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The Non-uniform in Time Small-Gain Theorem for a Wide Class of Control Systems with Outputs

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The notions of non-uniform in time robust global asymptotic output stability and non-uniform in time input-to-output stability (IOS) are extended to cover a wide class of control systems with outputs that includes (finite or infinite-dimensional) discrete-time and continuous-time control systems. A small-gain theorem, which makes use of the notion of non-uniform in time IOS property, is presented.

Keywords: Output Stability; Time-Varying Systems; Control Systems; Small-Gain Theorem

1. Introduction

The notion of non-uniform in time robust global asymptotic output stability (RGAOS) has been proved to be fruitful for the solution of several problems in Control Theory concerning finite-dimensional continuous-time systems (see, e.g. [14–17]). In this paper this notion is generalized in order to be applicable to a wide class of control systems with outputs that includes (finite or infinite-dimensional) discrete-time and continuous-time systems. The class of systems considered in this paper has the so-called "boundedness-implies-continuation" (BIC) property, which roughly speaking, means that

> the solution of the system can be continued as long as it remains bounded.

This property appears in all discrete-time systems and finite-dimensional continuous-time systems described by ordinary differential equations. It also appears in infinite-dimensional continuous-time systems described by retarded functional differential equations with completely continuous dynamics (e.g. systems that involve delays in their dynamics). Moreover, since every forward complete system has the BIC property, it is clear that the class of systems considered in this paper includes all forward complete infinite-dimensional continuous-time systems described by partial differential equations. The motivation for the extension of the notion of non-uniform in time RGAOS to a wide class of control systems that contains all discrete and continuous-time systems with delays is strong, since such systems are commonly used to model physical processes (see [2,21]).

A common feature of stability analysis is the application of small-gain results. Small-gain theorems for continuous-time finite-dimensional systems expressed in terms of "nonlinear gain functions" have a long history (see [6,7,24,25] and the references therein) that follows the fundamental work of Jiang–Teel–Praly in [5]. In particular most results make explicit use of the notion of uniform in time input-to-state stability (ISS), introduced by Sontag in [19], or the notion of uniform in time input-to-output stability (IOS), introduced by Sontag and Wang in [22,23] and extended in [3]. Recently, a non-uniform in time small-gain theorem for continuous-time finite-dimensional systems was presented in [15]. Small-gain theorems, converse Lyapunov theorems

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and the notion of uniform in time ISS for discrete-time finite-dimensional systems can be found in [8–12].

The present work generalizes the result in [15] to the wide class of control systems that possess the BIC property. Moreover, the framework used in this paper is more flexible compared to that used in [15], in the sense that the stability analysis of the present work incorporates outputs as well as "structured uncertainties". It is expected that the result presented in this paper will be a useful tool for the stability analysis of systems in the future.

The contents of this paper are presented as follows. In Section 2 we provide the notations and definitions of the notions used and several examples of systems that have the BIC property. In Section 3 we provide estimates of the transition maps expressed in norms of appropriate spaces as well as necessary and sufficient conditions for non-uniform in time RGAOS. The reader is introduced to the notion of non-uniform in time IOS property and the non-uniform in time small-gain theorem (Theorem 3.10) is presented. In Section 4 the proof of the non-uniform in time small-gain theorem is provided and numerical examples demonstrating the usefulness of the non-uniform in time small-gain theorem are also presented. Finally, Section 5 contains the conclusions of the paper. The proofs of some basic results are given in the appendix.

1.1. Notation

- By $\| \|_{\mathcal{X}}$, we denote the norm of the normed linear space $\mathcal{X}$. By $|$ $|$ we denote the euclidean norm of $\Re^n$.
- For definitions of classes K, K_∞, KL see [18]. K^+ denotes the class of positive continuous functions.
- A time set, denoted by $\mathcal{T}$, is either $\Re^+$ (the set of non-negative real numbers) or Z^+ (the set of non-negative integers). For any pair $a, b \in \mathcal{T}$ with $0 \le a \le b$ we define $[a,b] := \{t \in \mathcal{T}\colon a \le t \le b\}$. For any pair $a \in \mathcal{T}$, $b \in \mathcal{T} \cup \{+\infty\}$ with $0 \le a < b$ we define $[a,b) := \{t \in \mathcal{T}\colon a \le t < b\}$.
- For a given time set $\mathcal{T}$, by $\mathcal{M}(\mathcal{T}; U)$ we denote the set of all locally bounded functions $u : \mathcal{T} \to U$.
- By $C^j(A)$ ($C^j(A; \Omega)$), where $j \ge 0$ is a non-negative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of order j on A.

2. Control Systems with Outputs and the BIC Property

In this section, we introduce the reader to the notion of control systems with outputs used in this paper and the notion of a robust equilibrium point. Both concepts are defined in generality and capture the basic continuity properties needed in order to obtain non-trivial results. We would like to emphasize that the notion of the control systems adopted in this paper is similar to the notions of the topological dynamical systems used in [13,21], although there are some differences.

Definition 2.1. A control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs consists of

(i) a time set $\mathcal{T}$;
(ii) a set U which is a subset of a normed linear space $\mathcal{U}$ with $0 \in U$ and a set $M_U \subseteq \mathcal{M}(\mathcal{T}; U)$ which is called the "set of external inputs" and contains at least the identity zero input $u_0 \in M_U$, which satisfies $u_0(t) = 0 \in U$ for all $t \in \mathcal{T}$;
(iii) a set $M_D \subseteq \mathcal{M}(\mathcal{T}; D)$ which is called the "set of structured uncertainties";
(iv) a pair of normed linear spaces $\mathcal{X}, \mathcal{Y}$ called the "state space" and the "output space", respectively;
(v) a continuous map $H\colon \mathcal{T} \times \mathcal{X} \times U \to \mathcal{Y}$ that maps bounded sets of $\mathcal{T} \times \mathcal{X} \times U$ into bounded sets of $\mathcal{Y}$, called the "output map";
(vi) and the map $\phi\colon A_\phi \to \mathcal{X}$ where $A_\phi \subseteq \mathcal{T} \times \mathcal{T} \times \mathcal{X} \times M_U \times M_D$, called the "transition map", which has the following properties:

(1) *Existence*: For each $(t_0, x_0, u, d) \in \times \mathcal{T} \times \mathcal{X} \times M_U \times M_D$, there exists $t \in \mathcal{T}$, $t > t_0$ such that $[t_0, t] \times (t_0, x_0, u, d) \subseteq A_\phi$.
(2) *Identity property*: For each $(t_0, x_0, u, d) \in \mathcal{T} \times \mathcal{X} \times M_U \times M_D$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.
(3) *Causality*: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t)$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ with $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.
(4) *Semigroup property*: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t \ge t_0$ and for all $\tau \in [t_0, t]$, it holds that $(\tau, t_0, x_0, u, d) \in A_\phi$ with $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$.

A control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs is called a discrete-time system if $\mathcal{T} = Z^+$ and is called a continuous-time system if $\mathcal{T} = \Re^+$.

Among the control systems with outputs there is a class of control systems, which has a special property concerning the behavior of the transition map. This property is termed as the BIC property and is described below. There are many control systems that possess this property, as it will be shown by the examples in this section. The reader is also introduced to an important class of control systems with outputs that possess the BIC property – the class of forward complete control systems with outputs.

Definition 2.2. Consider a control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs. Let $U \subseteq U_1 \times U_2$, where U_1, U_2 is a pair of normed linear spaces. Let u_0 denote the identity zero input, that is, $u_0(t) = 0 \in U$ for all $t \in \mathcal{T}$ and let $u_{2,0}$ denote the identity zero input of the space U_2, that is, $u_{2,0}(t) = 0 \in U_2$ for all $t \in \mathcal{T}$. Let also $B_r := \{u_1 \in U_1; \|u_1\|_{U_1} \le r\}$ denote the closed sphere in U_1 with radius $r \ge 0$. We say that

(i) The system Σ has the BIC property if for each $(t_0, x_0, u, d) \in \mathcal{T} \times \mathcal{X} \times M_U \times M_D$, there exists a maximal existence time, that is, there exists $t_{\max} \in \mathcal{T} \cup \{+\infty\}$, such that $[t_0, t_{\max}) \times (t_0, x_0, u, d) \subseteq A_\phi$ and for all $t \ge t_{\max}$ it holds that $(t, t_0, x_0, u, d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$ then for every $M > 0$ there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > M$.

(ii) The system Σ is forward complete if for every $(t_0, x_0, u, d) \in \mathcal{T} \times \mathcal{X} \times M_U \times M_D$, $(t, t_0, x_0, u, d) \in A_\phi$ for all $t \ge t_0$. Clearly, every forward complete control system has the BIC property.

(iii) The system Σ is simply robustly forward complete (RFC) if it has the BIC property and for every $r \ge 0$, $T \ge 0$, it holds that

$$\sup\{\|\phi(t_0 + s, t_0, x_0, u_0, d)\|_{\mathcal{X}};\ s \in [0, T], \|x_0\|_{\mathcal{X}} \le r,\ t_0 \in [0, T], d \in M_D\} < +\infty.$$

(iv) The system Σ is RFC from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ if it has the BIC property and for every $r \ge 0$, $T \ge 0$, it holds that

$$\sup\{\|\phi(t_0 + s, t_0, x_0, (u_1, u_{2,0}), d)\|_{\mathcal{X}};\ u_1 \in \mathcal{M}(\mathcal{T}; B_r), (u_1, u_{2,0}) \in M_U,\ s \in [0, T], \|x_0\|_{\mathcal{X}} \le r, t_0 \in [0, T], d \in M_D\} < \infty.$$

In order to develop results concerning the stability of control systems we first need to define the notion of a robust equilibrium point.

Definition 2.3. Consider a control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with $0 \in U$ and $u_0 \in M_U$, where u_0 is the identity zero input, i.e., $u_0(t) = 0$ for all $t \in \mathcal{T}$. Suppose that $H(t, 0, 0) = 0$ for all $t \in \mathcal{T}$. We say that $0 \in \mathcal{X}$ is a robust equilibrium point for Σ if

(i) for every $(t, t_0, d) \in \mathcal{T} \times \mathcal{T} \times M_D$ with $t \ge t_0$ it holds that $\phi(t, t_0, 0, u_0, d) = 0$;

(ii) for every $\varepsilon > 0$, $T, h \in \mathcal{T}$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that if $\|x\|_{\mathcal{X}} < \delta$, $t_0 \in [0, T]$, $\tau \in [t_0, t_0 + h]$ then $(\tau, t_0, x, u_0, d) \in A_\phi$ for all $d \in M_D$ and

$$\sup\{\|\phi(\tau, t_0, x, u_0, d)\|_{\mathcal{X}}; d \in M_D,\ \tau \in [t_0, t_0 + h], t_0 \in [0, T]\} < \varepsilon.$$

The reader should not be surprised by the previous definition of a robust equilibrium point. The usual definition of equilibrium point does not require property (ii) of Definition 2.3 to hold. However, in most cases the control systems satisfy the property of *continuous dependence on the initial conditions* of the transition map, that is,

for each $(t, T, x_0, u) \in \mathcal{T} \times \mathcal{T} \times \mathcal{X} \times M_U$ with $t \ge T$ and for every $\varepsilon > 0$, there exists $\delta > 0$, such that if $\|x - x_0\|_{\mathcal{X}} < \delta$ then for every $(t_0, d) \in [0, T] \times M_D$ with $(t, t_0, x_0, u, d) \in A_\phi$ it follows that $(t, t_0, x, u, d) \in A_\phi$ and $\sup\{\|\phi(\tau, t_0, x, u, d) - \phi(\tau, t_0, x_0, u, d)\|_{\mathcal{X}};\ (t_0, d) \in [0, T] \times M_D, \tau \in [t_0, t]$ with $(t, t_0, x_0, u, d) \in A_\phi\} < \varepsilon$.

It can be immediately verified that if the transition map depends continuously on the initial conditions then the usual definition of an equilibrium point is equivalent to Definition 2.3 (since property (ii) of Definition 2.3 is automatically satisfied). Since, our effort is to provide results for systems that do not necessarily satisfy the property of continuous dependence on the initial conditions (e.g. discrete-time systems with discontinuous dynamics), we do not assume this property.

Definition 2.3 clarifies the main reason for which there is a distinction of the inputs acting on the system in Definition 2.1 of control systems ("external inputs" and "structured uncertainties"). The inputs that belong to the "set of structured uncertainties" ($M_D \subseteq \mathcal{M}(\mathcal{T}; D)$) do not alter the position of equilibrium points. On the other hand, inputs that belong to the "set of external inputs" ($M_U \subseteq \mathcal{M}(\mathcal{T}; U)$) are allowed to alter the position of equilibrium points. This reminds the difference between "additive" and "multiplicative" uncertainties in linear system theory.

The following examples show that the class of control systems with outputs and the BIC property is a wide class that contains all discrete-time systems, finite-dimensional systems described by ordinary differential equations, and infinite-dimensional systems described by retarded functional differential equations with completely continuous dynamics.

