INT. J. CONTROL, 2003, VOL. 76, NO. 15, 1536–1546

Taylor & Francis
Taylor & Francis Group

Non-uniform in time stabilization for linear systems and tracking control for non-holonomic systems in chained form

I. KARAFYLLIS† and J. TSINIAS†*

The paper contains certain results concerning non-uniform in time stabilization for linear time-varying systems by means of a linear time-varying feedback controller. These results enable us to present an explicit solution for the state feedback tracking control problem of non-holonomic systems in chained form under weaker hypotheses than those imposed in earlier existing works.

1. Introduction

The paper is a continuation of the authors' works (Tsinias and Karafyllis 1999, Tsinias 2000, 2001, Karafyllis 2002, Karafyllis and Tsinias 2003 a, b) on the non-uniform in time global stability and feedback stabilization of time-varying systems. In the present work we restrict ourselves to the linear case

$$\left.\begin{aligned} &\dot{x} = A(t)x + B(t)u \\ &x \in \Re^n, \quad u \in \Re^m, \quad t \geq 0 \end{aligned}\right\} \tag{1}$$

where $A(\cdot)$ and $B(\cdot)$ are C^0 matrices of dimensions $n \times n$ and $n \times m$, respectively and provide sufficient conditions for non-uniform in time feedback stabilization of (1) that enable us to derive a solution for the tracking control problem for non-holonomic systems under weaker hypotheses than these made in earlier existing works.

Particularly, the main purpose of § 2 is to provide sufficient conditions for non-uniform in time global stabilization for (1) by means of a linear time-varying feedback controller (Propositions 3 and 4). Among other things, we establish that:

- If the origin is globally asymptotically stable (GAS), generally, non-uniformly in time, for the linear time-varying system

$$\left.\begin{aligned} &\dot{x} = A(t)x \\ &x \in \Re^n, \quad t \geq 0 \end{aligned}\right\} \tag{2}$$

  where $A(\cdot)$ is a C^0 matrix of dimension $n \times n$, then the system above admits a quadratic time-varying Lyapunov function (Proposition 2).
- If system (1) is controllable, then it is asymptotically stabilized with an 'arbitrary fast' rate of convergence at the origin by means of a linear time-varying feedback (Propositions 3 and 4); particularly, the origin of the resulting closed-loop system is, in general, non-uniformly in time, asymptotically stable. The corresponding feedback stabilizer can be explicitly identified.

In § 3 we use the results of § 2 in order to derive sufficient conditions for the solvability of the state feedback tracking control problem by means of a linear feedback law for *non-holonomic* systems in chained form:

$$\left.\begin{aligned} &\dot{z} = u_1 \\ &\dot{x}_i = u_1 x_{i+1} \quad i = 1, \ldots, n-1 \\ &\dot{x}_n = u_2 \\ &z \in \Re, \quad x = (x_1, \ldots, x_n) \in \Re^n, \quad (u_1, u_2) \in \Re^2 \end{aligned}\right\} \tag{3}$$

Tracking control and feedback stabilization for non-holonomic systems (3) have been considered in many recent papers (Pomet 1992, Sordalen and Egeland 1995, Jiang and Nijmeijer 1997 a, b, Morin and Samson 1997, 1999 a, b, 2000, Jiang *et al.* 1998, Jiang and Nijmeijer 1999, Lefeber *et al.* 1999, Morin *et al.* 1999, Jiang 2000, Loria *et al.* 2001, Panteley *et al.* 2001, Sarychev 2001, Sun *et al.* 2001, Karafyllis and Tsinias 2003 a). We present here a generalization of the results that have appeared in the literature concerning the state feedback tracking control problem (Proposition 5), which is based on weaker hypotheses than those imposed in the previously mentioned works.

Notation: We denote by $|x|$ the usual Euclidean norm of a vector $x \in \Re^n$, by x' its transpose and by $|A| := \sup\{|Ax|; x \in \Re^n, |x| = 1\}$ the induced norm of a matrix $A \in \Re^{m \times n}$. By $\mathbf{L}^\infty(J)$ ($\mathbf{L}^\infty_{\mathrm{loc}}(J)$) we denote the space of measurable and essentially bounded (locally essentially bounded) functions $u(\cdot)$ defined on $J \subseteq \Re^+ := [0, +\infty)$ and taking values in $\Re^m$. Finally, $\Phi(t, t_0)$ is adopted throughout the paper to denote the fundamental solution matrix corresponding to (2) with $\Phi(t_0, t_0) = I$ (I denotes the identity matrix). For the reader's convenience we recall below some elementary well-known properties of $\Phi(t, t_0)$ that will be used

Received 9 October 2002. Revised 27 June 2003.
* Author for correspondence. e-mail: jtsin@central.ntua.gr
† Department of Mathematics, National Technical University of Athens, Zografou Campus 15780, Athens, Greece.

International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2003 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170310001604927

in the following sections and hold for any triple $t, t_0, \tau \geq 0$:

$$\frac{\partial \Phi}{\partial t}(t, t_0) = A(t)\Phi(t, t_0) \tag{4}$$

$$\frac{\partial \Phi}{\partial t_0}(t, t_0) = -\Phi(t, t_0)A(t_0) \tag{5}$$

$$\Phi(t, \tau)\Phi(\tau, t_0) = \Phi(t, t_0) \tag{6}$$

2. Results on non-uniform in time asymptotic stability

In this section we establish a Lyapunov characterization of (non-uniform in time) GAS for the particular linear case (2) and we prove that system (1) is controllable, if and only if it is stabilizable by means of a linear time-varying feedback in such a way that the trajectories of the closed-loop system approach zero with an arbitrary fast rate of convergence. However, this type of GAS is in general non-uniform with respect to initial values of time. The results constitute generalizations of well-known facts of linear systems theory, see for instance (Kalman 1960, Kailath 1980, Anderson and Moore 1990, Rugh 1996, Sontag 1998).

We provide below the precise notions of GAS and 'controllability' that have been adopted throughout the paper.

Definition 1: Consider the system

$$\left.\begin{aligned} &\dot{x} = f(t, x) \\ &x \in \Re^n, \quad t \geq 0 \end{aligned}\right\} \tag{7}$$

where f is measurable with respect to t, locally Lipschitz with respect to x, with $f(t, 0) = 0$ for all $t \geq 0$, and denote by $x(t) := x(t, t_0, x_0)$ its solution at time t, initiated from $x_0 \in \Re^n$ and time $t_0 \geq 0$. We say that $0 \in \Re^n$ is *globally asymptotically stable (GAS)* for (7), if:

I. For every $t_0 \geq 0$ and $x_0 \in \Re^n$ the corresponding solution $x(\cdot)$ exists for all $t \geq t_0$.

II. For every $\varepsilon > 0$ and $T \geq 0$, it holds that $\sup\{|x(t)|: t \geq t_0, |x_0| \leq \varepsilon, t_0 \in [0, T]\} < +\infty$ and there exists $\delta := \delta(\varepsilon, T) > 0$ such that

$$|x_0| \leq \delta, \quad t_0 \in [0, T] \Rightarrow |x(t)| \leq \varepsilon, \quad \forall t \geq t_0 \quad \text{(Stability)}$$

III. For every $\varepsilon > 0$, $R \geq 0$ and $T \geq 0$, there exists $\tau := \tau(\varepsilon, T, R) \geq 0$ such that

$$|x_0| \leq R, \quad t_0 \in [0, T] \Rightarrow |x(t)| \leq \varepsilon, \quad \forall t \geq t_0 + \tau \quad \text{(Attractivity)}$$

Definition 2: Consider the system

$$\left.\begin{aligned} &\dot{x} = f(t, x, u) \\ &x \in \Re^n, \quad u \in \Re^m, t \geq 0 \end{aligned}\right\} \tag{8}$$

where f is measurable with respect to t, locally Lipschitz with respect to (x, u), and denote by $x(t, t_0, x_0; u)$ its solution at time t that corresponds to the input $u(\cdot) \in \mathbf{L}^{\infty}_{\text{loc}}(\Re^+)$, initiated from $x_0 \in \Re^n$ at time $t_0 \geq 0$. System (8) is *controllable*, if for every $t_0 \geq 0$, there exists a time $T > t_0$ such that for every pair $(x_0, y) \in \Re^n \times \Re^n$, it holds $x(T, t_0, x_0; u) = y$ for some $u(\cdot) \in \mathbf{L}^{\infty}([t_0, T])$. Equivalently, (8) is controllable, if for every $t_0 \geq 0$, there exists $T > t_0$ such that (8) is *completely controllable* on the interval $[t_0, T]$ (see Sontag (1998) for the precise definition of complete controllability).

