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Predictor-based tracking for neuromuscular electrical stimulation‡

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SUMMARY

We present a new tracking controller for neuromuscular electrical stimulation (NMES), which is an emerging technology that artificially stimulates skeletal muscles to help restore functionality to human limbs. The novelty of our work is that we prove that the tracking error globally asymptotically and locally exponentially converges to zero for any positive input delay, coupled with our ability to satisfy a state constraint imposed by the physical system. Also, our controller only requires sampled measurements of the states instead of continuous measurements and allows perturbed sampling schedules, which can be important for practical purposes. Our work is based on a new method for constructing predictor maps for a large class of time-varying systems, which is of independent interest. Copyright © 2014 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Neuromuscular electrical stimulation (NMES) is a technology where skeletal muscles are artificially stimulated to help restore functionality to human limbs with motor neuron disorders [1, 2]. This is carried out using voltage excitation of skin or implanted electrodes, which produces muscle contraction, joint torque, and limb motion. NMES is an active area of research in biomedical and rehabilitation engineering, because it is key to developing neuroprosthetic devices. NMES control is challenging because of the nonlinear, time-varying, uncertain dynamics. The problem is compounded by the presence of time delays in the muscle response, due to finite propagation of chemical ions in the muscle, synaptic transmission delays, and other causes [1]. The simplest method for generating the desired limb motion is to apply the voltage signal via open-loop control using predefined stimulation schemes specific to the functionality being restored (e.g., walking) [3]. Not surprisingly, open-loop control was found to produce unsatisfactory results [3–6]. Despite this, most NMES controllers in clinical use are open loop [2, 6]. Classical feedback controllers (e.g., proportional integral derivative, or PID, control) have also produced unsatisfactory results [7], failing to guarantee closed-loop stability [6].

In parallel, considerable efforts have been devoted to understanding and modeling the nonlinear physiological and mechanical dynamics of muscle stimulation, activation, and contraction [3, 8–10].

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These models have enabled researchers to explore advanced, model-based feedback control methods to improve the effectiveness of NMES. Some work includes sliding mode control [6], adaptive control [11], neural network-based controllers [2, 12, 13], backstepping control [7, 14], and dynamic robust control [15].

Although previous efforts have advanced the field of nonlinear NMES control, the issue of compensation of time delays caused by the underlying (chemical) kinetics has received less attention. This is an important problem because of its potential destabilizing effect of delays on closed-loop stability [16]. Typically, the delay is modeled as an input delay to the musculoskeletal dynamics [1, 17] or to the muscle activation dynamics [6]. As noted in the significant work [1], most NMES controllers have not been designed to explicitly compensate for the time delay. Instead, some results have simply investigated the robustness of standard controllers to the input delay; see, for example, [6]. The first work to include time delay compensation in the design of the NMES control law was [1, 17]. In these papers, proportional derivative, or PD, and PID algorithms modified with a delay compensation term were designed using the predictor control approach [18]. In both, the tracking error for the knee joint angle was shown to be uniformly ultimately bounded using a Lyapunov–Krasovskii functional. Prediction uses dynamic controls to compensate arbitrarily long-time delays, and therefore may work better than delay compensating controllers that have upper bounds on the allowable delays (but see [19] for nonpredictive controls that compensate arbitrarily long input delays for nonlinear time-varying systems with no drift). See also [20] for inverse optimal NMES tracking without state constraints and without input delays.

There is a sizable literature on prediction. Although originally developed for linear systems, several recent papers provide predictive methods for nonlinear systems. See, for example, [21] for prediction of forward complete and feedforward systems, [22] for cases where the input delay is time-varying, and [23] for systems where the delay can depend on the state. However, it may be difficult to obtain continuous measurements of the state. Instead, only sampled state measurements may be available. To help overcome this challenge, [24] developed a predictor control design for time-invariant systems, based on sampled observations of the state and an Euler discretization scheme. Instead of continuous state observations, the controller in [24] uses an iterative process at each sampling time to find the control values for the next sampling interval.

