`Do Mathematical Magnitudes Exist in Nature?

How real are the Real Numbers?
Do Irrational Numbers Exist?
Do Mathematical Concepts have a Physical Meaning?

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Summary

In spite of the widely held view that the fundamental mathematical magnitudes are, somehow, inherent in Nature i.e., in the physical world, these magnitudes do not correspond directly to any physical phenomenon.

Neither real numbers, nor the trigonometric and other elementary functions exist in Nature. They are inventions of the human mind, which allow a concise and simplifying view of reality. Even the Natural Numbers considered as a set are a mental construct.

Introduction

A widely held view is that the mathematical concepts are, somehow, innate in the physical world. The mathematical description of various phenomena is today so familiar to us that we tend to consider the mathematical magnitudes used as, somehow, inherent (σὑμφυτες) /innate (εγγενεἰς) in Nature. Very often we identify a phenomenon with its mathematical description and so we think that the mathematical magnitudes, which are used, correspond to some existing physical magnitudes. But is this so?

Do all the conceivable points of a straight line exist in Nature?

In particular, are there any physical magnitudes that correspond to irrational numbers, like $\sqrt{2}$?[1]

Are there any physical phenomena described exactly by fundamental elementary functions like the sine or the cosine?

As we shall see, neither $\sqrt{2}$, nor the points of a straight line or phenomena described exactly by the common elementary functions (i.e. corresponding exactly to them), exist in the material world. All these concepts are only idealizations of reality, which never fully correspond to it.

Why don't we strive, then, to use more realistic mathematical magnitudes, i.e., magnitudes that describe exactly the physical phenomena?

As we shall see, the main reason is that this would complicate inconceivably the mathematical description. So, we prefer to idealize the phenomena and to identify mentally the parameters, upon which they depend, with mathematical magnitudes,

[1] The number $\sqrt{2}$ corresponds to the length of the diagonal of a square of side-length 1. It is called 'irrational', because its decimal representation :
$\sqrt{2}$=1,41421356237310… has no regularity. It has an infinite number of decimal characters which are not periodically repeated, as it happens for the decimal representation of fractions. For instance, it is 1/7=0,142857142857…. and the sequence of characters 142857 is continuously repeated with no ending.

which have been invented merely for the simplification of the description of these phenomena.

Whenever a more precise description of some phenomenon is needed, we simply use more complex mathematical descriptions, which correspond to the intended degree of precision.

These conclusions may seem strange to those accustomed to the naïve mathematical description of phenomena, but it is not new. They have been repeatedly emphasized by some important mathematicians and philosophers, but they are usually ignored by most mathematicians and physicists. Nevertheless, they are important when we try to understand nature, and improve our models for physical phenomena.

Let us consider at first the basic mathematical magnitudes, by means of which we construct all our physical models. Their lack of correspondence with physical phenomena, their mismatch to physical phenomena, has already been stressed by mathematicians like Felix Klein [1] and Alexandrov, Kolmogorov, Lavrent'ev [2]. Similar ideas develops also the philosopher Peter Smith [3], referring more specifically to chaotic phenomena.

We will discuss here their observations on the nature of various mathematical magnitudes and, then, we shall try to make clear why it is helpful to use such "unrealistic" magnitudes.

1. Do Irrational Numbers Exist in Nature?

Felix Klein [1, pp. 35-36] notes that our understanding of space has limited precision, even if we use the most sensitive instruments of observation. However, "In contrast with this property of empirical space perception which is restricted by limitations on exactness, abstract or ideal space perception demands unlimited exactness, by virtue of which … it corresponds exactly to the arithmetic definition of the number concept". So, we may distinguish 'mathematics of approximation' and 'mathematics of precision'.

According to Klein, the correspondence between the idealized conception of space and the set of real numbers is due to the, so called, Cantor's axiom (stated in 1872) which defines a one-to-one correspondence between the real numbers and the points on a straight line[2]. However, this 'real line' is an abstraction, which cannot be drawn. It does not correspond to any material shape. No matter how we draw a line, this will ideally correspond to a chain of atoms and will not be truly continuous.