Example 2.4. Consider a discrete-time control system with outputs $\Sigma := (Z^+, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$, with $M_U \equiv \mathcal{M}(Z^+; U)$ and $M_D \equiv \mathcal{M}(Z^+; D)$, that is, the sets of all sequences with values in U and D, respectively. We notice that by virtue of the causality and existence properties the map $f(t, x, u, d) := \phi(t + 1, t, x, u, d)$ is actually defined on $Z^+ \times \mathcal{X} \times U \times D$. We notice that the semigroup and the causality properties of the transition map imply that for every $(t, t_0, x_0, u, d) \in M_U \times M_D$, with $t \geq t_0$ it holds that:

$$\begin{aligned} x(t+1) &= \phi(t+1, t_0, x_0, \{u(\tau); \tau \in [t_0, t]\}, \\ &\quad \{d(\tau); \tau \in [t_0, t]\}) \\ &= f(t, \phi(t, t_0, x_0, \{u(\tau); \tau \in [t_0, t-1]\}, \\ &\quad \{d(\tau); \tau \in [t_0, t-1]\}), u(t), d(t)) \\ &= f(t, x(t), u(t), d(t)), \\ Y(t) &= H(t, \phi(t, t_0, x_0, \{u(\tau); \tau \in [t_0, t-1]\}, \\ &\quad \{d(\tau); \tau \in [t_0, t-1]\}), u(t)) \\ &= H(t, x(t), u(t)). \end{aligned}$$

This is the so-called evolution equation of the control system. Clearly, every discrete-time control system is forward complete. Moreover, if there exist funcions $a \in K_\infty$, $\beta \in K^+$ such that $\|f(t, x, 0, d)\|_{\mathcal{X}} \leq a(\beta(t)\|x\|_{\mathcal{X}})$ for all $(t, x, d) \in Z^+ \times \mathcal{X} \times D$ then $0 \in \mathcal{X}$ is a robust equilibrium point for Σ.

Example 2.5. Every pair of continuous mappings $f : \Re^+ \times \Re^n \times U \times D \to \Re^n$, $H : \Re^+ \times \Re^n \times U \to \Re^k$, where $0 \in U \subseteq \Re^m$, $D \subseteq \Re^l$, with $H(t, 0, 0) = 0$, $f(t, 0, 0, d) = 0$ for all $(t, d) \in \Re^+ \times D$ and such that the vector field $f : \Re^+ \times \Re^n \times U \times D \to \Re^n$, satisfies the following Lipschitz condition:

The function $f(t, x, u, d)$ is locally Lipschitz with respect to (x, u), uniformly in $d \in D$, in the sense that for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^n \times U$, there exists a constant $L \geq 0$ such that:

$$\begin{aligned} &|f(t, x, u, d) - f(t, y, v, d)| \leq L|(x-y, u-v)| \\ &\forall t \in I, \ \forall (x, u; y, v) \in S \times S, \ \forall d \in D \end{aligned}$$

defines a continuous-time control system $\Sigma := (\Re^+, \Re^n, \Re^k, M_U, M_D, \phi, H)$ with outputs and the BIC property, by the evolution equation:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t), d(t)), \\ Y(t) &= H(t, x(t), u(t)). \end{aligned}$$

This fact is also an immediate consequence of Proposition 3.7.2 in [21]. In this case M_U and M_D are the sets of all measurable and locally bounded functions with values in U and D, respectively. Notice that $0 \in \Re^n$ is a robust equilibrium point for Σ.

The following example is an immediate consequence of Theorems 2.2 and 3.2 in [2], concerning continuous dependence on initial conditions and continuation of solutions of retarded functional differential equations, respectively.

Example 2.6. Every pair of completely continuous mappings $f : \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D \to \Re^n$, $H : \Re^+ \times C^0([-r, 0]; \Re^n) \times U \to C^0([-r, 0]; \Re^k)$, where $r > 0$ is a constant, $0 \in U \subseteq \Re^m$, $D \subseteq \Re^l$ with $H(t, 0, 0) = 0$, $f(t, 0, 0, d) = 0$ for all $(t, d) \in \Re^+ \times D$ and such that the vector field $f : \Re^+ \times C^0([-r, 0]; \Re^n) \times U \times D \to \Re^n$, satisfies the following Lipschitz condition:

The function $f(t, x, u, d)$ is locally Lipschitz with respect to (x, u), uniformly in $d \in D$, in the sense that for every bounded interval $I \subset \Re^+$ and for every closed and bounded subset S of $C^0([-r, 0]; \Re^n) \times U$, there exists a constant $L \geq 0$ such that:

$$\begin{aligned} &|f(t, x, u, d) - f(t, y, v, d)| \leq L|u - v| \\ &\quad + L \max_{\tau \in [-r, 0]} |x(\tau) - y(\tau)|, \\ &\forall t \in I, \ \forall (x, y; y, v) \in S \times S, \ \forall d \in D \end{aligned}$$

defines a continuous-time control system $\Sigma := (\Re^+, C^0([-r, 0]; \Re^n), C^0([-r, 0]; \Re^k), M_U, M_D, \phi, H)$ with outputs and the BIC property, by the evolution equation:

$$\begin{aligned} \dot{x}(t) &= f(t, \{x(t+\tau); \tau \in [-r, 0]\}, u(t), d(t)), \\ Y(t) &= H(t, \{x(t+\tau); \tau \in [-r, 0]\}, u(t)). \end{aligned}$$

In this case M_U and M_D are the sets of measurable and locally bounded functions with values in U and D, respectively. Notice that $0 \in C^0([-r, 0]; \Re^n)$ is a robust equilibrium point for Σ.

3. Definitions of Stability Notions and Main Results

In this section we introduce the reader to the notion of a non-uniformly in time RGAOS system with outputs and the BIC property and we provide estimates for the transition maps of such systems. Notice that the definition of this property requires the external inputs acting on the control system to be identically equal to zero, that is, RGAOS is an "internal stability" property.

Definition 3.1. Consider a control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs that has the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point. Let $u_0 \in M_U$ be the identity zero input, that is, $u_0(t) = 0$ for all $t \in \mathcal{T}$. We say that Σ is non-uniformly in time RGAOS if Σ is RFC and the following properties hold:

P1 Σ is *Robustly Lagrange output stable*, that is, for every $\varepsilon > 0, T \in \mathcal{T}$, it holds that

$$\begin{aligned} &\sup\{\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}; t \in [t_0, +\infty), \\ &\quad \|x_0\|_{\mathcal{X}} \leq \varepsilon, t_0 \in [0, T], d \in M_D\} < +\infty. \end{aligned}$$

(*Robust Lagrange output stability*)

P2 Σ is *Robustly Lyapunov output stable*, that is, for every $\varepsilon > 0, T \in \mathcal{T}$ there exists a $\delta := \delta(\varepsilon, T) > 0$ such that:

$$\|x_0\|_{\mathcal{X}} \leq \delta, t_0 \in [0, T] \implies \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \varepsilon, \quad \forall t \in [t_0, +\infty), \ \forall d \in M_D.$$
(*Robust Lyapunov output stability*)

P3 Σ satisfies the *robust output attractivity property*, that is, for every $\varepsilon > 0, T \in \mathcal{T}$ and $R \geq 0$, there exists a $\tau := \tau(\varepsilon, T, R) \in \mathcal{T}$, such that:

$$\|x_0\|_{\mathcal{X}} \leq R, t_0 \in [0, T] \implies \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \varepsilon, \quad \forall t \in [t_0 + \tau, +\infty), \ \forall d \in M_D.$$

Moreover, if there exists $a \in K_\infty$ such that $a(\|x\|_{\mathcal{X}}) \leq \|H(t, x, 0)\|_{\mathcal{Y}}$ for all $(t, x) \in \mathcal{T} \times \mathcal{X}$, then we say that Σ is non-uniformly in time robustly globally asymptotically stable (RGAS).

The following four technical lemmas are proved in the appendix and are essential for the establishment of characterizations of RGAOS. Particularly, the following lemma is fundamental for the derivation of the basic estimates of the solutions of RGAOS control systems. Its proof is given in the appendix.

Lemma 3.2. Let $\mathcal{X}, \mathcal{Y}$ a pair of normed linear spaces and $\mathcal{T}$ a time set and let $H : \mathcal{T} \times \mathcal{X} \times D \to \mathcal{Y}$ a map that satisfies:

(i) for every bounded set $S \subset \mathcal{T} \times \mathcal{X}$ the set $H(S \times D)$ is bounded
(ii) $H(t, 0, d) = 0$ for all $(t, d) \in \mathcal{T} \times D$
(iii) for every $\varepsilon > 0, t \in \mathcal{T}$ there exists $\delta := \delta(\varepsilon, t) > 0$ such that

$$\sup\{\|H(\tau, x, d)\|_{\mathcal{Y}}; \tau \in \mathcal{T}, d \in D, |\tau - t| + \|x\|_{\mathcal{X}} < \delta\} < \varepsilon.$$

Then there exists a pair of functions $\zeta \in K_\infty$ and $\beta \in K^+$ such that:

$$\|H(t, x, d)\|_{\mathcal{Y}} \leq \zeta(\beta(t)\|x\|_{\mathcal{X}}), \quad \forall (t, x, d) \in \mathcal{T} \times \mathcal{X} \times D. \tag{3.1}$$

The next lemma shows an essential property of robust equilibrium points of control systems with outputs and the BIC property. Robust forward completeness and robust output attractivity guarantee robust Lyapunov and Lagrange output stability.

Lemma 3.3. Consider the control system $\Sigma = (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs and the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point. Suppose that Σ is RFC and satisfies the robust output attractivity property (property P3 of Definition 3.1). Then Σ is non-uniformly in time RGAOS.

The next lemma provides an estimate of the output behavior for non-uniformly in time RGAOS systems.

Lemma 3.4. Suppose that the control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs is non-uniformly in time RGAOS. Then there exist functions $\sigma \in KL, \beta \in K^+$ such that the following estimate holds for all $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$ and $t \in [t_0, +\infty)$:

$$\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \leq \sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0). \tag{3.2}$$

Finally, the following lemma provides an estimate for the transition map, which turns out to be a necessary and sufficient condition for robust forward completeness. It should be emphasized that the notion of robust forward completeness and its characterization provided by the following lemma apply also to control systems for which $0 \in \mathcal{X}$ is not necessarily an equilibrium point. Notice that similar characterizations are given in [1,17] for the case of finite-dimensional continuous-time systems with locally Lipschitz dynamics.

Lemma 3.5. Consider a control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs and the BIC property. Let $U \subseteq U_1 \times U_2$, where U_1, U_2 is a pair of normed linear spaces. Let $u_{2,0}$ denote the identity zero input of the space U_2, that is, $u_{2,0}(t) = 0 \in U_2$ for all $t \in \mathcal{T}$.

(i) Σ is RFC from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ if and only if there exist functions $\mu \in K^+, a \in K_\infty$ and a constant $R \geq 0$ such that the following estimate holds for all $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ with $(u_1, u_{2,0}) \in M_U$ and $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$:

$$\|\phi(t, t_0, x_0, (u_1, u_{2,0}), d)\|_{\mathcal{X}} \leq \mu(t) a\Big(R + \|x_0\|_{\mathcal{X}} + \sup_{\tau \in [t_0, t]} \|u_1(\tau)\|_{U_1}\Big), \quad \forall t \in [t_0, +\infty). \tag{3.3}$$

(ii) Σ is RFC, if and only if, there exist functions $\mu \in K^+$, $a \in K_\infty$ and a constant $R \geq 0$, such that for every $d \in M_D, (t_0, x_0) \in \mathcal{T} \times \mathcal{X}$, we have:

$$\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \mu(t) a(\|x_0\|_{\mathcal{X}} + R), \quad \forall t \in [t_0, +\infty). \tag{3.4}$$

Moreover, if $0 \in \mathcal{X}$ is a robust equilibrium point for Σ then inequality (3.4) holds with $R = 0$.

The following theorem combines the estimates provided by Lemmas 3.2–3.5 in order to obtain less conservative estimates for the transition map and the output and provide alternative characterizations of RGAOS.

Theorem 3.6. Consider a control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs and the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point. Let $u_0 \in M_U$ be the identity zero input, that is, $u_0(t) = 0$ for all $t \in T$. Then the following statements are equivalent:

(i) Σ is RGAOS.

(ii) There exist functions $\mu,\ \beta \in K^+$, $\sigma \in KL$, such that for every $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$, we have:

$$\begin{aligned} &\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \\ &\quad + \mu(t)\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \\ &\leq \sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0), \quad \forall t \in [t_0, +\infty). \end{aligned} \tag{3.5}$$

(iii) There exist functions $\mu,\ \beta \in K^+$, $\sigma \in KL$, $a \in K_\infty$ and a constant $r \geq 0$ such that for every $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$, we have:

$$\begin{aligned} &\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \\ &\quad \leq \sigma(\beta(t_0)(\|x_0\|_{\mathcal{X}} + r), t - t_0), \quad \forall t \in [t_0, +\infty), \end{aligned} \tag{3.6a}$$

$$\begin{aligned} &\|\phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{X}} \\ &\quad \leq \mu(t)a(\|x_0\|_{\mathcal{X}} + r), \quad \forall t \in [t_0, +\infty). \end{aligned} \tag{3.6b}$$

Proof. The implication (ii) $\Rightarrow$ (iii) is obvious. The implication (iii) $\Rightarrow$ (i) follows immediately by applying the results of Lemmas 3.3 and 3.5. To be more precise, notice that (by virtue of Lemma 3.5) estimate (3.6b) implies that Σ is RFC and (by virtue of the properties of the *KL* functions) estimate (3.6a) implies that Σ satisfies the robust output attractivity property. Consequently, since $0 \in \mathcal{X}$ is a robust equilibrium point, it follows by virtue of Lemma 3.3 that Σ is RGAOS.

