Global Exponential Stability for Discrete-Time Networks With Applications to Traffic Networks

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Abstract—This paper provides sufficient conditions for global asymptotic stability and global exponential stability, which can be applied to nonlinear, large-scale, uncertain discrete-time networks. The conditions are derived by means of vector Lyapunov functions. The obtained results are applied to traffic networks for the derivation of sufficient conditions for global exponential stability of the uncongested equilibrium point of the network. Specific results and algorithms are provided for freeway traffic models. Various examples illustrate the applicability of the obtained results.

Index Terms—Discrete-time systems, nonlinear systems, traffic networks.

I. INTRODUCTION

E XPONENTIAL stability is a very useful property for the equilibrium point of a given network. The purpose of this paper is three-fold:

- · to provide sufficient conditions for global asymptotic stability (GAS) and global exponential stability (GES), which can be easily applied to nonlinear, large-scale, uncertain discrete-time networks;
- to apply the aforementioned sufficient conditions to traffic networks and obtain conditions, which guarantee the GES of the uncongested equilibrium point;
- to study the stability properties of freeway traffic models and obtain easily checkable conditions which guarantee the GES of the uncongested equilibrium point.

Vector Lyapunov functions are useful to large-scale discretetime systems. Sufficient stability conditions by means of vector Lyapunov functions have been proposed in [11, pp. 792–798]. More recently, small-gain conditions have been proposed in [22], which can be expressed by means of a vector Lyapunov function formulation (as shown in [13, Ch. 5]). In this paper, we propose a set of conditions expressed by means of vector Lyapunov functions, which guarantee GAS and GES (Theorem 2.3) and can be applied easily to nonlinear, large-

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Digital Object Identifier 10.1109/TCNS.2014.2367364

scale, uncertain discrete-time systems. The basis for the applicability is the expression of the stability condition by means of a condition on the spectral radius of a non-negative matrix. Therefore, we can apply recent results on non-negative matrices that provide upper bounds for the spectral radius (see [3, Ch. 2]). The stability notions used in this paper are the standard stability notions for discrete-time systems used in [11, Ch. 13], [12], [13, Ch. 2], and [16, Ch. 4], but we also allow the discrete-time, uncertain system to be defined on a subset of a finite-dimensional space. Discrete-time systems defined on a subset of a finite-dimensional space were studied in [28, Ch. 1].

The conservatism of the obtained stability conditions can be reduced significantly if we have an accurate description of a trapping region of the system: this feature is exploited throughout this paper. A nonlinear system with a trapping region is a system for which all solutions enter a specific set after an initial transient period (for continuous-time systems without inputs the name "global uniform ultimate boundedness" is used in [14, p. 211] when the corresponding set is compact; the term "dissipative system" is used in the literature of the continuoustime systems with compact corresponding sets; see [28, p. 180].

The obtained stability results are applied to traffic networks (Section III). More specifically, we develop a general model for traffic networks, which consists of an arbitrary number of elementary components. The components can be interconnected to form any 2-D structure for the overall traffic network. This general formulation allows for a plethora of diverse traffic network infrastructures to be addressed on the basis of a unifying modeling approach; specific instances of the proposed general model may result in systems which are similar to other models in the literature (see, for example, [6], [8], and [25]). In particular, the traffic network structures and problems that can be considered as special cases of the proposed network model include: urban road networks consisting of interconnected links which are modelled as store-and-forward components [1] or cell-transmission links [4]; large urban networks consisting of smaller homogeneous subnetworks [2]; freeway stretches or networks consisting of a series of links which are modelled via the discretized Lighthill-Whitham-Richards (LWR) model [17] or its simplified cell transmission model (CTM) version [7]: large mixed (corridor) networks consisting of urban and freeway links [24]. As a matter of fact, the same generic approach may also be used for modeling water networks consisting of interconnected links which are modelled by discretized versions of the Lighthill-Whitham model [21], see [5] and [23]. Our main related result (Theorem 3.1) provides explicit formulas for the elements of a specific non-negative matrix whose spectral radius is critical for the GES of the uncongested

Manuscript received April 10, 2014; revised April 11, 2014 and August 29, 2014; accepted September 15, 2014. Date of publication November 12, 2014; date of current version March 13, 2015. The research leading to these results received funding from the European Research Council under the E.U.'s 7th Framework Programme (FP/2007-2013)/ERC Grant Agreement 321132 under project TRAMAN21. Recommended by Associate Editor F. Fagnani.

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equilibrium point of the traffic network. Therefore, our results can be used for the determination of the stability properties in a given traffic network within this framework (see Example 3.3).

The obtained results are specialized to the case of a freeway stretch (Section IV). The overall model in this specific constellation consists of a series of subsequent cells and is similar to the known first-order discrete Godunov approximations (see [9]) to the kinematic-wave partial differential equation of the LWR model (see [21] and [26]) with nonlinear [17] or piecewise linear (CTM, [7]) outflow functions. However, the presented framework can also accommodate recent modifications of the LWR model as in [18] to reflect the so-called capacity drop phenomenon. Our main related result (Corollary 4.3) provides an easily implementable algorithm for the determination of the stability properties of the uncongested equilibrium point of the freeway stretch. The results are different from other results in the literature on the CTM (see [6] and [10]), since our methodology is different from the methodology used in [6] and [10]. More specifically, in [10], the dynamical analysis is based on monotone systems theory and in [6], the results concerning the uncongested equilibrium point are local. On the other hand, in this paper, we provide global stability results based on a vector Lyapunov function analysis.

Notation:

- ℜ₊ := [0, +∞). For every set S, Sⁿ = <u>S × ... × S</u> for every positive integer n. ℜⁿ₊ := (ℜ₊)ⁿ. For every x ∈ ℜ, [x] denotes the integer part of x ∈ ℜ.
- We say that an increasing function ρ ∈ C⁰(ℜ₊; ℜ₊) is of class K_∞ if ρ(0) = 0 and lim_{s→+∞} ρ(s) = +∞. By KL, we denote the set of functions σ ∈ C⁰(ℜ₊ × ℜ₊; ℜ₊) with the properties: 1) for each t ≥ 0, the mapping σ(·, t) is increasing with σ(0, t) = 0; 2) for each s ≥ 0, the mapping σ(s, ·) is nonincreasing with lim_{t→+∞} σ(s, t) = 0.
- Let x, y ∈ ℝⁿ. We say that x ≤ y iff (y − x) ∈ ℝⁿ₊. By |x|, we denote the Euclidean norm of x ∈ ℝⁿ. Let A ∈ ℝ^{n×n} be a real matrix. By |A|, we denote the induced matrix norm. The spectral radius of A ∈ ℝ^{n×n} is denoted by ρ(A). When all elements of A are non-negative, then we say that A is non-negative and we write A ∈ ℝ^{n×n}₊.

II. VECTOR LYAPUNOV STABILITY CRITERIA FOR DISCRETE-TIME NETWORKS

Consider the discrete-time system

$$x^+ = F(d, x), \quad x \in S \subseteq \Re^n, \quad d \in D$$
 (2.1)

where $S \subseteq \Re^n$ is a nonempty closed set with $x^* \in S$, $D \subseteq \Re^l$ is a nonempty, compact set, $F: D \times S \to S$ is a locally bounded mapping, being continuous on the set $D \times \{x^*\}$ with $F(d, x^*) = x^*$ for all $d \in D$. We suppose that $\{x \in S : 0 < |x - x^*| \le \delta\} \neq \emptyset$ for every $\delta > 0$.

In order to develop the Vector Lyapunov Stability criteria, we need the notion of a trapping region (TR). A nonlinear system with a TR is a system for which all solutions enter a specific set after an initial transient period.

Definition 2.1: A TR for system (2.1) is a set $A \subseteq S$ for which there exists an integer $m \ge 0$ such that for every $x_0 \in S$, $\{d_i \in D\}_{i=0}^{\infty}$, the solution x(t) of (2.1) with initial condition $x(0) = x_0$ corresponding to input $\{d_i \in D\}_{i=0}^{\infty}$ satisfies $x(t) \in A$ for all $t \ge m$.

A direct consequence of Definition 2.1 is that every TR for (2.1) must contain all equilibrium points. We next define the robust stability notions used for (2.1).