Calvin Clawson [5, chapters 2 and 3] observes that the notion that a line is a "physical" magnitude, which consists of a set of points has inherent contradictions. If a line is nothing more than infinitely many points, each of which has no extension, then each point adds nothing to the length of the line and the length of any line segment will be zero. The sum of infinitely many zero lengths is still zero. The opposite view, that single points have indeed some tiny elementary size, leads to other contradictions. If every line segment consists of infinitely many points, then it will have infinite length. If, on the contrary, it consists of a finite number of points,

[2] Cantor defines as the 'real line' a straight line, on which a line section has been declared as having 'unit length'. He declares axiomatically, that to every rational or irrational number corresponds a point of this line. Conversely, every point of such a line is axiomatically considered as corresponding to a rational or irrational real number, which is called the abscissa of this point.

then the length of any line segment should be expressible as an integer multiple of this elementary length. Then, the ratio of the lengths of line segments should be expressible as a fraction, i.e., as the quotient of the integers, which express the length of each segment in terms of elementary lengths. However, this conclusion is not compatible with the Pythagorean theorem. According to it, the diagonal of a square is not commeasurable with its side, as the Pythagoreans already knew. The quotient of the lengths of these line segments is equal to $\sqrt{2}$, a number which cannot be written as a fraction[3].

Clawson concludes that these contradictions can only be avoided by the axiomatic definition of the "real line" by means of Cantor's axiom.

Similarly, A.D.Aleksandrov [2, pp.30-31] says:

"But ideally precise geometric forms and absolutely precise values for magnitudes represent abstractions. No concrete object has absolutely precise form nor can any concrete magnitude be measured with absolute accuracy, since it does not even *have* an absolutely accurate value. The length of a line segment, for example, has no sense if one tries to make it precise beyond the limits of atomic dimensions".

Here other questions may also arise: Do interatomic spaces exist or not? If they exist, how can they be described? Does space without any matter exist?

In any case, even for physical magnitudes, which we consider infinitely divisible, like time, it seems unreasonable to speak of irrational quantities. Can a physical phenomenon last $\sqrt{2}$ seconds? Such a time interval is not measurable.

What is more, in every case that we pass beyond well-known limits of quantitative accuracy, there appears a qualitative change in the physical magnitude, and in general it loses its original meaning. For example, the pressure of a gas cannot be made precise beyond the limits of the impact of a single molecule; electric charge ceases to be continuous when one tries to make it precise beyond the charge on an electron and so forth. The Nobel laureate Robert B. Laughlin goes even farther [5, pp.15-20]. He rejects even the existence of some physical magnitudes and physical objects in very small dimensions. (See Apendix 1).

In view of the absence in nature of objects of ideally precise form, the assertion that the ratio of the diagonal to the side of a square is equal to the $\sqrt{2}$ not only cannot be deduced with absolute accuracy from immediate measurement but does not even have any absolutely accurate meaning for an actual concrete square". (In Apendix 1 see also similar views of the philosopher Peter Smith).

The number √2 is, thus, an invention of our fantasy and the same is true for all irrational numbers. Only fractions can correspond to points on a straight line segment if we imagine it as a chain of sufficiently many "atoms". If we assume on such a line that the unit length corresponds to a segment of 500 atoms, then the fraction 3/5 corresponds to the 300th atom after the beginning of the unit segment, while 7/4 corresponds to the atom with number 875 (500*7/4=875).[4]

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[3] $\sqrt{2} = 1.4142135623730950488016887242096980785697...$ has infinitely many decimal digits without any regularity in their pattern.

[4] How "unreal" the irrational numbers are follows from the fact that we do not always know whether a number is irrational or not. E.g., this is not yet known for the numbers $2^e, 2^\pi, \pi^e$ and

But why do we need irrational numbers? Why did we invent them?

2. *Why do we need irrational numbers?*

For instance, why do we need the irrational number $\sqrt{2}$?

The only reasonable answer seems to be: In order to be able to determine the length of the sides of a square with area 2 (assuming, of course, that such a square can exist) or in order to be able to determine the length of the diagonal of an "ideal" square with side length 1. Nevertheless, according to the particle view of the composition of matter, such a material square (e.g. from material with constant thickness) cannot exist. For instance, we cannot have lengths extending over half an atom. The unit length, with which we measure all other lengths in Geometry, can only correspond to a chain of full atoms in the material world. Even its fractions should be expressible as a whole number of atoms as in the example given above.

All these considerations, of course, are based on of a naïve atomic theory, which refers to equidistant atoms of constant size with fixed positions. In reality no atom is perfectly immobile, even if it is embedded in a crystal lattice, and its diameter can change when all electrons of the outer layer are attached to a neighboring atom.

One may, of course, ask: If we have a body moving on a straight line, how can it move from position 1.41 to the position 1.42 without passing through position $\sqrt{2} = 1{,}414213562\ldots$?