In this paper, we introduce a related type of predictor control for time delay compensation in the NMES system. We consider the musculoskeletal dynamics for a human knee with an input delay as in [1, 2], but with the constraint that the knee joint angle cannot physically exceed certain limits. Our control is based on the hybrid, predictor feedback approach from [24]. Specifically, the approach in [24] is extended to account for the nonlinear time-varying nature of the NMES tracking control problem. Applying time-varying analogs of [24] to NMES is nontrivial, because one must first find a suitable nominal feedback and several other functions for the corresponding undelayed system. To overcome this challenge, we use a special change of coordinates that makes feedback linearization possible while ensuring that the physical state constraint on the knee position is satisfied. The control scheme uses sampled measurements and a numerical prediction of the state variables. Our control is model-based and ensures exponential tracking of the desired knee joint trajectory while satisfying the aforementioned state constraint under the input delay. This is an improvement over the existing significant NMES results [1, 17], which established the weaker ultimate boundedness condition on the tracking error under the input delay and which did not take the sampling and state constraint into account. Our result also guarantees robustness with respect to perturbations of the sampling schedule.

The rest of this paper is organized as follows. In Section 2, we present our notation. In Section 3, we review the NMES model and state our control objective. The control scheme and the main result are in Theorem 1. In Section 4, we state our general results on numerical approximation of solutions of time-varying systems, which generalize the corresponding results in [24] and use the step-size control ideas developed in [25]. The general results are used for the proof of our NMES tracking result, which is in Section 5. Section 6 demonstrates our controls in simulations, and Section 7 summarizes the value added by our work and suggests future research directions. The appendices prove certain claims and provide some formulas that are needed in the proofs of our theorems and for applying our hybrid controller.

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2. NOTATION AND DEFINITIONS

For each vector $x \in \mathbb{R}^n$, we let $|x|$ denote its usual Euclidean norm, and x' is its transpose. The norm $|\mathcal{M}|$ of a matrix $\mathcal{M} \in \mathbb{R}^{m\times n}$ is defined by $|\mathcal{M}| = \max\{|\mathcal{M}x| : x \in \mathbb{R}^n, |x| = 1\}$. We let $\mathbb{Z}_+$ denote the set of all non-negative integers. A partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $[0, +\infty)$ is any increasing sequence of times such that $T_0 = 0$ and $T_i \to +\infty$. For every real $x \geqslant 0$, we let $[x]$ denote its integer part, that is, $[x] = \max\{k \in \mathbb{Z}_+ : k \leqslant x\}$. An increasing continuous function $\gamma : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}$ provided $\gamma(0) = 0$. A class $\mathcal{K}$ function γ is of class of $\mathcal{K}_\infty$ provided $\lim_{s\to+\infty} \gamma(s) = +\infty$. By $\mathcal{KL}$, we denote the set of all continuous functions $\sigma : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that (i) for each $t \geqslant 0$, the mapping $\sigma(\cdot, t)$ is of class $\mathcal{K}$ and (ii) for each $s \geqslant 0$, the mapping $\sigma(s, \cdot)$ is non-increasing and satisfies $\lim_{t\to+\infty} \sigma(s, t) = 0$. Let m and n be any positive integers. Given any open subset $\mathcal{A} \subseteq \mathbb{R}^n$ and any integer $j \geqslant 0$, we let $C^j(\mathcal{A})$ denote the class of all functions having domain $\mathcal{A}$ that have continuous derivatives of order j. When we wish to restrict to functions taking values in a subset $\Omega \subseteq \mathbb{R}^m$, we denote the preceding set by $C^j(\mathcal{A}; \Omega)$. Given $x : [a - r, b) \to \mathbb{R}^n$ with $b > a \geqslant 0$ and $r \geqslant 0$, we let $\breve{T}_r(t)x$ denote the 'open history' of x from $t - r$ to t, that is, $\left(\breve{T}_r(t)x\right)(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0)$ and $t \in [a, b)$. Given any interval $I \subseteq [0, +\infty)$, we use $L^\infty(I; U)$ to denote the space of all measurable essentially bounded functions defined on I and taking values in $U \subseteq \mathbb{R}^m$. We also set $\|x\|_r = \sup_{\theta\in[-r,0)} |x(\theta)|$ for each function x. Notice that $\sup_{\theta\in[-r,0]} |x(\theta)|$ is not the essential supremum but the actual supremum and that is why the quantities $\sup_{\theta\in[-r,0]} |x(\theta)|$ and $\sup_{\theta\in[-r,0)} |x(\theta)|$ do not coincide in general. We use $|u|_{[a,b)}$ to denote the essential supremum of any function u over any interval $[a, b)$ in its domain. A function $h : \mathcal{A} \to \mathbb{R}$ where $0 \in \mathcal{A} \subseteq \mathbb{R}^n$ is called positive definite provided $h(0) = 0$ and $h(x) > 0$ for all $x \in \mathcal{A} \setminus \{0\}$. A function $h : \mathbb{R}^n \to \mathbb{R}$ is called radially unbounded provided that for each constant $M > 0$, the set $\{x \in \mathbb{R}^n : h(x) \leqslant M\}$ is bounded or empty. For any bounded function $\mathcal{F}$ defined on any subset of $\mathbb{R}$, we let $|\mathcal{F}|_\infty$ denote its supremum over its entire domain.