We next recall the following well-known result from linear control theory (see for instance Theorem 5 in Sontag 1998, p. 109).

Proposition 1: *System* (1) *is controllable, if and only if for every* $t_0 \geq 0$, *there exists a time* $T := T(t_0) > t_0$ *such that the controllability Gramian of* (1)

$$W(t, t_0) := \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)B'(\tau)\Phi'(t_0, \tau)\,\mathrm{d}\tau \tag{9}$$

is positive definite for every $t \geq T$, *i.e.*

$$W(t, t_0) > 0, \quad \forall t \geq T \tag{10}$$

The following elementary result will be used in Proposition 3 to establish equivalence between controllability and feedback stabilization capability for (1). Its proof is a direct consequence of the result of the previous proposition.

Lemma 1: *System* (1) *is controllable if and only if there exists a pair of non-decreasing* C^1 *functions* $a, \delta: \Re^+ \to (0, +\infty)$ *such that*

$$0 < W(t + \delta(t), t) \leq a(t)I, \qquad \forall t \geq 0 \tag{11}$$

Proof: Proposition 1 asserts that for every $t_0 \geq 0$ there is a positive constant $\delta_0 > 0$ such that $W(t_0 + \delta_0, t_0) > 0$, thus continuity of $W(\cdot)$ implies $W(t + \delta_0, t) > 0$ for all $t \in I_{t_0}$, where $I_{t_0} \subseteq \Re^+$ is a neighbourhood of t_0. It turns out that for any compact interval $I \subset \Re^+$ there exists a constant $\delta_I > 0$ such that $W(t + \delta_I, t) > 0$ for all $t \in I$. By employing this property, we can find a piecewise constant function $\bar{\delta}: \Re^+ \to (0, +\infty)$ such that $W(t + \bar{\delta}(t), t) > 0$ for all $t \geq 0$. For instance, we may take $\bar{\delta}(t) := \delta_{[n,n+1]}$, for $t \in [n, n+1)$, $n = 0, 1, 2, \ldots$. The left-hand side inequality of (11) follows by constructing a non-decreasing C^1 function $\delta: \Re^+ \to (0, +\infty)$ with $\delta(t) \geq \bar{\delta}(t)$ for all $t \geq 0$ and taking into account that $W(t + \delta(t), t) \geq W(t + \bar{\delta}(t), t) > 0$ for all $t \geq 0$. The existence of a C^1 non-decreasing function $a: \Re^+ \to (0, +\infty)$, such that the right-hand side inequality of (11) holds, is immediate. □

We next give a Lyapunov characterization of GAS for case (2).

Proposition 2: *The following statements are equivalent:*

(i) *Zero $0 \in \Re^n$ is GAS for (2).*

(ii) *There exist a C^0 function $l: \Re^+ \to (0, +\infty)$ with*

$$\int_0^{+\infty} l(s)\,\mathrm{d}s = +\infty \tag{12}$$

and a non-decreasing C^0 function $\theta: \Re^+ \to (0, +\infty)$, such that, if $x(t)$ denotes the solution of (2) with $x(t_0) = x_0$, the following holds

$$|x(t)| \le \exp\left\{-\int_0^t l(s)\,\mathrm{d}s\right\}\theta(t_0)|x_0|, \quad \forall t \ge t_0, \quad x_0 \in \Re^n \tag{13}$$

(iii) *There exists a positive definite C^1 matrix $P(\cdot) \in \Re^{n\times n}$ such that*

$$P(t) \ge I \tag{14}$$

$$\dot{P}(t) + P(t)A(t) + A'(t)P(t) + 2l(t)P(t) \le 0, \quad \forall t \ge 0 \tag{15}$$

where $l: \Re^+ \to (0, +\infty)$ is any C^0 function satisfying (12) and (13) and I is the unit matrix of dimension $n \times n$.

Proof: (i) $\Rightarrow$ (ii) Suppose that $0 \in \Re^n$ is GAS for (2). Then the fundamental solution matrix $\Phi(\cdot)$ of (2) satisfies $\lim_{t\to+\infty}|\Phi(t,0)| = 0$ and thus, we may define

$$\omega(t) := \sup_{\tau \ge t}|\Phi(\tau, 0)| \tag{16}$$

Obviously, $\omega: \Re^+ \to \Re^+$ is a C^0, non-increasing function and satisfies $\lim_{t\to+\infty}\omega(t) = 0$. Consequently, there exists a C^1, strictly decreasing, positive function $\phi: \Re^+ \to (0, +\infty)$, with $\omega(t) \le \phi(t)$, $\dot{\phi}(t) < 0$, for all $t \ge 0$ and in such a way that $\lim_{t\to+\infty}\phi(t) = 0$. Let

$$l(t) := -\frac{\dot{\phi}(t)}{\phi(t)} \tag{17}$$

Obviously, l is C^0 and satisfies (12). Moreover, from (16) and (17) it follows that

$$|\Phi(t,0)| \le \phi(t) = \phi(0)\exp\left\{-\int_0^t l(s)\,\mathrm{d}s\right\}, \quad \forall t \ge 0 \tag{18}$$

It turns out from (18) and by making use of (5) and (6) that

$$\begin{aligned}|\Phi(t,t_0)| &\le \phi(0)\exp\left\{-\int_0^t l(s)\,\mathrm{d}s\right\}|\Phi(0,t_0)| \\ &\le \theta(t_0)\exp\left\{-\int_0^t l(s)\,\mathrm{d}s\right\}, \quad \forall t \ge t_0\end{aligned} \tag{19}$$

where

$$\theta(t) := \phi(0)\exp\left\{\int_0^t |A(s)|\,\mathrm{d}s\right\} \tag{20}$$

The desired (13) is a direct consequence of (19).

(ii) $\Rightarrow$ (iii) Estimation (13) is equivalent to

$$|\Phi(t,0)| \le \theta(0)\exp\left\{-\int_0^t l(s)\,\mathrm{d}s\right\}, \quad \forall t \ge 0$$

which implies that

$$|\Phi(0,t)x| \ge \frac{1}{\theta(0)}\exp\left\{\int_0^t l(s)\,\mathrm{d}s\right\}|x|, \quad \forall t \ge 0, \quad \forall x \in \Re^n \tag{21}$$

Define

$$P(t) := \theta^2(0)\exp\left\{-2\int_0^t l(s)\,\mathrm{d}s\right\}\Phi'(0,t)\Phi(0,t) \tag{22}$$

Clearly, by (5), (21) and definition (22), it follows that $P(\cdot)$ is a C^1 positive definite matrix which satisfies both (14) and (15).

