Global exponential stabilization of freeway models

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SUMMARY

This work is devoted to the construction of feedback laws, which guarantee the robust global exponential stability of the uncongested equilibrium point for general discrete-time freeway models. The feedback construction is based on a control Lyapunov function approach and exploits certain important properties of freeway models. The developed feedback laws are tested in simulation, and a detailed comparison is made with an existing feedback law reported in the literature and employed in practice. The robustness properties of the corresponding closed-loop system with respect to measurement errors are also studied. Copyright © 2015 John Wiley & Sons, Ltd.

1. INTRODUCTION

Freeway traffic congestion has become a significant problem for modern societies, which leads to excessive delays, reduced traffic safety, increased fuel consumption and environmental pollution. The main traffic control measures employed to tackle traffic congestion are ramp metering (RM) and variable speed limits (VSL). RM is implemented by use of traffic lights positioned at on-ramps to control the entering traffic flow [1]. VSL are used for speed harmonization, but recent studies have demonstrated that it may be used as a mainstream metering device as well [2]. To achieve their goal, these control measures must be driven by appropriate control strategies. A branch of related research has considered nonlinear optimal control and model predictive control as network-wide freeway traffic control approaches; see, for example, [3–6]. However, none of the proposed methods has advanced to a field-operational tool. Another significant branch of freeway traffic control research has considered explicit feedback control approaches to tackle congestion problems. A pioneering development in this direction was the I-type local feedback RM regulator ALINEA [7], which has been used in hundreds of successful field implementations around the world; see, for example, [8, 9]. ALINEA controls the traffic entering from an on-ramp and targets a critical density in the mainstream merging segment so as to maximize the freeway throughput. Other proposed local feedback control algorithms for RM include [10–13], to mention just a few. Various extensions and modifications of ALINEA were proposed and field implemented over the years to address specific emerging needs. Most relevant in the present context is the extension to a PI-type regulator so as to efficiently address bottlenecks, which are located far downstream of the merge area [14]; and the parallel deployment of PI-type regulators to address multiple potential bottlenecks downstream of the metered on-ramp [15]. On the other hand, feedback control approaches for mainstream traffic

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control by use of VSL have been rather sparse; see [16]; see also [17] for a recent extension to the multiple bottleneck case.

To adequately address the increasing freeway traffic congestion problems, it is essential to investigate, develop and deploy the potentially most efficient methods, and recent control theory advances should be appropriately exploited to this end. In this work, we provide a rigorous methodology for the construction of explicit feedback laws that guarantee the robust global exponential stability of the uncongested equilibrium point for general nonlinear discrete-time freeway models. We focus on discrete-time freeway models that are generalized versions of the known first-order discrete Godunov approximations [18] to the kinematic-wave partial differential equation of the LWR-model [19, 20] with nonlinear [21] or piecewise linear (cell transmission model (CTM); [22–24]) outflow functions (fundamental diagram (FD)). Specifically, the constructed class of freeway models allows for the following: (i) consideration of generally nonlinear (including piecewise linear) FDs; (ii) consideration of all possible cases for the relative priorities of the inflows at freeway nodes [23], and even for time-varying and unknown priority rules; and (iii) modified demand functions according to [25], to account for the capacity drop phenomenon, which is not reflected in the classical LWR-model and its Godunov discretization. The construction of the robust global exponential feedback stabilizer is based on the control Lyapunov function (CLF) approach [26] as well as on certain important properties of freeway models. In summary, the contribution of the present work is threefold:

- a CLF is constructed for a class of freeway models; the formulas for the Lyapunov function are explicit (see formulas (2.19), (2.20) and (2.21)) and can be used in a straightforward way for various purposes;
- important properties of general nonlinear and uncertain discrete-time freeway models are proved (see properties (C1), (C2), (C3), (C4) and (C5) in Section 3);
- a parameterized family of global exponential feedback stabilizers for the uncongested equilibrium point of freeway models is constructed. The achieved stabilization is robust with respect to all priority rules that can be used for the inflows.

A comparison is made, by means of simulation, with an existing feedback law proposed in the literature and employed in practice. More specifically, we consider the random located bottleneck (RLB) PI-type regulator that was proposed in [15] and is the most sophisticated of the very few comparable feedback regulators that have been employed in field operations [9]. The simulations, presented in Section 4 of the present work, study the performance of the corresponding closed-loop systems, as well as their robustness under the effect of measurement errors. It was found that the performance and the robustness properties achieved by the proposed feedback law were better than or comparable with the performance and the robustness properties induced by the RLB PI regulator. Ongoing work and future work address further robustness issues in presence of modelling errors and persistent disturbances.

Definitions and notation. In this paper, we adopt the following notation and terminology:

* $\Re_+ := [0, +\infty)$. For every set S, $S^n = \underbrace{S \times \ldots \times S}_{n \ times}$ for every positive integer n.

* By $C^0(A \; ; \; \Omega)$, we denote the class of continuous functions on $A \subseteq \Re^n$, which take values in $\Omega \subseteq \Re^m$. By $C^k(A \; ; \; \Omega)$, where $k \geqslant 1$ is an integer, we denote the class of functions on $A \subseteq \Re^n$ with continuous derivatives of order k, which take values in $\Omega \subseteq \Re^m$.

* The transpose of $x \in \Re^n$ is denoted by x'. By $|x|$ we denote the Euclidean norm of $x \in \Re^n$.

Let $S \subseteq \Re^n$, $D \subseteq \Re^l$ be non-empty sets and consider the uncertain, discrete-time, dynamical system

$$x^+ = F(d, x) , \ x \in S , \ d \in D, \tag{1.1}$$

where $F : D \times S \to S$ is a mapping. The variable $x \in S$ denotes the state of (1.1), while here (and throughout the paper), x^+ denotes the value of the state at the next time instant, that is, (1.1) describes the recursive relation $x(t + 1) = F(d(t), x(t))$. Let $x^* \in S$ be an equilibrium point of (1.1), that is, $x^* \in S$ satisfies $x^* = F(d, x^*)$ for all $d \in D$. Notice that the requirement

$x^* = F(d, x^*)$ for all $d \in D$ implies that $d \in D$ denotes a vanishing perturbation, that is, a disturbance that does not change the position of the equilibrium point of the system. In general, the decision of which variables are vanishing perturbation or non-vanishing perturbation depends on the equilibrium point that we have in mind; for example, a perturbation may change the position of other equilibria but not the position of the equilibrium point that we intend to study.

We use the following definitions throughout the paper.

Definition 1

We say that $x^* \in S$ is robustly globally exponentially stable (RGES) for system (1.1) if there exist constants $M, \sigma > 0$ such that for every $x_0 \in S$ and for every sequence $\{d(t) \in D\}_{t=0}^{\infty}$ the solution $x(t)$ of (1.1) with initial condition $x(0) = x_0$ corresponding to input $\{d(t) \in D\}_{t=0}^{\infty}$ (i.e. the solution that satisfies $x(t + 1) = F(d(t), x(t))$ for all $t \geq 0$ and $x(0) = x_0$) satisfies the inequality $|x(t) - x^*| \leq M \exp(-\sigma t) |x_0 - x^*|$ for all $t \geq 0$.

Definition 2

A function $V : S \to \Re_+$ for which there exist constants $K_2 \geq K_1 > 0$, $p > 0$ and $\lambda \in [0, 1)$ such that the inequalities $K_1 |x - x^*|^p \leq V(x) \leq K_2 |x - x^*|^p$ and $V(F(d, x)) \leq \lambda V(x)$ hold for all $(d, x) \in D \times S$, is called a Lyapunov function with exponent $p > 0$ for (1.1).

Remark 1

If a Lyapunov function with exponent $p > 0$ exists for (1.1), then $x^* \in S$ is RGES. Indeed, if the state space were $\Re^n$ and not $S \subseteq \Re^n$ and if no disturbances were present, then we would be able to use Theorem 13.2 on pages 765–766 in [27] and conclude that the existence of a Lyapunov function with exponent $p > 0$ for (1.1) is a sufficient condition for RGES of $x^* \in S$. However, because the uncertain dynamical system (1.1) is defined on $S \subseteq \Re^n$ with disturbances $d \in D$, we cannot use the aforementioned theorem. On the other hand, we can use the inequality $V(F(d, x)) \leq \lambda V(x)$ inductively and obtain the estimate $V(x(t)) \leq \lambda^t V(x(0))$ for every solution of (1.1), for every sequence $\{d(t) \in [0, 1]^{n-1}\}_{t=0}^{\infty}$ and for every integer $t \geq 0$. The required exponential estimate of the solution is obtained by combining the previous estimate with the inequality $K_1 |x - x^*|^p \leq V(x) \leq K_2 |x - x^*|^p$, which directly implies $K_1 |x(t) - x^*|^p \leq \lambda^t K_2 |x(0) - x^*|^p$, or $|x(t) - x^*| \leq M \exp(-\sigma t) |x_0 - x^*|$, where $M = (K_2/K_1)^{1/p}$, $\sigma := -\frac{\ln(\lambda)}{p}$ for the case $\lambda > 0$ and arbitrary $\sigma > 0$ for the case $\lambda = 0$.

2. FREEWAY MODELS AND MAIN RESULT

2.1. Model derivation

We consider a freeway that consists of $n \geq 3$ components or cells; typical cell lengths may be 200–500 m. Each cell may have an external controllable inflow (e.g. from corresponding on-ramps), located near the cell's upstream boundary; and an external outflow (e.g. via corresponding off-ramps), located near the cell's downstream boundary (Figure 1). The number of vehicles at time $t \geq 0$ in component $i \in \{1, \dots, n\}$ is denoted by $x_i(t)$. The total outflow and the total inflow of vehicles of the component $i \in \{1, \dots, n\}$ at time $t \geq 0$ are denoted by $F_{i,out}(t) \geq 0$ and $F_{i,in}(t) \geq 0$, respectively. All flows during a time interval are measured in [veh]. Consequently, the balance of vehicles (conservation equation) for each component $i \in \{1, \dots, n\}$ gives:

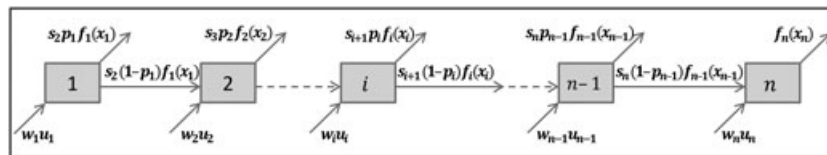$$x_i(t + 1) = x_i(t) - F_{i,out}(t) + F_{i,in}(t), \quad i = 1, \dots, n, \quad t \geq 0. \quad (2.1)$$

Figure 1. Scheme of the freeway model.

Each component of the network has storage capacity $a_i > 0 (i = 1, \ldots, n)$. Our first assumption states that the external (off-ramp) flows from each cell are constant percentages of the total exit flow, that is, there exist constants $p_i \in [0, 1]$, $i = 1, \ldots, n$, such that

$$\begin{pmatrix} \textit{flow of vehicles} \\ \textit{from component i to component i+1} \end{pmatrix} = (1 - p_i) F_{i,out}(t), \text{ for } i = 1, \ldots, n - 1 \quad (2.2)$$

$$\begin{pmatrix} \textit{flow of vehicles from} \\ \textit{component i to regions out of the freeway} \end{pmatrix} = p_i F_{i,out}(t), \text{ for } i = 1, \ldots, n. \quad (2.3)$$

The constants p_i are known as exit rates, that is, portions of $F_{i,out}(t)$ that are bound for the off-ramp of the i-th cell. Because the n-th cell is the last downstream cell of the considered freeway, we may assume that $p_n = 1$. We also assume that $p_i < 1$ for $i = 1, \ldots, n - 1$, and that all exits to regions out of the network can accommodate the respective exit flows.

Our second assumption is dealing with the attempted outflows $f_i(x_i)$, that is, the flows that will exit the cell if there is sufficient space in the downstream cell. We assume that there exist functions $f_i : [0, a_i] \to \Re_+$ with $0 < f_i(x_i) < x_i$ for all $x_i \in (0, a_i]$ and variables $s_i(t) \in [0, 1]$, $i = 2, \ldots, n$, so that

$$F_{i-1,out}(t) = s_i(t) f_{i-1}(x_{i-1}(t)), \quad i = 2, \ldots, n, \quad t \geqslant 0 \text{ and } F_{n,out}(t) = f_n(x_n(t)) \quad (2.4)$$

The variable $s_i(t) \in [0, 1]$, for each $i = 2, \ldots, n$, indicates the percentage of the attempted outflow from cell $(i-1)$ that becomes actual outflow from the same cell. The function $f_i : [0, a_i] \to \Re_+$ is called, in the specialized literature of traffic engineering (see, e.g. [21]), the demand part of the FD of the i-th cell, that is, the flow that will exit the cell i if there is sufficient space in the downstream cell $i+1$. Notice that (2.4) for $F_{n,out}(t)$ follows from our assumption that all exits to regions out of the network can accommodate the exit flows.

Let $u_i > 0 (i = 1, \ldots, n)$ denote the attempted external inflow to component $i \in \{1, \ldots, n\}$ from the region out of the freeway. Typically, u_i, $i = 2, \ldots, n$ correspond to external on-ramp flows that may be determined by a RM control strategy. For the very first cell 1, we assume, for convenience, that there is just one external inflow, u_1. Let the variables $w_i(t) \in [0, 1]$, $i = 1, \ldots, n$, indicate the percentage of the attempted external inflow to component $i \in \{1, \ldots, n\}$ that becomes actual inflow. Then, we obtain from (2.2) and (2.4):

$$F_{1,in}(t) = w_1(t) u_1(t) \text{ and } F_{i,in}(t) = w_i(t) u_i(t) + s_i(t)(1 - p_{i-1}) f_{i-1}(x_{i-1}(t)), i = 2, \ldots, n. \quad (2.5)$$

Our next assumption is derived from the Godunov discretization [21], which requires that the inflow of vehicles at the cell $i \in \{1, \ldots, n\}$ at time $t \geqslant 0$, denoted by $F_{i,in}(t) \geqslant 0$, cannot exceed the supply function of cell $i \in \{1, \ldots, n\}$ at time $t \geqslant 0$, that is,

$$F_{i,in}(t) \leqslant \min\left(q_i, c_i(a_i - x_i(t))\right), \quad i = 1, \ldots, n, \quad t \geqslant 0, \quad (2.6)$$

where $q_i \in (0, +\infty)$ denotes the maximum flow that the i-th cell can receive (or the capacity flow of the i-th cell) and $c_i \in (0, 1] (i = 1, \ldots, n)$ denotes the congestion wave speed of the i-th cell.