3. THE NMES MODEL AND MAIN RESULTS

Consider the following musculoskeletal model for a human using the leg extension machine from [1, 8]:

$$J\ddot{q} + b_1\dot{q} + b_2 \tanh(b_3\dot{q}) + k_1 q \exp(-k_2 q) + k_3 \tan(q) + mgl \sin(q) = U, \tag{1}$$

where $q(t) \in (-\pi/2, \pi/2)$ is the angular position of the lower leg limb about the knee joint, the positive constants J and m are the constant inertia and mass of the lower limb/machine combination, respectively, b_i and k_i for $i = 1, 2, 3$ are positive damping-related and elastic-related constants, respectively, g is the gravitational constant, and l is the distance between the knee joint and the center of the mass of the lower limb/machine. The leg extension machine in [1, 8] was designed with the user in sitting position such that the rest position for the free-swinging lower limb is $q = 0$. The quadricep muscles are stimulated using skin electrodes. We assume all system parameters are known (but see Section 6 for simulation results where they are uncertain). In [2], the $k_3 \tan(q)$ term is not present, because [2] does not consider the constraint on $q(t)$ that we will impose here. We use the term $k_3 \tan(q)$ to ensure forward completeness of the tracking system under a bounded torque. The control input U is the torque applied to the knee joint, and has the form $U = \zeta(q)\eta(q, \dot{q})\, v$, where $\zeta(q)$ is the known positive valued bounded moment arm, the positive valued bounded function $\eta(q, \dot{q})$ captures active and passive muscle characteristics and the dynamics of muscle recruitment, and v is the voltage potential across the quadriceps muscle applied through the electrodes.

We find it convenient to write the model in the form

$$\ddot{q}(t) = -\frac{dF}{dq}(q(t)) - H(\dot{q}(t)) + G(q(t), \dot{q}(t))\, v(t - \tau), \tag{2}$$

where q is valued in $(-\pi/2, \pi/2)$, v is valued in $\mathbb{R}$, $F : (-\pi/2, \pi/2) \to [0, +\infty)$ is a C^2 non-negative valued function satisfying $\lim_{q\to\pm\pi/2} F(q) = +\infty$, $H : \mathbb{R} \to \mathbb{R}$ is a C^1 function

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satisfying $\inf_{x \in \mathbb{R}} xH(x) \geqslant 0$, $G : (-\pi/2, \pi/2) \times \mathbb{R} \to (0, +\infty)$ is a C^1 positive valued and bounded function, and $\tau > 0$ is the constant time delay in the muscle response. The function $F : (-\pi/2, \pi/2) \to [0, +\infty)$ is the ratio of the potential and the inertia of the combined human shank-foot and machine, and $H : \mathbb{R} \to \mathbb{R}$ denotes the ratio of the viscous torque due to damping in the musculo-tendon complex and the inertia of the combined human shank-foot and machine. The preceding conditions hold using

$$\begin{aligned} F(q) &= \frac{mgl}{J}(1-\cos(q)) + \frac{k_1 \exp(-k_2 q)}{J k_2^2}(\exp(k_2 q) - 1 - k_2 q) + \frac{k_3}{J} \ln\left(\frac{1}{\cos(q)}\right), \\ H(\dot{q}) &= \frac{b_2}{J} \tanh(b_3 \dot{q}) + \frac{b_1}{J}\dot{q}, \text{ and } G(q, \dot{q}) = \frac{1}{J}\zeta(q)\eta(q, \dot{q}) \end{aligned} \tag{3}$$