Next we prove implication (i) $\Rightarrow$ (ii). Suppose that Σ is RGAOS. Then Lemmas 3.4 and 3.5 guarantee that there exist functions $\widetilde{\sigma} \in KL, \widetilde{\beta}, \widetilde{\mu} \in K^+, \widetilde{a} \in K_\infty$, such that the following estimates hold for all $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$ and $t \in [t_0, +\infty)$:

$$\begin{aligned} &\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \\ &\quad \leq \widetilde{\sigma}\big(\widetilde{\beta}(t_0)\|x_0\|_{\mathcal{X}}, t - t_0\big), \end{aligned} \tag{3.7a}$$

$$\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \leq \widetilde{\mu}(t)\widetilde{a}(\|x_0\|_{\mathcal{X}}). \tag{3.7b}$$

Estimates (3.7a) and (3.7b) imply (3.5) for $\sigma(s,t) := \widetilde{\sigma}(s,t) + \exp(-t)\widetilde{a}(s), \mu(t) := (\exp(-t)/\widetilde{\mu}(t)) \in K^+$ and $\beta(t) := \widetilde{\beta}(t) + 1$. The proof is complete. □

Next the reader is introduced to the notion of non-uniform in time IOS property for a control system with outputs and the BIC property. This notion is concerned with the qualitative behavior of a control system subject to the presence of external inputs acting on the control system (i.e. IOS is an "external stability" property).

Definition 3.7. Consider a control system $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ with outputs and the BIC property and for which $0 \in \mathcal{X}$ is a robust equilibrium point. Let $U \subseteq U_1 \times U_2$, where U_1, U_2 is a pair of normed linear spaces. Let $\mathrm{u}_{2,0}$ denote the identity zero input of the space U_2, that is, $u_{2,0}(t) = 0 \in U_2$ for all $t \in \mathcal{T}$. We say that Σ satisfies the *non-uniform in time IOS* property from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ if Σ is RFC from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ and there exist functions $\sigma \in KL,\ \beta,\ \gamma \in K^+,\ \rho \in K_\infty$ such that the following estimate holds for all $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ with $(u_1, u_{2,0}) \in M_U, (t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$ and $t \in [t_0, +\infty)$:

$$\begin{aligned} &\|H(t, \phi(t, t_0, x_0, (u_1, u_{2,0}), d), (u_1(t), 0))\|_{\mathcal{Y}} \\ &\quad \leq \max\Big\{\sigma(\beta(t_0)\|x_0\|_{\mathcal{X}}, t - t_0), \\ &\qquad \sup_{\tau \in [t_0, t]} \sigma\big(\beta(\tau)\rho\big(\gamma(\tau)\|u_1(\tau)\|_{U_1}\big), t - \tau\big)\Big\}. \end{aligned} \tag{3.8}$$

Moreover, if there exists $a \in K_\infty$ such that $a(\|x\|_{\mathcal{X}}) \leq \|H(t, x, (u_1, 0))\|_{\mathcal{Y}}$ for all $(t, x, u_1) \in \mathcal{T} \times \mathcal{X} \times U_1$ with $(u_1, 0) \in U$, then we say that Σ satisfies the *non-uniform in time ISS property* from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$.

It is clear by if Σ satisfies the non-uniform in time IOS property from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ then Σ is RGAOS and Σ is RFC from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$.The converse statement holds for autonomous finite-dimensional systems described by ordinary differential equations (see [15]) when the output is considered to be the whole state vector.

Usually the functions $\sigma \in KL$, $\beta, \gamma \in K^+$ and $\rho \in K_\infty$ which are involved in (3.8) are determined by using a Lyapunov functional for the system Σ (see [15] as well as the numerical examples of the following section of the present paper). However, this important issue will not be addressed directly in this paper. In the present paper the emphasis is placed on the existence of such functions.

The IOS property plays a fundamental role in the stability analysis of interconnected systems. We next give the notion of the interconnection or feedback connection of control systems.

Definition 3.8. Consider a pair of control systems $\Sigma_1 = (\mathcal{T}, \mathcal{X}_1, \mathcal{Y}_1, M_{S_2\times U}, M_D, \widetilde{\phi}_1, H_1), \Sigma_2 = (\mathcal{T}, \mathcal{X}_2, \mathcal{Y}_2, M_{S_1\times U}, M_D, \widetilde{\phi}_2, H_2)$ with outputs $H_1 : \mathcal{T} \times \mathcal{X}_1 \times U \to S_1 \subseteq \mathcal{Y}_1, H_2 : \mathcal{T} \times \mathcal{X}_2 \times \mathcal{Y}_1 \times U \to S_2 \subseteq \mathcal{Y}_2$ and the BIC property and for which $0 \in \mathcal{X}_i \quad i = 1, 2$ are robust equilibrium points. Suppose that there exists a *unique* map $\phi = (\phi_1, \phi_2) : A_\phi \to \mathcal{X}$ where $A_\phi \subseteq \mathcal{T} \times \mathcal{T} \times \mathcal{X} \times M_U \times M_D$ and $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, such that for every $(t, t_0, x_0, u, d) \in A_\phi$ with $t \geq t_0, x_0 = (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ it holds that:

> "there exists a pair of external inputs $\nu_i \subseteq \mathcal{M}(\mathcal{T}; S_i)\ i = 1, 2$ with $\nu_1(\tau) = H_1(\tau, \phi_1(\tau, t_0, x_0, u, d), u(\tau))$, $\nu_2(\tau) = H_2(\tau, \phi_2(\tau, t_0, x_0, u, d), \nu_1(\tau), u(\tau))$ for all $\tau \in [t_0, t]$, $(\nu_i, u) \in M_{S_i\times U}\ i = 1, 2$ and $\phi_1(\tau, t_0, x_0, u, d) = \widetilde{\phi}_1(\tau, t_0, x_1, (\nu_2, u), d), \phi_2(\tau, t_0, x_0, u, d) = \widetilde{\phi}_2(\tau, t_0, x_2, (\nu_1, u), d)$ for all $\tau \in [t_0, t]$."

Moreover, suppose that $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is a control system with outputs and the BIC property, where $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$, $H(t, x_1, x_2, u) := (H_1(t, x_1, u), H_2(t, x_2, H_1(t, x_1, u), u))$ for all $(t, x_1, x_2, u) \in \mathcal{T} \times \mathcal{X}_1 \times \mathcal{X}_2 \times U$, for which $0 \in \mathcal{X}$ is a robust equilibrium point and that there exists a constant $K > 0$ such that

$$K(\|Y_1\|_{\mathcal{Y}_1} + \|Y_2\|_{\mathcal{Y}_2}) \geq \|Y\|_{\mathcal{Y}} \geq \max\left\{\|Y_1\|_{\mathcal{Y}_1}, \|Y_2\|_{\mathcal{Y}_2}\right\}, \quad \text{for all } Y = (Y_1, Y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2, \tag{3.9a}$$

$$K(\|x_1\|_{\mathcal{X}_1} + \|x_2\|_{\mathcal{X}_2}) \geq \|x\|_{\mathcal{X}} \geq \max\left\{\|x_1\|_{\mathcal{X}_1}, \|x_2\|_{\mathcal{X}_2}\right\}, \quad \text{for all } x = (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2. \tag{3.9b}$$

Then system Σ is said to be the *feedback connection* or the *interconnection* of systems Σ_1 and Σ_2.

Remark 3.9. Notice that there is a difference between Definition 3.8 and 7.2.3 in [21]: we do not exclude interconnections of control systems that may have finite escape time. Moreover, notice that we can allow subsystem $\Sigma_2 = (\mathcal{T}, \mathcal{X}_2, \mathcal{Y}_2, M_{\mathcal{Y}_1\times U}, M_D, \widetilde{\phi}_2, H_2)$ to be just a continuous map from $\mathcal{T} \times \mathcal{Y}_1$ into $\mathcal{Y}_2$ (this is allowed as well by Definition 7.2.3 in [21]). Of course, usually continuity of this map is not enough to guarantee that there is an interconnection of two subsystems. More specifically, in order to guarantee the uniqueness of the map $\phi = (\phi_1, \phi_2) : A_\phi \to \mathcal{X}_1 \times \mathcal{X}_2$ other regularity properties must be satisfied as well depending on the nature of the overall system (e.g. for systems described by ordinary differential equations this map must be locally Lipschitz).

The following theorem generalizes the non-uniform in time small-gain theorem in [15], since it considers interconnections of the wide class of control systems that satisfy the BIC property and allows greater flexibility since the outputs of each subsystem are not required to be identically the state of each subsystem.

Theorem 3.10. Let u_0 denote the identity zero input, i.e., $u_0(t) = 0 \in U$ for all $t \in \mathcal{T}$. Suppose that $\Sigma := (\mathcal{T}, \mathcal{X}, \mathcal{Y}, M_U, M_D, \phi, H)$ is the feedback connection of systems $\Sigma_1 = (\mathcal{T}, \mathcal{X}_1, \mathcal{Y}_1, M_{S_2\times U}, M_D, \widetilde{\phi}_1, H_1)$ and $\Sigma_2 = (\mathcal{T}, \mathcal{X}_2, \mathcal{Y}_2, M_{S_1\times U}, M_D, \widetilde{\phi}_2, H_2)$ with outputs $H_1 : \mathcal{T} \times \mathcal{X}_1 \times U \to S_1 \subseteq \mathcal{Y}_1, H_2 : \mathcal{T} \times \mathcal{X}_2 \times \mathcal{Y}_1 \times U \to S_2 \subseteq \mathcal{Y}_2$. We assume that:

H1 Subsystem Σ_1 satisfies the non-uniform in time IOS property from the input $v_2 \in \mathcal{M}(\mathcal{T}; S_2)$. Particularly, assume that there exist functions $\sigma_1 \in KL, \beta_1, \gamma_1 \in K^+, \rho_1 \in K_\infty$ such that the following estimate holds for all $(t_0, x_1, (v_2, u_0), d) \in \mathcal{T} \times \mathcal{X}_1 \times M_{S_2\times U} \times M_D$ and $t \in [t_0, +\infty)$:

$$\|H_1(t, \widetilde{\phi}_1(t, t_0, x_1, (v_2, u_0), d), 0)\|_{\mathcal{Y}_1} \leq \max\left\{\sigma_1(\beta_1(t_0)\|x_1\|_{\mathcal{X}_1}, t - t_0), \sup_{\tau\in[t_0, t]} \sigma_1(\beta_1(\tau)\rho(\gamma_1(\tau)\|v_2(\tau)\|_{\mathcal{Y}_2}), t - \tau)\right\}. \tag{3.10}$$

H2 Subsystem Σ_2 satisfies the non-uniform in time IOS property from the input $v_1 \in \mathcal{M}(\mathcal{T}; S_1)$. Particularly, assume that there exist functions $\sigma_2 \in KL, \beta_2, \gamma_2 \in K^+, \rho_2 \in K_\infty$ such that the following estimate holds for all $(t_0, x_2, (v_1, u_0), d) \in \mathcal{T} \times \mathcal{X}_2 \times M_{S_1\times U} \times M_D$ and $t \in [t_0, +\infty)$:

$$\|H_2(t, \widetilde{\phi}_2(t, t_0, x_2, (v_1, u_0), d), v_1(t), 0)\|_{\mathcal{Y}_2} \leq \max\left\{\sigma_2(\beta_2(t_0)\|x_2\|_{\mathcal{X}_2}, t - t_0), \sup_{\tau\in[t_0, t]} \sigma_2(\beta_2(\tau)\rho(\gamma_2(\tau)\|v_1(\tau)\|_{\mathcal{Y}_1}), t - \tau)\right\}. \tag{3.11}$$

H3 In addition we assume that the following properties hold for all $t_0, s \geq 0$:

$$\lim_{t\to+\infty} \beta_1(t)\rho_1(\gamma_1(t)\sigma_2(s, t - t_0)) = 0, \tag{3.12a}$$

$$\lim_{t\to+\infty} \beta_2(t)\rho_2(\gamma_2(t)\sigma_1(s, t - t_0)) = 0. \tag{3.12b}$$

H4 Moreover, there exists a function a of class K_∞ with

$$a(s) \leq s, \quad \forall s \geq 0. \tag{3.13}$$

such that the following inequalities are satisfied for all $t_0 \geq 0$:

$$\sup_{t\in[t_0,+\infty)} \sigma_1(\beta_1(t)\rho_1(\gamma_1(t)\sigma_2(\beta_2(t_0)$$
$$\times \rho_2(\gamma_2(t_0)s), t-t_0)), 0) \leq a(s), \quad \forall s \geq 0 \tag{3.14a}$$
$$\sup_{t\in[t_0,+\infty)} \sigma_2(\beta_2(t)\rho_2(\gamma_2(t)\sigma_1(\beta_1(t_0)$$
$$\times \rho_1(\gamma_1(t_0)s), t-t_0)), 0) \leq a(s), \quad \forall s \geq 0 \tag{3.14b}$$

Then system Σ is non-uniformly in time RGAOS.

Remark 3.11

(i) Obviously, when $\beta_i\,\gamma_i$ $(i=1,2)$ bounded over $\Re^+$ (case of uniform in time ISS property), then Hypothesis H3 is automatically satisfied. Furthermore, if we define $i=1,2$ $r_i := \sup_{t\geq 0}\beta_i(t)$, $\mu_i := \sup_{t\geq 0}\gamma_i(t)$, $\zeta_i(s) := \sigma_i(r_i\rho_i(\mu_i s), 0)$, then it can be easily established that Hypothesis H4 is satisfied as well, provided that

$$\zeta_1(\zeta_2(s)) < s, \quad \forall s > 0 \tag{3.15}$$

which is exactly the same condition imposed in [5] for the Small-Gain Theorem in the uniform in time case. Indeed, if (3.15) holds, then by virtue of definitions of r_i, μ_i and ζ_i above, inequalities (3.14a,b) are satisfied with

$$a(s) := \max\left\{\zeta_1(\zeta_2(s)), \zeta_2(\zeta_1(s)), \tfrac{1}{2}s\right\}.$$

(ii) Hypothesis H3 is needed because the non-uniform in time IOS property does not guarantee the converging input converging output (CICO) property. Notice that this is an essential difference between the non-uniform in time and the uniform in time IOS properties.