Definition 2.2: We say that $x^* \in S$ is robustly globally asymptotically stable (RGAS) for system (2.1), if there exists a function $\sigma \in KL$ such that for every $x_0 \in S$, $\{d_i \in D\}_{i=0}^{\infty}$, the solution x(t) of (2.1) with $x(0) = x_0$ corresponding to $\{d_i \in D\}_{i=0}^{\infty}$ satisfies $|x(t) - x^*| \leq \sigma(|x_0 - x^*|, t)$ for all $t \geq 0$. We say that $x^* \in S$ is robustly globally exponentially stable (RGES) for system (2.1) if there exist constants $M, \sigma > 0$ such that for every $x_0 \in S$, $\{d_i \in D\}_{i=0}^{\infty}$, the solution x(t) of (2.1) with $x(0) = x_0$ corresponding to $\{d_i \in D\}_{i=0}^{\infty}$ satisfies $|x(t) - x^*| \leq M \exp(-\sigma t) |x_0 - x^*|$ for all $t \geq 0$.

We are now ready to state the main result of the section.

Theorem 2.3: Consider (2.1) and suppose that $A \subseteq S$ is a TR for (2.1). Moreover, suppose that there exist functions $a_1, a_2 \in K_{\infty}$ with $a_1(s) \leq a_2(s)$ for all $s \geq 0$, $V_i : A \to \Re_+$ (i = 1, ..., l) and a matrix $\Gamma = \{\gamma_{i,j} \geq 0, i, j = 1, ..., l\} \in \Re_+^{l \times l}$ such that the following inequalities hold for all $x \in A, d \in D$ and i = 1, ..., l:

$$a_1(|x - x^*|) \le \max_{\substack{i=1,\dots,l\\l}} (V_i(x)) \le a_2(|x - x^*|) \quad (2.2)$$

$$V_i(F(d,x)) \le \sum_{j=1}^{s} \gamma_{i,j} V_j(x).$$
 (2.3)

Moreover, suppose that the spectral radius $\rho(\Gamma)$ of the matrix Γ is less than 1. Then, $x^* \in S$ is RGAS for (2.1). Moreover, if there exist constants $L \ge 0$, $0 < K_1 \le K_2$, p > 0 such that $\sup\{|F(d, x) - x^*| : d \in D\} \le L|x - x^*|$ for all $x \in S \setminus A$ and if $a_i(s) = K_i s^p$ (i = 1, 2) for all $s \ge 0$, then $x^* \in S$ is RGES for (2.1).

Since the matrix Γ is non-negative, there are effective tools for the computation of its spectral radius [3, Ch. 2]. For example, if there exists $\varepsilon > 0$ such that

$$\max_{i=1,\dots,n} \left(\sum_{j=1}^{n} \gamma_{i,j} \right) < 1 \text{ or}$$
$$\max_{i=1,\dots,n} \left(\frac{\sum_{j=1}^{n} (\varepsilon + \gamma_{i,j}) \sum_{k=1}^{n} (\varepsilon + \gamma_{j,k})}{n\varepsilon + \sum_{j=1}^{n} \gamma_{i,j}} \right) < 1$$

then the spectral radius of Γ is less than 1. The aforementioned conditions can be used for large-scale systems easily.

It should be emphasized that the novelty of Theorem 2.3, with respect to existing results, lies in the presence of deterministic uncertainty and the exploitation of the TR.

Proof: Let $x_0 \in S$, $\{d_i \in D\}_{i=0}^{\infty}$ be given and consider the solution x(t) of (2.1) with $x(0) = x_0$ corresponding to $\{d_i \in D\}_{i=0}^{\infty}$. Let $m \ge 0$ be the integer in Definition 2.1. Let $j \in \{0, \ldots, m\}$ be the smallest integer for which it holds that $x(t) \in A$ for all $t \ge j$ [the fact that j exists and satisfies $j \in \{0, \ldots, m\}$ is a consequence of the fact that A is a TR for (2.1)]. We next show that there exists $b \in K_{\infty}$ such that

$$\max_{k=0,\dots,j} |x(k) - x^*| \le b \left(|x_0 - x^*| \right).$$
(2.4)

Indeed, if there exists a constant $L \ge 0$ such that $\max\{|F(d, x) - x^*| : d \in D\} \le L|x - x^*|$ for all $x \in S \setminus A$, then we may define $b(s) := \max(1, L^m)s$ for all $s \ge 0$. The fact that (2.4) holds is a consequence of the fact that $j \le m$, (2.1) and the resulting inequality $|x(t+1) - x^*| \le \max(1, L)|x(t) - x^*|$ which holds for all $t = 0, \ldots, j - 1$, for the case that $j \ge 1$.

For the general case, we define $F(d,x)=x^*$ for all $d\in D, x\in\Re^n\setminus S$ and

$$a(s) := \sup \{ |F(d,x) - x^*| : (d,x) \in D \times \Re^n, |x - x^*| \le s \}.$$
(2.5)

Clearly, a(s) is well defined by (2.5) for all $s \ge 0$, since F is a locally bounded mapping. The continuity of F on the set $D \times \{x^*\}$, in conjunction with the fact that $D \subseteq \Re^l$ is a compact set with $F(d, x^*) = x^*$ for all $d \in D$, implies that

$$\lim_{s \to 0^+} a(s) = a(0) = 0.$$
(2.6)

By virtue of [13, Lemma 2.4] there exists $\bar{a} \in K_{\infty}$ such that $s + a(s) \leq \bar{a}(s)$ for all $s \geq 0$. We define

$$b := \underbrace{\bar{a} \circ \ldots \circ \bar{a}}_{m \ times}. \tag{2.7}$$

Definition (2.7) shows that $b \in K_{\infty}$. Using the fact that $j \leq m$, inequality $|x(t+1) - x^*| \leq \bar{a}(|x(t) - x^*|)$, which holds for all $t = 0, \ldots, j - 1$ [a consequence of (2.5) and (2.1)] for the case that $j \geq 1$ and definition (2.7), we obtain (2.4). When j = 0, then (2.4) holds automatically.

Since $\rho(\Gamma) < 1$ there exist $M \ge 1, \sigma > 0$ such that

$$|\Gamma^t| \le M \exp(-\sigma t)$$
, for all integers $t \ge 0$ (2.8)

(see [27, p. 212, 231]). Next define

$$\xi(t) = (V_1(x(t+j)), \dots, V_l(x(t+j)))' \in \Re^l_+ \text{ for all } t \ge 0.$$
(2.9)

Equation (2.1), in conjunction with inequalities (2.3), implies that the following recursive relation holds for all $t \ge 0$:

$$\xi(t+1) \le \Gamma \xi(t). \tag{2.10}$$

Using the fact that Γ is a non-negative matrix (and consequently satisfies $\Gamma x \leq \Gamma y$ for all vectors $x, y \in \Re^l$ with $x \leq y$), we obtain from (2.10)

$$\xi(t) \le \Gamma^t \xi(0), \text{ for all } t \ge 0 \tag{2.11}$$

Using (2.8), (2.11) as well as definition (2.9) and (2.2), we obtain

$$a_1(|x(j+t) - x^*|) \le M \exp(-\sigma t) \sqrt{la_2(|x(j) - x^*|)},$$

for all $t \ge 0.$ (2.12)

Using (2.4) and (2.12), we obtain

$$a_1 \left(|x(j+t) - x^*| \right) \le M \exp(-\sigma t) \sqrt{la_2} \left(b \left(|x_0 - x^*| \right) \right),$$

for all $t \ge 0$. (2.13)

Since $a_1(s) \le a_2(s)$ for all $s \ge 0$, and since $M \ge 1$, $j \le m$, it follows from (2.4) and (2.13) that the following estimate holds for all $t \ge 0$:

$$a_1(|x(t) - x^*|) \le M \exp(-\sigma(t - m))\sqrt{l}a_2(b(|x_0 - x^*|)).$$
(2.14)

