the environment by people does not happen in a chaotic, anarchic way, but it is guided by motives, e.g. the survival instinct. "Thinking" robots must therefore acquire motives similar to those of people. An instruction "try anything and everything" is not enough, because the experiences being collected must be evaluated. Man does not hoard randomly experiences, but he evaluates and orders them according to their value for his biological survival and the control of the environment they allow him. This is why, like all mammals, he has a close relation with his parents by whom he is taught how to distinguish the essential from the inessential as well as the language which allows his communication with them. First of all, a robot does not have motives for the evaluation and hierarchic ordering of experiences. The "thinking robots" must, therefore, be equipped with motives similar to those of people. They should even have a close relationship with some educator or trainer. However, then they will be independent of our wishes and their behavior will be just as unpredictable as the behavior, the reactions of our fellow men. [5]

Appendix 1: What is a non-denumerable set? Are there different kinds of infinite?

The revolutionary discovery made by Georg Cantor in 1874 is that there are different kinds of infinite. More precisely, he proved in a simple and ingenious way that the real numbers of the interval between 0 and 1 are in a certain sense more numerous than the natural numbers.

He achieved this by "reductio ad absurdum":

Suppose that to every real number of the interval between 0 and 1, i.e., to every decimal number with integer part 0, like 0,3241597683…, corresponds a natural number. Then we can write all these real numbers in a table according to the order of their correspondence to NN.

For instance, suppose that the above table has the form

1<>0,135791113….
2<>0,2468101214….
3<>0,3282131415….
4<>0,4327253141 5….

Here x<>y means correspondence of x to y.

[5] The challenge for a learning robot is how to simulate the heuristic methods implicit in human understanding.

We assumed that all real numbers of the interval between 0 and 1 should be contained in this table. However, it is very easy to construct decimal numbers that do not belong to the above table. In order to construct such a number, say b, we choose as a first digit (after 0) one that differs from the first digit of the first decimal number in the table. As a second digit we choose one that differs from the second digit of the second decimal number in the table, and so on. Generally, we choose as digit for the n-th place (n=1, 2, 3, 4, …) a digit that differs from the digit at the n-th place of the n-th decimal number of the above table.

For instance, the number b, could have here the initial digits 0,2546... which are different from those that are underlined in the table. The first one differs from the first digit of the first real number in the table 0.135… The second one differs from the second digit of the second number 0,246… The third digit of b differs from the third digit of the third real number in the table and so on. Every such number b cannot belong to the above table, because every digit of it will differ from some digit of every decimal number in the table.

Thus, the set of real numbers of the interval between 0 and 1 necessarily has a contain much more numbers than the set **N** of natural numbers.

Such a set of numbers which cannot be put into correspondence to the natural numbers is called *uncountable.*

The proof that the set **P** of subsets of the natural numbers is uncountable is similar.

Suppose that to every subset of **N**, like the subset {0, 3, 24, 15, 9, 176, 283…}, corresponds a natural number. Then we may write all these sets in a table according to the order of their correspondence to the natural numbers. For instance, let

1<>{13, 57, 91, 113,….}
2<>{24, 68, 101, 214….}
3<>{13, 21, 28, 31, 415,….}
4<>{27, 43, 253,1415….}

NOT included in this table is every set, s, of the form: {s1, s2, s3, …} where the element s1 differs from the first element of the first set in this table, the second element s2 differs from the second element of the second set in this table and generally, the element sn (with n=1,2,3,4,…) differs from the n-th element of the n-th set in this table.

For instance, such a set may have elements larger by 1 from the corresponding element of every set in the table:

s1=13+1=14, s2=68+1=69, s3=28+1=29, s4=1415+1=1416, …. I.e., s={14, 69, 29, 1416, …}. This set differs from the first set in the table by its first element, from the second set by its second element and, generally, from the n-th set by its n-th element. Thus, the set **P** of the subsets of the natural numbers, which is called "powerset" of the natural numbers, has by all means a greater number of members than the set **N** of the natural numbers.

Why the set of properties of the Natural Numbers is uncountable.

Now we can also see why the set of properties of the natural numbers is uncountable:

From the standpoint of set theory we can see every property of the natural numbers as corresponding to the set of natural numbers, which have this property. Thus, for instance, the property "square of a natural number" corresponds to the set of squares of natural numbers: {1, 4, 9, 16, 25, 36, 49, 64, ….}. However, the same set corresponds also to the property (predicate) "sum of successive odd natural numbers" (see p.8) [6].

Of course, this means that each subset of natural numbers corresponds at least to one of their properties. Thus the set of their properties is at least as large as the set **P** of their subsets, which is uncountable.

OBSERVATION: In spite of the illusion that Skolem's set of ordered polynomials is more numerous than the set of natural numbers, it is a countable set. I.e., its elements can be placed into correspondence to the natural numbers.