Following [23], we assume that, when the total demand flow of a cell is lower than the supply of the downstream cell, that is, when $u_i(t) + (1 - p_{i-1}) f_{i-1}(x_{i-1}(t)) \leqslant \min\left(q_i, c_i(a_i - x_i(t))\right)$ for some $i \in \{2, \ldots, n\}$, then the demand flow can be fully accommodated by the downstream cell, and hence, we have $s_i(t) = w_i(t) = 1$. Similarly, when $u_1(t) \leqslant \min\left(q_1, c_1(a_1 - x_1(t))\right)$, then we have $w_1(t) = 1$. In contrast, when the total demand flow of a cell is higher than the supply of the downstream cell, that is, when $u_i(t) + (1 - p_{i-1}) f_{i-1}(x_{i-1}(t)) > \min\left(q_i, c_i(a_i - x_i(t))\right)$ for some $i \in \{2, \ldots, n\}$ (or when $u_1(t) > \min\left(q_1, c_1(a_1 - x_1(t))\right)$), then the demand flow cannot be fully accommodated by the downstream cell, and the actual flow is determined by the supply function, that

is, we have $F_{i,in}(t) = \min\left(q_i, c_i(a_i - x_i(t))\right)$ (or $F_{1,in}(t) = \min\left(q_1, c_1(a_1 - x_1(t))\right)$). Therefore, we obtain:

$$F_{1,in}(t) = \min\left(q_1, c_1(a_1 - x_1(t)), u_1(t)\right), \quad t \geqslant 0 \tag{2.7}$$

$$s_i(t) = (1 - d_i(t)) \min\left(1, \max\left(0, \frac{\min\left(q_i, c_i(a_i - x_i(t))\right) - u_i(t)}{(1 - p_{i-1})f_{i-1}(x_{i-1}(t))}\right)\right)$$
$$+ d_i(t) \min\left(1, \frac{\min\left(q_i, c_i(a_i - x_i(t))\right)}{(1 - p_{i-1})f_{i-1}(x_{i-1}(t))}\right),$$

$$i = 2, \ldots, n, \quad t \geqslant 0 \tag{2.8}$$

$$F_{i,in}(t) = \min\left(q_i, c_i(a_i - x_i(t)), u_i(t) + (1 - p_{i-1})f_{i-1}(x_{i-1}(t))\right), \quad i = 2, \ldots, n, \quad t \geqslant 0, \tag{2.9}$$

where

$$d_i(t) \in [0, 1], \quad i = 2, \ldots, n, \quad t \geqslant 0 \tag{2.10}$$

are time-varying parameters. Note that, if the supply is higher than the total demand, then (2.8) yields $s_i = 1$, irrespective of the value of d_i, because the total demand flow can be accommodated by the downstream cell. Thus, the parameter d_i determines the relative inflow priorities, when the downstream supply prevails. Specifically, when $d_i(t) = 0$, then the on-ramp inflow has absolute priority over the internal inflow; on the other hand, when $d_i(t) = 1$, then the internal inflow has absolute priority over the on-ramp inflow; while intermediate values of d_i reflect intermediate priority cases. The parameters $d_i(t) \in [0, 1]$ are treated as unknown parameters (disturbances). Notice that by introducing the parameters $d_i(t) \in [0, 1]$ (and by allowing them to be time-varying), we have taken into account all possible cases for the relative priorities of the inflows (and we also allow the priority rules to be time-varying); see [28–32] for freeway models with specific priority rules, which are special cases of our general approach.

All the aforementioned are illustrated in Figure 1. Combining (2.1), (2.2), (2.3), (2.4), (2.7) and (2.9), we obtain the following discrete-time dynamical system:

$$\begin{aligned} x_1^+ &= x_1 - s_2 f_1(x_1) + \min\left(q_1, c_1(a_1 - x_1), u_1\right) \\ &= x_1 - s_2 f_1(x_1) + w_1 u_1 \end{aligned} \tag{2.11}$$

$$\begin{aligned} x_i^+ &= x_i - s_{i+1} f_i(x_i) + \min\left(q_i, c_i(a_i - x_i), u_i + (1 - p_{i-1})f_{i-1}(x_{i-1})\right) \\ &= x_i - s_{i+1} f_i(x_i) + w_i u_i + s_i(1 - p_{i-1})f_{i-1}(x_{i-1}) \end{aligned}, \quad \text{for } i = 2, \ldots, n-1 \tag{2.12}$$

$$\begin{aligned} x_n^+ &= x_n - f_n(x_n) + \min\left(q_n, c_n(a_n - x_n), u_n + (1 - p_{n-1})f_{n-1}(x_{n-1})\right) \\ &= x_n - f_n(x_n) + w_n u_n + s_n(1 - p_{n-1})f_{n-1}(x_{n-1}), \end{aligned} \tag{2.13}$$

where $s_i \in [0, 1]$, $i = 2, \ldots, n$ are given by (2.8). The values of $w_i \in [0, 1]$, $i = 1, \ldots, n$, may also be similarly derived from (2.5) when $u_i > 0$, but they are not needed in what follows. Define $S = (0, a_1] \times (0, a_2] \times \ldots \times (0, a_n]$. Because the functions $f_i : [0, a_i] \to \Re_+$ satisfy $0 < f_i(x_i) < x_i$ for all $x_i \in (0, a_i]$, it follows that (2.11), (2.12) and (2.13) are uncertain control systems on S (i.e. $x = (x_1, \ldots, x_n)' \in S$) with inputs $u = (u_1, \ldots, u_n)' \in (0, +\infty) \times \Re_+^{n-1}$ and disturbances $d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$. We emphasize again that the uncertainty $d \in [0, 1]^{n-1}$ appears in the (2.11), (2.12) and (2.13) only when the supply function prevails, that is, only when $u_i(t) + (1 - p_{i-1})f_{i-1}(x_{i-1}(t)) > \min\left(q_i, c_i(a_i - x_i(t))\right)$ for some $i \in \{2, \ldots, n\}$.
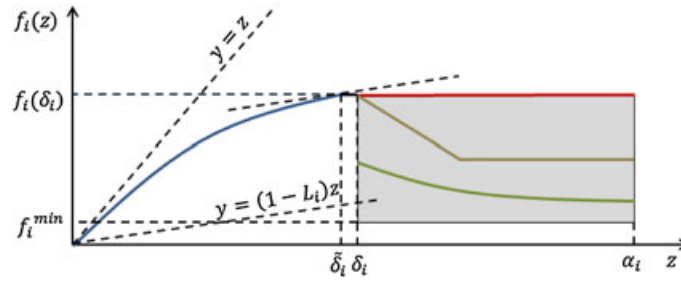
Figure 2. Implications of Assumption (H).

We make the following assumption for the functions $f_i : [0, a_i] \to \Re_+ (i = 1, \ldots, n)$:

(H) *The function* $f_i : [0, a_i] \to \Re_+$ *satisfies* $0 < f_i(z) < z$ *for all* $z \in (0, a_i]$. *There exists* $\delta_i \in (0, a_i]$ *such that* f_i *is continuous and increasing on* $[0, \delta_i]$. *Moreover, there exist constants* $L_i \in (0, 1), \tilde{\delta}_i \in (0, \delta_i]$ *such that* $f_i : [0, a_i] \to \Re_+$ *is* C^1 *on* $(0, \delta_i), 1 - L_i \leqslant f'_i(z)$ *for all* $z \in (0, \tilde{\delta}_i), f'_i(z) \leqslant 1$ *for all* $z \in (0, \delta_i)$. *Finally, there exists a positive constant* $f_i^{\min} > 0$ *such that* $f_i(\delta_i) \geqslant f_i(z) \geqslant f_i^{\min}$ *for all* $z \in [\delta_i, a_i]$.

Assumption (H) includes the basic properties of the so-called demand function [21] in the Godunov discretization; whereby δ_i is the critical density, where $f_i(x_i)$ achieves a maximum value. The implications of Assumption (H) for the demand function are illustrated in Figure 2. In words, the FD of cell i is composed of the increasing function $f_i(x_i)$ for $x_i \in [0, \delta_i]$; and by the non-increasing supply function $\min(q_i, c_i(a_i - x_i))$ for $x_i \in [\delta_i, a_i]$. Note, however, that Assumption (H) includes the possibility of reduced demand flow for overcritical densities (i.e. when $x_i(t) \geqslant \delta_i$), because $f_i(x_i)$ is allowed to be any arbitrary function (e.g. discontinuous or decreasing or, even, increasing), taking any values within the bounds mentioned in (H) (corresponding to the grey area in Figure 2), for $x_i \in [\delta_i, a_i]$; this could be used to reflect the capacity drop phenomenon, as it is treated in some recent works [33, 34]. Figure 2 presents, within the grey area of overcritical densities, three examples of demand functions, which satisfy assumption (H). In conclusion, the model (2.11)–(2.13) is a generalized version of the known first-order discrete Godunov approximation to the kinematic-wave partial differential equation of the LWR-model [19, 20] with nonlinear [21] or piecewise linear (CTM; [22, 23]) FD. However, the presented framework can also accommodate recent modifications of the LWR-model as in [25, 33, 34] to reflect the so-called capacity drop phenomenon. Notice that the piecewise smooth selections $f_i(z) = q_i \delta_i^{-1} z$ for $z \in [0, \delta_i]$ and $f_i(z) = q_i$ for $z \in (\delta_i, a_i]$ $(i = 1, \ldots, n)$ with $a_i \geqslant \delta_i + c_i^{-1} q_i$ allow us to obtain the standard CTM with: (i) triangular-FD (if $a_i = \delta_i + c_i^{-1} q_i$) and (ii) trapezoidal-FD (if $a_i > \delta_i + c_i^{-1} q_i$). In the latter case, Assumption (H) holds with arbitrary $\tilde{\delta}_i \in (0, \delta_i]$.

Define the vector field $\tilde{F} : [0, 1]^{n-1} \times S \times (0, +\infty) \times \Re_+^{n-1} \to S$ for all $x \in S := (0, a_1] \times \ldots (0, a_n], d = (d_2, \ldots, d_n) \in D = [0, 1]^{n-1}$ and $u = (u_1, \ldots, u_n) \in (0, +\infty) \times \Re_+^{n-1}$:

$$\tilde{F}(d, x, u) = (\tilde{F}_1(d, x, u), \ldots, \tilde{F}_n(d, x, u))' \in \Re^n$$

with $\tilde{F}_1(d, x, u) := x_1 - s_2 f_1(x_1) + \min(q_1, c_1(a_1 - x_1), u_1)$,

$$\tilde{F}_i(d, x, u) = x_i - s_{i+1} f_i(x_i) + \min(q_i, c_i(a_i - x_i), u_i + (1 - p_{i-1}) f_{i-1}(x_{i-1})),$$
$$\text{for } i = 2, \ldots, n-1,$$

$$\tilde{F}_n(d, x, u) = x_n - f_n(x_n) + \min(q_n, c_n(a_n - x_n), u_n + (1 - p_{n-1}) f_{n-1}(x_{n-1})) \text{ and}$$

$$s_i = (1 - d_i) \min\left(1, \max\left(0, \frac{\min(q_i, c_i(a_i - x_i)) - u_i}{(1 - p_{i-1}) f_{i-1}(x_{i-1})}\right)\right)$$
$$+ d_i \min\left(1, \frac{\min(q_i, c_i(a_i - x_i))}{(1 - p_{i-1}) f_{i-1}(x_{i-1})}\right), \text{ for } i = 2, \ldots, n.$$

$$(2.14)$$

Notice that, using definition (2.14), the control systems (2.11), (2.12) and (2.13) can be written in the following vector form

$$
\begin{aligned}
&x^+ = \tilde{F}(d, x, u) \\
&x \in S, \ d \in D, \ u \in (0, +\infty) \times \Re_+^{n-1}.
\end{aligned}
\tag{2.15}
$$

2.2. Main result

Consider the freeway model (2.15) under Assumption (H). We suppose that there exist $u_1^* > 0$, $u_i^* \geqslant 0 (i = 2, \ldots, n)$ and a vector $x^* = (x_1^*, \ldots, x_n^*) \in (0, \tilde{\delta}_1) \times \ldots (0, \tilde{\delta}_n)$ with

$$
f_1(x_1^*) = u_1^*, \quad f_i(x_i^*) = u_i^* + (1 - p_{i-1}) f_{i-1}(x_{i-1}^*) = u_i^* + \sum_{j=1}^{i-1} \left(\prod_{k=j}^{i-1} (1 - p_k) \right) u_j^* (i = 2, \ldots, n)
\tag{2.16}
$$

and

$$
u_1^* < \min \left(q_1, c_1 \left(a_1 - x_1^* \right) \right), u_i^* + (1 - p_{i-1}) f_{i-1}(x_{i-1}^*) < \min \left(q_i, c_i \left(a_i - x_i^* \right) \right) (i = 2, \ldots, n).
\tag{2.17}
$$

This is the uncongested equilibrium point of the freeway model (2.15). Notice that Assumption (H) guarantees that an uncongested equilibrium point always exists for the freeway model (2.15) when $u_1^* > 0$ and $u_i^* \geqslant 0 (i = 2, \ldots, n)$ are sufficiently small. The uncongested equilibrium point is not globally exponentially stable for arbitrary $u_1^* > 0, u_i^* \geqslant 0 (i = 2, \ldots, n)$; indeed, for relatively large values of external demands $u_1^* > 0, u_i^* \geqslant 0 (i = 2, \ldots, n)$ other equilibria for model (2.15) (congested equilibria) may appear, for which the cell densities are large and can attract the solution of (2.15) (see the numerical Example 4.2 in Section 4).