The answer is simple: Just as we can go from 1 to 2 and 3, when we consider integers, without going through intermediate values. Intermediate values do not always have a meaning. For instance, we cannot climb half a step on a stair. Intermediate positions *do not* exist necessarily, but are often introduced axiomatically in our considerations! Even the movement of a visual stimulus across the eye is not continuous. The brain simply "continuizes" the successive excitations of the cells of the retina by the photons reflected from the moving body. We are, simply, captives of our body, which compels us to see everything deterministically and cohesively.

Thus, a "natural" length $\sqrt{2}$ does not exist.

However, all our theoretical analyses would become extremely complicated if we rejected $\sqrt{2}$ as a tool of theorizing. For instance, the Pythagorean Theorem, which is a foundation of both Euclidean Geometry and Trigonometry, would not hold.

Thus, in Mathematics we assume that the "number" $\sqrt{2}$ has a meaning, although the unit square, which has $\sqrt{2}$ as length of its diagonal, does not exist in Nature.

But, if $\sqrt{2}$ has no physical meaning, what can be its mathematical meaning?

In reality, by $\sqrt{2}$ we mean the result (the limit) of an unending calculation process proceeding in steps (of an algorithm) of the following kind:

Euler's constant c (e is here Napier's constant and π the well known area of a unit circle). The same is true for e^e, π^π according to [E. Sondheimer -A. Rogerson: Numbers and Infinity, Dover, p.68].

We can answer such questions for particular categories of numbers, but have no general method for answering them directly.

$$x_0 = 1,\ x_{k+1} = \frac{1}{2}\left(x_k + \frac{2}{x_k}\right),\ k = 0,1,2,3,...$$

It yields successively the values $x_1 = 1.5,\ x_2 = 1.4166..,\ \mathrm{x}_3 = 1.414215...,$ and so on. Interrupting at will this process, we obtain increasingly accurate approximations of the positive solution of the equation $x^2=2$. However, none of these values corresponds exactly with this "number".

The same is true for all other irrational numbers, no matter whether they are algebraic, like $\sqrt{2}$ or transcendental as π=3.141592654....

But why don't we simplify mathematics accepting some rational approximations as values for these numbers?

3. Why do we care whether some numbers are rational or irrational?

This question is similar to the preceding one. It is related to the precision of mathematical relations.

What do we care, for instance, if the number π is rational or irrational?

An obvious answer is: Only then are such numerical relations as sin(π)=0 exactly valid (the angle is here measured in radians). Using approximations of π=3,141592654... on a hand calculator we have for instance:

sin(3.14)= 0.001592652 and sin(3.14159)=0.000002653 but not a zero value.

The same is true for all other simple relations we use in Trigonometry. More specifically, the Pythagorean relation $\cos^2\varphi + \sin^2\varphi = 1$, which is the basis of all Trigonometry, ceases to be valid if we replace the sine and cosine with rational approximations.

If we wish to have exact relations, we cannot avoid the use of irrational numbers. If we use finite approximations for all above magnitudes, then the Pythagorean equation is no more precisely valid and we cannot use it for further mathematical calculations.

4. Do the natural numbers exist in Nature?

Although it may seem strange to many people, a careful examination shows that not even these numbers exist in Nature, at least, as a set.

It is true that the first 5 to 7 natural numbers seem to correspond to inherent mental processes. The brain of many animals (men, apes, crows, pigeons and even fishes) can keep in its short term memory about 5 "chunks[5]" of information at the same time. They even remember the order, in which they appear. Thus, the origins of counting are inherent in the mind. They are extended by the people, by bringing the objects being counted into correspondence with one's fingers and toes. However, numbers exceeding 10 or 20 are conceived only as symbols, as arrays of decimal digits /characters, which we partially know how to handle in order to answer

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[5] This term has been used by Herbert Simon and G. A. Miller for pieces of information of the same kind. Five chunks of information can be, for instance, five objects of the same kind. They can also be five small numbers, five letters, five words and even five phrases (although the latter are often remembered roughly, i.e. not with their exact wording).

questions related to the comparison of one quantity with another, addition, multiplication, e.t.c.

How do we see that the natural numbers 0, 1, 2, 3, … are a mental construction and not inherent in Nature, i.e., in the physical world?

There are various observations, which support this view:

a. As Aleksandrov observes, the elementary operations with integers are not always meaningful. It is not always true that, e.g., 2+1=3. Two apples and a pear may, possibly, be considered as three fruits. But if we add two liters of hydrogen to one liter of oxygen the result is not three liters of gas, but a much smaller volume of water (in the form of steam drops, since the chemical reaction is extremely exothermic. If the numbers describe the multitude of some objects or the quantity of a certain material, then their addition is meaningful only when we consider similar objects. An apple and an orange are not two appleoranges! Their addition is meaningful only if we reduce them to a common category, the fruits. Then we say that we have two fruits. Similarly, it is not meaningful to add an idea and an apple. We do not usually say "I went to the park for a walk carrying two things; an idea and an apple". Addition is meaningful only if we add numbers or quantities of things of the same kind. But, which things are of the same kind is often a matter of logical abstraction. An idea and an apple can only be added if we consider them as objects of thought. Then we have two 'objects of thought'.