Why Skolem's Ordered Polynomials are a countable set

Although the set of Skolem's Ordered Polynomials is not isomorphic to the set of NN, it is, nevertheless, countable, because we can write it in the form:

$1 < 2 < 3 < 4 < \dots$

$< x < x+1 < x+2 < \dots$

$< x^2 < x^2+1 < x^2+2 < \dots$

$< x^3 < x^3+1 < x^3+2 < \dots$

$< x^4 < x^4+1 < x^4+2 < \dots$

Then we can rearrange it in the following ordering: (1),(2, x),(3, x+1, x^2),(4, x+2, x^2+1, x^3),… It follows the successive anti-diagonals of the above table (the diagonals that go from the top right to the bottom left). In this last arrangement we can place the members of this set into correspondence to the NN, i.e., we can count them all.

However, it can be shown that within the frame of the same formalism (i.e., of the Arithmetic with First Order Predicate Logic) there are also UNCOUNTABLE sets of magnitudes which fulfill the same axioms as well as all extensions of the axiom system.

Appendix 2: Second Order Logic and its Shortcomings.

The theorems of Löwenheim, Skolem and Gödel, have troubled /puzzled very much those who wish to reduce thinking to some formalism, no matter

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[6] Another example of how predicates are determined by a set of NN is the following: The relation "lengths of the sides of an orthogonal triangle" is satisfied by the set (3,4,5). However, (3, 4, 5) are also divisors of 60, or 120, or 180, … They are also successive natural numbers. Thus, there are more than one properties or relations, which characterize the same set of numbers. On the other hand, the relations of this example are also fulfilled by other natural numbers. Thus, there are many correspondences between sets of numerical relations and subsets of the natural numbers.

whether they are logicians or researchers of Artificial Intelligence. Of course, they understand that the non-uniqueness of the formal axiomatic systems is valid for the human as well as for the artificial brain. However, they are not concerned about the question if and why the human brain has a more clear "intuitive" conception of the natural numbers. They cannot accept the existence of meanings which do not accept a full formal description. They cannot conceive conceptual systems, which may not be mechanically reproducible, perhaps because they are dreaming of a future triumph of the mechanical reproduction of thinking, i.e. of the, so called, artificial intelligence.

Thus, many of them were led to the use of a system of formal logic which indeed rejects the non-uniqueness of the natural numbers. However, this logic, the, so called, Second Order Predicate Logic has many important inherent shortcomings, which we will describe subsequently.

It is an irrational logic!

So let us consider in short what a formal logic intends and what possibilities it offers. Initial goal of formal logic was to relieve our syllogisms from the inherent ambivalence of language. The replacement of words by symbols would allow the resolution of all disputes as Leibniz hoped in the 17th century. The aim of reducing all logic to a formalism was greatly enhanced in the 19th century by George Boole and Gotlob Frege. However, the further progress of mathematical logic in the 29th century led to a strange situation: The formal formulations of logic do not coincide with what we would call common logic. What is more, it does not seem that such a system exists. More specifically, no variation of the established Predicate Logic has all the properties we would wish a safe system of reasoning to have. Thus, we are obliged to settle with formal systems which, either lead us to unexpected new worlds of objects of thinking (like the so called "Nonstandard Number Systems"), or do determine exactly what we mean, but lead us to an world of unsafe conclusions and uncertainty about the validity of its propositions.

In particular, the theorems of Löwenheim and Skolem tell us that, if we use the basic complete[7] system of logic, the First Order Predicate Logic, then we cannot determine precisely what the natural numbers are. There are infinitely many systems of magnitudes which fulfill all the fundamental assumptions of Arithmetic (they are different from each other either with respect to the number or with respect to the arrangement of their elements). What is more, this result is even valid when we add as many as we like new axioms to the usual axioms of Arithmetic.

We can avoid this conclusion only when we use as a fundament a more demanding logic, the so called Second Order Logic. However, this logic has the unreasonable property that, contrary to the First Order Logic, it does not have a method for rejecting false propositions subjected to its judgement. Beyond this, it produces formulas with no meaning at all. They are not valid for any system of magnitudes, i.e. for no interpretation (called "model") of the formal expressions. We express these shortcomings more disguised and

[7] Complete system of axioms is called a system which allows the formal rejection of not satisfiable propositions, i.e., the formal proof of their unsatisfiability.

more elegantly by saying that “for Second Order Logic the syntactic (= formalistic) and semantic truth do not coincide”.

The relation of Second Order Logic to Semantics

In order to conduct formal proofs of numerical propositions it is necessary to include to the formal axioms of Arithmetic the axioms of an appropriate formal logic, which will allow the production of new propositions “without words”, i.e., expressed only by symbols. As we have said, in this way we will avoid ambivalent verbal expressions.

We speak about First Order Arithmetic when we include the usual system of axioms of Arithmetic to the set of axioms of First Order Logic, which refers to properties (predicates) of mental objects. Then we speak of First Order Arithmetic. If we include the axioms of Arithmetic to the (appropriately adjusted) system of axioms of Second Order Logic, which refers to properties of properties, i.e. sets of properties, then we speak of Second Order Arithmetic.