The following result is our main result in feedback design. The result shows that a continuous, robust, global exponential stabilizer exists for every freeway model of the form (2.15) under Assumption (H).

Theorem 2.1

Consider system (2.15) with $n \geqslant 3$ under Assumption (H) for each $i = 1, \ldots, n$. Then there exist a subset $R \subseteq \{1, \ldots, n\}$ of the set of all indices $i \in \{1, \ldots, n\}$ with $u_i^* > 0$, constants $\sigma \in (0, 1], b_i \in (0, u_i^*)$ for $i \in R$ and a constant $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$ the feedback law $k : S \to \Re_+^n$ defined by

$$
\begin{aligned}
&k(x) := (k_1(x), \ldots, k_n(x))' \in \Re^n \text{ with} \\
&k_i(x) := \max \left(u_i^* - \gamma_i \ \Xi(x), b_i \right), \text{ for all } x \in S, i \in R \text{ and } k_i(x) := u_i^*, \text{ for all } x \in S, i \notin R,
\end{aligned}
\tag{2.18}
$$

where $\gamma_i := \tau^{-1}(u_i^* - b_i)$ and

$$
\Xi(x) := \sum_{i=1}^n \sigma^i \max \left(0, x_i - x_i^* \right), \text{ for all } x \in S
\tag{2.19}
$$

achieves robust global exponential stabilization of the uncongested equilibrium point x^* of system (2.15), that is, x^* is RGES for the closed-loop system (2.15) with $u = k(x)$. Moreover, for every $\tau \in (0, \tau^*)$, there exist constants $Q, h, \theta, A, K > 0$ so that the function $V : S \to \Re_+$ defined by

$$
V(x) := \sum_{i=1}^n \sigma^i |x_i - x_i^*| + A\Xi(x) + K \max \left(0, \sum_{i=1}^n I_i(x) - P(x) \right), \text{ for all } x \in S,
\tag{2.20}
$$

where $I_j(x) := \sum_{i=1}^{j} x_i$ for $j = 1, \ldots, n$ and

$$P(x) := Q - \theta \min (h, \Xi(x)) \qquad (2.21)$$

is a Lyapunov function with exponent 1 for the closed-loop system (2.15) with $u = k(x)$.

Although Theorem 2.1 is an existence result, its proof is constructive and provides formulae for all constants and for the index set R (see following sections). Notice that the index set R is the set of all inflows that must be controlled in order to be able to guarantee that the uncongested equilibrium point is RGES; consequently, the knowledge of the index set R is critical.

The importance of Theorem 2.1 lies on the facts as follows:

- Theorem 2.1 provides a family of robust global exponential stabilizers (parameterized by the parameter $\tau \in (0, \tau^*)$) and an explicit formula for the feedback law (formula (2.18)).
- The achieved stabilization result is robust for all possible (and even time-varying) priority rules for the junctions that may apply at specific freeways; thus, there is no need to know or estimate the applied priority rules.
- Theorem 2.1 provides an explicit formula for the Lyapunov function of the closed-loop system. This is important, because the knowledge of the Lyapunov function allows the study of the robustness of the closed-loop system to various disturbances (measurement errors, modelling errors, etc.) as well the study of the effect of interconnections of freeways (by means of the small-gain theorem; see [26]).

The main idea behind the proof of Theorem 2.1 is the construction of the Lyapunov function of the closed-loop system, which acts as a CLF [26] for the open-loop system. The construction of the Lyapunov function is based on the observation that there are no congestion phenomena when the cell densities are sufficiently small, that is,

'There exists a set $\Omega \subset S$ of the form $\Omega = (0, \mu_1] \times \ldots \times (0, \mu_n]$, where $\mu_i > 0$ for $i = 1, \ldots, n$ are constants, such that no congestion phenomena are present when $x \in \Omega$'.

The existence of the set $\Omega \subset S$ is important because, when no congestion phenomena are present, then the freeway model admits the simple (cascade) form:

$$x_1^+ = x_1 - f_1(x_1) + u_1, x_i^+ = x_i - f_i(x_i) + (1-p_{i-1}) f_{i-1}(x_{i-1}) + u_{i+1}, \text{ for } i = 2, \ldots, n, x \in \Omega$$

and a Lyapunov function for the aforementioned form can be a function of the form $V_1(x) := \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| + A\Xi(x)$, where $\Xi(x) := \sum_{i=1}^{n} \sigma^i \max(0, x_i - x_i^*)$; and $\sigma \in (0, 1]$ and $A > 0$ are appropriate constants. The Lyapunov function for the freeway model is the linear combination of the 'Lyapunov function' for the uncongested model (i.e. $V_1(x) := \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| + A\Xi(x)$) and
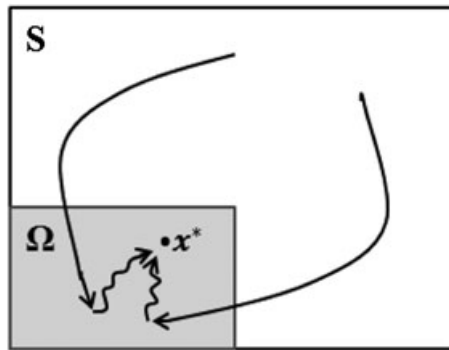

Figure 3. Idea behind Theorem 2.1.

a penalty term, that is, the term $\max\left(0, \sum_{i=1}^{n} I_i(x) - P(x)\right)$, that penalizes large cell densities (and thus penalizes the possibility of the state being out of the set $\Omega \subset S$). The appropriate selection of the weight of the penalty term $K > 0$ forces the selected control action to lead the state in the set $\Omega \subset S$ (Figure 3). In other words, the construction of the CLF guarantees that the control action will first eliminate all congestion phenomena and then will drive the state to the desired equilibrium.

3. PROOF OF MAIN RESULT

Assumption (H) has non-trivial consequences. A list of the most important consequences of Assumption (H) is given in the following. All following consequences are exploited in the proof of Theorem 2.1.

Consequences of Assumption (H):

(C1) *The mappings* $[0, \delta_i] \ni z \rightarrow (z - f_i(z)) \geqslant 0$ *are non-decreasing for* $i = 1, \ldots, n$.
Property (C1) is a direct consequence of the fact that $f_i'(z) \leqslant 1$ for all $z \in (0, \delta_i)$.

(C2) *For each* $i = 1, \ldots, n$, *there exist constants* $\lambda_i \in (0, 1)$, $G_i \in [0, 1]$ *such that*

$$|x_i - x_i^* - f_i(x_i) + f_i(x_i^*)| \leqslant \lambda_i |x_i - x_i^*| \text{ and}$$
$$|f_i(x_i) - f_i(x_i^*)| \leqslant G_i |x_i - x_i^*|, \text{ for all } x_i, x_i^* \in [0, \tilde{\delta}_i]. \tag{3.1}$$

Property (C2) is a direct consequence of the fact that there exist constants $L_i \in (0, 1)$, $\tilde{\delta}_i \in (0, \delta_i]$ such that $f_i : [0, a_i] \rightarrow \Re_+$ is C^1 on $(0, \delta_i)$ and $1 - L_i \leqslant f_i'(z)$ for all $z \in (0, \tilde{\delta}_i)$ and $f_i'(z) \leqslant 1$ for all $z \in (0, \delta_i)$. We conclude that (3.1) holds with $\lambda_i = L_i \in (0, 1)$ and $G_i = 1$.

(C3) *There exist constants* $\theta_i > 0 (i = 1, \ldots, n)$ *such that* $f_i(z) \geqslant \theta_i z$ *for all* $z \in [0, a_i]$ *and* $i = 1, \ldots, n$.
Property (C3) is a direct consequence of the fact that $f_i(z) \geqslant (1 - L_i)z$ for all $z \in (0, \tilde{\delta}_i]$ (a direct consequence of the fact that $1 - L_i \leqslant f_i'(z)$ for all $z \in (0, \tilde{\delta}_i)$), the fact that $f_i(z) \geqslant f_i(\tilde{\delta}_i)$ for all $z \in [\tilde{\delta}_i, \delta_i]$ and the fact that $f_i(z) \geqslant f_i^{\min}$ for all $z \in [\tilde{\delta}_i, \delta_i]$ (for example, the selection $\theta_i = \min\left(1 - L_i, \frac{f_i(\tilde{\delta}_i)}{\delta_i}, \frac{f_i^{\min}}{a_i}\right)$ satisfies (C3)).
The following consequence provides a useful linear lower bound on a weighted sum of exit rates. Its proof is provided in the Appendix.

(C4) *For every* $r_2, \ldots, r_n \geqslant 0$ *with* $r_i < \min(q_i, c_i a_i)$ *for* $i = 2, \ldots, n$, *there exists a constant* $C > 0$ *such that the following inequality holds for all* $x \in S := (0, a_1] \times \ldots (0, a_n], u \in U = (0, +\infty) \times [0, r_2] \times \ldots \times [0, r_n], d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$ *with* $p_n = 1 = s_{n+1}$:

$$\sum_{i=1}^{n} (1 + p_i(n - i)) s_{i+1} f_i(x_i) \geqslant C \sum_{i=1}^{n} (n + 1 - i) x_i. \tag{3.2}$$

Remark 3.1
The proof of Property (C4) implies that the constant $C > 0$ can be estimated in a straightforward way. We define the positive constants $Y_i > 0$ $(i = 1, \ldots, n)$ using the recursive formula:

$$Y_{k-1} = \min\left(Y_k, \frac{1 + p_{k-1}(n + 1 - k)}{n + 2 - k} l_k \theta_{k-1}, \right.$$
$$\frac{(n + 1 - k)\left(a_k - c_k^{-1} r_k\right) Y_k}{2(n + 1 - k)a_k + 2(n + 2 - k)a_{k-1}},$$
$$\left. \frac{q_k - r_k}{1 - p_{k-1}} \frac{1 + p_{k-1}(n + 1 - k)}{(n + 2 - k)a_{k-1}} \right) \tag{3.3}$$

for $k = n, n - 1, \ldots, 2$, with $Y_n = \theta_n$, where $\theta_i > 0$ $(i = 1, \ldots, n)$ are the constants involved in Property (C3) and $l_k = \min\left(1, \frac{c_k a_k - r_k}{2(1 - p_{k-1}) f_{k-1}(\delta_{k-1})}\right)$ for $k = 2, \ldots, n$. Then the constant $C > 0$ can be selected as $C = Y_1$. However, the estimation of the constant $C > 0$ by the recursive formula (3.3) is, in general, conservative.

Finally, the last consequence provides useful equalities and inequalities for a weighted sum of all vehicle densities of the freeway. Its proof is provided in the Appendix.

(C5) *The following equality holds for all* $x \in S := (0, a_1] \times \ldots (0, a_n], u = (u_1, \ldots, u_n)' \in (0, +\infty) \times \Re_+^{n-1}, d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$*with* $p_n = 1 = s_{n+1}$:

$$\sum_{i=1}^{n} I_i(x^+) = \sum_{i=1}^{n} I_i(x) + \sum_{i=1}^{n} (n + 1 - i) w_i u_i - \sum_{i=1}^{n} (1 + p_i(n - i)) s_{i+1} f_i(x_i), \quad (3.4)$$

where $I_j(x) := \sum_{i=1}^{j} x_i$ *for* $j = 1, \ldots, n$. *Moreover, for every* $r_2, \ldots, r_n \geqslant 0$ *with* $r_i < \min(q_i, c_i a_i)$ *for* $i = 2, \ldots, n$, *the following inequality holds:*

$$\sum_{i=1}^{n} I_i(x^+) \leqslant (1 - C) \sum_{i=1}^{n} I_i(x) + \sum_{i=1}^{n} (n + 1 - i) u_i, \quad \text{for all } (x, u, d) \in S \times U \times [0, 1]^{n-1},$$
$$(3.5)$$

where $U = (0, +\infty) \times [0, r_2] \times \ldots \times [0, r_n]$, *and* $C > 0$ *is the constant involved in (3.2).*

We are now ready to provide the proof of Theorem 2.1.

Proof of Theorem 2.1
Define $\beta_n = \tilde{\delta}_n$ and $\beta_i \in (0, \tilde{\delta}_i]$, for $i = 1, \ldots, n - 1$ to be the unique solution of the equation

$$f_i(\beta_i) = \min\left(f_i(\tilde{\delta}_i), \frac{q_{i+1} - u_{i+1}^*}{1 - p_i}\right). \quad (3.6)$$

Because the inequalities (2.17) and the fact that $\beta_n = \tilde{\delta}_n$, it follows that $\beta_i > x_i^*$ for $i = 1, \ldots, n$. Define $\omega_i = c_i(a_i - x_i^*) - u_i^* - (1 - p_{i-1}) f_{i-1}(x_{i-1}^*)$ for $i = 2, \ldots, n$ and $\omega_1 = c_1(a_1 - x_1^*) - u_1^*$. Next define:

$$\mu_i = \min\left(\beta_i, x_i^* + \frac{\omega_i}{2c_i}, x_i^* + \frac{\omega_{i+1}}{2(1 - p_i)}\right) \text{ for } i = 1, \ldots, n - 1 \text{ and } \mu_n = \min\left(\beta_n, x_n^* + \frac{\omega_n}{2c_n}\right). \quad (3.7)$$

Again, because the inequalities (2.17) and the fact that $\beta_i > x_i^*$ for $i = 1, \ldots, n$, it follows that $\mu_i > x_i^*$ for $i = 1, \ldots, n$.