The situation is more difficult for multiplication, because we cannot always multiply quantities of the same kind. Two meters of length and three meters of width may, possibly, make six square meters, but two oranges cannot be multiplied by three apples. We can, of course, multiply two oranges by something we call "times" or "repetitions". Three times two oranges correspond to six oranges.

Without a linguistic analysis and creation of appropriate visual representations the arithmetical operations have no meaning!

b. As a matter of fact, we are even unable to imagine somewhat large natural numbers like 123. The illusion, that we know what the number 123 means, is due to the fact that we have a handy system of writing it. However, let us try to imagine it as a queue (a line) of one hundred twenty three strokes: ||||….|||, as a primitive shepherd would, possibly, write it trying to count his sheep. We immediately notice that we are unable to imagine it, to grasp it mentally. The only way to compare numbers written in this way, e.g. 123 and 128, is to write them in two parallel lines in correspondence of one stroke to another. Then it shows which line has some strokes left.

c. In reality, the set of all natural numbers, which we can write is finite. No matter what arithmetic system of number representation we use, there will always be an inconceivably large number, which all the ink in the world is not enough to write down. How meaningful is it, then, to speak of its successor or its double?

In fact, we understand as the set of natural numbers (NN) the result of an unending process of adding each time one unit to the previous term. We are obliged to do this feat of imagination in order to be able to prove theorems of Arithmetic. The main instrument of proof in Arithmetic is the 'Axiom of Complete Induction'. It claims that if a mathematical relation (a property referring to natural numbers) holds for the first NN, 1, and it can be proved that it is transferred from an arbitrary NN, n, to its successor, n+1, then this relation is true for ALL natural numbers, i.e., up to infinity.

Nevertheless, we know that we will never be able to construct such a set of numbers, since our lifetime is finite. It is obvious, that the natural numbers are a mental conception, which does not care whether we can write them or not. In this respect we escape from any natural perception.

With respect to the natural numbers we must also note that their continuation to infinity raises the question whether they are uniquely defined as a whole, since the concept "infinite" is not uniquely determined. There are various kinds of "infinity", as Georg Cantor has shown in 1874. Correspondingly, Thoralf Skolem has shown with publications between 1929 and 1934, that the natural numbers can be extended in various ways beyond "infinity" considered now as a set, an independent entity and not as the limit of an unending dynamic process (see Appendix 2)

Similar to the previous ones are the following questions:

5. Do the usual elementary functions have a physical meaning?

Even the most common elementary functions we use, like the trigonometric functions $sinx$ and $cosx$, do not, correspond exactly to some physical reality, since their values are usually irrational numbers like √3/2. However, the equations they satisfy are much simpler than those, which would describe more precisely the physical phenomenon under consideration.

The above trigonometric functions satisfy the differential equation

$$y" = y$$

which describes oscillations and has the general solution

$$y = Asin(x) + Bcos(x)$$

where A and B are arbitrary constants.

Thus, we find it convenient to use such functions, since their equations have very simple form and can be easily "solved" or transformed to other convenient forms.

Mathematics are full of such extremely simplified models of physical reality, which allow us, however, to give easily solutions and to establish the approximate validity of basic properties of the phenomena (like the periodicity of the solutions of $y" = y$).

The models of physical processes usually considered in mathematics do not correspond to the physical reality. The equations, which mathematicians and physicists usually solve, do not describe exactly the physical world. However, without these simplifications, today's technology would not exist.

If we tried to create more precise mathematical models we would be still centuries behind, not only in mathematics, but also in technology. The physicists might not be able to find solutions or detect the basic properties of some phenomena. For instance, the solutions of the differential equation y"=y, which characterizes vibrations and oscillations, are periodic. But, if we replace this equation by one describing the phenomena more precisely the solutions cease to be periodic.[6] As another example, consider the orbit of a projectile. We usually told that it is a parabola. But this is not true if in addition to gravity and initial velocity we consider also the resistance of the air, and the fact that it is not still, but a moving fluid. The possibility to consider such variations of the original simplified model is given only by means of a computer. This became possible near the end of the 20th century, while the initial, simplified, description was discovered about the end of the 16th and the beginning of the 17th century, i.e. three centuries earlier.