Inequality (2.14) shows that the estimate $|x(t) - x^*| \leq \sigma(|x_0 - x^*|, t)$ holds for all $t \geq 0$ with $\sigma(s, t) := a_1^{-1}(M \exp(-\sigma(t-m)\sqrt{l}a_2(b(s))))$ (notice that $\sigma \in KL$) and, consequently, $x^* \in S$ is RGAS for system (2.1). If there exist constants $L \geq 0$, $0 < K_1 \leq K_2$, p > 0 such that $\max\{|F(d, x) - x^*| : d \in D\} \leq L|x - x^*|$ for all $x \in S \setminus A$ and if $a_i(s) = K_i s^p$ (i = 1, 2) for all $s \geq 0$, then inequality (2.13) implies that

$$K_1 |x(j+t) - x^*|^p \le M \sqrt{l} K_2 \exp(-\sigma t) \\ \times \max(1, L^{pm}) |x_0 - x^*|^p, \quad \text{for all } t \ge 0. \quad (2.15)$$

Here, we have used the fact that (2.4) holds with $b(s) := \max(1, L^m)s$ for all $s \ge 0$. It follows from (2.15) and (2.4) with $b(s) := \max(1, L^m)s$ and the facts that $j \le m$, $0 < K_1 \le K_2$ that the following estimate holds for all $t \ge 0$:

$$K_1 |x(t) - x^*|^p \le M \sqrt{l} K_2 \exp(-\sigma(t-m)) \max(1, L^{pm}) |x_0 - x^*|^p$$

which directly implies that $x^* \in S$ is RGES for system (2.1). The proof is complete.

III. GLOBAL STABILITY RESULTS FOR TRAFFIC NETWORKS

This section is devoted to the derivation of sufficient conditions that guarantee RGES for the equilibrium point of a traffic network. We consider a generic traffic network which consists of n components (see Section I for several specific instances of the generic model). The number of vehicles at time $t \ge 0$ in component $i \in \{1, ..., n\}$ is denoted by $x_i(t)$. The outflow and inflow of vehicles of the component $i \in \{1, ..., n\}$ at time $t \ge 0$ are denoted by $q_i(t) \ge 0$ and $F_i(t) \ge 0$, respectively. All flows during a time interval are measured in [veh]. Consequently, the balance of vehicles for each component $i \in \{1, ..., n\}$ gives

$$x_i(t+1) = x_i(t) - q_i(t) + F_i(t), \quad i = 1, \dots, n, \quad t \ge 0.$$
 (3.1)

Each component of the network has storage capacity $a_i > 0$ (i = 1, ..., n). Our first assumption states that the inflow of vehicles at the cell $i \in \{1, ..., n\}$ at time $t \ge 0$, denoted by $F_i(t) \ge 0$, cannot exceed the number of free positions for vehicles of cell $i \in \{1, ..., n\}$ at time $t \ge 0$, i.e.,

$$F_i(t) = \min\left(c_i\left(a_i - x_i(t)\right), \widetilde{F}_i(t)\right), \quad i = 1, \dots, n, \quad t \ge 0$$
(3.2)

where $F_i(t) \ge 0$ is the attempted inflow of vehicles at the component $i \in \{1, ..., n\}$ at time $t \ge 0$ and $c_i \in (0, 1]$ (i = 1, ..., n) are constants.

Our second assumption deals with the attempted outflows and inflows. We assume that there exist functions $f_i \in C^0(D \times [0, a_i]; \Re_+)$ with $f_i(d, x_i) \leq x_i$ for all $(d, x_i) \in D \times [0, a_i]$, where $D \subseteq \Re^l$ is a nonempty, compact set, constants $p_{i,j} \geq 0$, $i, j = 1, \ldots, n$, with $p_{i,i} = 0$ for $i = 1, \ldots, n$, and constants $Q_i \geq 0, i = 1, \ldots, n$ so that

(attempted flow of vehicles from component i to componentj)

$$= p_{i,j} f_i(d, x_i), \quad \text{for } i, j = 1, \dots, n$$
 (3.3)

(attempted flow of vehicles from component i to regions

out of the network)

$$= Q_i f_i(d, x_i), \quad \text{for } i = 1, \dots, n.$$
 (3.4)

We also assume that

$$\sum_{j=1}^{n} p_{i,j} + Q_i = 1.$$
(3.5)

Some explanations are needed at this point. The function $f_i: D \times [0, a_i] \to \Re_+$ is what in the specialized literature of traffic engineering is called the demand part of the fundamental diagram of the *i*th cell, that is, the flow that will exit the cell if there is sufficient space in the downstream cells; while $p_{i,j}$ are turning rates and Q_i are exit rates. The uncertainty $d \in D$ has been introduced in order to accommodate the uncertain nature of the fundamental diagram. Finally, (3.5) implies that the total attempted outflow from the *i*th cell is exactly equal to the demand part of the fundamental diagram $f_i(d, x_i)$.

Let $v_i > 0$ (i = 1, ..., n) denote the attempted inflow to component $i \in \{1, ..., n\}$ from the region out of the network. Our assumptions lead us to the following equations:

$$\widetilde{F}_{i}(t) = v_{i} + \sum_{j=1}^{n} p_{j,i} f_{j}(d(t), x_{j}(t)), \ i = 1, \dots, n, \ t \ge 0.$$
(3.6)

Equations (3.2) and (3.6) imply that the percentage of the attempted inflow of vehicles at cell *i* at time $t \ge 0$, which becomes the actual inflow of vehicles at cell *i* at time $t \ge 0$, denoted by $s_i(t) \in [0, 1]$ for $i = 1, ..., n, t \ge 0$ is given by

$$s_{i}(t) = \frac{\min\left(c_{i}\left(a_{i} - x_{i}(t)\right), v_{i} + \sum_{j=1}^{n} p_{j,i}f_{j}\left(d(t), x_{j}(t)\right)\right)}{v_{i} + \sum_{j=1}^{n} p_{j,i}f_{j}\left(d(t), x_{j}(t)\right)}.$$
(3.7)

Our final assumption relates the actual inflows with the outflows. Many rules for the outflows of road links have been proposed in the literature; see, for example, [6], [15], [19], and [20]. Here, we employ a similar rule to the so-called proportional priority, first-in-first-out (PP/FIFO) rule for junctions (see [6] and [15]). We assume that if cell i cannot accommodate

all inflows, then the actual inflows from the other cells of the network (or from regions out of the network) to cell *i* are equal percentages of the attempted inflows, i.e.,

(actual flow of vehicles from component j to component i)

 $s_i(t)$ (attempted flow of vehicles from component j

to component i)

$$i, j = 1, \dots, n. \tag{3.8}$$

Other assumptions could be accommodated in this modeling framework if required. Combining (3.3) with (3.8), we obtain

(actual flow of vehicles from component j to component i)

$$= s_i(t)p_{j,i}f_j(d,x_j), \qquad i,j = 1,\dots,n.$$
 (3.9)

Moreover, we assume that the actual flow of vehicles from cell $i \in \{1, ..., n\}$ to regions out of the network is equal to the corresponding attempted flow of vehicles. Thus, the outflow $q_i(t) \ge 0$ from cell $i \in \{1, ..., n\}$ is

$$q_i(t) = \left(Q_i + \sum_{j=1}^n s_j(t)p_{i,j}\right) f_i(d(t), x_i(t)). \quad (3.10)$$

Combining (3.1), (3.2), (3.6), (3.7), and (3.10), we obtain the following discrete-time dynamical system:

$$x_{i}^{+} = x_{i} + s_{i} \left(v_{i} + \sum_{j=1}^{n} p_{j,i} f_{j}(d, x_{j}) \right)$$
$$- \left(Q_{i} + \sum_{j=1}^{n} s_{j} p_{i,j} \right) f_{i}(d, x_{i}), \quad \text{for } i = 1, \dots, n. \quad (3.11)$$

Define $S = [0, a_1] \times \ldots \times [0, a_n]$. Since the functions f_i satisfy $f_i(d, x_i) \leq x_i$ for all $(d, x_i) \in D \times [0, a_i]$, it follows that (3.11) is an (uncertain) dynamical system on S.