(iii) $\Rightarrow$ (i) Consider the function

$$V(t,x) := x'P(t)x \tag{23}$$

where $P(\cdot)$ is defined by (14) and (15). It turns out that the derivative $\dot{V}\big|_{(2)}$ of V along the trajectories of (2) satisfies

$$\dot{V}\big|_{(2)} \le -2l(t)V(t,x) \tag{24}$$

Consequently, the solution $x(\cdot)$ of (2) initiated from $x_0 \in \Re^n$ at time $t_0 \ge 0$ satisfies

$$V(t,x(t)) \le \exp\left\{-2\int_{t_0}^t l(s)\,\mathrm{d}s\right\}V(t_0,x_0), \quad \forall t \ge t_0 \tag{25}$$

and thus by virtue of (14) and definition (23) it follows that

$$|x(t)| \le \exp\left\{-\int_{t_0}^t l(s)\,\mathrm{d}s\right\}|P(t_0)|^{1/2}|x_0|, \quad \forall t \ge t_0 \tag{26}$$

Inequality (26), in conjunction with (12), shows that $0 \in \Re^n$ is GAS for (2). The proof is complete. □

The next proposition establishes equivalence between controllability and non-uniform in time stabilizability for the linear case (1).

Proposition 3: *The following statements are equivalent:*

(i) *System (1) is controllable.*

(ii) *For every C^0 function $l: \Re^+ \to (0, +\infty)$ with $\int_0^{+\infty} l(s)\,\mathrm{d}s = +\infty$, there exists a C^0 function*

$\theta\colon \Re^+ \to (0,+\infty)$, such that $0 \in \Re^n$ is GAS for (1) with

$$u = k(t)x := -\tfrac{1}{2}B'(t)Q^{-1}(t)x \tag{27}$$

where

$$Q(t) := \int_t^{t+\delta(t)} \mu(t,\tau)\Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)\,\mathrm{d}\tau \tag{28}$$

$$\mu(t,\tau) := \exp\left(\int_\tau^t \left(2l(s) + \frac{\dot a(s)}{a(s)}\right)\mathrm{d}s\right) \tag{29}$$

for certain $a, \delta\colon \Re^+ \to (0,+\infty)$ non-decreasing C^1 functions (being independent of the choice of $l(\cdot)$) for which (11) is satisfied. Particularly, the solution $x(\cdot)$ of the closed-loop system (1) with (27) satisfies

$$|x(t)| \le \exp\left\{-\int_0^t l(s)\,\mathrm{d}s\right\}\theta(t_0)|x_0|, \quad \forall t \ge t_0, \quad x_0 \in \Re^n, \quad t_0 \ge 0 \tag{30}$$

Proof: (i) $\Rightarrow$ (ii) Let $a, \delta\colon \Re^+ \to (0,+\infty)$, be a pair of non-decreasing and C^1 functions for which (11) is satisfied, whose existence is guaranteed by Lemma 1. Since $\dot a(t) \ge 0$ for all $t \ge 0$, for each fixed $t \ge 0$ the function $\mu(t,\cdot)$ as defined by (29) is decreasing and satisfies $0 < \mu(t,\tau) \le 1$ for all $\tau \ge t$. Thus, by virtue of (11) and definitions (28) and (29), it follows that

$$\begin{aligned} 0 &< \mu(t, t+\delta(t))W(t+\delta(t), t) \\ &\le Q(t) \le W(t+\delta(t), t) \le a(t)I \end{aligned} \tag{31}$$

Moreover, using property (4), we evaluate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Q^{-1}(t) = {} & Q^{-1}(t)B(t)B'(t)Q^{-1}(t) \\ & - \left(2l(t) + \frac{\dot a(t)}{a(t)}\right)Q^{-1}(t) - Q^{-1}(t)A(t) \\ & - A'(t)Q^{-1}(t) - \big(1+\dot\delta(t)\big)\mu(t, t+\delta(t)) \\ & \times Q^{-1}(t)\Phi(t, t+\delta(t))B(t+\delta(t)) \\ & \times B'(t+\delta(t))\Phi'(t, t+\delta(t))Q^{-1}(t) \end{aligned} \tag{32}$$

Define

$$P(t) := a(t)Q^{-1}(t) \tag{33}$$

Clearly, from (31) and (33) we get

$$P(t) \ge I \tag{34}$$

Also, by virtue of (32) and (33) and using the fact that $\delta(\cdot)$ is non-decreasing, we obtain

$$\begin{aligned} & \dot P(t) + P(t)\big(A(t) - \tfrac{1}{2}B(t)B'(t)Q^{-1}(t)\big) \\ & \quad + \big(A(t) - \tfrac{1}{2}B(t)B'(t)Q^{-1}(t)\big)'P(t) + 2l(t)P(t) \\ & = a(t)\frac{\mathrm{d}}{\mathrm{d}t}Q^{-1}(t) + \dot a(t)Q^{-1}(t) + a(t)Q^{-1}(t)A(t) \\ & \quad - a(t)Q^{-1}(t)B(t)B'(t)Q^{-1}(t) \\ & \quad + a(t)A'(t)Q^{-1}(t) + 2l(t)a(t)Q^{-1}(t) \\ & = -a(t)\big(1+\dot\delta(t)\big)\mu(t, t+\delta(t))Q^{-1}(t)\Phi(t, t+\delta(t)) \\ & \quad \times B(t+\delta(t))B'(t+\delta(t))\Phi'(t, t+\delta(t))Q^{-1}(t) \\ & \le 0 \end{aligned} \tag{35}$$

The rest part of proof is a consequence of (34), (35) and the result of Proposition 2 (implication (iii) $\Rightarrow$ (ii)), for the closed-loop system (2) with (27).

(ii) $\Rightarrow$ (i) We again denote by $\Phi(t,t_0)$ the fundamental solution matrix of (2), namely, of system (1) with $u \equiv 0$, and recall the elementary property

$$\begin{aligned} \exp\left(-\int_{t_0}^t |A(s)|\,\mathrm{d}s\right)|x| &\le \left|\Phi'(t,t_0)x\right| \\ &\le \exp\left(\int_{t_0}^t |A(s)|\,\mathrm{d}s\right)|x|, \\ & \forall t \ge t_0 \ge 0, \quad x \in \Re^n \end{aligned} \tag{36}$$

Let

$$l(t) := 3|A(t)| + 1 \tag{37}$$

and let $k\colon \Re^+ \to \Re^{m\times n}$ be a C^0 mapping in such a way that the solution $x(\cdot)$ of the closed-loop system (1) with $u = k(t)x$ satisfies (30) for certain $\theta(\cdot)\colon \Re^+ \to (0,+\infty)$. Suppose on the contrary that (1) is not controllable. Then by invoking Theorem 5, p. 109 in Sontag (1998) there would exist $t_0 \ge 0$ such that for every $t \ge t_0$

$$p'(t)\Phi(t,\tau)b(\tau) = 0, \quad \forall \tau \in [t_0, t] \tag{38}$$

for certain non-zero vector $p(t) \in \Re^n$. Obviously, by (38) we have $p'(t)x(t) = p'(t)\Phi(t,t_0)x_0$ for all $x_0 \in \Re^n$ and $t \ge t_0$ and thus by invoking (30) it follows that

$$\left|p'(t)\Phi(t,t_0)x_0\right| \le |p(t)|\exp\left\{-\int_0^t l(s)\,\mathrm{d}s\right\}\theta(t_0)|x_0|, \quad \forall x_0 \in \Re^n, \quad t \ge t_0$$