Most concepts of mathematics are simplifications of reality. They are mental constructs that allow us to write particularly simple relations, so that we can process and study them easier.

Here it is worth citing an observation of Albert Einstein: "It seems, that the human reason must first construct forms independently, before we can find them in things. In Kepler's marvelous life-work we see particularly nicely, that understanding cannot blossom from mere experience, but from a comparison of invention of the intellect with observation" (Albert Einstein: On Kepler, Frankfurter Zeitung, Nov. 9, 1930).[7]

Einstein means here the curves called 'ellipses', which result from intersecting a cone by means of a plane. Conic sections were studied by Apollonius in the 3rd century B.C., while Kepler published the proof that the planetary orbits correspond to ellipses with the sun as one focal point in 1609.

6. What is the meaning of the classical differential equations of physics? How well do they describe physical phenomena?

Let us, for instance, consider the differential equation $y''-a^2y=0$ which characterizes both vibrations in space (vibrations of a string) and periodic variations in time (oscillations).

As a study of a phenomenon varying in time this equation is **a** much more faithful model of physical reality, since time as a mental conception is usually considered as infinitely divisible. Thus, we may define derivatives of functions of time like the second derivative, y", as limits taken for Δt tending to zero ($\Delta t \to 0$). On the contrary, space does not have literally infinitesimals. The equation describing the vibrations of a string fixed at its two ends should not be a differential equation, but rather a "difference equation" describing the motion of each material particle of the string. The string is not actually a continuum, but rather a chain of particles. Thus, the derivative $y'(t)=\lim_{\tau\to 0}\dfrac{y(t+\tau)-y(t)}{\tau}$ may have some "physical" meaning,

[6] Actually, there is no alternating current, whose intensity is given exactly by a sine function like $i(t) = I\sin(\omega t+\theta)$ with constant amplitude {εὐρος}, I, frequency, ω, and phase, θ. However, we usually assume that the variations of the electrical current's intensity follow with sufficient accuracy this mathematical description.

[7] The German original is: "Es scheint, dass die menschliche Vernunft die Formen erst selbständig konstruieren muss, eher wir sie in den Dingen nachweisen können. Aus Keplers wunderbarem Lebenswerk erkennen wir besonders schön, dass aus bloßer Empirie allein die Erkenntnis nicht erblühen kann, sondern nur aus dem Vergleich von Erdachtem mit dem Beobachteten." (Albert Einstein über Kepler. *Frankfurter Zeitung*, 9. November 1930)

See Einstein' s manuscript of this paper in Einstein Archives Online.

but $y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$, where x is a variable expressing distances in space, does <u>not</u> have a direct physical meaning and represents or constitutes only an idealized description of the slope of the, supposedly smooth, curve y(x) at every position x.[8]

Actually, the string is not even a chain of particles. The metal atoms are imbedded in a grid with elastic bonds between atoms, or rather, a succession of grids of varying orientation. So, the atoms do not constitute a string. Thus, it is not easy and useful to describe the motion of each atom separately.

The string-equation is rather based on simple assumptions related to experience (empirical observations) and not on atomic theory. Thus, as it happens often in physics and technology, the equation is independent of other physical principles[9].

Today's Physics consists of a set of independent theories based on empirical assumptions and observations and it is not a deductive theory based on first principles, as many theoretical physicists would have us believe.

But, even if we consider the description, for instance, of vibrations in space by means of differential equations as representing exactly the physical reality, the movement of a vibrating string is given more correctly by a complicated nonlinear differential equation[10]. All terms of this equation are indeed used during the study of various phenomena, but, usually, not all together. Depending on which aspects of a phenomenon we wish to emphasize, we usually omit some term and, thus, simplify the differential equation.

However, more often we use the extremely simplified form:

$$\alpha^2 y_{xx}(x,t) = y_{tt}(x,t)$$

Which is called *one dimensional wave equation*. Here y_{xx} denotes a second derivative vertical displacement y with respect to the space variable x, and y_{tt} a second derivative of y with respect to the time variable t.

Why do we do this simplification? This is done because the solution of the last equation is very simple:

$$y(x,t) = F(x - \alpha t) + G(x + \alpha t)$$

Here F and G are initially unknown functions, which are later determined by the initial and the boundary conditions. These functions are interpreted as representing two waves: If $a > 0$, the first one F, moves to the right, while the second one, G, moves to the left.

Such simplifications are, furthermore, acceptable because anything in physics and engineering can be observed and described only with a finite accuracy. Every material body has imperfections which are not taken in account when setting up an equation for the description of its behavior. Thus, the engineers do not usually care

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[8] The same is true for $y''(x) = \lim_{h \to 0} \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$

[9] A similar example is the differential equation of the bending of the, so called, "elastic line", which we use to study the bending of beams[9]. It is an equation set up based on immediate assumptions about bending and not derived from the atomic or other structure of the beam.