4. Proof of the Non-Uniform in Time Small-Gain Theorem and Numerical Examples

First, the proof of the non-uniform in time small-gain theorem (Theorem 3.10) is provided. Let us denote by $Y_1(t) := H_1(t, \phi_1(t, t_0, x_0, u_0, d), 0)$, $Y_2(t) := H_2(t, \phi_2(t, t_0, x_0, u_0, d), Y_1(t), 0)$ and $Y(t) = Y_1(t), Y_2(t))$ the outputs of system Σ for some $x_0 = (x_1, x_1) \in \mathcal{X}_1 \times \mathcal{X}_2, (t, t_0, x_0, d) \in A_\phi$ with $t \in [t_0, +\infty)$. The following claim is proved in the appendix and provides essential estimates for the trajectories of system Σ.

Claim. Under hypotheses H1–H4, Σ is RFC and there exist functions $B_i \in KL, \widetilde{\beta}_i \in K^+ (i=1,2)$ such that the following estimates hold for all) $t \in [t_0, +\infty)$:

$$\|Y_1(t)\|_{\mathcal{Y}_1} \leq \max\left\{B_1(\widetilde{\beta}_1(t_0)\|x_0\|_{\mathcal{X}}, t-t_0),\ a\Big(\sup_{\tau\in[t_0,t]}\|Y_1(t)\|_{\mathcal{Y}_1}\Big)\right\}, \tag{4.1a}$$
$$\|Y_2(t)\|_{\mathcal{Y}_2} \leq \max\left\{B_2(\widetilde{\beta}_2(t_0)\|x_0\|_{\mathcal{X}}, t-t_0),\ a\Big(\sup_{\tau\in[t_0,t]}\|Y_2(t)\|_{\mathcal{Y}_2}\Big)\right\}, \tag{4.1b}$$
$$\|Y_1(t)\|_{\mathcal{Y}} \leq B_1(\widetilde{\beta}_1(t_0)\|x_0\|_{\mathcal{X}}, 0), \tag{4.2a}$$
$$\|Y_2(t)\|_{\mathcal{Y}} \leq B_2(\widetilde{\beta}_2(t_0)\|x_0\|_{\mathcal{X}}, 0), \tag{4.2b}$$

where a is the function involved in (3.13) and (3.14a,b).

Next we prove that under hypotheses H1–H4, system Σ is non-uniformly in time RGAOS. Without loss of generality we may assume that the functions $\widetilde{\beta}_i$ $i=1,2$ (involved in (4.1a,b) and (4.2a,b)) are both non-decreasing. Let $K>0$ be the constant that satisfies $K(\|Y_1\|_{\mathcal{Y}_1} + \|Y_2\|_{\mathcal{Y}_2}) \geq \|Y\|_{\mathcal{Y}}$ for all $Y = (Y_1, Y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2$ Using the estimates (4.2a,b) we get:

$$\|Y(t)\|_{\mathcal{Y}} \leq KB_1(\widetilde{\beta}_1(t_0)\|x_0\|_{\mathcal{X}}, 0) + KB_2(\widetilde{\beta}_2(t_0)\|x_0\|_{\mathcal{X}}, 0), \quad \forall t \in [t_0, +\infty) \tag{4.3}$$

Inequality (4.3) shows that Σ is Robustly Lagrange and Lyapunov Output Stable. Next we establish that Σ satisfies the Robust Output Attractivity property. Consider the following functions defined for all $\tau \in \mathcal{T}, T \in \mathcal{T}$ and $s \geq 0$:

$$\chi_1(\tau, T, s) := \sup\left\{\|Y_1(t_0+\tau)\|; d \in M_D, \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T]\right\},$$
$$\chi_2(\tau, T, s) := \sup\left\{\|Y_2(t_0+\tau)\|; d \in M_D, \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T]\right\}. \tag{4.4}$$

In order to establish the robust output attractivity property, it suffices to show that $\lim_{\tau\to+\infty}\chi_i(\tau, T, s) = 0$ for $i=1,2$. Clearly, by virtue of (4.2a,b), both $\chi_i(\tau, T, s)$ are bounded, thus $\overline{\lim}_{\tau\to+\infty}\chi_i(\tau, T, s) = l_i < +\infty$ for $i=1,2$. It turns out that for every $\varepsilon > 0$ there exists $\xi := \xi(\varepsilon, T, s) \in \mathcal{T}$ such that for $i=1,2$ it holds:

$$l_i \leq \sup_{\tau\in[\xi,+\infty)} \chi_i(\tau, T, s) \leq l_i + \varepsilon. \tag{4.5}$$

Exploiting (4.1a,b), we get:

$$\| Y_1(t)\|_{\mathcal{Y}_1} \le \max\Big\{ B_1(\widetilde{\beta}_1(t_0+\xi)\|\phi(t_0+\xi,t_0,x_0,u_0,d)\|_{\mathcal{X}}, t-t_0-\xi), \; a\Big(\sup_{\tau\in[\xi,t-t_0]} \| Y_1(t_0+\tau)\|_{\mathcal{Y}_1}\Big)\Big\}; \tag{4.6a}$$

$$\| Y_2(t)\|_{\mathcal{Y}_2} \le \max\Big\{ B_2(\widetilde{\beta}_2(t_0+\xi)\|\phi(t_0+\xi,t_0,x_0,u_0,d)\|_{\mathcal{X}}, t-t_0-\xi), \; a\Big(\sup_{\tau\in[\xi,t-t_0]} \| Y_2(t_0+\tau)\|_{\mathcal{Y}_2}\Big)\Big\},$$
$$\forall t\in[t_0+\xi,+\infty). \tag{4.6b}$$

By (4.4), (4.5), (4.6a,b) and the fact that Σ is RFC, it then follows:

$$l_i \le a(l_i+\varepsilon), \quad i=1,2. \tag{4.7}$$

But we have assumed in (3.13) that $a(s)<s$ for all $s>0$ and since $\varepsilon>0$ is arbitrary, we conclude from (4.7) that $l_i = \overline{\lim}_{\tau\to+\infty}\chi_i(\tau,T,s)=0$, for $i=1,2$, thus system Σ satisfies the robust output attractivity property.

The non-uniform in time Small-Gain Theorem 3.10 is an important tool for analyzing the stability properties of control systems that possess the BIC property and particularly control systems that involve delays. In Example 2.6, we showed that if the dynamics of such systems are locally Lipschitz and completely continuous then they satisfy the BIC property.

Example 4.1. Consider the nonlinear planar system

$$\begin{aligned} \dot{x}(t) &= -x^3(t) + b_1(t)x(t)y(t-\tau_1(t)),\\ \dot{y}(t) &= b_2(t)x^2(t-\tau_2(t)) - y(t) + v(t), \end{aligned} \tag{4.8}$$

where $\tau_i \in C^0(\Re^+;[0,c])$ $i=1,2$ for some $c\ge 0$. For the case $\tau_i(t)\equiv 0$ $i=1,2$, $b_1(t)\equiv 1$ and $b_2(t)\equiv b_2$ (constant), the stability behavior of (4.8) is studied in [4] by applying the small-gain theorem of Jiang–Teel–Praly. It is proved that, if $|b_2|<\frac{1}{2}$, then system (4.8) satisfies the uniform in time ISS property from the input v. This system is also studied in [15], by applying the non-uniform in time small-gain theorem in [15] for the case $\tau_i(t)\equiv 0$ $i=1,2$, under the following more general hypothesis:

A1 b_1 and b_2: $\Re^+\to\Re$ are C^0 functions and there exist a constant $K\ge 0$ and a positive non-decreasing function $\zeta\in C^1(\Re^+;(0,+\infty))$ such that

$$r := \sup_{t\ge 0}\frac{\dot{\zeta}(t)}{\zeta(t)} < 1, \tag{4.9a}$$

$$|b_1(t)|\le\zeta(t),\ |b_2(t)|\le\frac{K}{\zeta(t)}, \quad \forall t\ge 0. \tag{4.9b}$$

It is proved that if $K<\frac{1}{2}(1-r)$ then system (4.8) satisfies the non-uniform in time ISS property from the input v.

Here, we prove that under hypothesis A1 and if $K\exp(c)<\frac{1}{2}(1-r)$ then there exists $\varepsilon\in(0,1)$ such that system (4.8) with

$$\begin{aligned} v(t) &:= \varepsilon\exp(-c)\Bigg(d_1(t)\sup_{\theta\in[-c,0]}|y(t+\theta)| + \frac{d_2(t)}{\zeta(t)}\sup_{\theta\in[-c,0]}|x(t+\theta)|^2\Bigg)\\ &= \varepsilon\exp(-c)\Bigg(d_1(t)\,\|y(t)\| + \frac{d_2(t)}{\zeta(t)}\,\|x(t)\|^2\Bigg), \end{aligned}$$

where $d=(d_1,d_2)\in M_D$ (M_D is the class of all continuous functions $d:\Re^+\to[-1,1]^2$), namely

$$\begin{aligned} \dot{x}(t) &= -x^3(t) + b_1(t)x(t)y(t-\tau_1(t)),\\ \dot{y}(t) &= b_2(t)x^2(t-\tau_2(t)) - y(t)\\ &\quad + \varepsilon\exp(-c)d_1(t)\|y(t)\|\\ &\quad + \varepsilon\exp(-c)\frac{d_2(t)}{\zeta(t)}\|x(t)\|^2 \end{aligned} \tag{4.10}$$

is RGAS (i.e. we consider system (4.10) with output the whole state $(x,y)\in C^0([-c,0];\Re^2)$).

First, we consider the auxiliary subsystem $\Sigma_1^{\mathrm{aux}} := (\Re^+,\ C^0([-c,0];\Re),\ C^0([-c,0];\Re),\ M_U,\ M_D,\ \widetilde{\phi}_1,\ H_1)$ where $C^0([-c,0];\Re)$ is a normed linear space with the sup norm, which is described by the following evolution equation:

$$\begin{aligned} \dot{x}(t) &= -x^3(t) + b_1(t)x(t)v_2(t),\\ Y_1(t) &= H_1(t,x(t)) := \{x(t+\theta);\theta\in[-c,0]\}\\ &\in C^0([-c,0];\Re), \end{aligned} \tag{4.11}$$

where $U=\Re, M_U$ is the class of all continuous functions v_2: $\Re^+\to\Re$ and M_D is irrelevant. We consider the Lyapunov function for this subsystem $V(t,x):=x^2/2$ and by virtue of (4.9b) we find that the derivative of this function along the trajectories of Σ_1 satisfies for all $\varepsilon_1\in(0,1)$:

$$\dot{V}\le -4(1-\varepsilon_1)V^2, \quad \forall (t,x)\in\Re^+\times\Re,$$
$$\text{provided that } \left(\frac{\zeta(t)}{\varepsilon_1}|v_2|\right)^{1/2}\le |x|.$$

This inequality shows that the system $\dot{x}(t) = -x^3(t)+\varepsilon_1 d(t)(b_1(t)/\zeta(t))x^3(t)$, where $d(t)\in[-1,1]$ is RGAS and using equivalence of statements (ii) and (iii) of Proposition 2.5 in [15] in conjunction with equivalence of statements (ii) and (iii) of Proposition 3.1 in [15], we establish the following estimate:

$$\begin{aligned} &|x(t)| \\ &\quad \leq \max\left\{\widetilde{\sigma}_1(\beta_1(t_0)|x(t_0)|, t-t_0), \sup_{\tau\in[t_0,t]} \widetilde{\sigma}(\beta_1(\tau)\rho_1(\gamma_1(\tau)|v_2(\tau|), t-\tau)\right\}, \\ &\quad \forall t \geq t_0 \end{aligned} \tag{4.12a}$$

with $\gamma_1(t) := \zeta(t)$, $\beta(t) := 1$, $\rho_1(s) := (s/\varepsilon_1)^{1/2}$ and $\widetilde{\sigma}(s,t) := s/\sqrt{1+2(1-\varepsilon_1)s^2 t}$ for arbitrary choice of $\varepsilon_1 \in (0,1)$. Estimate (4.12a) implies that the solutions of subsystem (4.11) satisfy the estimate:

$$\begin{aligned} &\|x(t)\| \\ &\quad \leq \max\left\{\sigma_1(\beta_1(t_0)\|x(t_0)\|, t-t_0), \sup_{\tau\in[t_0,t]} \sigma_1(\beta_1(\tau)\rho_1(\gamma_1(\tau)|v_2(\tau|), t-\tau)\right\}, \\ &\quad \forall t \geq t_0. \end{aligned} \tag{4.12b}$$

Estimate (4.12b) in conjunction with the fact that ζ is non-decreasing implies (3.3) and (3.10) with

$$\begin{aligned} &\mu(t) := \exp(\varepsilon_1 c)\max\left\{1, \sqrt{\tfrac{\zeta(t)}{\varepsilon_1}}\right\}, \quad \gamma_1(t) := \zeta(t), \\ &\beta_1(t) := 1, \quad \rho_1(s) := \left(\tfrac{s}{\varepsilon_1}\right)^{1/2}, \\ &a(s) := s + \sqrt{s}, \quad R := 0 \end{aligned}$$

and

$$\sigma_1(s,t) := \begin{cases} \exp(\delta(c-t))s & \text{if } 0 \leq t \leq c \\ \frac{s}{\sqrt{1+2(1-\varepsilon_1)s^2(t-c)}} & \text{if } t > c \end{cases}$$
$$\text{for arbitrary } \delta > 0.$$

Next we consider the subsystem $\Sigma_1 := (\Re^+, C^0([-c,0];\Re), C^0([-c,0];\Re), M_U, M_D, \widetilde{\phi}_1 H_1)$, where $C^0([-c,0];\Re)$ is a normed linear space with the sup norm, which is described by the following evolution equation:

$$\begin{aligned} \dot{x}(t) &= -x^3(t) + b_1(t)x(t)v_2(t-\tau_1(t)), \\ Y_1(t) &= H_1(t, x(t)) := x(t) \\ &= \{x(t+\theta); \theta\in[-c,0]\} \\ &\in C^0([-c,0];\Re), \end{aligned} \tag{4.13}$$

where $U = C^0([-c,0];\Re)$, M_U is the class of all continuous functions $u\colon \Re^+ \to C^0([-c,0];\Re)$ and M_D is irrelevant. By virtue of the previous analysis for the auxiliary subsystem Σ_1^{aux}, it follows that system (4.13) satisfies estimate (4.12b) with $\sup_{\theta\in[-c,0]}|v_2(\tau+\theta)|$ in place of $|v_2(\tau)|$. Thus, subsystem Σ_1 satisfies (3.3) and (3.10) with the same $\mu, \gamma_1, \beta_1, \rho_1, a, R, \sigma_1$ and arbitrary $\varepsilon_1 \in (0,1)$, $\delta > 0$. It follows from Lemma 3.5 that subsystem Σ_1 is RFC from the input v_2. Thus subsystem Σ_1 satisfies the non-uniform in time IOS property from the input v_2.