A component $i \in \{1, ..., n\}$ of the traffic network (3.11) is said to be "congested" at time t if $c_i(a_i - x_i(t)) < v_i + \sum_{j=1}^n p_{j,i} f_j(d(t), x_j(t))$ (or, equivalently, if $s_i(t) < 1$). In this case, the actual inflow to component $i \in \{1, ..., n\}$ is less than the attempted inflow. We will assume next that there exists an equilibrium point for which no congestion phenomena are present: the uncongested equilibrium point of the network.

(H) The matrix $P = \{p_{i,j} : i, j = 1, ..., n\}$ satisfies $det(I - P') \neq 0$. There exists a point $x^* = (x_1^*, ..., x_n^*)' \in S$ that satisfies the following for all $d \in D$ and i = 1, ..., n:

$$v_i + c_i x_i^* + \sum_{j=1}^n p_{j,i} f_j \left(d, x_j^* \right) \le c_i a_i$$
 (3.12)

$$f_i(d, x_i^*) = f_i^* = v_i + \sum_{j=1}^n p_{j,i} f_j(d, x_j^*).$$
(3.13)

We are now in a position to prove the following theorem.

Theorem 3.1: Consider system (3.7), (3.11) under assumption (H). Assume that there exist constants $L \ge 0, 0 \le \underline{b}_i < 0$ $\overline{b}_i \leq a_i \ \lambda_i, \mu_i \geq 0, \ \omega_i \in [x_i^*, a_i) \ (i = 1, \dots, n)$ such that the set $A = [\underline{b}_1, \overline{b}_1] \times \ldots \times [\underline{b}_n, \overline{b}_n]$ is a TR for system (3.7) and (3.11), and such that the following inequalities hold for all i = 1, ..., n:

$$\begin{aligned} |f_{i}(d, x_{i}) - f_{i}^{*}| &\leq L |x_{i} - x_{i}^{*}|, \\ \text{for all } (d, x_{i}) &\in D \times [0, a_{i}] \\ |x_{i} - x_{i}^{*} - G_{i} (\theta, f_{i}(d, x_{i})) f_{i}(d, x_{i}) \\ &+ \min \left(c_{i}(a_{i} - x_{i}), v_{i} + \sum_{j=1}^{n} p_{j,i} f_{j}^{*} \right) \\ &\leq \lambda_{i} |x_{i} - x_{i}^{*}|, \\ \text{for all } (d, x_{i}) &\in D \times [\underline{b}_{i}, \overline{b}_{i}], \\ \theta &= (\theta_{1}, \dots, \theta_{n}) \in [0, 1]^{n}, i = 1, \dots, n \quad (3.15) \\ |f_{i}^{*} - f_{i}(d, x_{i})| &\leq \mu_{i} |x_{i} - x_{i}^{*}|, \\ \text{for all } (d, x_{i}) \in D \times [\underline{b}_{i}, \overline{b}_{i}] \text{ and } i = 1, \dots, n \quad (3.16) \end{aligned}$$

where

$$G_i(\theta, y) := Q_i$$

+
$$\sum_{j=1}^n \frac{\min\left(c_j(a_j - \theta_j \omega_j), v_j + p_{i,j}y + \sum_{k \neq i} p_{k,j}f_k^*\right)}{v_j + p_{i,j}y + \sum_{k \neq i} p_{k,j}f_k^*} p_{i,j}$$

Define $F_i = \max\{f_i(d, s) : s \in [\underline{b}_i, \overline{b}_i], d \in D\} \ (i = 1, ..., n)$ and assume that

$$f_j^* + p_{i,j} \left(F_i - f_i^* \right) \le c_j a_j \text{ for all } i, j = 1, \dots, n.$$
 (3.17)

Define the matrix $\Gamma = \{\gamma_{i,j} : i, j = 1, \dots, n\}$ by

$$\begin{aligned} \gamma_{i,i} &:= \lambda_i, \text{ for } i = 1, \dots, n \\ \gamma_{i,j} &:= \frac{F_i p_{i,j} c_j \max(0, \bar{b}_j - \omega_j)}{\left(f_j^* + p_{i,j} \left(F_i - f_i^*\right)\right) \left(\bar{b}_j - x_j^*\right)} \\ &+ \left(p_{j,i} + \sum_{k=1}^n \frac{F_i p_{i,k} p_{j,k}}{f_k^* + p_{i,k} \left(F_i - f_i^*\right)}\right) \mu_j \\ &\quad \text{ for } i, j = 1, \dots, n \text{ with } i \neq j. \end{aligned}$$
(3.18)

If $\rho(\Gamma)$ is less than 1, then x^* is RGES for (3.11).

Remark 3.2: (a) Assumption (3.17) is not restrictive: since we are studying the properties of the uncongested equilibrium point, the equilibrium flow values f_i^* for i = 1, ..., n are far smaller than the quantities $c_i a_i$, and condition (3.17) holds.

(b) It should be pointed out that Theorem 3.1 is based on the estimation of the constants $\lambda_i, \mu_i \ge 0$ (i = 1, ..., n) which satisfy inequalities (3.15), (3.16). The numerical evaluation of the magnitude of the constants $\lambda_i, \mu_i \geq 0$ (i = 1, ..., n) can be performed independently for each cell, no matter how many interconnections are present. This implies that the computational complexity for the evaluation of the constants $\lambda_i, \mu_i \geq 0$ (i = 1, ..., n) is of order n and is independent of the number of interconnections. This feature is important for the analysis of large-scale networks.

Proof of Theorem 3.1: We use Theorem 2.3 for

$$V_i(x) := |x_i - x_i^*| \quad (i = 1, \dots, n)$$
 (3.20)

and the dynamical system (3.7) and (3.11). Since the inequality

$$\frac{1}{\sqrt{n}}|x - x^*| \le \max_{i=1,\dots,n} (V_i(x)) \le |x - x^*|$$

holds for all $x \in A$ and since (3.14) implies the condition $\max\{|F(d, x) - x^*| : d \in D\} \le L|x - x^*|$ for all $x \in S$, for a certain constant $L \ge 0$, where $F(d, x) = (F_1(d, x), \dots, F_n(d, x))$ $(x))' \in \Re^n$ and

$$F_i(d,x) := x_i + s_i \left(v_i + \sum_{j=1}^n p_{j,i} f_j(d,x_j) \right)$$
$$- \left(Q_i + \sum_{j=1}^n s_j p_{i,j} \right) f_i(d,x_i)$$

it suffices to show that (2.3) holds for all $x \in A$, i = 1, ..., n.

The remaining part of proof is devoted to the proof of (2.3). Indeed, using (3.7) and (3.20) we obtain for all $(d, x) \in D \times A$, $\theta \in [0, 1]^n$ and i = 1, ..., n

$$V_{i}(F(d,x)) = \left|x_{i}^{+} - x_{i}^{*}\right| \leq f_{i}(d,x_{i}) \sum_{j=1}^{n} p_{i,j} |w_{i,j}| + |x_{i} - x_{i}^{*} - G_{i}(\theta, f_{i}(d,x_{i})) f_{i}(d,x_{i}) + \min(c_{i}(a_{i} - x_{i}), f_{i}^{*})| + \left|s_{i}\left(v_{i} + \sum_{j=1}^{n} p_{j,i}f_{j}(d,x_{j})\right) - \min(c_{i}(a_{i} - x_{i}), f_{i}^{*})| \right|$$

$$(3.21)$$

where

(2,10)

$$w_{i,j} := \frac{\min\left(c_j(a_j - x_j), v_j + \sum_{k=1}^n p_{k,j} f_k(d, x_k)\right)}{v_j + \sum_{k=1}^n p_{k,j} f_k(d, x_k)} - \frac{\min\left(c_j(a_j - \theta_j \omega_j), v_j + p_{i,j} f_i(d, x_i) + \sum_{k \neq i} p_{k,j} f_k^*\right)}{v_j + p_{i,j} f_i(d, x_i) + \sum_{k \neq i} p_{k,j} f_k^*}.$$
 (3.22)

Using (3.15) and (3.16) and the fact that $|\min(a, x)|$ – $\min(a, y) \le |x - y|$ for all $a, x, y \in \Re$, we obtain from (3.21) for all $(d, x) \in D \times A$, $\theta \in [0, 1]^n$ and $i = 1, \dots, n$

$$V_{i}(F(d,x)) \leq \lambda_{i} |x_{i} - x_{i}^{*}| + f_{i}(d,x_{i}) \sum_{j=1}^{n} p_{i,j} |w_{i,j}| + \sum_{j=1}^{n} p_{j,i} \mu_{j} |x_{j} - x_{j}^{*}|. \quad (3.23)$$