[10] $\alpha^2 \frac{\partial}{\partial x}\left[\frac{y_x(x,t)}{\sqrt{1 + y_x^2(x,t)}}\right] - g\sqrt{1 + y_x^2(x,t)} + \frac{1}{\delta} f(x,t) = y_{tt}(x,t)$

for more than three to four decimal digits of a solution. Higher accuracy would be meaningful only for structures consisting of perfectly homogeneous and pure materials. The equations we set up are simplifications of the exact phenomena assuming a nonexistent perfect homogeneity and purity of materials. From this point of view a numeric approximation is, then, as good as the exact solution of an equation, because the equation itself is only approximating reality.

7. Against Platonism

Platonism believes that the abstract concepts, as the concepts of Geometry and in general the basic concepts of Mathematics exist in an immaterial world, the world of ideas, with which our thought communicates and is lead by it in creating appropriate concepts for the description of reality.

However, the conclusion of the above considerations is that the numbers as well as other mathematical magnitudes are creations of a process of evolution and simplification of the models by means of which we approximate the description of reality. They are not inherent in Nature and they do not exist in some immaterial world of ideas, as Plato imagined. Since they steadily evolve due to the progress in Mathematics, they have no a priori given fixed form. We, simply, imagine that their last, more recent and further developed form is their "true" or "real" form, because we are used to handle them by means of it.

Even the so called "natural numbers" have acquired a more clear foundation, when their presentation changed from an additive form (the Greek or Roman form) to the decimal (i.e., polynomial, or place-value) representation. This happened in the West gradually from the 13th to the 16th century, while real and complex numbers were clearly defined only in the 19th century.

As Einstein correctly observes, human understanding must, first, create independently the forms, before it can establish their existence in the material world.

Our impression that certain concepts are "natural", because we always use them, is merely due to habit. It is also due to the fact that certain concepts appear to us to be absolutely indispensable for thought. Equally indispensable for our life seems today to be the cell phone, but forty years ago no one could imagine that it will exist. The fact that something seems absolutely indispensable does not mean that this is inherent in the world that surrounds us.

8. How "objective" are Mathematics?

The contribution of Mathematics to the understanding of Nature.

The rejection of a physical existence of various mathematical magnitudes may raise the impression that Mathematics is an arbitrary creation like Poetry. However, this is a misconception! Mathematics provides a logically safe (the only logically safe) framework for a comprehensive outlook of reality[11]. Many other theoretical sciences provide important insights into various aspects of reality. However, they lack logical coherence. In social and economical sciences there is no absolute logical

[11] Mathematics is the only somewhat logically safe /secure framework on which we can base a logical outlook of reality.

interconnection of all concepts. So we cannot always reach valid conclusions from the existing theories no matter how brilliant they may be. We can achieve this indirectly to some extent by mathematizing these theories.

In short we may note:

1. The creation of abstract mathematical concepts is an idealization of empirical concepts, i.e. concepts corresponding to direct experiences.
2. Mathematics is the art and science of parsimonious logical modeling. We may idealize experience, but we build up logically coherent theories with these idealized concepts. This allows physical theories to achieve /attain logical coherence by adopting models corresponding to mathematical structures and satisfactorily certain physical phenomena.

Concluding Remarks

The essential consequence of this analysis is that most mathematical concepts are inventions and not somehow inherent in Nature. However, they are inventions, which considerably simplify the description of natural phenomena.

We should not consider fundamental concepts as unchanging /immutable. What is more, only certain origins of them are somehow inborne, i.e. given by the way we integrate in our mind outward experiences as we grow up. They are part of our steady dialogue with Nature. All limiting processes are inventions, but the first five to seven positive integers seem to be inherent in neural mechanisms. The geometrical concepts are also somehow a consensus between our perception of the environment and our understanding of it. What grants to these concepts persuasiveness is the logical coherence of geometric and other mathematical theories; not their supposed innate character.

In any case, our conception of the real numbers is not related to the way Nature operates, but rather to what seems convenient to us. Most mathematical quantities have no direct relation to natural phenomena. They are rather related to simplifications of some physical phenomena and have reached their present form by cultural evolution.