The following step is to consider subsystem $\Sigma_2 := (\Re^+, C^0([-c,0];\Re), C^0([-c,0];\Re), M_U, M_D, \widetilde{\phi}_2, H_2)$, where $C^0([-c,0];\Re)$, is a normed linear space with the sup norm, which is described by the following evolution equation:

$$\begin{aligned} \dot{y}(t) &= b_2(t)v_1^2(t-\tau_2(t)) - y(t) \\ &\quad + \varepsilon\exp(-c)\mathrm{d}_1(t)\|y(t)\| \\ &\quad + \varepsilon\exp(-c)\frac{\mathrm{d}_2(t)}{\zeta(t)}\|v_1(t)\|^2, \\ Y_2(t) &= H_2(t, y(t)) := \{y(t+\theta); \theta\in[-c,0]\} \\ &\quad \in C^0([-c,0];R), \end{aligned} \tag{4.14}$$

where $U = C^0([-c,0];\Re)$, M_U is the class of all continuous mappings $u : \Re^+ \to C^0([-c,0];\Re)$ and M_D is the class of all continuous functions $d : \Re^+ \to [-1,1]^2$. It is clear from (4.9b) that the solution of (4.14) satisfies the following estimate for all $t \geq t_0$ and $d = (d_1, d_2) \in M_D$:

$$\begin{aligned} \exp(t)|y(t)| &\leq \exp(t_0)|y(t_0)| \\ &\quad + \varepsilon\exp(-c)\int_{t_0}^t \exp(\tau)\|y(\tau)\|\mathrm{d}\tau \\ &\quad + (K\exp(c)+\varepsilon)\exp(-c) \\ &\quad \times \int_{t_0}^t \exp(\tau)\frac{\|v_1(\tau)\|^2}{\zeta(\tau)}\mathrm{d}\tau. \end{aligned}$$

Thus for every $y_0 \in C^0([-c,0];\Re)$ and $t_0 \geq 0$ we obtain:

$$\begin{aligned} &\exp(t)\|y(t)\| \\ &\quad \leq \exp(c+t_0)\|y_0\| + \varepsilon\int_{t_0}^t \exp(\tau)\|y(\tau)\|\mathrm{d}\tau \\ &\quad + (K\exp(c)+\varepsilon) \times \int_{t_0}^t \exp(\tau)\frac{\|v_1(\tau)\|^2}{\zeta(\tau)} - \mathrm{d}\tau, \\ &\quad \forall t \geq t_0. \end{aligned}$$

Applying Gronwall's inequality, we obtain the following estimate:

$$\begin{aligned} \|y(t)\| &\leq \exp(-(1-\varepsilon)(t-t_0))\exp(c)\|y_0\| \\ &\quad + (K\exp(c)+\varepsilon)\int_{t_0}^t \exp(-(1-\varepsilon)(t-\tau)) \\ &\quad \times \frac{\|v_1(\tau)\|^2}{\zeta(\tau)}\mathrm{d}\tau, \quad \forall t \geq t_0, \end{aligned}$$

which implies

$$\|y(t)\| \leq \exp(-(1-\varepsilon)(t-t_0)) \exp(c)\|y_0\| + \sup_{r\in[t_0,t]} \frac{(K\exp(c)+\varepsilon)}{\varepsilon_2} \times \exp(-(1-\varepsilon-\varepsilon_2)(t-\tau)) \frac{\|v_1(\tau)\|^2}{\zeta(\tau)}, \quad \forall t \geq t_0, \tag{4.15}$$

for all $0<\varepsilon_2<1-\varepsilon$. Using the elementary inequality $s_1+s_2 \leq \max\{2s_1, 2s_2\}$, in conjunction with estimate (4.15), we conclude that system (4.14) satisfies (3.3) and (3.11) with $a(s):=s+s^2$, $R:=0$,

$$\gamma_2(t) := \left(\frac{1}{\zeta(t)}\right)^{1/2}, \quad \beta_2(t) := 1,$$

$$\rho_2(s) := \frac{(K\exp(c)+\varepsilon)}{\varepsilon_2} \exp(-c)s^2,$$

$$\mu(t) := \exp(c)\left(1+\frac{K+\varepsilon}{\varepsilon_2\zeta(0)}\right) \text{ and}$$

$$\sigma_2(s,t) := 2s\ \exp(c-(1-\varepsilon-\varepsilon_2)t)$$

for arbitrary $0<\varepsilon_2<1-\varepsilon$. It follows from Lemma 3.5 that subsystem Σ_2 is RFC from the input v_1. Thus subsystem Σ_2 satisfies the non-uniform in time IOS property from the input v_1.

It is clear that system (4.10) is the interconnection of systems Σ_1 and Σ_2. Moreover, notice that by virtue of (4.9a) we have $\zeta(t) \leq \zeta(t_0)\exp(r(t-t_0))$ for all $t \geq t_0 \geq 0$. This observation, in conjunction with the fact that ζ is non-decreasing and the definitions of γ_i, σ_i, β_i, ρ_i $i=1,2$ implies that (3.12a,b) are satisfied if

$$\varepsilon+\varepsilon_2 < 1-r. \tag{4.16a}$$

On the other hand, setting $a(s):=Ls$ for certain $L\in(0,1)$, we obtain by evaluating (3.14a,b) and assuming that (4.16a) holds:

$$\varepsilon \leq \tfrac{1}{2}\varepsilon_1\varepsilon_2 L^2 \exp(-\delta c) - K\exp(c). \tag{4.16b}$$

Inequalities (4.16a,b) in conjunction with the fact that the constants $\delta>0$, $\varepsilon_1\in(0,1)$ and $L\in(0,1)$ are arbitrary, show that if $K\exp(c) < \frac{1}{2}(1-r)$ then there exists $\varepsilon>0$ such that system (4.10) is RGAS.

The following example is an application of the non-uniform small-gain theorem in discrete-time finite-dimensional systems.

Example 4.2. Consider the planar continuous-time system

$$\frac{dx}{dt_r}(t_r) = -x(t_r) + a_1\exp(-t_r)y(t_r), \tag{4.17a}$$

$$\frac{dy}{dt_r}(t_r) = a_2(1+t_r)x(t_r) - Ky(t_r), \tag{4.17b}$$

where a_i $(i=1,2)$ and $K>0$ are constants satisfying

$$|a_1a_2| < \frac{K}{8}\exp\left(\frac{3}{2}\right). \tag{4.18}$$

We have used the notation $t_r \in \Re^+$ to denote the time of the continuous-time system instead of t which is used to denote the time of the discrete-time system that we are going to study. In [15] we showed that under hypothesis (4.18), the linear continuous-time system (4.17) is non-uniformly in time RGAS. If one tries to simulate system (4.17) using the explicit Euler method with constant step size $\min\left\{\frac{1}{2},\frac{1}{K}\right\} > h > 0$, initiated at time $t_{r,0}=h\ t_0$ (for certain $t_0\in Z^+$), from $(x(t_{r,0}), y(t_{r,0})) = (x_0,y_0) \in \Re^2$ then the following discrete-time planar system arises:

$$x(t+1) = (1-h)x(t) + h\,a_1\exp(-ht)y(t), \tag{4.19a}$$

$$y(t+1) = ha_2(1+ht)x(t) + (1-hK)y(t) \tag{4.19b}$$

with initial condition $(x(t_0), y(t_0)) = (x_0, y_0)$. Here we prove that the discrete-time system (4.19) is RGAS, provided that

$$|a_1a_2|\frac{1+\sqrt{1-h}}{(1-h)(1-Kh)} < \frac{K}{8}. \tag{4.20}$$

This is an important information for simulation purposes, since the simulated system must have the same qualitative properties with the original system. We set $c_1:=-\log(1-h)>0$ and $c_2:=-\log(1-Kh)>0$. Then it can be inductively proved that subsystems (4.19a,b) satisfy the following estimates for all $t\in[t_0,+\infty)$:

$$|x(t)| \leq \exp(-c_1(t-t_0))|x_0| + h|a_1|\exp(c_1) \times \sum_{\tau=t_0}^{t}\exp(-c_1(t-\tau))\exp(-h\tau)|y(\tau)|, \tag{4.21a}$$

$$|y(t)| \leq \exp(-c_2(t-t_0))|y_0| + h|a_2|\exp(c_2) \times \sum_{\tau=t_0}^{t}\exp(-c_2(t-\tau))(1+h\tau)|x(\tau)|. \tag{4.21b}$$

Let $\varepsilon_1 \in (0, c_1)$ and $\varepsilon_2 \in (0, c_2)$ be arbitrary constants. Estimates (4.21a,b) imply the following estimates for all $t \in [t_0, +\infty)$:

$$|x(t)| \leq \exp(-(c_1-\varepsilon_1)(t-t_0))|x_0| + \frac{h|a_1|\exp(c_1+\varepsilon_1)}{\exp(\varepsilon_1)-1} \sup_{\tau\in[t_0,t]} (\exp(-(c_1-\varepsilon_1)(t-\tau)) \times \exp(-h\tau)|y(\tau)|), \tag{4.22a}$$

$$|y(t)| \leq \exp(-(c_2-\varepsilon_2)(t-t_0))|y_0| + \frac{h|a_2|\exp(c_2+\varepsilon_2)}{\exp(\varepsilon_2)-1} \sup_{\tau\in[t_0,t]} (\exp(-(c_2-\varepsilon_2)(t-\tau)) \times (1+h\tau)|x(\tau)|). \tag{4.22b}$$

Using the elementary inequality $s_1+s_2 \leq \max\{2s_1, 2s_2\}$, in conjunction with estimates (4.22a,b), we conclude that inequalities (3.10) and (3.11) are satisfied with $\sigma_i(s, t) := 2 \exp(-(c_i-\varepsilon_i)t)s$, $\rho_i(s) := ((h|a_i|\exp(c_i+\varepsilon_i))/(\exp(\varepsilon_i)-1))s$, $\beta_i(t) := 1$, $i = 1, 2$, $\gamma_1(t) := \exp(-h\,t)$ and $\gamma_2(t) := 1 + h\,t$. Moreover, it follows from Lemma 3.5 that subsystem (4.19a) is RFC from the input y, since it satisfies (3.3) with $\mu(t) := 1 + ((h|a_1|\exp(c_1+\varepsilon_1))/(\exp(\varepsilon_1)-1))$, $a(s) := s$ and $R := 0$. Similarly, it follows from Lemma 3.5 that subsystem (4.19b) is RFC from the input x, since it satisfies (3.3) with $\mu(t) := 1 + ((h|a_2|\exp(c_2+\varepsilon_2))/(\exp(\varepsilon_2)-1))\,(1+h\,t)$, $a(s) := s$ and $R := 0$. Consequently, both subsystems (4.19a) and (4.19b) satisfy the non-uniform in time IOS property from the inputs y and x, respectively. Taking into account the previous definitions we conclude that properties (3.12a,b) hold. On the other hand, using a similar approach as in Example 5.2 in [15], we guarantee that inequalities (3.14a,b) are satisfied with $a(s) := L\,s$ for certain $L \in (0, 1)$, provided that the following inequality is satisfied:

$$\sup_{t_0\in\tau}\left\{\frac{4h^2|a_1a_2|\exp(c_1+c_2+\varepsilon_1+\varepsilon_2)}{(\exp(\varepsilon_1)-1)(\exp(\varepsilon_2)-1)} \times \sup_{t\geq t_0}(\exp(-h\,t)\exp(-(c_2-\varepsilon_2)(t-t_0))). \times \sup_{t\geq t_0}((1+h\,t)\exp(-(c_1-\varepsilon_1)(t-t_0)))\right\} < 1.$$

Making use of the elementary inequality

$$\sup_{t\geq t_0}\left((1+ht)\exp\left(-\frac{c_1}{2}t\right)\right) \leq \frac{2h}{c_1}\exp(-1)\exp(c_1/2h),$$

as well as the facts $0 < h < \frac{1}{2}$ and $c_1 := -\log(1-h)$ (which imply $c_1 \geq h > (c_1/2)$), it follows that the previous inequality is satisfied for $\varepsilon_1 = c_1/2$, provided that the following inequality is satisfied:

$$\frac{8h^2|a_1a_2|\exp((3c_1/2)+2c_2)}{(\exp(c_1/2)-1)(\exp(c_2)-1)} < 1.$$

By making use of the definitions $c_1 := -\log(1-h)$ and $c_2 := -\log(1-K\,h)$, it can be verified that the latter inequality is equivalent with (4.20). It is clear that inequality (4.20) is "more demanding" than inequality (4.18) and this shows the limitations of the explicit Euler method with constant step size.