We next show that for every $(d, x) \in D \times A$ and $i = 1, \ldots, n$, we can select $\theta_i \in [0, 1]$ in a way so that we can minimize the values of $|w_{i,j}|$ $(j=1,\ldots,n)$. Continuity of the mapping $[0,1] \ni \theta_j \rightarrow \min(c_j(a_j - \theta_j \omega_j)/(v_j + p_{i,j}f_i(d,x_i) + \sum_{k \neq i} p_{k,j}f_k^*)), 1)$ implies the existence of $\theta_j \in [0,1]$ with $w_{i,j} = 0$, provided that

$$\min\left(\frac{c_{j}(a_{j}-\omega_{j})}{v_{j}+p_{i,j}f_{i}(d,x_{i})+\sum_{k\neq i}p_{k,j}f_{k}^{*}},1\right) \leq s_{j}$$

$$\leq \min\left(\frac{c_{j}a_{j}}{v_{j}+p_{i,j}f_{i}(d,x_{i})+\sum_{k\neq i}p_{k,j}f_{k}^{*}},1\right) \quad (3.24)$$

where s_j is defined in (3.7). If (3.24) does not hold, then $\min(c_j(a_j - \omega_j)/(v_j + p_{i,j}f_i(d, x_i) + \sum_{k \neq i} p_{k,j}f_k^*), 1) > s_j$. This follows from $\min(c_j a_j/(v_j + p_{i,j}f_i(d, x_i) + \sum_{k \neq i} p_{k,j}f_k^*), 1) = 1$ (a consequence of (3.13), (3.17)) and $s_j \leq 1$. Consequently, (3.22) implies

$$|w_{i,j}| = \frac{\min\left(c_j(a_j - \omega_j), v_j + p_{i,j}f_i(d, x_i) + \sum_{k \neq i} p_{k,j}f_k^*\right)}{v_j + p_{i,j}f_i(d, x_i) + \sum_{k \neq i} p_{k,j}f_k^*} - \frac{\min\left(c_j(a_j - x_j), v_j + \sum_{k=1}^n p_{k,j}f_k(d, x_k)\right)}{v_j + \sum_{k=1}^n p_{k,j}f_k(d, x_k)}$$

when (3.24) does not hold. Moreover, since $s_j = (\min(c_j(a_j - x_j), v_j + \sum_{k=1}^{n} p_{k,j} f_k(d, x_k))/v_j + \sum_{k=1}^{n} p_{k,j} f_k(d, x_k)) < 1$, we obtain $c_j(a_j - x_j) < v_j + \sum_{k=1}^{n} p_{k,j} f_k(d, x_k)$ and consequently

$$|w_{i,j}| = \frac{\min\left(c_j(a_j - \omega_j), v_j + p_{i,j}f_i(d, x_i) + \sum_{k \neq i} p_{k,j}f_k^*\right)}{v_j + p_{i,j}f_i(d, x_i) + \sum_{k \neq i} p_{k,j}f_k^*} - \frac{c_j(a_j - x_j)}{v_j + \sum_{k=1}^n p_{k,j}f_k(d, x_k)}$$
(3.25)

provided that (3.24) does not hold.

Hence, when (3.24) does not hold, we have from (3.25), (3.16), and (3.24)

$$\begin{split} w_{i,j} & \left| \left(v_j + p_{i,j} f_i(d, x_i) + \sum_{k \neq i} p_{k,j} f_k^* + \sum_{k \neq i} p_{k,j} \mu_k | x_k - x_k^* | \right) \right. \\ & \leq \min \left(c_j(a_j - \omega_j), v_j + p_{i,j} f_i(d, x_i) + \sum_{k \neq i} p_{k,j} f_k^* \right) \\ & \times - c_j(a_j - x_j) \\ & + \frac{\min \left(c_j(a_j - \omega_j), v_j + p_{i,j} f_i(d, x_i) + \sum_{k \neq i} p_{k,j} f_k^* \right)}{v_j + p_{i,j} f_i(d, x_i) + \sum_{k \neq i} p_{k,j} f_k^*} \\ & \times \sum_{k \neq i} p_{k,j} \mu_k | x_k - x_k^* | \, . \end{split}$$

Using (3.13), the inequalities $x_j - \omega_j \leq \max(0, x_j - \omega_j)$ and $\min(c_j(a_j - \omega_j), v_j + p_{i,j}f_i(d, x_i) + \sum_{k \neq i} p_{k,j}f_k^*) \leq c_j(a_j - \omega_j)$, we obtain

$$|w_{i,j}| \le \frac{c_j \max(0, x_j - \omega_j) + \sum_{k \ne i} p_{k,j} \mu_k |x_k - x_k^*|}{f_j^* + p_{i,j} \left(f_i(d, x_i) - f_i^*\right) + \sum_{k \ne i} p_{k,j} \mu_k |x_k - x_k^*|}$$

The above inequality holds when (3.24) holds as well. Using the above inequality in conjunction with (3.23), we obtain for all $(d, x) \in D \times A$ and i = 1, ..., n

$$V_{i}(F(d,x)) \leq \lambda_{i} |x_{i} - x_{i}^{*}| + \sum_{j=1}^{n} p_{j,i}\mu_{j} |x_{j} - x_{j}^{*}| + f_{i}(d,x_{i})$$

$$\times \sum_{j=1}^{n} p_{i,j} \frac{c_{j} \max(0, x_{j} - \omega_{j}) + \sum_{k \neq i} p_{k,j}\mu_{k} |x_{k} - x_{k}^{*}|}{f_{j}^{*} + \sum_{k \neq i} p_{k,j}\mu_{k} |x_{k} - x_{k}^{*}| + p_{i,j} (f_{i}(d,x_{i}) - f_{i}^{*})}.$$
(3.26)

Using the facts that $x_j \in [\underline{b}_j, \overline{b}_j]$ and $\omega_j \ge x_j^*$, we obtain $\max(0, x_j - \omega_j) \le (\max(0, \overline{b}_j - \omega_j)/(\overline{b}_j - x_j^*))|x_j - x_j^*|$. Therefore, we obtain from (3.26) for all $(d, x) \in D \times A$ and $i = 1, \ldots, n$

$$V_{i}(F(d,x)) \leq \lambda_{i} |x_{i} - x_{i}^{*}| + \sum_{j=1}^{n} p_{j,i}\mu_{j} |x_{j} - x_{j}^{*}|$$

$$+ \sum_{j=1}^{n} \frac{p_{i,j}f_{i}(d,x_{i})c_{j}\max(0,\bar{b}_{j} - \omega_{j})}{\left(f_{j}^{*} + p_{i,j}\left(f_{i}(d,x_{i}) - f_{i}^{*}\right)\right)\left(\bar{b}_{j} - x_{j}^{*}\right)}$$

$$\times |x_{j} - x_{j}^{*}|$$

$$+ \sum_{j=1}^{n} \frac{p_{i,j}f_{i}(d,x_{i})}{f_{j}^{*} + p_{i,j}\left(f_{i}(d,x_{i}) - f_{i}^{*}\right)} \sum_{k \neq i} p_{k,j}\mu_{k}$$

$$\times |x_{k} - x_{k}^{*}|.$$
(3.27)

Finally, using definitions (3.20), the fact that $p_{i,j}f_i(d, x_i)/(f_j^* + p_{i,j}(f_i(d, x_i) - f_i^*)) \le p_{i,j}F_i/(f_j^* + p_{i,j}(F_i - f_i^*))$ for all $(d, x) \in D \times A$, where $F_i = \max\{f_i(d, s) : s \in [\underline{b}_i, \overline{b}_i], d \in D\}$ and the fact that $p_{i,i} = 0$, we obtain (2.3).