By letting $x_0 = \Phi'(t,t_0)p(t)$ in the previous inequality we obtain

$$\left|\Phi'(t,t_0)p(t)\right|^2 \le |p(t)|^2\exp\left\{-\int_0^t l(s)\,\mathrm{d}s\right\}\theta(t_0)\left|\Phi'(t,t_0)\right|, \quad \forall t \ge t_0$$

and the latter, in conjunction with (36) and (37), implies that

$$\exp\left(t + 3\int_0^{t_0} |A(s)|\,\mathrm{d}s\right) \leq \theta(t_0), \quad \text{for all } t \geq t_0$$

a contradiction. □

Remark 1: The result of Proposition 3, as well as the corresponding stabilization methodology, generalizes a well-known fact from linear control theory (see, i.e. Rugh 1996), which asserts that system (1) is *globally exponentially stabilizable* with 'arbitrary fast' rate of convergence, under the hypothesis of *uniform controllability*, namely, under the assumption

$$a_1 I \leq W(t+\delta, t) \leq a_2 I, \quad \forall t \geq 0 \tag{39}$$

for certain positive constants $a_1, a_2, \delta > 0$. To be more precise, equations (27)–(29) coincide with the formula given in Rugh (1996), with $\delta(t) \equiv \delta > 0$, $a(t) \equiv a_2 > 0$, where a_2, δ are the constants defined in (39) and $l(t) \equiv l > 0$ (constant). Moreover, the solution $x(\cdot)$ of the corresponding closed-loop system satisfies $|x(t)| \leq K\exp\{-l(t-t_0)\}|x_0|$ for all $t \geq t_0$, with $l > 0$ (constant) as above and for certain constant $K > 0$.

The next proposition provides a linear feedback controller, which globally asymptotically stabilizes (1) at zero and simultaneously is a solution of an infinite horizon optimal control problem. The corresponding feedback design is based on the solvability of a time-varying Riccati differential equation and constitutes a generalization of standard optimal control procedures (see, i.e. Kalman 1960, Kailath 1980, Anderson and Moore 1990, Amato *et al.* 1996).

Proposition 4: *Suppose that* (1) *is controllable. Let* $l\colon \Re^+ \to (0, +\infty)$ *be any* C^0 *function with* $\int_0^{+\infty} l(s)\,\mathrm{d}s = +\infty$. *Consider the infinite horizon optimal control problem*

$$\min_{u(\cdot)\in C^0([t_0,+\infty))} J(t_0, x_0, u(\cdot));$$

$$J(t_0, x_0, u(\cdot)) := \int_{t_0}^{+\infty} \big(x'(t, t_0, x_0; u(\cdot))Q(t)x(t, t_0, x_0; u(\cdot)) + u'(t)R(t)u(t)\big)\,\mathrm{d}t \tag{40}$$

where $x(t, t_0, x_0; u(\cdot))$ *above denotes the solution of* (1) *corresponding to input* $u(\cdot) \in C^0([t_0, +\infty))$ *initiated from* $x_0 \in \Re^n$ *at time* $t_0 \geq 0$, $R(\cdot) \in \Re^{m\times m}$ *is a* C^0 *positive definite matrix and* $Q(\cdot) \in \Re^{n\times n}$ *is a* C^0 *positive semi-definite matrix that satisfies for all* $t \geq 0$

$$Q(t) \geq \exp\left(2\int_0^t l(s)\right)\left(2l(t)I + A(t) + A'(t) + \exp\left(2\int_0^t l(s)\right)B(t)R^{-1}(t)B'(t)\right) \tag{41}$$

Then there exist a C^0 *function* $\theta : \Re^+ \to (0, +\infty)$ *and a* C^0 *mapping* $k : \Re^+ \to \Re^{m\times n}$ *such that:*

- *Zero is GAS for the closed-loop system* (1) *with* $u = k(t)x$ *and particularly its solution* $x(\cdot)$ *satisfies* (30).
- *For any* $(t_0, x_0) \in \Re^+ \times \Re^n$, *the control* $u(t) = k(t)x(t)$, *where* $x(t)$ *denotes the solution of the closed-loop system* (1) *with* $u = k(t)x$ *initiated from* $x_0 \in \Re^n$ *at time* $t_0 \geq 0$, *minimizes the performance index given by* (40).

Proof: The proof is based on a well-known optimal control result (see, e.g. Kalman 1960, Anderson and Moore 1990), which asserts that, under the controllability assumption for (1), for any pair of continuous positive semi-definite matrices $Q(\cdot) \in \Re^{n\times n}$ and $R(\cdot) \in \Re^{m\times m}$ in such a way that for each $t \geq 0$ the matrix $R(t)$ is positive definite, the infinite horizon optimal control problem given by (40) is solvable; namely, for every $(t_0, x_0) \in \Re^+ \times \Re^n$ there exists a $u^*(\cdot) \in C^0([t_0, +\infty))$ such that

$$J(t_0, x_0, u^*(\cdot)) = \min_{u(\cdot)\in C^0([t_0,+\infty))} J(t_0, x_0, u(\cdot))$$

Particularly, there exists a C^1 positive semi-definite matrix $P(\cdot) \in \Re^{n\times n}$ that satisfies the Riccati equation

$$\dot{P}(t) + P(t)A(t) + A'(t)P(t) - P(t)B(t)R^{-1}(t)B'(t)P(t) + Q(t) = 0, \quad \forall t \geq 0 \tag{42}$$

with the properties:

P1. $x_0' P(t_0)x_0 = \min_{u(\cdot)\in C^0([t_0,+\infty))} J(t_0, x_0, u(\cdot)), \quad \forall (t_0, x_0) \in \Re^+ \times \Re^n$

P2. $u^*(t) = -R^{-1}(t)B'(t)P(t)x(t, t_0, x_0; u^*(\cdot)), \quad$ for $t \geq t_0$

It turns out that the feedback controller

$$\left.\begin{aligned} u^* &= k(t)x \\ k(t) &:= -R^{-1}(t)B'(t)P(t) \end{aligned}\right\} \tag{43}$$

minimizes the cost function $J(t_0, x_0, u(\cdot))$.

We next show that the origin of the closed-loop system (1) with (43) is GAS and particularly estimation (30) holds, provided that (41) is satisfied. Let

$$M(t) := \exp\left(2\int_0^t l(s)\,\mathrm{d}s\right)I, \quad t \geq 0 \tag{44}$$

It follows, by invoking (41) and (44), that for every $u(\cdot) \in \mathbf{L}^\infty_{\mathrm{loc}}(\Re^+)$ it holds

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(x'(t,t_0,x_0;u(\cdot))M(t)x(t,t_0,x_0;u(\cdot))\big)\\&\quad= x'(t,t_0,x_0;u(\cdot))\big(\dot{M}(t)+M(t)A(t)\\&\qquad+A'(t)M(t)\big)x(t,t_0,x_0;u(\cdot))\\&\qquad+2u'(t)B'(t)M(t)x(t,t_0,x_0;u(\cdot))\\&\quad\le x'(t,t_0,x_0;u(\cdot))Q(t)x(t,t_0,x_0;u(\cdot))+u'(t)R(t)u(t)\end{aligned} \tag{45}$$