5. Conclusions

The notions of non-uniform in time RGAOS and non-uniform in time IOS are extended to cover a wide class of control systems with outputs that includes a wide class of discrete- and continuous-time control systems that possess the property that "the solution of the system can be continued as long as it remains bounded" (BIC property). A non-uniform in time small-gain theorem, which makes use of the notion of non-uniform in time IOS property, is presented. The results are illustrated by examples that show the usefulness of the non-uniform in time small-gain theorem for the stability analysis of interconnected systems.

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Appendix

Proof of Lemma 3.2. First notice that without loss of generality we may assume that $\mathcal{T} = \Re^+$, since for the case $\mathcal{T} = Z^+$ we may replace $H : \mathcal{T} \times \mathcal{X} \times D \to \mathcal{Y}$, by the map $\widetilde{H} : \Re^+ \times \mathcal{X} \times D \to \mathcal{Y}$, which satisfies $\widetilde{H}(t,x,d) = H(t,x)$ for all $(t,x) \in Z^+ \times \mathcal{X} \times \mathcal{D}$ and $\widetilde{H}(t,x,d) := (1 - t + [t])H([t],x,d) + (t - [t])H([t]+1, x, d)$ for all $(t,x) \in (\Re^+ \setminus Z^+) \times \mathcal{X} \times D$, where $[t] := \max\{\tau \in Z^+; \tau \le t\}$ denotes the integer part of $t \in \Re^+$. Notice that the map $\widetilde{H} : \Re^+ \times \mathcal{X} \times D \to \mathcal{Y}$ satisfies the properties (i)–(iii) with $\mathcal{T} = \Re^+$.

Let $a : \Re^+ \times \Re^+ \to \Re^+$ be defined as

$$a(t,s) := \sup\{\|H(\tau,x,d)\|_{\mathcal{Y}}; \\ d \in D, \tau \in [0,t], \|x\|_{\mathcal{X}} \le s\}.$$

Since for every $(t,s) \in \Re^+ \times \Re^+$ the set $\{(\tau,x) \in \Re^+ \times \mathcal{X}; \tau \in [0,t], \|x\|_{\mathcal{X}} \le s\} \subset \Re^+ \times \mathcal{X}$ is bounded, it follows from assumption (i) that the set $\{y \in \mathcal{Y}; d \in D, \tau \in [0,t], \|x\|_{\mathcal{X}} \le s, y = H(\tau,x,d)\} \subset \mathcal{Y}$ is bounded, and thus $a(t,s) < +\infty$ for all $(t,s) \in \Re^+ \times \Re^+$. Moreover, we have that:

(a) $a(t,0) = 0$ for all $t \ge 0$.
(b) For each fixed $t \ge 0$, the functions $a(t,\cdot)$ and $a(\cdot,t)$ are non-decreasing.
(c) $\|H(t,x,d)\|_{\mathcal{Y}} \le a(t,\|x\|_{\mathcal{X}})$ for all $(t,x,d) \in \Re^+ \times \mathcal{X} \times D$.

We finally show that $\lim_{s\to 0^+} a(t,s) = 0$ for all $t \ge 0$. Equivalently, we show that for every $t \ge 0$, $\varepsilon > 0$ there exists $\overline{\delta}(\varepsilon,t) > 0$ such that $a(t,\overline{\delta}(\varepsilon,t)) \le \varepsilon$. Let $t \ge 0$ and $\varepsilon > 0$ be arbitrary. It follows from assumption (iii) that for every $t_i \in [0,t]$ there exists $\delta(\varepsilon,t_i) > 0$ such that: if $(\tau,x) \in \Re^+ \times \mathcal{X}$ with $|\tau - t_i| + \|x\|_{\mathcal{X}} < \delta(\varepsilon,t_i)$ then $\sup_{d\in D}\|H(\tau,x,d)\|_{\mathcal{Y}} < \varepsilon$. Let $\mathcal{N}(t_i,\varepsilon) := \{\tau \in \Re^+; |\tau - t_i| < \frac{1}{2}\delta(\varepsilon,t_i)\}$. Clearly, there exists a finite positive integer N and a sequence of times $t_i \in [0,t], i = 1,\ldots,N$ such that $[0,t] = \cup_{i=1,\ldots,N} \mathcal{N}(t_i,\varepsilon)$. Let $\overline{\delta}(\varepsilon,t) := \frac{1}{2}\min_{i=1,\ldots,N}\delta(\varepsilon,t_i)$. Obviously, $\overline{\delta}(\varepsilon,t) > 0$ and for every $\tau \in [0,t]$ there exists $t_i \in [0,t], i = 1,\ldots,N$ such that $\tau \in \mathcal{N}(t_i,\varepsilon)$. Thus, if $\|x\|_{\mathcal{X}} < \overline{\delta}(\varepsilon,t)$ we have $|\tau - t_i| + \|x\|_{\mathcal{X}} < \delta(\varepsilon,t_i)$, and consequently $\sup_{d\in D}\|H(\tau,x,d)\|_{\mathcal{Y}} < \varepsilon$. This property in conjunction with the definition of $a : \Re^+ \times \Re^+ \to \Re^+$ implies $a(t,\overline{\delta}(\varepsilon,t)) \le \varepsilon$.

Clearly, $a : \Re^+ \times \Re^+ \to \Re^+$ satisfies all requirements of Lemma 2.3 in [14] and consequently there exists a pair of functions $\zeta \in K_\infty$ and $\beta \in K^+$ such that: $a(t,s) \le \zeta(\beta(t)s)$ for all $(t,s) \in \Re^+ \times \Re^+$. The latter inequality in conjunction with property (c) above, gives the desired inequality (3.1).

Proof of Lemma 3.3. It suffices to show that Σ is robustly Lagrange and Lyapunov output stable, that is, it satisfies properties P1 and P2 of Definition 3.1. First we show that Σ is robustly Lagrange output stable, by showing that $a(T,s) < +\infty$ for all $T \in \mathcal{T}$, $s \ge 0$, where

$$a(T,s) := \sup\Big\{\|H(t,\phi(t,t_0,x_0,u_0,d),0)\|_{\mathcal{Y}}: \\ d \in M_D, t \in [t_0,+\infty), \|x_0\|_{\mathcal{X}} \le s, \\ t_0 \in [0,T]\Big\}.$$

Notice that by virtue of robust output attractivity property we have for every $\varepsilon > 0$:

$$a(T,s) \le \varepsilon + \sup\Big\{\|H(t,\phi(t,t_0,x_0,u_0,d),0)\|_{\mathcal{Y}}: \\ d \in M_D, t \in [t_0, t_0 + \tau(\varepsilon,T,s)], \\ \|x_0\|_{\mathcal{X}} \le s, t_0 \in [0,T]\Big\},$$

where $\tau := \tau(\varepsilon, T, R) \in \mathcal{T}$ is the time involved in the robust output attractivity property of Definition 3.1. Notice that, by virtue of robust forward completeness (which implies that the set $\{\phi(t, t_0, x_0, u_0, d); t \in [t_0, t_0, +\tau], \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T], d \in M_D\}$ is bounded) and since $H : \mathcal{T} \times \mathcal{X} \times U \to \mathcal{Y}$ maps bounded sets of $\mathcal{T} \times \mathcal{X} \times U$ into bounded sets of $\mathcal{Y}$, we obtain

$$\sup\Big\{\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}; \\ d \in M_D,\ t \in [t_0, t_0 + \tau(\varepsilon, T, s)], \\ \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T]\Big\} < +\infty.$$

Combining the previous inequalities we obtain that $a(T, s) < +\infty$ for all $T \in \mathcal{T}, s \geq 0$, or equivalently that Σ is robustly Lagrange output stable.

Next we show that Σ is robustly Lyapunov output stable. Since the map $H(t, x, 0)$ is continuous and maps bounded sets of $\mathcal{T} \times \mathcal{X}$ into bounded sets of $\mathcal{Y}$ with $H(t, 0, 0) = 0$ for all $t \in \mathcal{T}$, by virtue of Lemma 3.2, there exist functions $\zeta \in K_\infty$ and $\gamma \in K^+$ such that

$$\|H(t, x, 0)\|_{\mathcal{Y}} \leq \zeta(\gamma(t)\|x\|_{\mathcal{X}}), \\ \forall (t, x) \in \mathcal{T} \times \mathcal{X}. \tag{A1}$$

Let arbitrary $\varepsilon > 0$ and $T \in \mathcal{T}$. By virtue of the robust output attractivity property there exists $\tau := \tau(\varepsilon, T) \in \mathcal{T}$, such that:

$$\|x_0\|_{\mathcal{X}} \leq \varepsilon, t_0 \in [0, T] \ \Rightarrow\ \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}} \\ \leq \varepsilon, \quad \forall t \in [t_0 + \tau, +\infty), \quad \forall d \in M_D. \tag{A2}$$

Since $0 \in \mathcal{X}$ is a robust equilibrium point for Σ, it follows that for every $\varepsilon > 0$ and $T \in \mathcal{T}$ there exists $\delta' > 0$ such that

$$\sup\Big\{\|\phi(\tau, t_0, x, u_0, d)\|_{\mathcal{X}}; d \in M_D, \\ \tau \in [t_0, t_0 + \tau(\varepsilon, T)], t_0 \in [0, T]\Big\} \\ < \frac{\zeta^{-1}(\varepsilon)}{\max\limits_{0 \leq t \leq \tau(\varepsilon, T)} \gamma(t)} \\ \text{provided that} \|x\|_{\mathcal{X}} < \delta',$$

where $\tau := \tau(\varepsilon, T) \in \mathcal{T}$ is the time involved in (A2). The latter inequality in conjunction with (A1) gives:

$$\sup\Big\{\|H(\tau, \phi(\tau, t_0, x, u_0, d), 0)\|_{\mathcal{Y}}; \\ d \in M_D, \tau \in [t_0, t_0 + \tau(\varepsilon, T)], t_0 \in [0, T]\Big\} < \varepsilon \\ \text{provided that} \|x\|_{\mathcal{X}} < \delta'. \tag{A3}$$

It is clear from (A2) and (A3), that robust Lyapunov output stability property is satisfied for $\delta(\varepsilon, T) = \min\{\varepsilon, \delta'\}$. The proof is complete. □

Proof of Lemma 3.4. As in the proof of Proposition 2.2 in [14], let $\xi, T \in \mathcal{T}, s \geq 0$ and define:

$$a(T, s) := \sup\Big\{\|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}: \\ d \in M_D, t \in [t_0, +\infty), \\ \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T]\Big\}, \tag{A4}$$

$$M(\xi, T, s) := \sup\Big\{\|H(t_0 + \xi, \phi(t_0 \\ + \xi, t_0, x_0, u_0, d), 0)\|_{\mathcal{Y}}: d \in M_D, \\ \|x_0\|_{\mathcal{X}} \leq s, t_0 \in [0, T]\Big\}. \tag{A5}$$

First notice that by virtue of robust Lagrange output stability a is well defined, that is, $a(T, s) < +\infty$ for every $T \in \mathcal{T},\ s \geq 0$. Furthermore, notice that M is well defined, since by definitions (A4) and (A5) the following inequality is satisfied for all $\xi, T \in \mathcal{T}$ and $s \geq 0$:

$$M(\xi, T, s) \leq a(T, s). \tag{A6}$$

Notice also that, for the case $\mathcal{T} = Z^+$ we may extend the domain of $a : \mathcal{T} \times \Re^+ \to \Re^+$, to $\Re^+ \times \Re^+$, using the continuous extension $a(t, s) := (1 - t + [t])a([t], s) + (t - [t])a([t] + 1, s)$ for all $(t, s) \in (\Re^+ \backslash Z^+) \times \Re^+$, where $[t] := \max\{\tau \in Z^+; \tau \leq t\}$ denotes the integer part of $t \in \Re^+$. Moreover, a satisfies all hypotheses of the Lemma 2.3 in [14], namely for each fixed $s \geq 0$ $a(\cdot, s)$ is non-decreasing, for each fixed $T \geq 0$, $a(T, \cdot)$ is non-decreasing and satisfies $a(\cdot, 0) = 0$. Furthermore, robust Lyapunov output stability asserts that for every $T \geq 0$ $\lim_{s \to 0^+} a(T, s) = 0$. It turns out from Lemma 2.3 in [14], that there exist functions $\zeta_1 \in K_\infty$ and $\gamma \in K^+$ such that

$$a(T, s) \leq \zeta_1(\gamma(T)s), \quad \forall (T, s) \in (\Re^+)^2. \tag{A7}$$

Next we proceed exactly as in the proof of Proposition 2.2 in [14] to establish that:

$$M(\xi, T, s) \leq \mu(\xi)\theta(T, s), \quad \forall s \in \Re^+, \quad T, \xi \in \mathcal{T}, \tag{A8}$$

where $\theta(T, s) := p(T)g(\zeta_1(\gamma(T)s))$, $\mu \in K^+$ is a strictly decreasing function with $\lim_{\xi \to +\infty} \mu(\xi) = 0, p \in K^+$ is a non-decreasing function with $p(0) = 1$ and $\lim_{t \to +\infty} p(t) = +\infty$ and $g(s) := \sqrt{s} + s^2$. Applying again Lemma 2.3 in [14], we conclude that there exist functions $\zeta_2 \in K_\infty$ and $\beta \in K^+$ such that

$$\theta(T, s) \leq \zeta_2(\beta(T)s), \quad \forall (T, s) \in (\Re^+)^2.$$

Define the KL function $\sigma(s, t) := \mu(t)\zeta_2(s)$. The desired (3.2) is a consequence of (A5), (A8) and the inequality above. □