The seeking of the creation of a formal theory of meaning of the natural numbers, which could become part of a more general formal theory of meaning, has ended up after investigations which lasted for one century to the following impasse:

The First Order Arithmetic (FOA) is indeed a system of conducting and verifying proofs, but it does not determine uniquely the objects (the magnitudes) to which it refers. There are radically different interpretations, radically different models, which satisfy this formal system. This is even true when we add to the original ones as many fundamental assumptions (axioms).

The Second Order Arithmetic (SOA) avoids this non-uniqueness; it defines uniquely (up to isomorphy) the objects to which it refers. However, in SOA no effective proof tools are available beyond those of FOA and, in general, it cannot judge the validity of a given supposed proof.

Ultimate purpose of using SOA was to establish a formal semantics of natural numbers, i.e. to define uniquely the objects of arithmetic considerations, i.e., at least, of arithmetic thinking. However, this is counterbalanced by an inability to verify proofs and by an incompatibility of syntactically produced formulas with semantically true propositions, i.e. propositions which are verifiable for some kind of arithmetic magnitudes (some model of arithmetic).

Formal semantics is refers here to the existence or absence of a mental model which fulfills a formula produced within the formalism.

Why do these happen?

The most basic difference of First Order and Second Order Formal Arithmetic, is the form which the, so called, “Axiom of Induction” takes. This axiom is the basic instrument for proving a relation for all natural numbers. In simple words, it says that if a relation holds for 1 and from its validity for the number k follows its validity also for its successor, k+1, then this relation is valid for all natural numbers.

In First Order Formal Arithmetic the induction axiom is not a single formula, but, in reality a formula corresponding to each constructible formal expression, φ, i.e., an expression we can construct formally in such way that it can be computed for every given natural number. This is why it is called, not an axiom, but an "axiom schema". On the other hand, in Second Order Formal Arithmetic it is a single formula which refers to any predicate (property of NN), P, no matter whether we can construct it with our formalism or not!

However this has important consequences for the whole system. When we try to interpret the Second Order Axiom of Induction, we must indicate to what predicates, P, the term "for all P", which is contained in it, refers (where P symbolizes some property of natural numbers). I.e., we must define the universe of predicates, P, which refer to natural numbers (NN). If we choose this universe to consist of all possible predicates constructible or not (all possible relations of NN), then, according to the set theoretic interpretation of natural number properties (predicates) as subsets of natural numbers, their set is "uncountable", just as the real numbers between 0 and 1 (see Appendix 1).

There is no way to arrange them all in a sequence which can be scanned through. But an uncountable set of properties of the NN is extremely problematic. There is no systematic way to test if a formula is valid or not, because there is no systematic way of running through all the predicates.

We cannot arrange the set of predicates in a table and run through them testing, for instance, whether there is a property (predicate) that fulfills the formal expression under consideration or not.

When a formula says that there exists a predicate of the NN which fulfills a certain proposition, we must run through all possible propositions until we find one that fulfills the proposition. But there is no way to do so, because they are uncountable. When a formula says that a certain proposition is fulfilled by all predicates of the NN, we must run through all possible propositions verifying this statement. But, again, there is no way to do so, because they are uncountable.

This lies ultimately the fundamental shortcoming of Second Order Logic and of the Arithmetic which is based on it. Thus, we have uniqueness of the natural Numbers in Second Order Arithmetic, but only with by admitting as a counterbalance the acceptance of an unconceivable and not constructible set of properties of them. If we restrict again the universe of predicates to a countable and constructible set, then we have again nonstandard models.

Thus, the second order Arithmetic is necessarily incomplete, not only because its formalism is unable to construct appropriate formal proofs, but because most of its relations are not constructible. There is no systematically (algorithmically) constructible set, which determines them.

The deeper cause of the formal indeterminacy of NN

This is the fact that all possible arithmetic predicates (properties) are, according to their set theoretic interpretation NOT COUNTABLE. Therefore, also all propositions that refer to relations of these predicates are not countable. On the contrary, all possible formal expressions with finite length which are produced by means of a finite or countably infinite set of symbols are ONLY COUNTABLE, because they can be ordered lexicographically[8].

Thus, all elementary expressions which result from a formal proposition in combination with the given axioms can be produced systematically one by one and we can examine each time if each one is in agreement with the proposition or not.

What did Skolem and Gödel add to this observation?

Gödel says something more specific: That for every finite system of formal axioms there are finite formal expressions which can neither be proved nor disproved!

On the other hand, Skolem says something else: That no finite (or countably infinite) system of axioms is enough in order to characterize uniquely the set of natural numbers. There are always other sets of Hyperintegers (countable as well as uncountable) which satisfy these axioms. This leaves open the possibility to distinguish such number systems by adding axioms which are valid for one, but not for another!

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[8] We can, simply, give to all of them a unique symbolism s1, s2, s3, s4, … Then we can count all the chains of certain length (with a certain number of symbols) by arranging them in order of the size of the chain of their indices, where in order to separate the indices we use the digit 0, which has been omitted in the unified symbolism. For instance, the symbol chain s1s3s15s1s2 will have the composite index 1030150102. Thus, all symbol chains of a certain length, n, will be ordered in the order of the size of their composite indices.