Example 3.3: Consider the traffic network shown in Fig. 1, for which the matrix $P = \{p_{i,j} : i, j = 1, ..., 5\}$ is

where $p, \tilde{p} > 0$ are constants with $p + \tilde{p} \leq 1$. The external inflows and the capacities of the network are

$$v_1 = v_2 = v_3 = v > 0, \quad v_4 = v_5 = \tilde{v} > 0,$$

$$c_i = 1, \quad a_i = a > 0 \quad (i = 1, \dots, 5)$$
(3.29)



Fig. 1. Traffic network of Example 3.3.

where $v, \tilde{v}, a > 0$ are constants. Finally, we assume that all functions $f_i (i = 1, ..., 5)$ are given by

$$f_i(x) = f(x) := \begin{cases} rx \text{ for } x \in [0, \delta] \\ r\delta - q(x - \delta) \text{ for } x \in (\delta, a] \end{cases} (i = 1, \dots, 5)$$
(3.30)

where $\delta \in (0, a)$, $r \in (0, 1]$, $q \in [0, \delta r/(a - \delta)]$ are constants. Note that the lower part of the right-hand side of (3.30) allows for the modeling of capacity drop at the outflow of congestion according to [18]. The network has the (uncongested) equilibrium point

$$x^* = (c, c, c, \kappa, \kappa) \tag{3.31}$$

where c := v/(r(1-p)), $\kappa := (\tilde{v}(1-p) + \tilde{p}v)/(r(1-p))$, which satisfies (H) provided that $c \le \delta$, $c(r+1) \le a$, $\kappa \le \delta$, and $(r+1)\kappa \le a$. We next apply Theorem 3.1 under the assumption

$$v + pr\delta \le a \text{ and } \widetilde{v} + \widetilde{p}r\delta \le a$$
 (3.32)

with $A = S = [0, a]^5$. Assumption (3.32) is assumption (3.17) for the given network. The matrix Γ is equal to

$$\Gamma = \begin{bmatrix} \lambda_1 & (a - \omega_2)\varphi & p\mu & 0 & 0\\ p\mu & \lambda_2 & (a - \omega_3)\varphi & 0 & (a - \omega_5)\zeta\\ (a - \omega_1)\varphi & p\mu & \lambda_3 & (a - \omega_4)\zeta & 0\\ 0 & 0 & \widetilde{p}\mu & \lambda_4 & 0\\ 0 & \widetilde{p}\mu & 0 & 0 & \lambda_5 \end{bmatrix}$$
(3.33)

where

$$\varphi = \frac{pr^2\delta(1-p)}{(pr\delta+v)(ar(1-p)-v)}$$

$$\zeta = \frac{r^2\delta\widetilde{p}(1-p)}{(\widetilde{v}+\widetilde{p}r\delta)(ar(1-p)-\widetilde{v}(1-p)-\widetilde{p}v)}$$

$$\mu := \sup\left\{r\frac{|v-(1-p)f(s)|}{|rs(1-p)-v|}: s \in [0,a], s \neq \frac{v}{r(1-p)}\right\}$$
(3.34)
$$\lambda_4 = \lambda_5 = \sup\left\{\frac{|s-\kappa-f(s)+\min(a-s,r\kappa)|}{|s-\kappa|}: s \in [0,a], s \neq \kappa\right\}$$

$$\lambda_4 = \max(u_i, w_i), \quad i = 1, 2, 3$$
(3.36)

where

h

 \widetilde{h}

$$\begin{aligned} (\omega, s) &:= \left(1 - p + \frac{\min\left(a - \omega, v + pf(s)\right)}{v + pf(s)}p\right)f(s) \\ (\omega, s) &:= \left(1 - \widetilde{p} + \frac{\min\left(a - \omega, \widetilde{v} + \widetilde{p}f(s)\right)}{\widetilde{v} + \widetilde{p}f(s)}\widetilde{p}\right)f(s) \\ g(s) &:= s - c + \min(a - s, cr), \quad B = [0, a] \setminus \{c\} \\ u_1 &:= \sup\left\{\frac{g(s) - h(\omega_2, s)}{|s - c|} : s \in B\right\} \\ w_1 &= \sup\left\{\frac{h(0, s) - g(s)}{|s - c|} : s \in B\right\} \\ u_2 &= \sup\left\{\frac{h(0, s) - g(s)}{|s - c|} : s \in B\right\} \\ w_2 &= w_3 = \sup\left\{\frac{h(0, s) - f(s) - h(\omega_3, s) - \widetilde{h}(\omega_5, s)}{|s - c|} : s \in B\right\} \\ u_3 &= \sup\left\{\frac{g(s) + f(s) - h(\omega_1, s) - \widetilde{h}(\omega_4, s)}{|s - c|} : s \in B\right\} \end{aligned}$$

and $\omega_i \in [c, a)$ $(i = 1, 2, 3), \omega_i \in [\kappa, a)$ (i = 4, 5) are constants. For $a = 10, v = 0.4, \tilde{v} = 0.4, \delta = 5, p = 0.2, \tilde{p} = 0.1, r = 0.55, q = 0.1$, the selection $\omega_1 = 9.14, \omega_2 = 8.53, \omega_3 = 9.559, \omega_4 = 9.37, \omega_5 = 9.329$ gives

$$\Gamma = \begin{bmatrix} 0.7905 & 0.0281 & 0.11 & 0 & 0 \\ 0.11 & 0.8166 & 0.0281 & 0 & 0.0298 \\ 0.0548 & 0.11 & 0.7905 & 0.028 & 0 \\ 0 & 0 & 0.055 & 0.7869 & 0 \\ 0 & 0.055 & 0 & 0 & 0.7869 \end{bmatrix}.$$

Since $\max_{i=1,...,5} (\sum_{j=1}^{5} \gamma_{i,j}) = 0.9845 < 1$, we can conclude that $\rho(\Gamma) < 1$ and, consequently, Theorem 3.1 implies that the (uncongested) equilibrium point is GES.

IV. GLOBAL EXPONENTIAL STABILITY FOR FREEWAYS

A freeway divided in $n \ge 3$ sections or cells is a traffic network of the form (3.7), (3.11) with $p_{i,j} = 0$ for all i, j = 1, \ldots, n with $j \ne i + 1$. Defining $p_{i,i+1} = p_i$ for $i = 1, \ldots, n-1$ and if we further suppose that $v_i = 0$ for $i = 2, \ldots, n, p_i = 1$, $f_i(d, x_i) = f_i(x_i)$ for $i = 1, \ldots, n, v_1 = v > 0$, we obtain from (3.5), (3.7), and (3.11)

$$x_1^+ = x_1 - \min\left(c_2(a_2 - x_2), f_1(x_1)\right) + \min\left(c_1(a_1 - x_1), v\right)$$
(4.1)

$$x_i^+ = x_i - \min(c_{i+1}(a_{i+1} - x_{i+1}), f_i(x_i))$$

+ min(c_i(a_i - x_i), f_{i-1}(x_{i-1})), for i = 2, ..., n - 1

$$x_n^+ = x_n - f_n(x_n) + \min\left(c_n(a_n - x_n), f_{n-1}(x_{n-1})\right). \quad (4.3)$$

Again, $c_i \in (0, 1]$, $f_i \in C^0([0, a_i]; \Re_+)$ (i = 1, ..., n) are functions with $f_i(s) \leq s$ for all $s \in [0, a_i]$. We suppose that there exists a vector $x^* = (x_1^*, ..., x_n^*) \in [0, a_1] \times ... [0, a_n]$ with $f_i(x_i^*) = v$ and $c_i x_i^* + v < c_i a_i$ (i = 1, ..., n). It follows that assumption (H) holds for the equilibrium point $x^* \in \Re^n$. The following corollary is a direct consequence of Theorem 3.1 (although Theorem 3.1 was applied to the model (3.7), (3.11) which required $v_i > 0$ for i = 1, ..., n, all arguments in the proof of Theorem 3.1 can be repeated).