Proof of Lemma 3.5 **(i)** Suppose first that Σ is RFC from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$. Let also $B_r := \{u_1 \in U_1; \|u_1\|_{U_1} \leq r\}$ denote the closed sphere in U_1 with radius $r \geq 0$. We define:

$$\begin{aligned}\omega(T, r) := \sup\Big\{ &||\phi(t_0 + h, t_0, x_0, (u_1, u_{2,0}), d)||_{\mathcal{X}};\\ &||x_0||_{\mathcal{X}} \leq r, u_1 \in \mathcal{M}(\mathcal{T}; B_r),\\ &(u_1, u_{2,0}) \in M_U, t_0 \in [0, T],\\ &h \in [0, T], d \in M_D \Big\}. \end{aligned} \tag{A9}$$

Notice that by virtue of robust forward completeness from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$, the function $\omega : \mathcal{T} \times \Re^+ \to \Re^+$ is finite valued and for each fixed $t \geq 0$ the mappings $\omega(t, \cdot)$ and $\omega(\cdot, t)$ are non-decreasing. Notice that, for the case $\mathcal{T} = Z^+$ we may extend the domain of $\omega : \mathcal{T} \times \Re^+ \to \Re^+$, to $\Re^+ \times \Re^+$, using the continuous extension $\omega(t, s) := (1 - t + [t])\omega([t], s) + (t - [t])\,\omega\,([t] + 1, s)$ for all $(t, s) \in (\Re^+ \backslash Z^+) \times \Re^+$, where $[t] := \max\{\tau \in Z^+;\ \tau \leq t\}$ denotes the integer part of $t \in \Re^+$. Moreover, definition (A9) in conjunction with the causality property for system Σ, implies that for every $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ with $(u_1, u_{2,0}) \in M_U$ and $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$, the transition map satisfies:

$$\begin{aligned}&||\phi(t, t_0, x_0, (u_1, u_{2,0}), d)||_{\mathcal{X}}\\ &\quad \leq \omega(t, ||x_0||_{\mathcal{X}} + \sup_{\tau \in [t_0, t]} ||u_1(\tau)||_{U_1}),\\ &\quad \forall t \in [t_0, +\infty). \end{aligned} \tag{A10}$$

Since for each fixed $t > 0$ the mappings $\omega(t, \cdot)$ and $\omega(\cdot\,, t)$ are non-decreasing, inequality (A10) implies:

$$\begin{aligned}&\|\phi(t, t_0, x_0, (u_1, u_{2,0}), d)\|_{\mathcal{X}}\\ &\quad \leq \omega(t, t) + \omega(\|x_0\|_{\mathcal{X}} + \sup_{\tau \in [t_0, t]} \|u_1(\tau)\|_{U_1}, \|x_0\|_{\mathcal{X}}\\ &\qquad + \sup_{\tau \in [t_0, t]} \|u_1(\tau)\|_{U_1}), \quad \forall t \geq t_0. \end{aligned} \tag{A11}$$

Let $\mu \in K^+$ be a non-decreasing continuous function that satisfies $1 + \omega(s, s) \leq \mu\,(s)$ for all $s \geq 0$ and define

$$a(s) := s + \begin{cases} s(1 + \mu(0)), & \text{for } 0 \leq s < l, \\ 1 + \mu(s - 1), & \text{for } s \geq 1 \end{cases}$$

with $a \in K_\infty$. It follows that the following inequalities hold for all $s, t \geq 0$:

$$\begin{aligned}&\omega(t, t) + \omega(s, s) \leq \mu(t) + \mu(s) \leq \mu(t)(1 + \mu(s))\\ &\quad \leq \mu(t)a(1 + s). \end{aligned} \tag{A12}$$

Inequalities (A11) and (A12) imply inequality (3.3) with $R = 1$.

Conversely, suppose that (3.3) is satisfied for the transition map of Σ. Clearly, by virtue of the BIC property, for each $(t_0, x_0, d, (u_1, u_{2,0})) \in \mathcal{T} \times \mathcal{X} \times M_D \times M_U$ with $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$, there exists a maximal existence time, that is, there exists $t_{\max} \in \mathcal{T} \cup \{+\infty\}$, such that $[t_0,\ t_{\max}) \times (t_0,\ x_0,\ u_0,\ d) \subseteq A_\phi$ and for all $t \geq t_{\max}$ it holds that $(t,\ t_0,\ x_0,\ u_0,\ d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$ then for every $M > 0$ there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0,\ x_0, u, d)\|_{\mathcal{X}} > M$. Suppose that $t_{\max} < +\infty$, for some $(t_0, x_0, d, (u_1, u_{2,0})) \in \mathcal{T} \times \mathcal{X} \times M_D \times M_U$. Then there exists $t \in [t_0,\ t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathcal{X}} > a(\|x_0\|_{\mathcal{X}} + \sup_{\tau \in [t_0, t_{\max}]} \|u_1(\tau)\|_{U_1} + R)\ \max_{t \in [0, t_{\max}]} \mu(t)$. On the other hand since (3.3) holds we obtain

$$\begin{aligned}&\|\phi(t, t_0, x_0, (u_1, u_{1,0}), d)\|_{\mathcal{X}}\\ &\quad \leq a\Big(\|x_0\|_{\mathcal{X}} + \sup_{\tau \in [t_0, t_{\max}]} \|u_1(\tau)\|_{U_1} + R\Big) \max_{t \in [0, t_{\max}]} \mu(t),\end{aligned}$$

which is clearly a contradiction. Thus, since (3.3) holds we must have $t_{\max} = +\infty$ for all $(t_0, x_0, d, (u_1, u_{2,0})) \in \mathcal{T} \times \mathcal{X} \times M_D \times M_U$ with $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$. Moreover, using (3.3) for all $\varepsilon \geq 0$, $T \in \mathcal{T}$, we obtain

$$\begin{aligned}&\sup\Big\{ \|\phi(t_0 + s, t_0, x_0, (u_1, u_{2,0}), d)\|_{\mathcal{X}}; s \in [0, T],\\ &\qquad u_1 \in \mathcal{M}(\mathcal{T}; B_\varepsilon), (u_1, u_{2,0}) \in M_U, \|x_0\|_{\mathcal{X}}\\ &\qquad \leq \varepsilon, t_0 \in [0, T], d \in M_D \Big\}\\ &\quad \leq a(R + 2\varepsilon) \max_{t \in [0, 2T]} \mu(t) < +\infty,\end{aligned}$$

that is, the property of robust forward completeness from the input $u_1 \in \mathcal{M}(\mathcal{T}; U_1)$ is satisfied.

The proof statement (ii) is identical with proof of statement (i) (simply set $u_1 \equiv 0$ in the previous inequalities). For the case that $0 \in \mathcal{X}$ is a robust equilibrium point for Σ, then we follow the following procedure. Suppose that (3.4) is satisfied for appropriate $\mu \in K^+, a \in K_\infty$ and constant $R > 0$. Consider the control system $\Sigma' := (\mathcal{T}, \mathcal{X}, \mathcal{X}, M_U, M_D, \widetilde{\phi}, \widetilde{H})$, where

$$\begin{aligned}&\widetilde{\phi}(t, t_0, x_0, u, d)\\ &\quad := \frac{\exp(-t)}{\mu(t)} \phi(t, t_0, \mu(t_0) \exp(t_0) x_0, u, d), \end{aligned} \tag{A13}$$

$$\widetilde{H}(t, x, u) := x. \tag{A14}$$

It can be immediately verified that $\Sigma' := (\mathcal{T}, \mathcal{X}, \mathcal{X}, M_U, M_D, \widetilde{\phi}, \widetilde{H}$ is a control system with outputs and the BIC property, for which $0 \in \mathcal{X}$ is an equilibrium point. Moreover, by virtue of (3.4) we obtain the following estimate for all $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$

and $t \in [t_0, +\infty)$:

$$\begin{aligned}&\|\widetilde{\phi}(t, t_0, x_0, u_0, d)\|_{\mathcal{X}}\\&\quad= \|\widetilde{H}(t, \widetilde{\phi}(t, t_0, x_0, u_0, d), 0)\|_{\mathcal{X}}\\&\quad\le \exp(-(t-t_0))a(\omega(t_0)\|x_0\|_{\mathcal{X}} + R) \qquad \text{(A15)}\end{aligned}$$

where $\omega(t) := \mu(t)\exp(t) \in K^+$. Estimate (A15) shows that $\Sigma' := (\mathcal{T}, \mathcal{X}, \mathcal{X}, M_U, M_D, \widetilde{\phi}, \widetilde{H})$ is RFC and satisfies the property of robust output attractivity. Thus, since $0 \in \mathcal{X}$ is an equilibrium point for Σ', it follows by Lemma 3.3 that $\sum'$ is RGAOS. Consequently, Lemma 3.4 guarantees the existence of functions $\bar{\sigma} \in KL, \bar{\beta} \in K^+$ such that the following estimate holds for all $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$ and $t \in [t_0, +\infty)$:

$$\begin{aligned}&\|\widetilde{H}(t, \widetilde{\phi}(t, t_0, x_0, u_0, d), 0\|_{\mathcal{X}}\\&\quad= \|\widetilde{\phi}(t, t_0, x_0, u_0, d)\|_{\mathcal{X}}\\&\quad\le \bar{\sigma}(\bar{\beta}(t_0)\|x_0\|_{\mathcal{X}}, t-t_0). \qquad \text{(A16)}\end{aligned}$$

Defining $\widetilde{\mu}(t) := (\exp(-t)/\mu(t)) \in K^+$ and using definition (A13) in conjunction with estimate (A16), we obtain the following estimate for all $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$ and $t \in [t_0, +\infty)$:

$$\begin{aligned}&\widetilde{\mu}(t)\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}}\\&\quad\le \bar{\sigma}(\bar{\beta}(t_0)\widetilde{\mu}(t_0)\|x_0\|_{\mathcal{X}}, t-t_0). \qquad \text{(A17)}\end{aligned}$$

Applying Corollary 10 and Remark 11 in [20] to the function $\overline{\sigma}(s, 0)$, we obtain a function $\widetilde{a} \in K_\infty$ that satisfies $\overline{\sigma}(rs, 0) \le \widetilde{a}(r)\widetilde{a}(s)$ for all $r, s \ge 0$. By virtue of (A17), we conclude that (3.4) is satisfied for Σ with $R = 0$, $\mu(t) := \widetilde{a}(\max_{0 \le \tau \le t} \beta(\tau)\widetilde{\mu}(\tau))/\widetilde{\mu}(t)$ and $a(s) = \widetilde{a}(s)$.

Proof of Claim made in the proof of Theorem 3.10. Since subsystem Σ_1 is RFC from the input $v_2 \in \mathcal{M}(\mathcal{T}; \mathcal{Y}_2)$ and subsystem Σ_2 is RFC from the input $v_1 \in \mathcal{M}(\mathcal{T}; \mathcal{Y}_1)$, it follows from Lemma 3.5 that there exists functions $\mu_1, \mu_2 \in K^+$, $a_1, a_2 \in K_\infty$ and constants $R_1, R_2 \ge 0$ such that the following estimates hold for all $(t_0, x_1, (v_2, u_0), x_2, (v_1, u_0), d) \in \mathcal{T} \times \mathcal{X}_1 \times M_{\mathcal{Y}_2 \times U} \times \mathcal{X}_2 \times M_{\mathcal{Y}_1 \times U} \times M_D$ and $t \in [t_0, +\infty)$:

$$\begin{aligned}&\|\widetilde{\phi}_1(t, t_0, x_1, (v_2, u_0), d\|_{\mathcal{X}_1}\\&\quad\le \mu_1(t)a_1\Big(R_1 + \|x_1\|_{\mathcal{X}_1} + \sup_{\tau\in[t_0,t]}\|v_2(\tau)\|_{\mathcal{Y}_2}\Big), \qquad \text{(A18)}\end{aligned}$$

$$\begin{aligned}&\|\widetilde{\phi}_2(t, t_0, x_2, (v_1, u_0), d\|_{\mathcal{X}_2}\\&\quad\le \mu_2(t)a_2\Big(R_2 + \|x_2\|_{\mathcal{X}_2} + \sup_{\tau\in[t_0,t]}\|v_1(\tau)\|_{\mathcal{Y}_1}\Big). \qquad \text{(A19)}\end{aligned}$$

We next proceed by first establishing the following property:

Property. Under Hypothesis H3 of Theorem 3.10, there exists functions $B_i \in KL$ and $\widetilde{\beta}_i \in K^+$ $(i = 1, 2)$ such that for all $s_1, s_2 \ge 0$ and $t \ge t_0$ the following inequalities hold:

$$\begin{aligned}&\sigma_1(\beta_1(t_0)s_1, t-t_0) + \sup_{\tau\in[t_0,t]} \sigma_1(\beta_1(\tau)\rho_1(\gamma_1(\tau)\\&\quad\times \sigma_2(\beta_2(t_0)s_2, \tau-t_0)), t-\tau)\\&\qquad\le B_1(\widetilde{\beta}_1(t_0)\max(s_1, s_2), t-t_0), \qquad \text{(A20)}\end{aligned}$$

$$\begin{aligned}&\sigma_2(\beta_2(t_0)s_1, t-t_0) + \sup_{\tau\in[t_0,t]} \sigma_2(\beta_2(\tau)\rho_2(\gamma_1(\tau)\\&\quad\times \sigma_1(\beta_1(t_0)s_2, \tau-t_0)), t-\tau)\\&\quad\le B_2(\widetilde{\beta}_2(t_0)\max(s_1, s_2), t-t_0). \qquad \text{(A21)}\end{aligned}$$