It follows from (2.5), (2.7), (2.8), (2.9), (2.11), (2.12), (2.13), (3.6) and (3.7) that the following equations hold when $x \in \Omega = (0, \mu_1] \times \ldots \times (0, \mu_n]$ and $u_i \in [0, u_i^*]$ for $i = 1, \ldots, n$:

$$w_i = 1, \text{ for } i = 1, \ldots, n \text{ and } s_i = 1, \text{ for } i = 2, \ldots, n \quad (3.8)$$

$$x_1^+ = x_1 - f_1(x_1) + u_1, x_i^+ = x_i - f_i(x_i) + (1 - p_{i-1}) f_{i-1}(x_{i-1}) + u_{i+1}, \text{ for } i = 2, \ldots, n. \quad (3.9)$$

To see this, notice that for all $x \in \Omega = (0, \mu_1] \times \ldots \times (0, \mu_n]$ and $u \in [0, u_1^*] \times \ldots \times [0, u_n^*]$ we have $u_1 \leqslant \min(q_1, c_1(a_1 - x_1))$ and $u_i + (1 - p_{i-1}) f_{i-1}(x_{i-1}) \leqslant \min(q_i, c_i(a_i - x_i))$ for $i = 2, \ldots, n$. Indeed, Assumption (H) in conjunction with (2.17) and definition (3.7) implies that

$$u_i + (1 - p_{i-1}) f_{i-1}(x_{i-1}) \leqslant u_i^* + (1 - p_{i-1}) f_{i-1}(\mu_{i-1}) \leqslant u_i^* + (1 - p_{i-1}) f_{i-1}(\beta_{i-1}) \leqslant q_i$$

for $i = 2, \ldots, n$ and for all $x \in \Omega = (0, \mu_1] \times \ldots \times (0, \mu_n]$ and $u \in [0, u_1^*] \times \ldots \times [0, u_n^*]$. The inequality $u_1 \leqslant u_1^* \leqslant q_1$ is directly implied by (2.17). Moreover, Assumption (H) (and particularly the fact that f_i is increasing on $[0, \delta_i]$ with $f_i'(z) \leqslant 1$ for all $z \in (0, \delta_i)$ for $i = 1, \ldots, n$) in conjunction with (2.17) and definition (3.7) implies that

$$
\begin{aligned}
u_i &+ (1 - p_{i-1}) f_{i-1}(x_{i-1}) \leqslant u_i^* + (1 - p_{i-1}) f_{i-1}(x_{i-1}^*) \\
&+ (1 - p_{i-1}) \big(f_{i-1}(x_{i-1}) - f_{i-1}(x_{i-1}^*) \big) \\
&\leqslant u_i^* + (1 - p_{i-1}) f_{i-1}(x_{i-1}^*) + (1 - p_{i-1}) \max \big(0, x_{i-1} - x_{i-1}^* \big) \\
&\leqslant u_i^* + (1 - p_{i-1}) f_{i-1}(x_{i-1}^*) + \frac{\omega_i}{2} = \frac{1}{2} c_i \big(a_i - x_i^* \big) \\
&+ \frac{1}{2} u_i^* + \frac{1}{2} (1 - p_{i-1}) f_{i-1}(x_{i-1}^*) = c_i \big(a_i - x_i^* \big) - \frac{\omega_i}{2} \\
&\leqslant c_i \big(a_i - x_i^* \big) - c_i \max \big(0, x_i - x_i^* \big) \leqslant c_i \big(a_i - x_i^* \big) + c_i \big(x_i^* - x_i \big) = c_i \big(a_i - x_i \big)
\end{aligned}
$$

for $i = 2, \ldots, n$ and for all $x \in \Omega = (0, \mu_1] \times \ldots \times (0, \mu_n]$ and $u \in [0, u_1^*] \times \ldots \times [0, u_n^*]$. The inequality $u_1 \leqslant u_1^* \leqslant c_1(a_1 - x_1)$ is a consequence of (2.17), definition (3.7) and the inequalities

$$
\begin{aligned}
u_1^* &\leqslant \tfrac{1}{2} u_1^* + \tfrac{1}{2} c_1 \big(a_1 - x_1^* \big) = c_1 \big(a_1 - x_1^* \big) - \tfrac{\omega_1}{2} \leqslant c_1 \big(a_1 - x_1^* \big) - c_1 \max \big(0, x_1 - x_1^* \big) \\
&\leqslant c_1 \big(a_1 - x_1^* \big) + c_1 \big(x_1^* - x_1 \big) = c_1 \big(a_1 - x_1 \big).
\end{aligned}
$$

Let $\lambda_i \in (0, 1)$, $G_i \in [0, 1]$ $(i = 1, \ldots, n)$, be the constants involved in Property (C2). Let $\sigma \in (0, 1]$ be a constant so that

$$
L := \max \left(\lambda_n, \max_{i=1,\ldots,n-1} (\lambda_i + \sigma G_i (1 - p_i)) \right) < 1. \tag{3.10}
$$

Notice that $\max \left(\lambda_n, \max_{i=1,\ldots,n-1} (\lambda_i + \sigma G_i (1 - p_i)) \right) < 1$ for all $\sigma \in (0, 1)$. In what follows, we have $p_n = 1 = s_{n+1}$. Let $r_i = u_i^*$ for $i = 2, \ldots, n$, and let $C > 0$ be the constant involved in (3.2). Let $R \subseteq \{1, \ldots, n\}$ be a subset of the set of all indices $i \in \{1, \ldots, n\}$ for which $u_i^* > 0$ and such that

$$
\begin{aligned}
\sum_{i \notin R} (n + 1 - i) u_i^* &< \min_{i=1,\ldots,n} \big(((n - i) p_i + 1) f_i(x_i^*) \big) \text{ and} \\
\sum_{i \notin R} (n + 1 - i) u_i^* &< C \min_{i=1,\ldots,n} \big((n + 1 - i) \mu_i \big),
\end{aligned} \tag{3.11}
$$

where $\mu_i > x_i^*$ for $i = 1, \ldots, n$ are the constants defined by (3.7). Such a set $R \subseteq \{1, \ldots, n\}$ always exists (for example, $R \subseteq \{1, \ldots, n\}$ can be the set of all indices $i \in \{1, \ldots, n\}$ for which $u_i^* > 0$). Inequalities (3.11) imply that there exist constants $\varepsilon \in (0, 1)$ and $b_i \in (0, u_i^*)$ for $i \in R$ such that

$$
\begin{aligned}
\sum_{i \in R} (n + 1 - i) b_i + \sum_{i \notin R} (n + 1 - i) u_i^* &\leqslant \min_{i=1,\ldots,n} \big(((n - i) p_i + 1) f_i(x_i^*) \big) \text{ and} \\
\sum_{i \in R} (n + 1 - i) b_i + \sum_{i \notin R} (n + 1 - i) u_i^* &\leqslant \varepsilon C \min_{i=1,\ldots,n} \big((n + 1 - i) \mu_i \big).
\end{aligned} \tag{3.12}
$$

We next define the following parameters:

- Define $h := \min_{i=1,\ldots,n} \big(\sigma^i \big(\mu_i - x_i^* \big) \big)$.

- Define

$$Q := \max \left(\min_{i=1,\ldots,n} (\mu_i (n+1-i)), (1-C) \sum_{i=1}^{n} I_i(x^*) \right.$$

$$\left. +(1-C)h \max_{i=1,\ldots,n} ((n+1-i)\sigma^{-i}) + \sum_{i=1}^{n} (n+1-i) u_i^* \right)$$

and $\theta := h^{-1} \left(Q - \varepsilon \min_{i=1,\ldots,n} ((n+1-i)\mu_i) \right)$.

- Define $\tau^* := \min \left(h, (\theta L)^{-1} \sum_{i \in R} (n+1-i)(u_i^*-b_i) \right)$ and let $\tau \in (0, \tau^*)$.
- Define $A := 1 + (1-L)^{-1} \sum_{i \in R} \sigma^i \gamma_i$, where $\gamma_i := \tau^{-1}(u_i^*-b_i)$ for $i \in R$.
- Define $K := \dfrac{\sum_{i=1}^{n} \sigma^i \max(a_i-x_i^*, x_i^*) + A \sum_{i=1}^{n} \sigma^i (a_i-x_i^*) - (A+L)h}{(1-\varepsilon)C \min_{i=1,\ldots,n} ((n+1-i)\mu_i)}$.

We next prove the implication:

$$\text{If } x \in \Omega = (0, \mu_1] \times \ldots \times (0, \mu_n], d \in [0,1]^{n-1} \text{ and } u \in [0, u_1^*] \times \ldots \times [0, u_n^*] \text{ then } \Xi(x^+) \leqslant L \Xi(x), \tag{3.13}$$

where $L \in (0, 1)$ is defined by (3.10) and $x^+ = \tilde{F}(d, x, u)$. Indeed, using (3.9) and definition (2.19), we acquire for all $x \in \Omega = (0, \mu_1] \times \ldots \times (0, \mu_n], d \in [0,1]^{n-1}$ and $u \in [0, u_1^*] \times \ldots \times [0, u_n^*]$:

$$\begin{aligned} \Xi(x^+) &= \sum_{i=2}^{n} \sigma^i \max \left(0, x_i - f_i(x_i) + (1-p_{i-1})f_{i-1}(x_{i-1}) + u_i - x_i^* \right) \\ &\quad + \sigma \max \left(0, x_1 - f_1(x_1) + u_1 - x_1^* \right) \\ &\leqslant \sum_{i=1}^{n} \sigma^i \max \left(0, x_i - f_i(x_i) + f_i(x_i^*) - x_i^* \right) \\ &\quad + \sum_{i=2}^{n} \sigma^i (1-p_{i-1}) \max \left(0, f_{i-1}(x_{i-1}) - f_{i-1}(x_{i-1}^*) \right). \end{aligned} \tag{3.14}$$

Using (3.1), the fact that $\mu_i \leqslant \tilde{\delta}_i$ for $i = 1, \ldots, n$ (a consequence of (3.6) and (3.7)) and the fact that f_i is increasing on $[0, \tilde{\delta}_i]$ for $i = 1, \ldots, n$ (a consequence of Assumption (H)), we obtain

$$\max \left(0, f_i(x_i) - f_i(x_i^*) \right) \leqslant G_i \max \left(0, x_i - x_i^* \right), \text{ for all } x_i \in [0, \mu_i], i = 1, \ldots, n. \tag{3.15}$$

Using Properties (C1) and (C2) and the fact that $\mu_i \leqslant \tilde{\delta}_i$ for $i = 1, \ldots, n$ (a consequence of (3.6) and (3.7)), we obtain

$$\max \left(0, x_i - f_i(x_i) + f_i(x_i^*) - x_i^* \right) \leqslant \lambda_i \max \left(0, x_i - x_i^* \right), \text{ for all } x_i \in [0, \mu_i], i = 1, \ldots, n. \tag{3.16}$$

Combining (3.10), (3.14), (3.15) and (3.16), we obtain implication (3.13).

Next, we show the implication:

$$\text{If } x \in S, d \in [0,1]^{n-1} \text{ and } u \in [0, u_1^*] \times \ldots \times [0, u_n^*] \text{ then } P(x^+) \geqslant P(x), \tag{3.17}$$

where $x^+ = \tilde{F}(d, x, u)$. Indeed, (3.17) is a direct consequence of (3.13) and definition (2.21) when $x \in \Omega = (0, \mu_1] \times \ldots \times (0, \mu_n]$. On the other hand, when $x \in S \setminus \Omega$, there exists at least one $i \in \{1, \ldots, n\}$ for which $x_i > \mu_i$. Therefore, definition (2.19) implies $\Xi(x) > \min_{i=1,\ldots,n} \left(\sigma^i (\mu_i - x_i^*) \right)$, and consequently definition (2.21) gives $P(x) = Q - \theta h$ (a consequence of the fact that

$h = \min\limits_{i=1,...,n} \left(\sigma^i \left(\mu_i - x_i^*\right)\right)$. Because $P(x) \geqslant Q - \theta h$ for all $x \in S$ (a consequence of (2.21)), we obtain $P(x^+) \geqslant Q - \theta h = P(x)$ when $x \in S \backslash \Omega$.

In what follows, we have $x^+ = \tilde{F}(d, x, k(x))$. Next, we make the following claims. Their proofs can be found in the Appendix.

(Claim 1): *For all $x \in S, d \in [0, 1]^{n-1}$, the following inequality holds:*

$$V(x^+) \leqslant V(x) - (1 - L) \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| . \tag{3.18}$$

(Claim 2): *There exist constants $K_2 \geqslant K_1 > 0$ such that the following inequality holds:*

$$K_1 |x - x^*| \leqslant V(x) \leqslant K_2 |x - x^*| \text{ for all } x \in S. \tag{3.19}$$

Using (3.18), the fact that $\sigma \in (0, 1]$, and (3.19), we obtain for all $x \in S, d \in [0, 1]^{n-1}$:

$$V(x^+) \leqslant V(x) - (1-L) \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| \leqslant V(x) - (1-L)\sigma^n |x - x^*| \leqslant \left(1 - (1 - L)\sigma^n K_2^{-1}\right) V(x).$$

The aforementioned inequality implies that the inequality

$$V(\tilde{F}(d, x, k(x))) \leqslant \tilde{L} V(x) \text{ for all } x \in S, d \in [0, 1]^{n-1} \tag{3.20}$$

holds with $\tilde{L} := 1 - (1 - L)\sigma^n K_2^{-1}$. Notice that $\tilde{L} \in (0, 1)$. Inequalities (3.19) and (3.20) show that the function $V : S \rightarrow \Re_+$ is a Lyapunov function with exponent 1 for the closed-loop system (2.15) with $u = k(x)$. Remark 1.3 guarantees that x^* is RGES for the closed-loop system (2.15) with $u = k(x)$. The proof is complete.