Corollary 4.1: Consider (4.1), (4.2), and (4.3) with $n \ge 3$. Assume that there exist constants $0 \le \underline{b}_i < \overline{b}_i \le a_i$ $(i = 1, \ldots, n)$ such that the set $A = [\underline{b}_1, \overline{b}_1] \times \ldots \times [\underline{b}_n, \overline{b}_n]$ is a TR for (4.1), (4.2), and (4.3). Moreover, assume that there exists $L \ge 0$ such that the following inequalities hold for $i = 1, \ldots, n$:

$$|f_i(x) - v| \le L |x - x_i^*|$$
, for all $x \in [0, a_i]$. (4.4)

Furthermore, assume that there exist constants $\lambda_i \ge 0$ $(i = 1, ..., n), \ \mu_i \ge 0 \ (i = 1, ..., n - 1), \ \omega_i \in [x_i^*, a_i) \ (i = 2, ..., n)$ such that

$$|s - x_i^* - \min(c_{i+1}(a_{i+1} - \omega_{i+1}), f_i(s)) + \min(c_i(a_i - s), v)|$$

$$\leq \lambda_i |s - x_i^*|, \quad \text{for } s \in [\underline{b}_i, \overline{b}_i], \ i = 1, \dots, n-1$$
(4.5)

$$|s - x_i^* - \min(c_{i+1}a_{i+1}, f_i(s)) + \min(c_i(a_i - s), v)|$$

$$\leq \lambda_i |s - x_i^*|, \quad \text{for } s \in [\underline{b}_i, \overline{b}_i], \ i = 1, \dots, n-1$$
(4.6)

$$|s - x_n^* - f_n(s) + \min \left(c_n(a_n - s), v \right) | \le \lambda_n |s - x_n^*|,$$

for all $s \in [\underline{b}_n, \overline{b}_n]$ (4.7)

$$|v-f_i(s)| \le \mu_i |s-x_i^*|$$
, for all $s \in [\underline{b}_i, \overline{b}_i]$, $i=1,\ldots,n-1$.

(4.8)

Assume that $\max\{f_i(s) : s \in [\underline{b}_i, b_i]\} \leq c_{i+1}a_{i+1}$ for all $i = 1, \ldots, n-1$. Define the tridiagonal matrix $\Gamma = \{\gamma_{i,j} : i, j = 1, \ldots, n\}$

$$\gamma_{i,i} := \lambda_i, \text{ for } i = 1, \dots, n \tag{4.9}$$

$$\gamma_{i,i+1} := \frac{c_{i+1} \max(0, \bar{b}_{i+1} - \omega_{i+1})}{\bar{b}_{i+1} - x_i^*},$$

for
$$i = 1, \dots, n-1$$
 (4.10)

$$\gamma_{i,i-1} := \mu_{i-1}, \text{ for } i = 2, \dots, n.$$
 (4.11)

If $\rho(\Gamma) < 1$, then x^* is GES for (4.1), (4.2), and (4.3).

Corollary 4.1 shows that the TR is crucial for the stability properties of the system (4.1), (4.2), (4.3). Indeed, if $\omega_i \geq \overline{b}_i$ for $i = 2, \ldots, n$, then $\rho(\Gamma) = \max_{i=1,\ldots,n} (\lambda_i)$ (because, in this case, Γ is lower triangular). The crudest TR that can be used is $A = [0, a_1] \times \ldots [0, a_n]$. However, we can generate "smaller" TRs by means of the following proposition. Proposition 4.2: Suppose that there exist constants $0 \le \underline{b}_i < \overline{b}_i \le a_i$ (i = 1, ..., n) such that the set $A = [\underline{b}_1, \overline{b}_1] \times ... \times [\underline{b}_n, \overline{b}_n]$ is a TR for (4.1), (4.2), and (4.3) with $n \ge 3$. Let $i \in \{1, ..., n\}, \delta \in [0, \overline{b}_i]$ be a constant such that one of the following holds:

If
$$i = 1$$
 and $\delta \ge x_1^*$ then

$$\min_{\delta \le s \le \overline{b}_1} \left(\min\left(c_2(a_2 - \overline{b}_2), f_1(s)\right) - \min\left(c_1(a_1 - s), v\right) \right) > 0$$
and
$$\max_{\underline{b}_1 \le s \le \delta} \left(s - \min\left(c_2(a_2 - \overline{b}_2), f_1(s)\right) + \min\left(c_1(a_1 - s), v\right) \right) \le \delta.$$
(4.12)

If
$$i \in \{2, ..., n-1\}$$
 and $\delta \ge x_i^*$ then

$$\min_{\delta \le s \le \overline{b}_i} \left(\min \left(c_{i+1}(a_{i+1} - \overline{b}_{i+1}), f_i(s) \right) - \min \left(c_i(a_i - s), F_{i-1} \right) \right) > 0 \text{ and}$$

$$\max_{\underline{b}_i \le s \le \delta} \left(s - \min \left(c_{i+1}(a_{i+1} - \overline{b}_{i+1}), f_i(s) \right) + \min \left(c_i(a_i - s), F_{i-1} \right) \right) \le \delta,$$

where $F_{i-1} = \max \left\{ f_{i-1}(s) : s \in [\underline{b}_{i-1}, \overline{b}_{i-1}] \right\}.$ (4.13)

If
$$i = n$$
 and $\delta \ge x_n^*$ then

$$\min_{\delta \le s \le \overline{b}_n} (f_n(s) - \min(c_n(a_n - s), F_{n-1})) > 0$$
and
$$\max_{\underline{b}_n \le s \le \delta} (s - f_n(s) + \min(c_n(a_n - s), F_{n-1})) \le \delta,$$

where
$$F_{n-1} = \max\left\{f_{n-1}(s) : s \in [\underline{b}_{n-1}, \overline{b}_{n-1}]\right\}.$$
 (4.14)

Then, the set $B \subseteq A$, which results from the replacement of b_i by δ in the formula $[\underline{b}_1, \overline{b}_1] \times \ldots \times [\underline{b}_n, \overline{b}_n]$, is a TR for (4.1), (4.2), and (4.3).

Proof: We consider the case i = 1 (all other cases are similar). We consider the case $\delta < \overline{b}_1$. Notice that since $A = [\underline{b}_1, \overline{b}_1] \times \ldots \times [\underline{b}_n, \overline{b}_n]$ is a TR for (4.1), (4.2), and (4.3), there exists $m \ge 0$ such that for every $x_0 \in S$, the solution x(t) of (4.1), (4.2), and (4.3) with $x(0) = x_0$ satisfies $x(t) \in A$ for all $t \ge m$. Consequently, (4.1) implies for all $t \ge m$

$$x_1(t+1) \le x_1(t) - \min\left(c_2(a_2 - \bar{b}_2), f_1(x_1(t))\right) + \min\left(c_1(a_1 - x_1(t)), v\right). \quad (4.15)$$

It follows from (4.12) and (4.15) that if $x_1(t) \leq \delta$ for certain $t \geq m$, then $x_1(t+1) \leq \delta$. Thus, the following property holds:

(**P**): If there exists $T \ge m$ with $x_1(T) \le \delta$, then it holds that $x_1(t) \le \delta$ for all $t \ge T$.

Let $\varepsilon := \min_{\delta \le s \le \overline{b_1}} (\min(c_2(a_2 - \overline{b_2}), f_1(s)) - \min(c_1(a_1 - s), v)) > 0$. We claim that the solution x(t) of (4.1), (4.2), and (4.3) with arbitrary initial condition $x(0) = x_0 \in S$ satisfies $x_1(t) \le \delta$ for all $t \ge m + [(\overline{b_1} - \delta)/\varepsilon] + 1$. The proof is made by contradiction. Suppose that there exists $x_0 \in S$ and

 $t \geq m + [(\overline{b}_1 - \delta)/\varepsilon] + 1$ such that $x_1(t) > \delta$. Notice that property (P) guarantees that $x_1(j) > \delta$ for all $j = m, \ldots, t$. It follows from (4.15) and definition $\varepsilon := \min_{\delta \leq s \leq \overline{b}_1} (\min(c_2(a_2 - \overline{b}_2), f_1(s)) - \min(c_1(a_1 - s), v)) > 0$ that the following inequality holds for all $j = m, \ldots, t$:

$$x_1(j+1) \le x_1(j) - \varepsilon. \tag{4.16}$$

Inequality (4.16) implies that $x_1(t) \leq x_1(m) - (t-m)\varepsilon$. The previous inequality, in conjunction with $x_1(t) > \delta$ and the fact that $x_1(m) \leq \overline{b}_1$, implies $(t-m)\varepsilon < \overline{b}_1 - \delta$ which contradicts the fact that $t \geq m + [(\overline{b}_1 - \delta)/\varepsilon] + 1$.