Their features, their form, is not intrinsic in nature but an attempt to model physical observations by means of concepts we form in order to perceive and control our immediate environment. They interpret phenomena of the microcosm (the world of atomic phenomena) and the macrocosm (the world of astronomic observations) in terms of concepts formed for use in the 'mesocosm', the world of directly perceptible objects and phenomena which have our order of magnitude. So, we should not expect to find for interatomic or astronomic phenomena descriptions that necessarily conform to common sense! Whether we should consider space and time as grainy (κοκκώδες) or continuous is not a matter of the supposed "true" nature of these concepts. It depends only on how convenient the mathematical model that results may be.

The ultimate goal of physics is to provide a parsimonious and mathematically handy account for the known phenomena. This is the essence of what we call "beauty" in a theory and not merely how strange it seems. As Pericles said one must "strife for beauty with thriftiness (parsimoniousness)" («Φιλοκαλούμεν μετ' ευτελείας») both in Science and in Art.

One must always bear in mind that the ultimate goal in a physical theory is not strangeness, but comprehensiveness (περιεκτικότητα) and parsimoniousness. The ultimate criterion of what is possible in nature is experimental verifiability and not some preconceived theoretical construction[12].

Today's physical theories seem to most people strange and unintelligible. What they would like to see is a theory, which would restore the basic concepts to their "normal" usage, the way we are accustomed to use them. However, this is unlikely. Physical theories may become even stranger, if this would allow a more parsimonious account of all experimental data.

We should not consider physical concepts as absolute. We should not speak of the 'true nature of space or time', but rather be aware that all theories are a matter of convenience. We cannot ignore experimental results, but we have always to be careful how we interpret them. Particularly, we must be aware that certain aspects of physical concepts are due to the mathematical instruments we use and not inherent in Nature.

The natural Sciences as a Shadow Theatre

The view of Nature provided by scientific theories can be likened with a foggy landscape or a shadow theatre. We see some hazy shadows /shapes, we give them names and describe the way they move or behave. However, we don't know and cannot even guess who or what they are. We will never know the "thing as such" (das Ding an sich) as Immanuel Kant observes. We should not forget that, even if we think that we perceive it clearly, what we see is determined by the structure and functioning of our brain and our sensory organs. For instance, there are insects and animals who see a different spectrum of light frequencies. The objects have for these animals different colours, which we cannot even imagine.

Reality is not so clear and unique as we like to believe. Nevertheless, what is important is the logical coherence of our scientific theories, the simplicity of their foundations and their verifiability. Our descriptions must describe faithfully the role that each shadow plays so that our conclusions may be trustworthy.

Literature

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2. A. D. Aleksandrov, A. N. Kolmogorov, M. A. Lavrentev: Mathematics, Its Contents, Methods and Meaning, Dover, 1999.

3. Peter Smith, Explaining Chaos, Cambridge U. P. 1998.

4. Robert B. Laughlin: A Different Universe, Basic Books, 2005.
(R. B. Laughlin: Ένα Διαφορετικό Σύμπαν, Κάτοπτρο, 2008).

5. Calvin Clawson: Mathematical Sorcery, Revealing the Secrets of Numbers, Perseus Publishing, 1999 (Calvin C. Clawson: Η Μαγεία των Μαθηματικών, Κέδρος 2008).

[12] Especially with respect to discrete versus continuous models one must bear in mind that discrete models seem from a certain point of view more appropriate for matter consisting of particles but are often far more difficult to analyze mathematically. But a continuous model can be in a sense just as well, since the solution of the equations, to which it leads, is usually done by discretization, i.e., by means of numerical approximations which can go up to the accuracy that measuring instruments can yield.

6. Robert B. Laughlin – David Pines: The Theory of Everything, 28-31, PNAS, January 4, 2000, vol.97, no.1. (to be found in the internet).
7. P. W. Anderson: More Is Different, *Science*, New Series, Vol. 177, No. 4047. (Aug. 4, 1972), pp. 393-396.
8. Stephen Blundell, Superconductivity, Oxford U.P., 2009.

Appendix 1: Robert Laughlin's views about the conditional existence of certain physical magnitudes

The view that Physics can be reduced to describing the behavior of elementary particles is rejected today by many scientists like the Nobel laureates Phillip A. Anderson [7] and Robert B. Laughlin [4, pp.15-20]. According to Laughlin, even the view that there are individual electrons is wrong.

What we consider to be as fundamental elements of the discretization of the structure of matter, like the concept "electron" [4, p.17], are not genuinely reductionist concepts, because the measurement of their properties (e.g. the measurement of the charge of an electron) is based on collective phenomena, i.e., on macroscopic measurements.