Remark 3.2
The proof of Theorem 2.1 provides a methodology for obtaining an estimation of the set $R \subseteq \{1, ..., n\}$, the constant $\sigma \in (0, 1]$ and the critical constant $\tau^* > 0$. Let $r_i = u_i^*$ for $i = 2, ..., n$ and let $C > 0$ be the constant involved in (3.2). Select $R \subseteq \{1, ..., n\}$ to be a subset of the set of all indices $i \in \{1, ..., n\}$, for which $u_i^* > 0$ and for which there exist $b_i \in (0, u_i^*)$ such that

$$\sum_{i \in R} (n + 1 - i) b_i + \sum_{i \notin R} (n + 1 - i) u_i^* \leqslant \min_{i=1,...,n} \left(((n - i) p_i + 1) f_i(x_i^*)\right) \text{ and}$$

$$\sum_{i \in R} (n + 1 - i) b_i + \sum_{i \notin R} (n + 1 - i) u_i^* < C \min_{i=1,...,n} ((n + 1 - i) \mu_i),$$

where $\mu_i > x_i^*$, for $i = 1, ..., n$, are the constants defined by (3.6). Let $\varepsilon \in (0, 1)$ be a constant that satisfies $\sum\limits_{i \in R} (n + 1 - i) b_i + \sum\limits_{i \notin R} (n + 1 - i) u_i^* \leqslant \varepsilon C \min\limits_{i=1,...,n} ((n + 1 - i) \mu_i)$. The estimation of the critical constant $\tau^* > 0$ may be carried out in the following way:

- Select $\sigma \in (0, 1]$ so that $L = \max \left(\lambda_n, \max\limits_{i=1,...,n-1} (\lambda_i + \sigma G_i (1 - p_i))\right) < 1$, where $\lambda_i \in (0, 1), G_i \in [0, 1] (i = 1, ..., n)$, are the constants involved in Property (C2).
- Define $h := \min\limits_{i=1,...,n} \left(\sigma^i \left(\mu_i - x_i^*\right)\right)$.
- Define $Q := \max \left(\min\limits_{i=1,...,n} (\mu_i (n + 1 - i)),\right.$

$$(1 - C) \sum_{i=1}^{n} I_i(x^*) + (1 - C)h \max_{i=1,...,n} \left((n + 1 - i)\sigma^{-i}\right) + \sum_{i=1}^{n} (n + 1 - i) u_i^*\right) \text{ and}$$

$\theta := h^{-1} \left(Q - \varepsilon \min\limits_{i=1,...,n} ((n+1-i) \mu_i)\right)$, where $I_j(x) := \sum\limits_{i=1}^{j} x_i$ for $j = 1, ..., n$.

The estimated value of $\tau^* > 0$ is given by $\tau^* := \min\left(h, (\theta L)^{-1} \sum_{i \in R} (n + 1 - i)\left(u_i^* - b_i\right)\right)$. However, the estimated value of $\tau^* > 0$, which is obtained by applying the aforementioned methodology, may be conservative (significantly smaller than the actual value).

4. ILLUSTRATIVE EXAMPLES

The issue of the selection of which specific inflows must be controlled for the stabilization of the uncongested equilibrium point of a freeway is crucial. The following example illustrates how Theorem 2.1 can be used for such a selection.

Example 4.1
Consider a freeway model of the form (2.11), (2.12), (2.13) and (2.8) with $n = 4$ cells. The first and the third cell have one on-ramp, while there are no intermediate off-ramps (i.e. $p_i = 0$ for $i = 1, 2, 3$). Each cell has jam density $a_i = 80$ [veh] and is characterized by the same demand functions that are given by

$$f_i(z) = \begin{cases} (9/10)z & for\ z \in [0, 40] \\ (9/10)(80 - z) & for\ z \in (40, 370/9] \\ 35 & for\ z \in (370/9, 80] \end{cases} (i = 1, \ldots, 4).$$

Assumptions (H) hold with $\delta_i = \tilde{\delta}_i = 40$ [veh], $c_i = 9/10$, $q_i = 36$ [veh] (leading to a triangular-shaped FD) and $L_i = 1/10$ ($i = 1, \ldots, 4$). Property (C3) holds with $\theta_i = 7/16$ for $i = 1, \ldots, 4$. The value of constant $C > 0$ that satisfies (3.2) was estimated by applying the procedure described in Remark 3.1 with $r_1 = 35.9$ [veh], $r_2 = r_4 = 0$ [veh] and $r_3 = 1$ [veh]. It was found that $C \geqslant 0.005$.

Next, we consider inflows $u_1^* = 35.5$ [veh], $u_2^* = u_4^* = 0$ [veh] and $u_3^* > 0$ [veh]. The uncongested equilibrium point exists for all $u_3^* < 0.5$ [veh]. However, for constant inflow $u_1 = 35.5$ [veh], the uncongested equilibrium point is not globally exponentially stable because of the existence of congested equilibria. Therefore, there is a need for controlling the main inflow u_1. At this point, the following question becomes crucial:

'For what values of $u_3^* \in (0, 0.5)$ can the uncongested equilibrium point be globally exponentially stabilized by controlling only the inflow u_1, i.e., for what values of $u_3^* \in (0, 0.5)$ do we have $R = \{1\}$?'.

We checked numerically the inequalities (3.11) by computing the values of μ_i from (3.6) for given values of u_3^*. It was found that inequalities (3.11) hold for $R = \{1\}$, provided that $u_3^* < 0.1$ [veh]. Therefore, we conclude that the uncongested equilibrium point can be globally exponentially stabilized by controlling only the inflow u_1 for $u_3^* < 0.1$ [veh]. The answer may be conservative, since the estimation of the constant $C > 0$ that satisfies (3.2) is conservative.

Example 4.2
Consider a freeway model of the form (2.11), (2.12), (2.13) and (2.8) with $n = 5$ cells. The freeway stretch considered for the simulation test is 2.5 km long and has three lanes. The cells are homogeneous; thus, each cell is 0.5 km, and it has no intermediate on/off-ramps (i.e. $u_i(t) \equiv u_i^* = 0$ for $i = 2, 3, 4, 5$ and $p_i = 0$ for $i = 1, \ldots, 4$) (Figure 4). Consequently, the only control possibility is the inflow u_1 of the first cell. The simulation time step is set to $T = 15$ s. However, because all flows and densities are measured in [veh] (as imposed by the form of the model (2.11), (2.12), (2.13) and (2.8)), the cell length, the time step and the number of lanes do not appear explicitly, but they are only reflected implicitly in the values of every variable and every constant (e.g. critical density, jam density, flow capacity and wave speed) corresponding to density or flow. Each cell has the same critical density $\delta_i = 55$ [veh] ($i = 1, \ldots, 5$) (corresponding to 36.7 [veh/km/lane] with the aforementioned settings) and the same jam density $a_i = 170$ [veh] ($i = 1, \ldots, 5$) (corresponding to 113.3 [veh/km/lane]). We also suppose that the cell flow capacities are $q_i = 25$ [veh] $i = 1, 2, 3, 4$
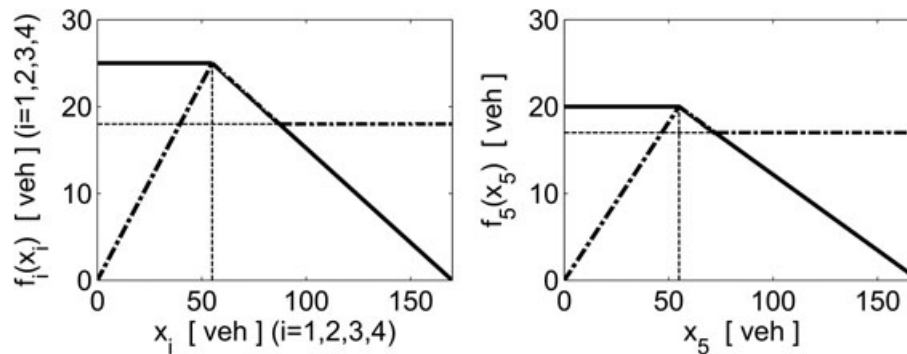
Figure 4. Freeway stretch.

Figure 5. Fundamental diagram of every cell.

and $q_5 = 20$ [veh] (corresponding to 2000 [veh/h/lane] and 1600 [veh/h/lane], respectively). Note that the last cell has 20% lower flow capacity (e.g. due to grade or curvature or tunnel or bridge) than the first four cells and is therefore a potential bottleneck for the freeway. Furthermore, the congestion wave speed is $c_i = 25/115 (i = 1, \ldots, 4)$ and $c_5 = 20/115$ (corresponding to 26 and 20.9 [km/h], respectively).

Figure 5 depicts the triangular FDs for the aforementioned model. The dash-dotted line corresponds to the demand function, while the solid line corresponds to the supply function. More precisely, the demand part for every cell is given by the following functions

$$f_i(z) = \begin{cases} (5/11)z & z \in [0, 55] \\ (25/115)(170 - z) & z \in (55, 87.2] \\ 18 & z \in (87.2, 170] \end{cases}$$

$$(i = 1, \ldots, 4), f_5(z) = \begin{cases} (4/11)z & z \in [0, 55] \\ (20/115)(170 - z) & z \in (170, 72.25] \\ 17 & z \in (72.25, 170]. \end{cases}$$

Notice that the capacity drop phenomenon has been taken into account by considering a partly decreasing demand function for overcritical densities $x_i \in (55, 170]$.

Assumption (H) holds with $\delta_i = \tilde{\delta}_i = 55$ [veh] $(i = 1, \ldots, 5)$, $L_i = 6/11$ $(i = 1, \ldots, 4)$, $L_5 = 7/11$. The uncongested equilibrium point $x_i^* = 11u_1^*/5$ [veh] $(i = 1, \ldots, 4)$, $x_5^* = 11u_1^*/4$ [veh], exists for $u_1^* < 20$ [veh]. Simulations showed that the open-loop system converges to an uncongested equilibrium point for main inflow u_1^* less than 17 [veh]. For higher values of the main inflow, the uncongested equilibrium point is not globally exponentially stable due to the existence of additional (congested) equilibria. This is shown in Figure 6, where the evolution of the Euclidean norm of the deviation of the solution of the open-loop system, with constant inflow $u_1^* = 19.99$ [veh], from the uncongested equilibrium point is depicted. In this test, the solution is attracted by the congested equilibrium $(91.8, 91.8, 91.8, 91.8, 72.25)'$ [veh] for which the value of the Euclidean norm of its deviation from the uncongested equilibrium point is 97.12 [veh]. The components of the uncongested equilibrium for $u_1^* = 19.99$ [veh], are $x_i^* = 43.978$ [veh] $(i = 1, \ldots, 4)$ and $x_5^* = 54.9725$ [veh]. Therefore, if the objective is the operation of the freeway with large flows, then a control strategy will be needed.

We next notice that Property (C2) holds with $\lambda_i = 6/11, G_i = 5/11 (i = 1, \ldots, 4), \lambda_5 = 7/11$ and $G_5 = 4/11$. Therefore, we are in a position to achieve global exponential stabilization of the
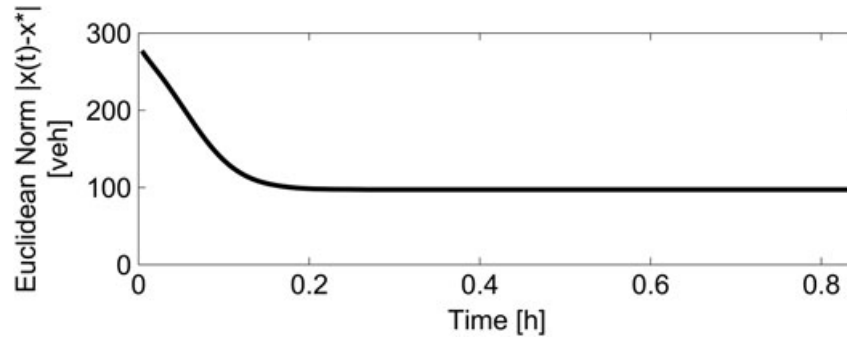
Figure 6. The evolution of the Euclidean norm of the deviation $x(t) - x^*$ of the state from the uncongested equilibrium, i.e., $|x(t) - x^*|$, for the open-loop system with constant inflow $u_1^* = 19.99$ [veh] and with fully congested initial condition $x_0 = (a_1, a_2, a_3, a_4, a_5)'$.

uncongested equilibrium point for model by using Theorem 2.1. Indeed, Theorem 2.1 guarantees that for every $\sigma \in (0, 1]$, there exists a constant $b_1 \in (0, u_1^*)$ and a constant $\gamma > 0$ such that the feedback law $k : (0, 10]^5 \to \Re_+$ defined by

$$u_1 = \max \left(u_1^* - \gamma \sum_{i=1}^{5} \sigma^i \max \left(0, x_i - x_i^* \right), b_1 \right) \tag{4.1}$$

achieves robust global exponential stabilization of the uncongested equilibrium point $x^* = (x_1^*, \ldots, x_5^*)' \in (0, 55)^5$ for the closed-loop system.

We selected $u_1^* = 19.99$ [veh], which is very close to 20 [veh], the capacity flow of cell 5. The value of the constant $b_1 \in (0, u_1^*)$ was chosen to be 0.2 [veh]; this is a rather low minimum flow value in practice but allows us here to study the dynamic properties of the regulators in a broader feasible control area. Various values of the constants $\sigma \in (0, 1]$ and $\gamma > 0$ were tested by performing a simulation study with respect to various initial conditions. Low values for $\sigma \in (0, 1]$ require large values for $\gamma > 0$ in order to have global exponential stability for the closed-loop system. Moreover, in order to evaluate the performance of the controller, we used as a performance criterion the total number of vehicles exiting the freeway (*VEF*) on the interval $[0, N]$, that is,

$$VEF_N = \sum_{k=0}^{N} f_5(x_5(k \cdot T)). \tag{4.2}$$

Notice that the freeway performs best (and total delays are minimized) if *VEF* is maximized; the maximum theoretical value for *VEF* is $20 \cdot (N + 1)$, which is achieved if cell 5 is operating at capacity flow ($q_5 = 20$ [veh]) at all times. For $N = 200$, the maximum theoretical value of *VEF* is 4020 [veh].

All the following tests of the proposed regulator (4.1) were conducted with the same values $\sigma = 0.7$ and $\gamma = 0.6$. The response of the density of the fifth cell for the closed-loop system with the proposed feedback regulator (4.1) and initial condition $x_0 = (60, 57, 58, 60, 62)'$ is shown in Figure 7 (solid line). The densities of all the other cells have similar responses and therefore are omitted. Notice that all initial cell density values are slightly over-critical (slightly congested). For this case, we had $VEF_{200} = 3979.8$ [veh]. The feedback regulator is seen to respond very satisfactorily in this test and achieves an accordingly high performance.