Let us denote by $x(\cdot)$ the optimal solution of the closed-loop system (1) with (43). It turns out from (40), (44), (45) and recalling property P1 that

$$\begin{aligned}x_0'P(t_0)x_0 &= \int_{t_0}^{+\infty}\Big(x'(\tau)Q(\tau)x(\tau)+u^{*'}(\tau)R(\tau)u^*(\tau)\Big)\,\mathrm{d}\tau\\&\ge \int_{t_0}^{t}\Big(x'(\tau)Q(\tau)x(\tau)+u^{*'}(\tau)R(\tau)u^*(\tau)\Big)\,\mathrm{d}\tau\\&\ge \int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\big(x'(\tau)M(\tau)x(\tau)\big)\,\mathrm{d}\tau\\&= \exp\bigg(2\int_0^t l(s)\,\mathrm{d}s\bigg)|x(t)|^2\\&\quad-\exp\bigg(2\int_0^{t_0} l(s)\,\mathrm{d}s\bigg)|x_0|^2\end{aligned}$$

and thus the trajectory $x(\cdot)$ of the closed-loop system (1) with (43) initiated from x_0 at time t_0 satisfies

$$|x(t)|^2\le \exp\bigg(-2\int_0^t l(s)\,\mathrm{d}s\bigg)x_0'\bigg(P(t_0)+\exp\bigg(2\int_0^{t_0} l(s)\,\mathrm{d}s\bigg)I\bigg)x_0 \tag{46}$$

Inequality (46) implies (30) with

$$\theta(t) := |P(t)|^{1/2}+\exp\bigg\{\int_0^t l(s)\,\mathrm{d}s\bigg\} \qquad \square$$

Remark 2: The drawback in the methodology employed in the proof of Proposition 4 is that the construction of the feedback stabilizer is reduced to the difficult problem of finding an appropriate solution of the time-varying Riccati differential equation (42). On the other hand, the advantage of the existence result of Proposition 4 compared to the approach of Proposition 3 is that the control action is square integrable; particularly, for $R(\cdot)>0$ being arbitrary, the corresponding stabilizer $u^*(\cdot)$ satisfies

$$\int_{t_0}^{+\infty} u^{*\prime}(t)R(t)u^*(t)\,\mathrm{d}t<+\infty$$

3. Application to tracking problems

The result of this section constitutes generalization of earlier contributions on the subject and is based on extremely simple hypotheses regarding reference inputs of system (3). We next recall the precise definition of the state feedback tracking problem at a reference trajectory for the non-holonomic case (3).

Problem formulation: Consider a reference trajectory $(z_r(t),x_r(t))=(z_r(t);x_{1r}(t),\dots,x_{nr}(t))'\in\Re^{n+1}$, $t\ge 0$ of system (3), namely

$$\dot{z}_r=u_{1r};\dot{x}_{ir}=u_{1r}x_{(i+1)r},\quad 1\le i\le n-1;\quad \dot{x}_{nr}=u_{2r} \tag{47}$$

for certain reference control inputs u_{1r} and u_{2r}. Denote the tracking error as

$$(z_e(t),e(t)) := (z(t)-z_r(t),x(t)-x_r(t))$$

where $(z(t),x(t))$ is any arbitrary solution of (3). Then $(z_e(t),e(t))$ satisfies

$$\begin{aligned}\dot{z}_e=v_1;\quad \dot{e}_i=(u_{1r}(t)+v_1)e_{i+1}+v_1x_{(i+1)r}(t),\\ 1\le i\le n-1;\quad \dot{e}_n=v_2\end{aligned} \tag{48}$$

$$v_1 := u_1-u_{1r}(t);\quad v_2 := u_2-u_{2r}(t) \tag{49}$$

The state feedback tracking control problem is said to be *globally solvable*, if there exists a pair of time-varying feedback controllers of the form

$$v_1=U_1(t,z_e,e),\quad v_2=U_2(t,z_e,e) \tag{50}$$

such that $0\in\Re^{n+1}$ is GAS for the closed-loop system (48) with (50).

The following proposition is the main result of our work and its proof is based on the results of §2 (Propositions 2 and 3).

Proposition 5: *Consider the system* (48) *and suppose that:*

A1 *The functions $u_{1r}(\cdot)$ and $u_{2r}(\cdot)$ are of class $C^0(\Re^+)$ and $\mathbf{L}^\infty_{\mathrm{loc}}(\Re^+)$, respectively.*

A2 *There is no time $t_0\ge 0$ such that $u_{1r}(t)=0$ for all $t\ge t_0$.*

Then there is a pair of C^0 mappings $k_i (i=1,2)$, such that the linear time-varying feedback law:

$$v_1=U_1(t,z_e):=k_1(t)z_e,\qquad v_2=U_2(t,e):=k_2(t)e \tag{51}$$

solves the state feedback tracking problem globally.

For the proof of Proposition 5 we need the following technical lemma, which provides a criterion for the controllability of the time-varying case

$$\left.\begin{aligned}&\dot{x}=a(t)Ax+Bu\\&x\in\Re^n,\quad u\in\Re^m,\quad t\ge 0\end{aligned}\right\} \tag{52}$$

where A,B are constant matrices of dimensions $n\times n$ and $n\times m$, respectively, with $n>m$ and $a:\Re^+\to\Re$ is a C^0 function.

Lemma 2: *For the system* (52) *we assume that the pair* (A, B) *satisfies the controllability rank condition. Then the following statements are equivalent:*

(i) *The function* $a(\cdot)$ *satisfies hypothesis A2 of Proposition 5, namely, there is no time* $t_0 \geq 0$ *for which* $a(t) = 0$ *for all* $t \geq t_0$.

(ii) *System* (52) *is controllable (in the sense of Definition 2).*

(iii) *System* (52) *is stabilizable (by means of a linear time-varying feedback), in the sense of Statement (ii) of Proposition 3; particularly, stabilization is exhibited with an 'arbitrary fast' rate of convergence at zero for the trajectories of the corresponding closed-loop system.*

Proof: (i) $\Rightarrow$ (ii) We first show that, if (52) is not controllable then there exists some time $t_0 \geq 0$ such that

$$a(t) = 0, \quad \forall t \geq t_0 \tag{53}$$

Indeed, if (52) is not controllable, there exists some time $t_0 \geq 0$ such that for every $T \geq t_0$, there is a non-zero vector $p \in \Re^n$ with

$$p' \Phi(T, t) B = 0, \qquad \forall t \in [t_0, T] \tag{54}$$

where $\Phi(t, t_0)$ is the fundamental solution matrix for (52) with $u \equiv 0$. It is a matter of calculations to verify that

$$\Phi(t, t_0) = \exp\left(A \int_{t_0}^{t} a(s)\, \mathrm{d}s\right) = I + \sum_{k=1}^{\infty} \frac{1}{k!} \left(A \int_{t_0}^{t} a(s)\, \mathrm{d}s\right)^k$$

and therefore (54) is equivalent to

$$p' \exp\left(A \int_{t}^{T} a(s)\, \mathrm{d}s\right) B = 0, \qquad \forall t \in [t_0, T] \tag{55}$$

We claim that (55) implies that $a(t) = 0$ for all $t \in [t_0, T]$, hence, since T is arbitrary, equation (53) holds. Suppose on the contrary that there exists time $\tilde{t}$ with $a(\tilde{t}) \neq 0$, or equivalently, by continuity of $a(\cdot)$, there exists an open interval $(t_1, t_2) \subset [t_0, T]$ with $a(t) \neq 0$ for all $t \in (t_1, t_2)$.