On p.18 he says: "Insofar as our knowledge of the physical world rests on experimental certainty, it is logical that we should associate the greatest truth with the most certain measurement. But this would seem to imply that a collective effect can be more true than the microscopic rules from which it descends. In the case of temperature, a quantity that never had a reductionist definition in the first place, this is easy to understand and accept. … But the electron charge is another matter. We are accustomed to thinking of this charge as a building block of nature requiring no collective context to make sense. The experiments in question, of course refute this idea. They reveal that the electron charge makes sense *only* in a collective context…"

On p. xv of the Preface he says: "I am increasingly persuaded that all physical law we know about has collective origins, not just some of it. In other words, the distinction between fundamental laws and the laws descending from them is a myth, as is the idea of mastery of the universe through mathematics alone. Physical law cannot generally be anticipated by pure thought, but must be discovered experimentally, because control of nature is achieved only when nature allows this through a principle of organization".

In the article "The Theory of Everything" written by R. Laughlin and David Pines, on p.28, it is noted that Schrödinger's differential equations cannot be solved with precision for a system of more than ten particles. No electronic computer that exists *or will ever exist* can exceed this limit, because the memory required increases exponentially with the number of particles. If the memory needed for the presentation of the wave equation is N, then the memory needed for the presentation of the wave equation of k particles is N^k. The, the equations of Quantum Theory, which seem to describe with very high accuracy the behavior of a small number of particles (single atoms and molecules) cannot be solved exactly for a system of more than 10 particles. No computer that can ever exist will be able to break this barrier, because it constitutes a "dimensional catastrophe". Thus, we are unable to predict the behavior of large systems of particles by calculating and have to take refuge in observation and experiment. In fact, certain approximation methods, which seem to describe the phenomena satisfactorily, are unable to predict the outcome of such experiments and are only useful for their retrospective explanation. Similarly, Stephen Blundell [8, p. 45 and 139-140] referring to the phenomenon of superconductivity at extremely low temperatures says that this inability to predict the behavior of a system is particularly obvious in Suprrconductor Physics, which is characterized by "*macroscopic quantum phenomena*".

The views of Peter Smith about mathematics and reality

Peter Smith [3, pp. 39-40] discusses the fact that the mathematical description of chaotic phenomena has no physical meaning for very small dimensions. The magnitudes, like the pressure of a gas or the velocity of a fluid, which certain differential equations are supposed to describe, lose their meaning then.

Peter Smith notes also correctly, that magnitudes like the velocity of a fluid, can never acquire a precise meaning. If it is assumed to be the mean value of the velocities of the particles in some region of space, then its value depends on the size of this region. If the region is too large, then the mean value of the particles is essentially constant. If the region is too small, then it may contain no particles of the fluid.

Appendix 2: The non-uniqueness of the set of Natural Numbers

The set of natural numbers is supposed to continue up to "infinity". The fact that this concept is in some sense the right end of this ordered set and is symbolized by the special symbol ∞, leads often to the misconception that it is something uniquely determined. However, already Cantor has shown that there are many kinds of "infinity", all the cardinal numbers and all the ordinal numbers.

Thoralf Skolem has shown that, in addition to such "infinities", we can also define a countable multitude of distinguishable "numbers", which can be characterized as "static infinities", because in the ordered queue of natural numbers they all come beyond any finite natural number.

Let us consider functions of the Natural Numbers (1, 2, 3, 4, …), which have only Natural Numbers as values. By definition write for two such functions $[f(x)]<[g(x)]$ if $f(x)<g(x)$ is true not necessarily for all, but for infinitely many positive integers x. This is a way of arranging such magnitudes, which is arbitrarily chosen by us.[13]

Then, we may extend the set of natural numbers in the following way, by including in the set such functions considered as entities:

$$1<2<3<\ldots<[x]<[x+1]<[x+3]<\ldots<[2x]<[2x+1]<\ldots<[x^2]<[x^2+1]<\ldots<[2x^2]< [2x^2+1]< \ldots<[x^3]<\ldots.$$

In this ordered set the expressions [x], [x+1], … may be called "numbers" since they are placed in the same serial arrangement as the usual natural numbers. Furthermore, in spite of the fact that they are not finite, they are distinct (different from each other). We may call them 'hyper-finite' or 'meta-finite' (because they come after the finite ones). They are, thus, distinct and different from each other

[13] In this arrangement it is $[5x+2]<[x^2]$, because it is $5x+2<x^2$ for all x that exceed 5.

static kinds of infinity (in contrast to the dynamic infinity of quantities tending to infinite). This extended set of natural numbers is often called “hyperintegers”, and played an important role in the creation of Nonstandard Analysis by Abraham Robinson. The term “infinite” has, thus, no definite meaning. It characterizes, thus, various kinds of mathematical magnitudes, determined by appropriate procedures.