A detailed comparison of the proposed feedback regulator (4.1) was made with the RLB PI regulator, which was proposed in [15]. The RLB PI regulator for the present system is implemented as follows:

$$v_i(t) = \min \left(u_{\max}, \min \left(q_1, c_1(a_1 - x_1(t-1)), u_1(t-1) \right) \right.$$
$$\left. + \psi, \max \left(u_{\min}, v_i(t-1) - K_p \left(x_i(t) - x_i(t-1) \right) + K_I \left(\delta_i - x_i(t) \right) \right) \right) \tag{4.3}$$
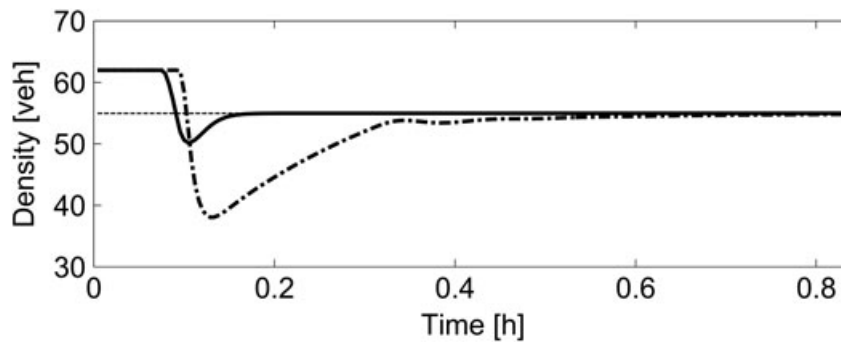
Figure 7. The response of the density of the fifth cell for the closed-loop system with initial condition $x_0 = (60, 57, 58, 60, 62)'$ using: the proposed feedback regulator (4.1) (solid line); and the random located bottleneck PI regulator (dash-dotted line).

for $i = 1, \ldots, 5$,

$$v_i^{sm}(t) = \vartheta v_i(t) + (1 - \vartheta) v_i^{sm}(t - 1), \text{ for } i = 1, \ldots, 5 \qquad (4.4)$$

$$j(t) = \min \left\{ l \in \{1, 2, 3, 4, 5\} : v_l^{sm}(t) = \min_{i=1,\ldots,5} \left(v_i^{sm}(t) \right) \right\} \qquad (4.5)$$

$$u_1(t) = v_{j(t)}, \qquad (4.6)$$

where $\psi, K_p, K_I > 0$ and $\vartheta \in (0, 1)$ are constant parameters. Essentially, (4.3) reflects the parallel (independent) operation of five bounded PI-type regulators, one for each cell, while (4.4) performs an exponential smoothing of the respective obtained inflows (with smoothing parameter ϑ). Eventually, the smoothed inflow values are compared in (4.5) in order to pick the currently most conservative regulator; whose (unsmoothed) inflow is finally actually activated as a control input in (4.6); see [15] for the background and detailed reasoning for this approach. The parameters for the RLB PI regulators are set (as proposed in [15] – with the suitable transformation in the current units) to be $K_p = 5/18, K_I = 1/90$, while $\psi = 4$ [veh], $\vartheta = 0.5, u_{\min} = 0.2$ [veh] and $u_{\max} = 25$ [veh]. These values were indeed tested, before being adopted, and were indeed found to be near optimal. Notice that all PI regulators were given the same gain values for simplicity and convenience, as suggested in [15]. In all reported tests, the initial condition for the RLB PI regulator was $v_i(-1) = v_i^{sm}(-1) = u_1(-1) = 20$ [veh], for $i = 1, \ldots, 5$, and $x(-1) = x(0) = x_0$, where x_0 is the vector of the initial values for the densities of every cell.

When applied to the same initial condition $x_0 = (60, 57, 58, 60, 62)'$, the RLB PI regulator (Figure 7, dash-dotted line), led to slower convergence compared with the proposed regulator (4.1) (Figure 7, solid line). This is also reflected in the computed value of $VEF_{200} = 3785.9$ [veh] for RLB PI regulator. In general, conducting a simulation study with various levels of initial conditions, the proposed regulator (4.1) exhibited faster performance than the RLB PI regulator. For example, Figure 8 shows the evolution of the Euclidean norm $|x(t) - x^*|$ for the closed-loop system with the proposed feedback regulator (4.1) (solid line) and for the closed-loop system with the RLB PI regulator (4.3), (4.4), (4.5) and (4.6) (dash-dotted line), when starting from the initial condition $x_0 = (a_1, a_2, a_3, a_4, a_5)'$, reflecting a fully congested original state. It is again clear that the proposed feedback regulator (4.1) achieves faster convergence and higher performance of $VEF_{200} = 3845.2$ [veh], while $VEF_{200} = 3007.8$ [veh] resulted for the RLB PI regulator.

We next investigated the robustness of the proposed feedback regulator with respect to measurement errors. The applied formula for the measurements is
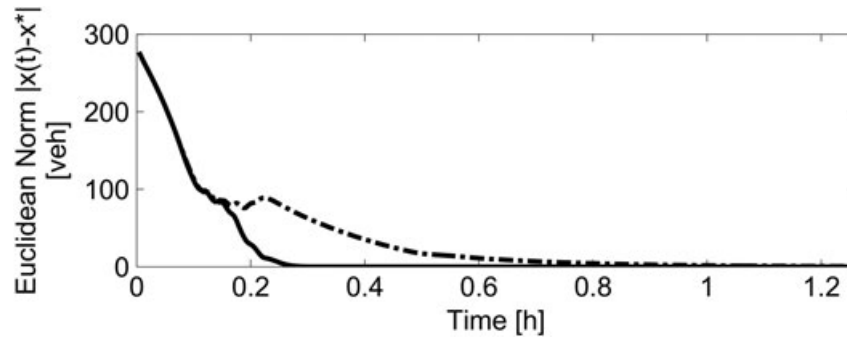
$$\tilde{x}(t) = P (x(t) + Ae(t)), \qquad (4.7)$$

Figure 8. The evolution of the Euclidean norm of the deviation $x(t) - x^*$ of the state from the uncongested equilibrium, that is, $|x(t) - x^*|$, for the closed-loop system and initial condition $x_0 = (a_1, a_2, a_3, a_4, a_5)'$ for two cases, for the proposed feedback regulator (4.1) (solid line) and for the random located bottleneck PI regulator (4.3), (4.4), (4.5), (4.6) (dash-dotted line).

where P is the projection operator on the closure of S, $e(t)$ is a normalized vector and $A \geqslant 0$ is the magnitude of the measurement error. In this case, the feedback law (4.1) was implemented based on the state measurement $\tilde{x}(t)$ given by (4.7), that is,

$$u_1(t) = \max \left(u_1^* - \gamma \sum_{i=1}^{5} \sigma^i \max \left(0, \tilde{x}_i(t) - x_i^* \right), b_1 \right). \tag{4.8}$$

For comparison purposes, we also present the performance of the RLB PI regulator for the same system, under the same measurement errors. In this case, (4.3) is replaced by the equation

$$
\begin{aligned}
v_i(t) = \min \left(u_{\max}, \min \left(q_1, c_1(a_1 - x_1(t-1)), u_1(t-1) \right) \right. \\
+ \psi, \max \left(u_{\min}, v_i(t-1) \right. \\
- K_p \left(\tilde{x}_i(t) - \tilde{x}_i(t-1) \right) \\
\left. \left. + K_I \left(\delta_i - \tilde{x}_i(t) \right) \right) \right)
\end{aligned}
\tag{4.9}
$$

for $i = 1, \ldots, 5$, where the state measurement $\tilde{x}(t)$ is given by (4.7).

Figure 9 shows the evolution of the Euclidean norm of the deviation of the state from the uncongested equilibrium point for two cases: for the closed-loop system with the proposed feedback regulator (4.8) (solid line); and for the closed-loop system with the RLB PI regulator (4.9), (4.4), (4.5) and (4.6) (dash-dotted line), where the state measurement in both cases is given by (4.7) with $A = 10$ [veh], $e(t) = \frac{\cos(\omega t)}{\sqrt{5}} (1, 1, \ldots, 1)$, $\omega = \pi$. The initial condition was the uncongested equilibrium point.

In this test, the RLB PI regulator is less sensitive to measurement errors than the proposed feedback regulator (4.8), the latter producing a higher offset (Figure 9, solid line). This is also reflected in the computed values of $VEF_{200} = 3789$ [veh] for the proposed feedback regulator (4.9) (which is 6% less than the maximum value of VEF_{200}) and $VEF_{200} = 4016.8$ [veh] for the RLB PI regulator (which is 0.8% less than the maximum value of VEF_{200}) because of the measurement error. The ultimate mean values of the states are much closer to the equilibrium values for the RLB PI regulator than for the proposed feedback regulator (4.8), indicating that the RLB PI regulator achieves a much smaller mean offset in this case. It should be noted at this point that various frequencies ω were tested for measurement errors. While Figure 9 is typical for medium and high frequencies (the RLB PI regulator achieves a smaller mean offset than the proposed feedback regulator (4.8)), the results indicate higher sensitivity of the RLB PI regulator with respect to measurement errors at low frequencies (Figure 10). For low frequency measurement errors, the proposed feedback regulator (4.8) achieves a smaller mean offset than the RLB PI regulator, as shown in Figure 10. Figure 10 shows the evolution of the Euclidean norm of the deviation of the state from the uncongested equilibrium point for two cases: for the closed-loop system with the proposed feedback regulator
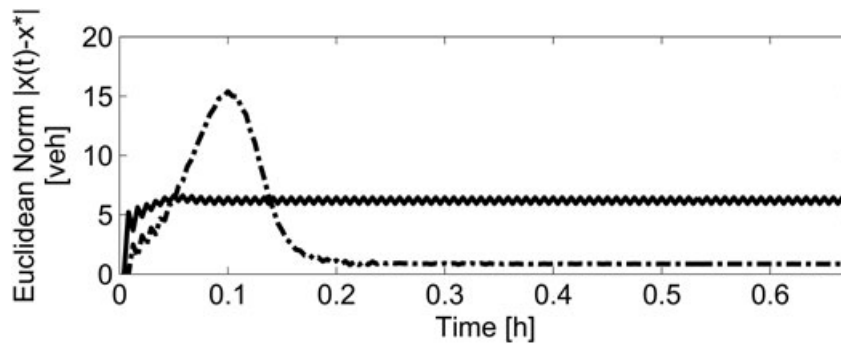
Figure 9. The evolution of the Euclidean norm of the deviation $x(t) - x^*$ of the state from the uncongested equilibrium, i.e., $|x(t) - x^*|$, for the closed-loop system and initial condition the uncongested equilibrium point for two cases: for the proposed feedback regulator (4.8) (solid line); and for the random located bottleneck PI regulator (4.9), (4.4), (4.5) and (4.6) (dash-dotted line). In both cases, the state measurement is given by (4.7) with $A = 10$ [veh], $e(t) = \left(\cos(\omega t)/\sqrt{5}\right)(1, 1, \ldots, 1)$ and $\omega = \pi$.

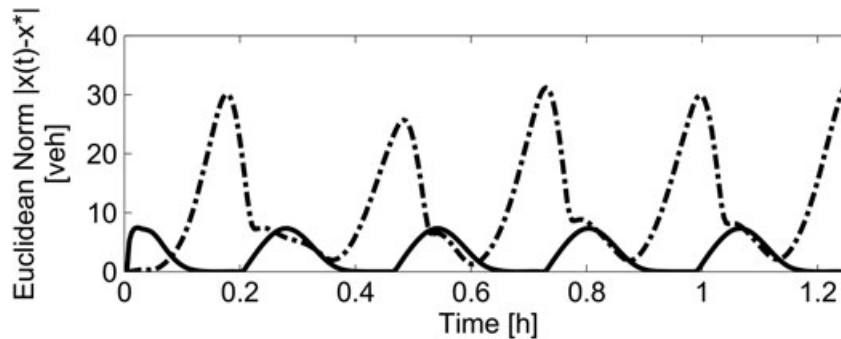
Figure 10. The evolution of the Euclidean norm of the deviation $x(t) - x^*$ of the state from the uncongested equilibrium, i.e., $|x(t) - x^*|$, for the closed-loop system and initial condition the uncongested equilibrium point for two cases: for the proposed feedback regulator (4.8) (solid line); and for the random located bottleneck PI regulator (4.9), (4.4), (4.5) and (4.6) (dash-dotted line). In both cases, the state measurement is given by (4.7) with $A = 10$ [veh], $e(t) = \left(\cos(\omega t)\Big/\sqrt{5}\right)(1, 1, \ldots, 1)$ and $\omega = 0.1$.

(4.8) (solid line), and for the closed-loop system with the RLB PI regulator (4.9), (4.4), (4.5) and (4.6) (dash-dotted line), where the state measurement in both cases is given by (4.7) with $A = 10$ [veh], $e(t) = \left(\cos(\omega t)\Big/\sqrt{5}\right)(1, 1, \ldots, 1)$, $\omega = 0.1$. The initial condition is the uncongested equilibrium point.

The conclusions of this simulation study are as follows:

- The proposed feedback regulator (4.1) can achieve a faster convergence of the state to the equilibrium compared with the RLB PI regulator in the absence of measurement errors.
- The proposed feedback regulator (4.1) is quite robust to measurement errors. However, it is more sensitive to measurement errors with high frequency than the RLB PI regulator, but it is less sensitive to low-frequency measurement errors than the RLB PI regulator. Intended future extensions are expected to improve the properties of the proposed feedback regulator in this respect, as well as in cases of modelling errors or persisting disturbances.

5. CONCLUSIONS

This work provided a rigorous methodology for the construction of a parameterized family of explicit feedback laws that guarantee the robust global exponential stability of the uncongested equilibrium point for general nonlinear and uncertain discrete-time freeway models. The construction

of the global exponential feedback stabilizer was based on the CLF approach as well as on certain important properties of freeway models.