On the other hand, by (55) we get that

$$\frac{\mathrm{d}}{\mathrm{d}t} p' \exp\left(A \int_{t}^{T} a(s)\, \mathrm{d}s\right) B = 0$$

for all $t \in [t_0, T]$, which implies

$$a(t) p' A \exp\left(A \int_{t}^{T} a(s)\, \mathrm{d}s\right) B = 0, \quad \forall t \in [t_0, T] \tag{56}$$

Since $a(\cdot)$ is non-zero on (t_1, t_2), it follows by (56) that

$$p' A \exp\left(A \int_{t}^{T} a(s)\, \mathrm{d}s\right) B = 0, \quad \forall t \in (t_1, t_2)$$

If we keep differentiating on the interval (t_1, t_2) we get

$$p' A^j \exp\left(A \int_{t}^{T} a(s)\, \mathrm{d}s\right) B = 0, \qquad \forall t \in (t_1, t_2), \quad \text{for } j = 0, \ldots, n-1$$

or equivalently

$$p' \exp\left(A \int_{t}^{T} a(s)\, \mathrm{d}s\right) A^j B = 0, \qquad \forall t \in (t_1, t_2), \quad \text{for } j = 0, \ldots, n-1 \tag{57}$$

Hence, for each $t \in (t_1, t_2)$ it follows from (57)

$$p' \exp\left(A \int_{t}^{T} a(s)\, \mathrm{d}s\right) \left(B, AB, \ldots, A^{n-1} B\right) = 0$$

and, since the pair of matrices (A, B) is controllable, the latter implies $p = 0$, a contradiction. We conclude that (52) is controllable.

The rest part of proof is immediate. Particularly, implication (ii) $\Rightarrow$ (iii) is a consequence of Proposition 3 and finally (iii) $\Rightarrow$ (i) is obvious. $\square$

We are now in a position to establish Proposition 5.

Proof of Proposition 5: By Lemma 2 it follows that, for any C^0 mapping $u_{1r}(\cdot)\colon \Re^+ \to \Re$ satisfying hypothesis A2, the system

$$\left.\begin{aligned} &\dot{e}_i = u_{1r}(t) e_{i+1} \quad 1 \leq i \leq n-1 \\ &\dot{e}_n = v \\ &\text{with } v \text{ as input} \end{aligned}\right\} \tag{58}$$

is controllable (in the sense of Definition 2) and thus, according to Proposition 3, there exists a C^0 mapping $k\colon \Re^+ \to \Re^{1\times n}$ such that $0 \in \Re^n$ is GAS for the closed-loop system (58) with

$$v = k(t) e \tag{59}$$

Thus, by Proposition 2, there exists a C^1 positive definite matrix $P(\cdot) \in \Re^{n\times n}$ and a positive C^0 function $l\colon \Re^+ \to \Re^+$ with

$$\int_0^{+\infty} l(s)\, \mathrm{d}s = +\infty$$

such that, if we define $V(t, e) := e' P(t) e$, the following inequalities hold

$$|e|^2 \leq V(t, e) \tag{60}$$

$$\dot{V}\big|_{(58), v=k(t)e} \leq -2 l(t) V(t, e), \quad \text{for all } (t, e) \in \Re^+ \times \Re^n \tag{61}$$

Let $\phi\colon \Re^+ \to (0, +\infty)$ a C^0 function with

$$\int_0^{+\infty} \phi(s)\, \mathrm{d}s = +\infty$$

yet to be selected and define

$$k_1(t) := -\phi(t), \qquad k_2(t) := k(t) \tag{62}$$

Consider the resulting system (48) with (51) and (62), namely, the system

$$\left.\begin{aligned} &\dot{z}_e = v_1 \\ &\dot{e}_i = (u_{1r}(t) + v_1)e_{i+1} + v_1 x_{(i+1)r}(t), \qquad 1 \le i \le n-1 \\ &\dot{e}_n = v_2 \\ &\text{with } v_1 = -\phi(t)z_e, \qquad v_2 = k(t)e \end{aligned}\right\} \tag{63}$$

and let $(z_e(\cdot), e(\cdot))$ be its corresponding solution initiated from $(z_e(t_0), e(t_0))$ at time $t_0 \ge 0$. Obviously, we have

$$z_e(t) = \exp\left\{-\int_{t_0}^{t} \phi(s)\,\mathrm{d}s\right\} z_e(t_0) \tag{64}$$

$$v_1(t) = -\phi(t)\exp\left\{-\int_{t_0}^{t} \phi(s)\,\mathrm{d}s\right\} z_e(t_0) \tag{65}$$

Furthermore, by (60), (61) and (65) it follows that the time derivative $\dot{V}$ of $V(\cdot)$ along the trajectories of (63) satisfies

$$\begin{aligned} \dot{V} \le &-(2l(t) - (2 + L(t))|P(t)||v_1(t)|)V(t,e) \\ &+ L(t)|P(t)||v_1(t)| \end{aligned} \tag{66}$$

$$L(t) := |u_{1r}(t)| + \sum_{i=2}^{n} |x_{ir}(t)| \tag{67}$$

$x_{ir}(\cdot)$ being the ith component of the solution of (47). Let $a\colon \Re^+ \to \Re^+$ be any C^0 function with

$$\sup_{t\ge 0} |P(t)|(2 + L(t))\exp\left\{t + 2\int_0^t l(s)\,\mathrm{d}s - \int_0^t a(s)\,\mathrm{d}s\right\} < +\infty \tag{68}$$

and define

$$\phi(t) := \frac{\exp\left\{-\int_0^t (s + a(s))\,\mathrm{d}s\right\}}{\int_t^{+\infty} \exp\left\{-\int_0^{\tau} (s + a(s))\,\mathrm{d}s\right\}\mathrm{d}\tau} \tag{69}$$

Note that $\lim_{t\to+\infty} \phi(t) = +\infty$ and consequently

$$\int_0^{+\infty} \phi(s)\,\mathrm{d}s = +\infty$$

By (68) and definition (69), there exists a constant $R > 0$ such that

$$\phi(t)|P(t)|(2 + L(t))\exp\left\{t + \int_0^t (2l(s) - \phi(s))\,\mathrm{d}s\right\} \le R, \quad \forall t \ge 0 \tag{70}$$

Then by (65) and (70) we estimate

$$(2 + L(t))|P(t)||v_1(t)| \le R\exp\left\{-t + \int_0^{t_0} \phi(s)\,\mathrm{d}s\right\}|z_e(t_0)| \tag{71}$$

$$L(t)|P(t)||v_1(t)| \le R\exp\left\{-t - 2\int_0^t l(s)\,\mathrm{d}s + \int_0^{t_0} \phi(s)\,\mathrm{d}s\right\}|z_e(t_0)| \tag{72}$$

By (60), (66), (71), (72) and use of the comparison principle we get

$$\begin{aligned} |e(t)| \le &\exp\left\{\frac{1}{2}h(t_0, |z_e(t_0)|)\right\}\exp\left\{-\int_{t_0}^t l(s)\,\mathrm{d}s\right\} \\ &\times \left[|P(t_0)|^{1/2}|e(t_0)| + (h(t_0, |z_e(t_0)|))^{1/2}\right] \end{aligned} \tag{73}$$

$$h(t, w) := Rw\exp\left\{\int_0^t \phi(s)\,\mathrm{d}s\right\} \tag{74}$$

Relations (64) and (73), (74) imply that $0 \in \Re^{n+1}$ is GAS for the closed-loop system (63). We conclude that the mapping (51), where $k_i(\cdot)$ $(i = 1, 2)$ are defined by (62) is a solution of the tracking problem. □