Using Proposition 4.2 and Corollary 4.1, we can construct an algorithm that provides easily checkable sufficient conditions for the GES of x^* .

Corollary 4.3: Consider system (4.1), (4.2), and (4.3) with $n \ge 3$. Suppose that $0 < f_n(s)$ for all $s \in (0, a_n]$, $0 < f_i(s) < c_{i+1}a_{i+1}$ for all $s \in (0, a_i]$ and i = 1, ..., n-1. Perform the following algorithm:

Step 1: Find $k_n \in [x_n^*, a_n)$ such that $\min_{k_n \le s \le a_n} (f_n(s) - \min(c_n(a_n - s), F_{n-1})) > 0$ and $\max_{0 \le s \le k_n} (s - f_n(s) + \min(c_n(a_n - s), F_{n-1})) \le k_n$, where $F_{n-1} := \max_{s \in [0, a_{n-1}]} (f_{n-1}(s))$.

Step n+1-i, where $i \in \{2, \ldots, n-1\}$: Find $k_i \in [x_i^*, a_i)$ such that $\min_{k_i \le s \le a_i} (\min(c_{i+1}(a_{i+1}-k_{i+1}), f_i(s)) - \min(c_i(a_i-s), F_{i-1})) > 0$ and $\max_{0 \le s \le k_i} (s - \min(c_{i+1}(a_{i+1}-k_{i+1}), f_i(s)) + \min(c_i(a_i-s), F_{i-1})) \le k_i$, where $F_{i-1} := \max_{s \in [0, a_{i-1}]} (f_{i-1}(s))$.

Step n: Find $\overline{b}_1 \in [x_1^*, a_1)$ such that $\min_{\overline{b}_1 \le s \le a_1} (\min(c_2(a_2 - k_2), f_1(s)) - \min(c_1(a_1 - s), v)) > 0$ and $\max_{0 \le s \le \overline{b}_1} (s - \min(c_2(a_2 - k_2), f_1(s)) + \min(c_1(a_1 - s), v)) \le \overline{b}_1$.

Step n+i-1, where $i \in \{2, \ldots, n-1\}$: Find $\overline{b}_i \in [x_i^*, k_i]$ such that $\min_{\overline{b}_i \leq s \leq k_i} (\min(c_{i+1}(a_{i+1}-k_{i+1}), f_i(s)) - \min(c_i(a_i-s), F_{i-1})) > 0$ and $\max_{0 \leq s \leq \overline{b}_i} (s - \min(c_{i+1}(a_{i+1}-k_{i+1}), f_i(s)) + \min(c_i(a_i-s), F_{i-1})) \leq \overline{b}_i$, where $F_{i-1} := \max_{s \in [0, \overline{b}_{i-1}]} (f_{i-1}(s))$.

Step 2n - 1: Find $\overline{b}_n \in [x_n^*, k_n]$ such that $\min_{\overline{b}_n \le s \le k_n} (f_n(s) - \min(c_n(a_n - s), F_{n-1})) > 0$ and $\max_{0 \le s \le \overline{b}_n} (s - f_n(s) + \min(c_n(a_n - s), F_{n-1})) \le \overline{b}_n$, where $F_{n-1} := \max_{s \in [0, \overline{b}_{n-1}]} (f_{n-1}(s))$.

Assume that there exist $\lambda_i \in [0, 1)$ (i = 1, ..., n) such that

$$\begin{aligned} \left| s - x_i^* - \min(c_{i+1}(a_{i+1} - \overline{b}_{i+1}), f_i(s)) + \min(c_i(a_i - s), v) \right| \\ \leq \lambda_i \left| s - x_i^* \right|, \quad \text{for } s \in [0, \overline{b}_i], \ i = 1, \dots, n-1 \quad (4.17) \end{aligned}$$

$$s - x_i^* - \min(c_{i+1}a_{i+1}, f_i(s)) + \min(c_i(a_i - s), v))$$

$$\leq \lambda_i |s - x_i^*|, \quad \text{for } s \in [0, \overline{b}_i], \ i = 1, \dots, n-1 \quad (4.18)$$

$$|s - x_n^* - f_n(s) + \min(c_n(a_n - s), v)| \le \lambda_n |s - x_n^*|,$$

for all $s \in [0, \bar{b}_n].$ (4.19)

Finally, assume that inequalities (4.4) hold for certain constant $L \ge 0$. Then, x^* is GES for (4.1), (4.2), and (4.3).

All steps of the algorithm can be performed since $0 < f_n(s)$ for all $s \in (0, a_n]$, $0 < f_i(s) < c_{i+1}a_{i+1}$ for all $s \in (0, a_i]$ and $i = 1, \ldots, n-1$.

Example 4.4: Consider the network (4.1), (4.2), and (4.3) with n = 5, $c_i = 1$, $a_i = 10$ (i = 1, ..., 5)

$$f_i(s) = f(s) := \begin{cases} 0.5s \text{ for } s \in [0,5] \\ 3 - 0.1s \text{ for } s \in (5,10] \end{cases} \quad (i = 1, \dots, 4),$$

$$f_5(s) := \begin{cases} 0.4s \text{ for } s \in [0,5]\\ 2 - p(s-5) \text{ for } s \in (5,10] \end{cases}, \quad v = 1$$
(4.20)

where $p \in [0, 0.4)$. We have $x_i^* = 2$ (i = 1, ..., 4), $x_5^* = 2.5$ and (H) holds. We consider the following question: "For what values of $p \in [0, 0.4)$ x^* is GES?". The algorithm of Corollary 4.3 was performed for values of $p \in [0, 0.4)$ in the following way: for a given integer N > 0, a grid of points $s_i = ia/N$ (i = 0, 1, ..., N) was generated. Then, \overline{b}_i (i = 1, ..., 5) and k_i (i = 2, ..., 5) were chosen to be the smallest grid points $s_j = ja/N$ so that $\min_{j \le l \le N}(q(la/N)) > 0$ and $\max_{0 \le l \le j}(la - Nq(la/N)) < ja$, where:

- $q(s) := f_5(s) \min(10 s, 2.5)$ and F := 2.5 for k_5 ;
- $q(s) := \min(10 k_{i+1}, f(s)) \min(10 s, 2.5)$, for k_i (i = 4, 3, 2);
- $q(s) := \min(10 k_2, f(s)) \min(10 s, 1)$ for \overline{b}_1 ;

•
$$q(s) := \min(10 - k_{i+1}, f(s)) - \min(10 - s, F_{i-1})$$

 $F_{i-1} := \max_{s \in [0, \overline{b}_{i-1}]} (f(s)) \text{ for } \overline{b}_i \ (i = 2, 3, 4);$

• $q(s) := f_5(s) - \min(10 - s, F_4)$ and $F_4 := \max_{s \in [0, c_4]} (f(s))$, for \overline{b}_5 .

For all N > 0, there exists $p_N > 0$ such that the assumptions of Corollary 4.3 hold with $\lambda_i = 0.5$ (i = 1, ..., 4) and $\lambda_5 =$ 0.6 for all $p \in [0, p_N]$. We obtained $p_{100} = 0.189$, $p_{1000} =$ 0.244, $p_{2000} = 0.247$, indicating a sequence that tends to 0.25 as $N \to +\infty$. For p = 0.25, there exist additional equilibria and, therefore, x^* cannot be GES. The results show that the sufficient conditions of Corollary 4.3 are virtually exact in this case.

V. CONCLUDING REMARKS

Sufficient conditions for GAS and GES have been given, by means of vector Lyapunov functions. The results were applied to traffic networks for the derivation of sufficient conditions for GES of the uncongested equilibrium point. Specific results were provided for freeway models.

The results of this paper can be used for different purposes for future research:

- for the derivation of feedback laws which stabilize the uncongested equilibrium point;
- for the study of the dynamic behavior of traffic networks under the effect of external disturbances (varying inflows);
- for the study of complicated freeway models divided in n ≥ 3 cells, each with one on-ramp and one off-ramp.

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