Simulation-based comparisons were made with an existing feedback law, which was proposed in the literature and has been in practical use. More specifically, we compared the performance and some robustness properties of the closed-loop system under the effect of the proposed feedback law and under the effect of the RLB PI regulator [15]. In most cases, it was found that the performance and the robustness properties guaranteed by the implementation of the proposed feedback law were good and comparable with or better than the performance and the robustness properties induced by the RLB PI regulator.

Future research will address robustness issues in a rigorous way: the knowledge of a Lyapunov function for the closed-loop system can be exploited to this purpose. Finally, the present approach does not consider the impact of inflow control on upstream traffic flow conditions (e.g. queue forming at on-ramps); future extensions will address these issues appropriately. Testing of the proposed feedback approach with other (e.g. second-order) traffic simulation models is also under way.

APPENDIX

Proof of (C4): We prove the following claim:

(Claim): For all $m = 1, \ldots, n - 1$, there exists a constant $C_m > 0$ such that the following inequality holds for all $x \in S := (0, a_1] \times \ldots \times (0, a_n], u \in U = (0, +\infty) \times [0, r_2] \times \ldots \times [0, r_n]$, $d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$:

$$\sum_{i=m}^{n} (1 + p_i(n - i)) s_{i+1} f_i(x_i) \geq C_m \sum_{i=m}^{n} (n + 1 - i) x_i. \tag{A.1}$$

Property (C4) is a direct consequence of the aforementioned claim.

First, we prove the claim for $m = n - 1$. Define $l_n = \min\left(1, \frac{c_n a_n - r_n}{2(1 - p_{n-1}) f_{n-1}(\delta_{n-1})}\right)$. Indeed, using Property (C3), we acquire for all $(x_{n-1}, x_n) \in (0, a_{n-1}] \times (0, a_n], u \in U = (0, +\infty) \times [0, r_2] \times \ldots \times [0, r_n], d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$ with $s_n \geq l_n$:

$$\begin{aligned} f_n(x_n) + (1 + p_{n-1}) s_n f_{n-1}(x_{n-1}) \\ \geq l_n (1 + p_{n-1}) \theta_{n-1} x_{n-1} + \theta_n x_n \\ \geq (x_n + 2x_{n-1}) \min\left(\theta_n, \frac{1 + p_{n-1}}{2} l_n \theta_{n-1}\right). \end{aligned} \tag{A.2}$$

On the other hand, for all $(x_{n-1}, x_n) \in (0, a_{n-1}] \times (0, a_n], u \in U = (0, +\infty) \times [0, r_2] \times \ldots \times [0, r_n], d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$ with $s_n < l_n$, it follows from (2.8) that $(1 - p_{n-1}) f_{n-1}(x_{n-1}) + u_n > \min(q_n, c_n(a_n - x_n))$. We distinguish the cases:

- $\min(q_n, c_n(a_n - x_n)) = c_n(a_n - x_n)$. In this case, we have $a_n - c_n^{-1} q_n \leq x_n$. Assumption (H) in conjunction with the fact that $u_n \leq r_n < c_n a_n$ implies that $l_n(1 - p_{n-1}) f_{n-1}(\delta_{n-1}) + r_n > c_n(a_n - x_n)$, which by virtue of definition $l_n = \min\left(1, \frac{c_n a_n - r_n}{2(1 - p_{n-1}) f_{n-1}(\delta_{n-1})}\right)$, gives $x_n > \frac{1}{2}\left(a_n - c_n^{-1} r_n\right)$. Using Property (C3), we acquire

$$\begin{aligned} f_n(x_n) + (1 + p_{n-1}) s_n f_{n-1}(x_{n-1}) \\ \geq \theta_n x_n \geq \theta_n \frac{1}{2}\left(a_n - c_n^{-1} r_n\right) \\ \geq \theta_n \frac{1}{2}\left(a_n - c_n^{-1} r_n\right) \frac{x_n + 2x_{n-1}}{a_n + 2a_{n-1}}. \end{aligned}$$

- $\min(q_n, c_n(a_n - x_n)) = q_n$. In this case, we obtain from (2.8) and the fact that $u_n \leq r_n < q_n$ the inequality $s_n \geq \frac{q_n - r_n}{(1 - p_{n-1}) f_{n-1}(x_{n-1})}$. Using Property (C3), we acquire

$$f_n(x_n) + (1 + p_{n-1}) s_n f_{n-1}(x_{n-1}) \geq \theta_n x_n + (1 + p_{n-1}) \frac{q_n - r_n}{1 - p_n}$$

$$\geq \theta_n x_n + (1 + p_{n-1}) \frac{q_n - r_n}{1 - p_n} \frac{2 x_{n-1}}{2 a_{n-1}} \geq (x_n + 2 x_{n-1}) \min \left(\theta_n, \frac{q_n - r_n}{1 - p_n} \frac{1 + p_{n-1}}{2 a_{n-1}} \right).$$

It follows from (A.2) and the aforementioned inequalities that (A.1) holds with $C_{n-1} :=$ $\min \left(\theta_n, \frac{1 + p_{n-1}}{2} l_n \theta_{n-1}, \frac{\theta_n (a_n - c_n^{-1} r_n)}{2(a_n + 2a_{n-1})}, \frac{1 + p_{n-1}}{1 - p_{n-1}} \frac{q_n - r_n}{2 a_{n-1}} \right).$

Next, we suppose that the claim holds for $m = k \in \{2, \ldots, n-1\}$, and we show that the claim holds for $m = k - 1$. Define $l_k = \min \left(1, \frac{c_k a_k - r_k}{2(1 - p_{k-1}) f_{k-1}(\delta_{k-1})} \right)$. Using (A.1) for $m = k$ and Property (C3), we obtain for all $x \in S := (0, a_1] \times \ldots \times (0, a_n], u \in U = (0, +\infty) \times [0, r_2] \times \ldots \times [0, r_n], d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$ with $s_k \geq l_k$:

$$\sum_{i=k-1}^{n} (1 + p_i(n - i)) s_{i+1} f_i(x_i)$$

$$= \sum_{i=k}^{n} (1 + p_i(n - i)) s_{i+1} f_i(x_i) + (1 + p_{k-1}(n + 1 - k)) s_k f_{k-1}(x_{k-1})$$

$$\geq C_k \sum_{i=k}^{n} (n + 1 - i) x_i + (1 + p_{k-1}(n + 1 - k)) l_k \theta_{k-1} x_{k-1} \geq$$

$$\geq \min \left(C_k, \frac{1 + (n + 1 - k) p_{k-1}}{n + 2 - k} l_k \theta_{k-1} \right) \sum_{i=k-1}^{n} (n + 1 - i) x_i. \tag{A.3}$$

On the other hand, for all $x \in S := (0, a_1] \times \ldots \times (0, a_n], u \in U = (0, +\infty) \times [0, r_2] \times \ldots \times [0, r_n], d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$ with $s_k < l_k$, it follows from (2.8) that $l_k(1 - p_{k-1}) f_{k-1}(x_{k-1}) + u_k > \min (q_k, c_k(a_k - x_k))$. We distinguish the cases:

- $\min (q_k, c_k(a_k - x_k)) = c_k(a_k - x_k)$. Assumption (H) in conjunction with the fact that $u_k \leq r_k < c_k a_k$ implies that $l_k(1 - p_{k-1}) f_{k-1}(\delta_{k-1}) + r_k > c_k(a_k - x_k)$, which by virtue of definition $l_k = \min \left(1, \frac{c_k a_k - r_k}{2(1 - p_{k-1}) f_{k-1}(\delta_{k-1})} \right)$, gives $x_k > \frac{1}{2} \left(a_k - c_k^{-1} r_k \right)$. Using (A.1) for $m = k$ and Property (C3), we acquire

$$\sum_{i=k-1}^{n} (1 + p_i(n - i)) s_{i+1} f_i(x_i) \geq \sum_{i=k}^{n} (1 + p_i(n - i)) s_{i+1} f_i(x_i)$$

$$\geq C_k \sum_{i=k}^{n} (n + 1 - i) x_i = C_k \sum_{i=k+1}^{n} (n + 1 - i) x_i + C_k(n + 1 - k) x_k$$

$$\geq C_k \sum_{i=k+1}^{n} (n + 1 - i) x_i + C_k(n + 1 - k) \frac{1}{2} \left(a_k - c_k^{-1} r_k \right)$$

$$\geq C_k \sum_{i=k+1}^{n} (n + 1 - i) x_i + C_k(n + 1 - k) \frac{1}{2} \left(a_k - c_k^{-1} r_k \right) \frac{(n + 1 - k) x_k + (n + 2 - k) x_{k-1}}{(n + 1 - k) a_k + (n + 2 - k) a_{k-1}}$$

$$\geq C_k \min \left(1, \frac{(n + 1 - k) \left(a_k - c_k^{-1} r_k \right)}{2(n + 1 - k) a_k + 2(n + 2 - k) a_{k-1}} \right) \sum_{i=k-1}^{n} (n + 1 - i) x_i.$$

- $\min (q_k, c_k(a_k - x_k)) = q_k$. In this case, we obtain from (2.8) and the fact that $u_k \leq r_k < q_k$ the inequality $s_k \geq \frac{q_k - r_k}{(1 - p_{k-1}) f_{k-1}(x_{n-1})}$. Consequently, we acquire

$$\sum_{i=k-1}^{n} (1 + p_i(n-i)) s_{i+1} f_i(x_i)$$

$$= \sum_{i=k}^{n} (1 + p_i(n-i)) s_{i+1} f_i(x_i) + (1 + p_{k-1}(n+1-k)) s_k f_{k-1}(x_{k-1})$$

$$\geqslant C_k \sum_{i=k}^{n} (n+1-i)x_i + (1 + p_{k-1}(n+1-k)) \frac{q_k - r_k}{1 - p_{k-1}}$$

$$\geqslant C_k \sum_{i=k}^{n} (n+1-i)x_i + (1 + p_{k-1}(n+1-k)) \frac{q_k - r_k}{1 - p_{k-1}} \frac{(n+2-k)x_{k-1}}{(n+2-k)a_{k-1}}$$

$$\geqslant \min\left(C_k, \frac{q_k - r_k}{1 - p_{k-1}} \frac{1 + p_{k-1}(n+1-k)}{(n+2-k)a_{k-1}} \right) \sum_{i=k-1}^{n} (n+1-i)x_i.$$

It follows from (A.3) and the aforementioned inequalities that the claim holds for $m = k - 1$ with

$$C_{k-1} = \min\left(C_k, \frac{1 + p_{k-1}(n+1-k)}{n+2-k} l_k \theta_{k-1}, \right.$$

$$\times \frac{(n+1-k)C_k \left(a_k - c_k^{-1} r_k \right)}{2(n+1-k)a_k + 2(n+2-k)a_{k-1}},$$

$$\left. \times \frac{q_k - r_k}{1 - p_{k-1}} \frac{1 + p_{k-1}(n+1-k)}{(n+2-k)a_{k-1}} \right).$$

Proof of (C5): The following equations hold for all $x \in S := (0, a_1] \times \ldots (0, a_n], u = (u_1, \ldots, u_n)' \in (0, +\infty) \times \Re_+^{n-1}, d = (d_2, \ldots, d_n) \in [0, 1]^{n-1}$ with $p_n = 1 = s_{n+1}$ and are direct consequences of (2.11), (2.12), (2.13) and definitions $I_j(x) := \sum_{i=1}^{j} x_i$ for $j = 1, \ldots, n$:

$$I_j(x^+) = I_j(x) + \sum_{i=1}^{j} w_i u_i - \sum_{i=1}^{j-1} s_{i+1} p_i f_i(x_i) - s_{j+1} f_j(x_j), \text{ for } j = 2, \ldots, n-1 \quad \text{(A.4)}$$

$$I_n(x^+) = I_n(x) + \sum_{i=1}^{n} w_i u_i - \sum_{i=1}^{n-1} s_{i+1} p_i f_i(x_i) - f_n(x_n). \quad \text{(A.5)}$$

Equality (3.4) is a consequence of (2.7), (A.4), (A.5) and definitions $p_n = 1 = s_{n+1}$.
   Combining (3.4) and (3.2), we acquire

$$\sum_{i=1}^{n} I_i(x^+) \leqslant \sum_{i=1}^{n} I_i(x) - C \sum_{i=1}^{n} (n+1-i) x_i$$

$$+ \sum_{i=1}^{n} (n+1-i) w_i u_i, \text{ for all } (x, u, d) \in S \times U \times [0, 1]^{n-1}. \quad \text{(A.6)}$$

   Because $w_i \in [0, 1]$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^{n} I_i(x) = \sum_{i=1}^{n} (n+1-i) x_i$, it follows from (A.6) that (3.5) holds.

Proof of the Claim 1 made in the proof of Theorem 2.1: We distinguish two cases.
   Case 1: $x \in \Omega = (0, \mu_1] \times \ldots \times (0, \mu_n], d \in [0, 1]^{n-1}$.