Remark 3: Lemma 2 guarantees that Assumption A2 made in Proposition 5 is equivalent to the property that system (58) is controllable (in the sense of Definition 2). This again, according to Proposition 1, is equivalent with the property that for every $t \ge 0$, there exists $\delta(t) > 0$ such that the controllability Gramian of system (58)

$$W(t + \delta(t), t) := \int_t^{t+\delta(t)} \Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)\,\mathrm{d}\tau$$

is positive definite, where

$$B(t) := (0, \ldots, 0, 1)'$$

$$\Phi(t, t_0) := \{\phi_{i,j}(t, t_0);\, 1 \le i, j \le n\}$$

$$\phi_{i,j}(t, t_0) := \begin{cases} 0 & \text{for } j < i \\ \dfrac{1}{(j-i)!}\left(\displaystyle\int_{t_0}^t u_{1r}(\tau)\,\mathrm{d}\tau\right)^{j-i} & \text{for } j \ge i \end{cases}$$

$\Phi(t, t_0)$ being the fundamental solution matrix of (58) with $v \equiv 0$. Direct evaluation of the controllability Gramian gives the equivalence between hypothesis A2 and the following one:

A2′ *For every* $t \ge 0$, *there exists* $\delta(t) > 0$ *such that*

$$\int_t^{t+\delta(t)} w(\tau, t)w'(\tau, t)\,\mathrm{d}\tau > 0 \tag{75}$$

with $w(t,t_0) := (w_1(t,t_0),\dots,w_n(t,t_0))$ *, where each component* $w_i(t,t_0)$ *is defined as*

$$w_i(t,t_0) := \frac{1}{(n-i)!}\left(\int_t^{t_0} u_{1r}(\tau)\,\mathrm{d}\tau\right)^{n-i} \tag{76}$$

Remark 4: *Comparisons with some earlier existing works*. We make some comparison with some earlier existing works for the tracking problem for the case (3). In Jiang (2000, Theorem 1) it is assumed that the reference trajectory $x_{ir}(\cdot)$ $(i=2,\dots,n)$ and the mappings $u_{1r}(\cdot)$, $\dot{u}_{1r}(\cdot)$, $u_{2r}(\cdot)$ are bounded over $\Re^+$ and $u_{1r}(\cdot)$ does not converge to zero as $t\to+\infty$. Note that the results in Jiang (2000) generalize those in Jiang and Nijmeijer (1997 a, b, 1999) and Jiang *et al.* (1998). In our recent work (Karafyllis and Tsinias 2003 a) it is assumed that the mapping $u_{1r}(\cdot)$ is C^1 and the mapping $u_{2r}(\cdot)$ is measurable and locally essentially bounded and there exist constants $p\ge 1$, $K>0$ and $c,r,M,\lambda\ge 0$ such that the following hold for all $t\ge 0$

$$\left.\begin{aligned} &\sum_{i=2}^{n}|x_{ir}(t)|\le M\exp\{\lambda t\}\\ &|u_{1r}(t)|+|\dot{u}_{1r}(t)|\le K\exp\{ct\}\\ &\lim_{t\to+\infty}\frac{1}{t}\int_0^t |u_{1r}(\tau)|^p\exp\{r\tau\}\,\mathrm{d}\tau=+\infty \end{aligned}\right\} \tag{77}$$

where $x_{ir}(\cdot)$ denotes the reference trajectory of (3). We note here that the above set of hypotheses imposed in Karafyllis and Tsinias (2003 a) leads to solvability of the tracking problem for a larger class of nonholonomic systems that includes the case (3). Finally, in Lefeber *et al.* (1999) it is assumed that the mapping $u_{1r}(\cdot)$ is continuous, the reference trajectory $x_{ir}(\cdot)$ $(i=2,\dots,n)$ is bounded over $\Re^+$ and there exist positive constants δ_0, ε_1 and ε_2 such that the following matrix inequality holds for all $t\ge 0$

$$\varepsilon_1 I\le\int_t^{t+\delta_0} w(\tau,t)w'(\tau,t)\,\mathrm{d}\tau\le\varepsilon_2 I \tag{78}$$

where $w(\tau,t)$ is defined by (76). This is equivalent to the assumption that (58) is uniformly controllable, namely, its controllability Gramian satisfies (39). Proposition 5 of the present paper clearly generalizes the results obtained in the papers mentioned above for the case (3), since it is based on weaker hypotheses. Particularly, by taking into account the equivalence between hypotheses A2 and A2′, it follows that assumption (78), imposed in Lefeber *et al.* (1999) is indeed stricter than (75).

The following numerical example illustrates the nature of Proposition 5 and shows that indeed our methodology exhibits asymptotic tracking under weaker hypotheses for the reference control input $u_{1r}(t)$ than those imposed in the previously mentioned works.

Example 1: Consider the three-dimensional system (3) and its reference trajectory

$$\begin{aligned}(z_r(t);x_{1r}(t),x_{2r}(t))'\\ :=\left(\int_0^t\exp(-\tau^2)\,\mathrm{d}\tau,t,\exp(t^2)\right),\quad t\ge 0\end{aligned}$$

corresponding to the reference control inputs $u_{1r}(t):=\exp(-t^2)$ and $u_{2r}(t):=2t\exp(t^2)$. Note that the state feedback tracking control problem for this reference trajectory *cannot be solved* by the proposed methodologies in earlier existing works. For example, Theorem 1 in Jiang (2000) cannot be applied, since $u_{1r}(\cdot)$ converges to zero as $t\to+\infty$ and the component of the reference trajectory $x_{2r}(t)$ as well as the mapping $u_{2r}(\cdot)$ are not bounded over $\Re^+$. Neither does the result in Karafyllis and Tsinias (2003 a) work in this case, since the first and the last condition in (77) do not hold. Finally, note that for $p':=(1,0)$, $|p|=1$, we obtain

$$\begin{aligned}p'\int_t^{t+\delta} w(\tau,t)w'(\tau,t)\,\mathrm{d}\tau\, p\\ =\int_t^{t+\delta}\left(\int_t^\tau u_{1r}(s)\,\mathrm{d}s\right)^2\mathrm{d}\tau\le\delta^3\exp(-2t^2),\\ \text{for every }\delta>0\end{aligned}$$

(where $w(\tau,t)$ is defined by (76)), thus the above inequality asserts that a constant $\delta_0>0$ for which (78) holds does not exist. Moreover, the component of the reference trajectory $x_{2r}(t)$ is not bounded over $\Re^+$. Consequently, the result in Lefeber *et al.* (1999) is not applicable. On the other hand, Proposition 5 of the present paper guarantees that there exists a linear time- varying feedback law (51), which solves the state feedback tracking control problem globally. In order to construct such a feedback law, we first construct a linear time-varying stabilizer for system (58). We may use the result of Proposition 3 to determine an explicit formula for this feedback law. In the above case however, we can directly proceed (by applying an elementary backstepping design approach) considering the quadratic Lyapunov function

$$\begin{aligned}V(t,e_1,e_2)&:=16\exp\left(t+2t^2\right)e_1^2+2\left(e_2+2(t+1)\exp(t^2)e_1\right)^2\\ &=(e_1,e_2)P(t)\begin{pmatrix}e_1\\ e_2\end{pmatrix}\end{aligned} \tag{79}$$

$$P(t):=\begin{bmatrix}8\exp(2t^2)\left(2\exp(t)+(t+1)^2\right) & 4(t+1)\exp(t^2)\\ 4(t+1)\exp(t^2) & 2\end{bmatrix} \tag{80}$$

which satisfies

$$\left(e_1^2+e_2^2\right) \le V(t,e_1,e_2) \le 32\exp\left(2(t+t^2)\right)\left(e_1^2+e_2^2\right),$$
$$\forall (t,e_1,e_2) \in \Re^+ \times \Re^2 \quad (81)$$

namely (60) holds and it is a matter of calculations to verify that (61) is satisfied with $l(t) \equiv 1$ and

$$\begin{aligned} v &= k(t)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \\ &:= -2\exp(t^2)[1+(2t+1)(t+1)+4\exp(t)]e_1 \\ &\quad -(2t+3)e_2 \end{aligned} \quad (82)$$