Proof: By using Fact VII in [15] and Hypothesis H3, there exists functions $S_i \in KL$ and $\delta_i \in K^+$ $(i = 1, 2)$ such that

$$\begin{aligned}&\beta_1(t)\rho_1(\gamma_1(\tau)\sigma_2(s, t-t_0)) \le S_1(\delta_1(t_0)s, t-t_0),\\&\quad\forall t \ge t_0,\quad s \ge 0, \qquad \text{(A22)}\end{aligned}$$

$$\begin{aligned}&\beta_2(t)\rho_2(\gamma_1(\tau)\sigma_1(s, t-t_0)) \le S_2(\delta_2(t_0)s, t-t_0),\\&\quad\forall t \ge t_0,\quad s \ge 0. \qquad \text{(A23)}\end{aligned}$$

Furthermore, by using Fact VI in [15], there exists functions $R_i \in KL$ $(i = 1, 2)$ such that

$$\begin{aligned}&\sup_{\tau\in[t_0,t]} \sigma_1(S_1(s, \tau-t_0), t-\tau) \le R_1(s, t-t_0),\\&\quad\forall t \ge t_0,\quad s \ge 0, \qquad \text{(A24)}\end{aligned}$$

$$\begin{aligned}&\sup_{\tau\in[t_0,t]} \sigma_2(S_2(s, \tau-t_0), t-\tau) \le R_2(s, t-t_0),\\&\quad\forall t \ge t_0,\quad s \ge 0. \qquad \text{(A25)}\end{aligned}$$

We define $\widetilde{\beta}_1(t) := \delta_1(t)\beta_2(t) + \beta_1(t)$ and $\widetilde{\beta}_2(t) := \delta_2(t)\beta_1(t) + \beta_2(t)$. Inequalities (A22)–(A25) along with previous definitions, imply that for all $s \ge 0$ and $t \ge t_0$ the following inequalities hold:

$$\begin{aligned}&\sup_{\tau\in[t_0,t]} \sigma_1(\beta_1(\tau)\rho_1(\gamma_1(\tau)\sigma_2(\beta_2(t_0)s, \tau-t_0)), t-\tau)\\&\quad\le R_1(\widetilde{\beta}_1(t_0)s, t-t_0), \qquad \text{(A26)}\end{aligned}$$

$$\begin{aligned}&\sup_{\tau\in[t_0,t]} \sigma_2(\beta_2(\tau)\rho_2(\gamma_2(\tau)\sigma_1(\beta_1(t_0)s, \tau-t_0)), t-\tau)\\&\quad\le R_2(\widetilde{\beta}_2(t_0)s, t-t_0). \qquad \text{(A27)}\end{aligned}$$

We define for $i = 1, 2$, the functions of class KL, $B_i(s, t) := R_i(s, t) + \sigma_i(s, t)$. The previous definitions in conjunction with inequalities (A26) and (A27) imply the desired (A20) and (A21).

We are now in a position to prove our claim. Clearly, since Σ has the BIC property, it follows that for every $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$, there exists a maximal existence time, that is, there exists $t_{\max} \in \mathcal{T} \cup \{+\infty\}$, such that $[t_0, t_{\max}) \times (t_0, x_0, u_0, d) \subseteq A_\phi$

and for all $t \geq t_{\max}$ it holds that $(t, t_0, x_0, u_0, d) \notin A_\phi$. In addition, if $t_{\max} < +\infty$ then for every $M > 0$ there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d\|_{\mathcal{X}} > M$. Let $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times M_D$ with $x_0 = (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ and $t < t_{\max}$ be arbitrary. Since $\phi_1(\tau, t_0, x_0, u_0, d) = \widetilde{\phi}_1(\tau, t_0, x_1, (Y_2, u_0), d)$, $\phi_2(\tau, t_0, x_0, u_0, d) = \widetilde{\phi}_2(\tau, t_0, x_2, (Y_1, u_0), d)$ for all $\tau \in [t_0, t]$, by exploiting (3.10) and (3.11), we obtain the following estimates:

$$
\begin{aligned}
&\| Y_1(t)\|_{\mathcal{Y}_1} \leq \max\{a_1, a_2, a_3\}, \\
&a_1 := \sigma_1(\beta_1(t_0)\|x_1\|_{\mathcal{X}_1}, t - t_0), \\
&a_2 := \sup_{\tau\in[t_0,t]} \sigma_1(\beta_1(\tau)\rho_1(\gamma_1(\tau) \\
&\qquad \times \sigma_2(\beta_2(t_0)\|x_2\|_{\mathcal{X}_2}, \tau - t_0)), t - \tau), \\
&a_3 := \sup_{\tau\in[t_0,t]} \sigma_1\big(\beta_1(\tau)\rho_1\big(\gamma_1(\tau) \sup_{\xi\in[t_0,\tau]} \sigma_2(\beta_2(\xi) \\
&\qquad \times \rho_2(\gamma_2(\xi)\| Y_1(\xi)\|_{Y_1}), \tau - \xi)\big), t - \tau\big),
\end{aligned} \tag{A28}
$$

$$
\begin{aligned}
&\| Y_2(t)\|_{\mathcal{Y}_2} \leq \max\{b_1, b_2, b_3\}, \\
&b_1 := \sigma_2(\beta_2(t_0)\|x_2\|_{\mathcal{X}_2}, t - t_0), \\
&b_2 := \sup_{\tau\in[t_0,t]} \sigma_2(\beta_2(\tau)\rho_2(\gamma_2(\tau) \\
&\qquad \times \sigma_1(\beta_1(t_0)\|x_1\|_{\mathcal{X}_1}, \tau - t_0)), t - \tau), \\
&b_3 := \sup_{\tau\in[t_0,t]} \sigma_2\Big(\beta_2(\tau)\rho_2\Big(\gamma_2(\tau) \sup_{\xi\in[t_0,\tau]} \sigma_1(\beta_1(\xi) \\
&\qquad \times \rho_1(\gamma_1(\xi)\| Y_2(\xi)\|_{Y_2}), \tau - \xi\big)\Big), t - \tau\Big).
\end{aligned} \tag{A29}
$$

Exploiting inequalities (3.14a,b) we obtain:

$$
\begin{aligned}
&\sup_{\tau\in[t_0,t]} \sigma_1(\beta_1(\tau)\rho_1(\gamma_1(\tau) \sup_{\xi\in[t_0,\tau]} \sigma_2(\beta_2(\xi)\rho_2 \\
&\quad \times (\gamma_2(\xi)\| Y_1(\xi)\|_{\mathcal{Y}_1}), \tau - \xi)), t - \tau) \\
&\quad \leq a\Big(\sup_{\tau\in[t_0,t]} \| Y_1(\tau)\|_{\mathcal{Y}_1}\Big),
\end{aligned} \tag{A30}
$$

$$
\begin{aligned}
&\sup_{\tau\in[t_0,t]} \sigma_2(\beta_2(\tau)\rho_2(\gamma_2(\tau) \sup_{\xi\in[t_0,\tau]} \sigma_1(\beta_1(\xi)\rho_1 \\
&\quad \times (\gamma_1(\xi)\| Y_2(\xi)\|_{\mathcal{Y}_2}), \tau - \xi)), t - \tau) \\
&\quad \leq a\Big(\sup_{\tau\in[t_0,t]} \| Y_2(\tau)\|_{\mathcal{Y}_2}\Big),
\end{aligned} \tag{A31}
$$

where a is the function involved in (3.13) and (3.14a,b). Furthermore, by virtue of (A20), (A21), (A30) and (A31) and since $\|x_1\|_{\mathcal{X}_1} \leq \|x_0\|_{\mathcal{X}}$, $\|x_2\|_{\mathcal{X}_2} \leq \|x_0\|_{\mathcal{X}}$, we obtain:

$$
\| Y_1(t)\|_{\mathcal{Y}_1} \leq \max\Big\{B_1(\widetilde{\beta}_1(t_0)\|x_0\|_{\mathcal{X}}, t - t_0), \; a\Big(\sup_{\tau\in[t_0,t]} \| Y_1(t)\|_{\mathcal{Y}_1}\Big)\Big\}, \tag{A32}
$$

$$
\| Y_2(t)\|_{\mathcal{Y}_2} \leq \max\Big\{B_2(\widetilde{\beta}_2(t_0)\|x_0\|_{\mathcal{X}}, t - t_0), \; a\Big(\sup_{\tau\in[t_0,t]} \| Y_2(t)\|_{\mathcal{Y}_2}\Big)\Big\}. \tag{A33}
$$

Clearly, the above inequalities imply that:

$$
\sup_{\tau\in[t_0,t]} \| Y_1(\tau)\|_{\mathcal{Y}_1} \leq \max\Big\{B_1\Big(\widetilde{\beta}_1(t_0)\|x_0\|_{\mathcal{X}}, 0\Big), \; a\Big(\sup_{\tau\in[t_0,t]} \| Y_1(t)\|_{Y_1}\Big)\Big\},
$$

$$
\sup_{\tau\in[t_0,t]} \| Y_2(\tau)\|_{\mathcal{Y}_2} \leq \max\Big\{B_2\Big(\widetilde{\beta}_2(t_0)\|x_0\|_{\mathcal{X}}, 0\Big), \; a\Big(\sup_{\tau\in[t_0,t]} \| Y_2(t)\|_{Y_2}\Big)\Big\},
$$

which in conjunction with our hypothesis (3.13) gives:

$$
\sup_{\tau\in[t_0,t]} \| Y_1(\tau)\|_{\mathcal{Y}_1} \leq B_1\Big(\widetilde{\beta}_1(t_0)\|x_0\|_{\mathcal{X}}, 0\Big), \tag{A34}
$$

$$
\sup_{\tau\in[t_0,t]} \| Y_2(\tau)\|_{\mathcal{Y}_2} \leq B_2\Big(\widetilde{\beta}_2(t_0)\|x_0\|_{\mathcal{X}}, 0\Big). \tag{A35}
$$

Let $K > 0$ be the constant that satisfies $K(\|x_1\|_{\mathcal{X}1} + \|x_2\|_{\mathcal{X}2}) \geq \|x_0\|_{\mathcal{X}}$ for all $x_0 = (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. Defining $\widetilde{a}(s) := a_1(s) + a_2(s), \widetilde{a} \in K_\infty$, $\widetilde{\beta}(t) := 1 + \widetilde{\beta}_1(t) + \widetilde{\beta}_2(t)$, $\widetilde{\beta} \in K^+$, $\mu(t) := K\max\{\mu_1(t), \; \mu_2(t)\}$, $\mu \in K^+$ and $B(s) := \widetilde{a}^{-1}(2s + B_1(s, \; 0) + B_2(s, 0)), B \in K_\infty, R := \max\{R_1, \; R_2\}$ and since $\phi_1(\tau, t_0, x_0, u_0, d) = \widetilde{\phi}_1(\tau, t_0, x_1, (Y_2, u_0), d), \phi_2(\tau, t_0, x_0, u_0, d) = \widetilde{\phi}_2(\tau, t_0, x_2, (Y_1, u_0), d)$ for all $\tau \in [t_0, t]$, we obtain by exploiting (A18) and (A19) and inequalities (A34) and (A35):

$$
\begin{aligned}
&\|\phi(t, t_0, x_0, u_0, d)\|_{\mathcal{X}} \\
&\quad \leq \mu(t)B(R + \widetilde{\beta}(t_0)\|x_0\|_{\mathcal{X}}).
\end{aligned} \tag{A36}
$$

Suppose that $t_{\max} < +\infty$, for some $(t_0, x_0, d) \in \mathcal{T} \times \mathcal{X} \times \mathcal{M}_D$. Then there exists $t \in [t_0, \; t_{\max})$ with $\|\phi(t, \, t_0, \, x_0, \, u, \, d)\|_{\mathcal{X}} > B(R + \widetilde{\beta}(t_0)\|x_0\|_{\mathcal{X}}) \max_{t\in[0, \, t_{\max}]} \mu(t)$. On the other hand since (A36) holds we obtain $\|\phi(t, \, t_0, \, x_0, \, u, \, d)\|_{\mathcal{X}} \leq B(R + \widetilde{\beta}(t_0)\|x_0\|_{\mathcal{X}}) \max_{t\in[0, \, t_{\max}]} \mu(t)$, which is clearly a contradiction. Thus, since (A36) holds we must have $t_{\max} = +\infty$ for all $(t_0, \, x_0, \, d) \in \mathcal{T} \times \mathcal{X} \times \mathcal{M}_D$. Moreover, using (A36) for all $\varepsilon, \, T \geq 0$ we obtain

$$
\begin{aligned}
&\sup\{\|\phi(t_0, +s, t_0, x_0, u_0, d)\|_{\mathcal{X}}; s \in [0, T], \|x_0\|_{\mathcal{X}} \\
&\quad \leq \varepsilon, t_0 \in [0, T], d \in M_D\} \\
&\quad \leq B\Big(R + \varepsilon \max_{t_0\in[0, \, T]} \widetilde{\beta}(t_0)\Big) \max_{t\in[0, \, 2T]} \mu(t) < +\infty,
\end{aligned}
$$

which directly implies that Σ is RFC.Thus we may conclude that inequalities (A32)–(A35) are satisfied for all $t \in [t_0, +\infty)$.