Definition (2.20) and equations (3.4), (3.8), (3.9) with $u_i = k_i(x) \leqslant u_i^*$ give:

$$V(x^+) = \sigma |x_1 - f_1(x_1) + u_1 - x_1^*| + \sum_{i=2}^{n} \sigma^i |x_i - f_i(x_i) + (1 - p_{i-1}) f_{i-1}(x_{i-1}) + u_i - x_i^*|$$

$$+ K \max \left(0, \sum_{i=1}^{n} I_i(x) - \sum_{i=1}^{n} (1 + (n-i)p_i) f_i(x_i) + \sum_{i=1}^{n} (n+1-i) u_i - P(x^+) \right)$$

$$+ A \Xi(x^+) \tag{A.7}$$

with $p_n = 1$. Using (3.13), Property (C2), the fact that $\mu_i \leqslant \tilde{\delta}_i$ for $i = 1, \ldots, n$ (a consequence of (3.6) and (3.7)) and definition (3.10), we obtain from (A.7)

$$V(x^+) \leqslant L \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| + \sum_{i=1}^{n} \sigma^i |u_i - u_i^*| + LA\Xi(x)$$

$$+ K \max \left(0, \sum_{i=1}^{n} I_i(x) - \sum_{i=1}^{n} (1 + (n-i)p_i) f_i(x_i) + \sum_{i=1}^{n} (n+1-i) u_i - P(x^+) \right). \tag{A.8}$$

It follows from the combination of (2.18) and inequality (A.8) that the following inequality holds for all $x \in \Omega$

$$V(x^+) \leqslant L \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| + \sum_{i \in R} \sigma^i \min \left(\gamma_i \Xi(x), u_i^* - b_i \right) + LA\Xi(x)$$

$$+ K \max \left(0, \sum_{i=1}^{n} I_i(x) - \sum_{i=1}^{n} (1 + (n-i)p_i) f_i(x_i) + \sum_{i=1}^{n} (n+1-i) u_i - P(x^+) \right). \tag{A.9}$$

Inequality (3.2) and (3.8) imply that:

$$\sum_{i=1}^{n} (1 + (n-i)p_i) f_i(x_i) \geqslant C \sum_{i=1}^{n} I_i(x). \tag{A.10}$$

Using (A.8) and (A.10), we acquire

$$V(x^+) \leqslant L \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| + \sum_{i \in R} \sigma^i \min \left(\gamma_i \Xi(x), u_i^* - b_i \right) + LA\Xi(x)$$

$$+ K \max \left(0, (1 - C) \sum_{i=1}^{n} I_i(x) + \sum_{i=1}^{n} (n+1-i) u_i - P(x^+) \right). \tag{A.11}$$

We next distinguish two cases:

Case 1(i): $\Xi(x) \leqslant \tau$. In this case, we have $\gamma_i \Xi(x) \leqslant u_i^* - b_i$ for all $i \in R$. Because $\Xi(x) \leqslant h$ (a consequence of $\tau < \tau^* \leqslant h$), we acquire from (3.13) and definition (2.18) that $u_i = k_i(x) = u_i^* - \gamma_i \Xi(x) \geqslant b_i$ for all $i \in R$ and $\min(h, \Xi(x^+)) \leqslant L \min(h, \Xi(x))$. Using the definitions $I_j(x) := \sum_{i=1}^{j} x_i$ for $j = 1, \ldots, n$, $P(x) := Q - \theta \min(h, \Xi(x))$ and the facts

- $Q \geqslant (1 - C) \sum_{i=1}^{n} I_i(x^*) + (1 - C)h \max_{i=1,\ldots,n} \left((n+1-i)\sigma^{-i} \right) + \sum_{i=1}^{n} (n+1-i) u_i^*$;
- $\sum_{i \in R} (n+1-i) \gamma_i = \tau^{-1} \sum_{i \in R} (n+1-i) \left(u_i^* - b_i \right) \geqslant (\tau^*)^{-1} \sum_{i \in R} (n+1-i) \left(u_i^* - b_i \right) \geqslant \theta L$ (a consequence of $\tau \leqslant \tau^* \leqslant (\theta L)^{-1} \sum_{i \in R} (n+1-i) \left(u_i^* - b_i \right)$);

- $\sum_{i=1}^{n} (n+1-i)(x_i - x_i^*) \leqslant \sum_{i=1}^{n} \left((n+1-i)\sigma^{-i}\right) \sigma^i \max(0, x_i - x_i^*) \leqslant \max_{i=1,\ldots,n} \left((n+1-i)\sigma^{-i}\right) \Xi(x)$ for $i = 1,\ldots,n$ (a consequence of definition (2.19));

we acquire

$$Q \geqslant (1-C) \sum_{i=1}^{n} I_i(x^*) + (1-C)h \max_{i=1,\ldots,n} \left((n+1-i)\sigma^{-i}\right) + \sum_{i=1}^{n} (n+1-i) u_i^* \Rightarrow$$

$$Q \geqslant (1-C) \sum_{i=1}^{n} I_i(x^*) + (1-C)\Xi(x) \max_{i=1,\ldots,n} \left((n+1-i)\sigma^{-i}\right) + \sum_{i=1}^{n} (n+1-i) u_i^* \Rightarrow$$

$$Q \geqslant \theta L \Xi(x) - \sum_{i \in R} (n+1-i)\gamma_i \Xi(x) + (1-C) \sum_{i=1}^{n} I_i(x^*)$$

$$+ (1-C) \sum_{i=1}^{n} (n+1-i)(x_i - x_i^*) + \sum_{i=1}^{n} (n+1-i) u_i^* \Rightarrow$$

$$Q \geqslant \theta \min(h, \Xi(x^+)) + (1-C) \sum_{i=1}^{n} I_i(x)$$

$$+ \sum_{i \in R} (n+1-i)(u_i^* - \gamma_i \Xi(x)) + \sum_{i \notin R} (n+1-i) u_i^* \Rightarrow$$

$$0 \geqslant \sum_{i=1}^{n} (n+1-i) u_i + (1-C) \sum_{i=1}^{n} I_i(x) - P(x^+).$$

Combining (A.11) with the aforementioned inequality, we obtain

$$V(x^+) \leqslant L \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| + \sum_{i \in R} \sigma^i \gamma_i \Xi(x) + LA \Xi(x). \tag{A.12}$$

It follows from (A.12) and the fact that $A \geqslant (1-L)^{-1} \sum_{i \in R} \sigma^i \gamma_i$ that (3.18) holds when $\Xi(x) \leqslant \tau$.

Case 1(ii): $\Xi(x) > \tau$. In this case, $\gamma_i \Xi(x) > u_i^* - b_i$ for all $i \in R$. Definition (2.18) implies that $k_i(x) = b_i$ for all $i \in R$. Moreover, in this case, there exists at least one $i \in \{1,\ldots,n\}$ for which $x_i > x_i^*$. Because f_i is increasing on $[0, \mu_i]$ for $i = 1,\ldots,n$ (a consequence of (H) and the fact that $\mu_i \leqslant \tilde{\delta}_i$), we conclude that there exists at least one $i \in \{1,\ldots,n\}$ for which $f_i(x_i) > f_i(x_i^*)$. Consequently, we obtain from (3.12) and the fact that $u_i = k_i(x) = b_i$ for all $i \in R$:

$$\sum_{i=1}^{n} (n+1-i) u_i = \sum_{i \in R} (n+1-i) b_i + \sum_{i \notin R} (n+1-i) u_i^*$$

$$\leqslant \min_{i=1,\ldots,n} \left(((n-i) p_i + 1) f_i(x_i^*)\right)$$

$$\leqslant \sum_{i=1}^{n} ((n-i) p_i + 1) f_i(x_i).$$

Combining (3.17) and (A.9) with the aforementioned inequality and using the fact $A \geqslant (1-L)^{-1} \sum_{i \in R} \sigma^i \gamma_i$, we conclude that (3.18) holds when $\Xi(x) > \tau$.

Case 2: $x \in S \backslash \Omega, d \in [0, 1]^{n-1}$.

In this case, there exists at least one $i \in \{1,\ldots,n\}$ for which $x_i > \mu_i$. Therefore, definition (2.19) implies $\Xi(x) > h = \min_{i=1,\ldots,n} \left(\sigma^i (\mu_i - x_i^*)\right)$, and consequently, definition (2.21) gives $P(x) = Q - \theta h$. Moreover, definition (2.18) gives $k_i(x) = b_i$ for all $i \in R$ (a direct consequence

of the facts that $\tau < \tau^* \leqslant h$ and $\gamma_i = \tau^{-1}\left(u_i^* - b_i\right) \geqslant h^{-1}\left(u_i^* - b_i\right)$). Combining, we obtain from definitions (2.20) and (3.17)

$$
\begin{aligned}
V(x^+) &= \sum_{i=1}^{n} \sigma^i \left|x_i^+ - x_i^*\right| + A\,\Xi(x^+) + K \max\left(0, \sum_{i=1}^{n} I_i(x^+) - P(x^+)\right) \\
&\leqslant \sum_{i=1}^{n} \sigma^i \left|x_i^+ - x_i^*\right| + K \max\left(0, \sum_{i=1}^{n} I_i(x^+) - Q + \theta h\right) + A\,\Xi(x^+).
\end{aligned}
\tag{A.13}
$$

Using (3.5), the facts that $u_i = k_i(x) = b_i$ for all $i \in R$, $Q - \theta h = \varepsilon \min\limits_{i=1,\dots,n}\left((n+1-i)\,\mu_i\right)$ and $\sum\limits_{i \in R}(n+1-i)\,b_i + \sum\limits_{i \notin R}(n+1-i)\,u_i^* \leqslant \varepsilon\,C \min\limits_{i=1,\dots,n}\left((n+1-i)\,\mu_i\right)$ (which both imply that $\sum\limits_{i \in R}(n+1-i)\,b_i + \sum\limits_{i \notin R}(n+1-i)\,u_i^* \leqslant C\,(Q - \theta h)$), we acquire

$$
\max\left(0, \sum_{i=1}^{n} I_i(x^+) - Q + \theta h\right) \leqslant (1-C) \max\left(0, \sum_{i=1}^{n} I_i(x) - Q + \theta h\right).
\tag{A.14}
$$

Combining (A.13) and (A.14), we obtain

$$
V(x^+) \leqslant \sum_{i=1}^{n} \sigma^i \left|x_i^+ - x_i^*\right| + K(1-C) \max\left(0, \sum_{i=1}^{n} I_i(x) - Q + \theta h\right) + A\,\Xi(x^+).
\tag{A.15}
$$

Definition (2.19) in conjunction with (A.15) implies that the following inequality holds

$$
V(x^+) \leqslant \sum_{i=1}^{n} \sigma^i \max\left(a_i - x_i^*, x_i^*\right) + K(1-C) \max\left(0, \sum_{i=1}^{n} I_i(x) - Q + \theta h\right) + A\sum_{i=1}^{n} \sigma^i \left(a_i - x_i^*\right).
\tag{A.16}
$$

The fact that there exists at least one $i \in \{1, \dots, n\}$ for which $x_i > \mu_i$, implies that

$$
\sum_{i=1}^{n} I_i(x) = \sum_{i=1}^{n} (n+1-i)\,x_i \geqslant \min_{i=1,\dots,n}\left((n+1-i)\,\mu_i\right).
\tag{A.17}
$$

Using (A.16), (A.17) and the fact that $Q - \theta h = \varepsilon \min\limits_{i=1,\dots,n}\left((n+1-i)\,\mu_i\right)$, we obtain

$$
\begin{aligned}
V(x^+) &\leqslant \sum_{i=1}^{n} \sigma^i \max\left(a_i - x_i^*, x_i^*\right) + K \max\left(0, \sum_{i=1}^{n} I_i(x) - Q + \theta h\right) \\
&\quad + A\sum_{i=1}^{n} \sigma^i \left(a_i - x_i^*\right) - KC\,(1-\varepsilon) \min_{i=1,\dots,n}\left((n+1-i)\,\mu_i\right).
\end{aligned}
\tag{A.18}
$$

Because $K \geqslant \dfrac{\sum_{i=1}^{n} \sigma^i \max(a_i - x_i^*, x_i^*) + A\sum_{i=1}^{n} \sigma^i (a_i - x_i^*) - (A+L)h}{(1-\varepsilon)C \min\limits_{i=1,\dots,n}((n+1-i)\mu_i)}$, $\sum\limits_{i=1}^{n} \sigma^i \left|x_i - x_i^*\right| \geqslant \Xi(x) > h$, we conclude from (A.18) and definition (2.20) that (3.18) holds. The proof is complete.

Proof of the Claim 2 made in the proof of Theorem 2.1: Because $\sigma \in (0,1]$, we acquire for all $x \in S$

$$
\sigma^n |x - x^*| \leqslant \sum_{i=1}^{n} \sigma^i |x_i - x_i^*| \leqslant |x - x^*| \sum_{i=1}^{n} \sigma^i.
\tag{A.19}
$$

Similarly, using definition (2.19), we obtain for all $x \in S$

$$0 \leqslant \Xi(x) \leqslant \sum_{i=1}^{n} \sigma^i \, |x_i - x_i^*| \leqslant |x - x^*| \sum_{i=1}^{n} \sigma^i. \tag{A.20}$$

Using (A.20), the fact that $I_j(x) := \sum_{i=1}^{j} x_i$ for $j = 1, \ldots, n$, definition (2.21) and the fact that $Q \geqslant \sum_{i=1}^{n} I_i(x^*)$ (a consequence of (3.5) and the fact that $Q \geqslant (1 - C) \sum_{i=1}^{n} I_i(x^*) + \sum_{i=1}^{n} (n + 1 - i) u_i^*$), we obtain for all $x \in S$

$$\max\left(0, \sum_{i=1}^{n} I_i(x) - P(x)\right) \leqslant \max\left(0, \sum_{i=1}^{n} I_i(x) - \sum_{i=1}^{n} I_i(x^*)\right) + \max\left(0, \sum_{i=1}^{n} I_i(x^*) - P(x)\right)$$

$$\leqslant \max\left(0, \sum_{i=1}^{n} (n + 1 - i)(x_i - x_i^*)\right) + \max\left(0, \sum_{i=1}^{n} I_i(x^*) - Q + \theta \min(h, \Xi(x))\right)$$

$$\leqslant \sum_{i=1}^{n} (n + 1 - i) \, |x_i - x_i^*| + \max\left(0, \sum_{i=1}^{n} I_i(x^*) - Q\right) + \theta \min(h, \Xi(x))$$

$$\leqslant \sum_{i=1}^{n} (n + 1 - i) \, |x_i - x_i^*| + \theta \, \Xi(x) \leqslant |x - x^*| \sum_{i=1}^{n} (n + 1 - i) + \theta \, |x - x^*| \sum_{i=1}^{n} \sigma^i.$$

$$\tag{A.21}$$

It follows from definition (2.20), (A.19), (A.20) and (A.21) that there exist constants $K_2 \geqslant K_1 > 0$ such that inequality (3.19) holds. The proof is complete.

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