It turns out that the linear time-varying feedback given by (82) globally asymptotically stabilizes (58) at the origin. Next, by taking into account definition (67) and the right-hand side inequality (81), we can determine a C^0 function $a: \Re^+ \to \Re^+$ such that (68) holds. For example, we may select $a(t) := 5 + 7t$ and then consider ϕ as defined by (69)

$$\phi(t) := \frac{\exp\left(-5t-4t^2\right)}{\int_t^{+\infty}\exp(-5\tau-4\tau^2)\,\mathrm{d}\tau} \quad (83)$$

We conclude, according to the procedure employed in the proof of Proposition 5, that the linear time-varying feedback law

$$v_1 := -\phi(t)z_e$$
$$v_2 := k(t)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

where

$$(z_e,e_1,e_2) := \left(z-\int_0^t \exp(-\tau^2)\,\mathrm{d}\tau, x_1-t, x_2-\exp(t^2)\right)$$

and $k(t)$ and $\phi(t)$ are determined by (82) and (83), respectively, globally solves the state feedback tracking control problem for this case.

4. Conclusions

We have established a Lyapunov characterization of (non-uniform in time) global asymptotic stability for linear time-varying systems and we have proved that a linear time-varying control system is controllable, if and only if it is (non-uniformly in time) stabilizable by means of a linear time-varying feedback in such a way that the trajectories of the closed-loop system approach zero with an arbitrary fast rate of convergence. The results constitute generalizations of well-known facts of linear systems theory. We have also derived sufficient conditions for the solvability of the state feedback tracking control problem for non-holonomic systems in chained form, which constitute a generalization of the results that have appeared in the literature concerning this problem.

References

Amato, F., Pironti, A., and Scala, S., 1996, Necessary and sufficient conditions for quadratic stability and stabilizability of uncertain linear time-varying systems. *IEEE Transactions on Automatic Control*, **41**, 125–128.

Anderson, B., and Moore, J., 1990, *Optimal Control: Linear Quadratic Methods* (Englewood Cliffs, NJ: Prentice-Hall).

Jiang, Z. P., 2000, Lyapunov design of global state and output feedback trackers for nonholonomic control systems. *International Journal of Control*, **73**, 744–761.

Jiang, Z. P., Lefeber, E., and Nijmeijer, H., 1998, Stabilization and tracking of a nonholonomic mobile robot with saturating actuators. In *Proceedings of CONTROLO '98, 3rd Portuguese Conference on Automatic Control*, **2**, 315–320.

Jiang, Z. P., and Nijmeijer, H., 1997 a, Backstepping-based tracking control of nonholonomic chained systems. In *Proceedings of the 4th European Control Conference*, Brussels, Belgium, Paper 672 (TH-M A2).

Jiang, Z. P., and Nijmeijer, H., 1997 b, Tracking control of mobile robots: a case study in backstepping. *Automatica*, **33**, 1393–1399.

Jiang, Z.P., and Nijmeijer, H., 1999, A recursive technique for tracking control of nonholonomic systems in chained form. *IEEE Transactions on Automatic Control*, **44**, 265–279.

Kailath, T., 1980, *Linear Systems* (Englewood Cliffs, NJ: Prentice-Hall).

Kalman, R. E., 1960, Contributions to the theory of optimal control, *Bol. Soc. Mat. Mex.*, 102–119.

Karafyllis, I., 2002, Non-uniform stabilization of control systems. *IMA Journal of Mathematical Control and Information*, **19**, 419–444.

Karafyllis, I., and Tsinias, J., 2003 a, Global stabilization and asymptotic tracking for a class of nonlinear systems by means of time-varying feedback. *International Journal of Robust and Nonlinear Control*, **13**, 559–588.

Karafyllis, I., and Tsinias, J., 2003 b, A converse Lyapunov theorem for non-uniform in time global asymptotic stability and its application to feedback stabilization. To appear in *SIAM Journal on Control and Optimization*.

Lefeber, E., Robertsson, A., and H. Nijmeijer, 1999, Linear controllers for tracking chained-form systems. In D. Aeyels, F. Lamnabhi-Lagarrigue and A. van der Schaft (Eds), *Stability and Stabilization of Nonlinear Systems*, Lecture Notes in Control and Information Sciences 246 (London: Springer-Verlag), pp. 183–199.

Loria, A., Panteley, E., and Melhem, K., 2001, UGAS of skew-symmetric systems: application to the nonholonomic integrator. *Proceedings of NOLCOS* 2001, St Petersburg, Russia, pp. 1016–1021.

Morin, P., Pomet, J.-B., and Samson, C., 1999, Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of Lie brackets in closed loop. *SIAM Journal on Control and Optimization*, **38**, 22–49.

Morin, P., and Samson, C., 1997, Application of backstepping techniques to the time-varying exponential stabilization of chained form systems. *European Journal of Control*, **3**, 15–36.

Morin, P., and Samson, C., 1999 a, Exponential stabilization of nonlinear driftless systems with robustness to unmodeled dynamics. *ESAIM: Control, Optimisation and Calculus of Variations*, **4**, 1–35.

Morin, P., and Samson, C., 1999 b, Robust point-stabilization of nonlinear affine control systems. In D. Aeyels, F. Lamnabhi-Lagarrigue and A. van der Schaft (Eds), *Stability and Stabilization of Nonlinear Systems*, Lecture Notes in Control and Information Sciences 246 (London: Springer-Verlag), pp. 215–237.

Morin, P., and Samson, C., 2000, Control of nonlinear chained systems: from the Routh–Hurwitz stability criterion to time-varying exponential stabilizers. *IEEE Transactions on Automatic Control*, **45**, 141–146.

Panteley, E., Loria, A., and Teel, A., 2001, Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Transactions on Automatic Control*, **46**, 1874–1886.

Pomet, J.-P., 1992, Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *System Control Letters*, **18**, 147–158.

Rugh, W. J., 1996, *Linear Systems Theory* (Upper Saddle River: Prentice-Hall).

Sarychev, A. V., 2001, Lie- and chronologico-algebraic tools for studying stability of time-varying systems. *Systems and Control Letters*, **43**, 59–76.

Sontag, E. D., 1998, *Mathematical Control Theory*, 2nd edn (New York: Springer-Verlag).

Sordalen, O. J., and Egeland, O., 1995, Exponential stabilization of nonholonomic chained systems. *IEEE Transactions on Automatic Control*, **40**, 35–49.

Sun, Z., Ge, S. S., Huo, W., and Lee, T. H., 2001, Stabilization of nonholonomic chained systems via nonregular feedback linearization. *Systems and Control Letters*, **44**, 279–289.

Tsinias, J., 2000, Backstepping design for time-varying nonlinear systems with unknown parameters. *Systems and Control Letters*, **39**, 219–227.

Tsinias, J., 2001, A converse Lyapunov theorem for nonuniform in time, global exponential robust stability. *Systems and Control Letters*, **44**, 373–384.

Tsinias, J., and Karafyllis, I., 1999, ISS property for time-varying systems and application to partial-static feedback stabilization and asymptotic tracking. *IEEE Transactions on Automatic Control*, **44**, 2179–2185.