The control objective is the asymptotic tracking of any desired C^2 reference signal $q_d(t)$ satisfying

$$\ddot{q}_d(t) = -\frac{dF}{dq}(q_d(t)) - H(\dot{q}_d(t)) + G(q_d(t), \dot{q}_d(t))\, v_d(t-\tau) \tag{4}$$

for all $t \geqslant 0$, where $v_d \in C^1([-\tau, +\infty); \mathbb{R})$, and such that the following holds:

$$\sup_{t \geqslant 0} |\dot{q}_d(t)| + \sup_{t \geqslant 0} |v_d(t-\tau)| + \sup_{t \geqslant 0} |\dot{v}_d(t-\tau)| < +\infty \text{ and } \sup_{t \geqslant 0} |q_d(t)| < \frac{\pi}{2} \tag{5}$$

The last inequality in (5) is the physical constraint that the knee cannot bend more than $\pm\pi/2$ from the straight down rest position $q = 0$.

We will design our NMES controller with the help of a time-varying analog of [24]. We use a hybrid predictor feedback control that guarantees global asymptotic and local exponential convergence of the tracking error to 0. Our controller does not require continuous measurement of the state variables but rather sampled measurements. The latter feature is important for practical purposes. To describe our results, we set

$$\zeta_{1,d}(t) = \tan(q_d(t)) \text{ and } \zeta_{2,d}(t) = \frac{\dot{q}_d(t)}{\cos^2(q_d(t))} \tag{6}$$

for all $t \geqslant 0$. We also define the function $\Omega : [0, +\infty) \times [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Omega(t_0, h, x; v) = \begin{bmatrix} \Omega_1(t_0, h, x; v) \\ \Omega_2(t_0, h, x; v) \end{bmatrix} \tag{7}$$

for all $(t_0, h, x) \in [0, +\infty) \times [0, +\infty) \times \mathbb{R}^2$ using the formulas

$$\begin{aligned} \Omega_1(t_0, h, x; v) &= x_1 + h x_2 \\ \Omega_2(t_0, h, x; v) &= x_2 + \int_{t_0}^{t_0+h} g_1(\zeta_d(s) + x)\mathrm{d}s + \int_{t_0}^{t_0+h} g_2(\zeta_d(s) + x) v(s-\tau)\mathrm{d}s \\ &\quad + \zeta_{2,d}(t_0) - \zeta_{2,d}(t_0 + h) \end{aligned} \tag{8}$$

where the input $v \in L^\infty([-\tau, +\infty); \mathbb{R})$ will be specified below after we introduce the necessary notation,

$$\begin{aligned} g_1(x) &= -\left(1 + x_1^2\right) \frac{dF}{dq}(\tan^{-1}(x_1)) + \frac{2x_1}{1 + x_1^2} x_2^2 - \left(1 + x_1^2\right) H\left(\frac{x_2}{1 + x_1^2}\right) \text{ and} \\ g_2(x) &= \left(1 + x_1^2\right) G\left(\tan^{-1}(x_1), \frac{x_2}{1 + x_1^2}\right). \end{aligned} \tag{9}$$

Let $\{T_i\}_{i=0}^{\infty}$ be any partition of $[0, +\infty)$ such that $\sup_{i \geqslant 0}(T_{i+1} - T_i) < +\infty$. Given any initial time $t_0 \geqslant 0$, the sampling times will be $t_0 + T_i$ for $i = 0, 1, 2, \ldots$. At each sampling time $t_0 + T_i$, we measure $(q(t_0 + T_i), \dot{q}(t_0 + T_i)) \in (-\pi/2, \pi/2) \times \mathbb{R}$. Next, we perform the calculation

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$$z_{k+1} = \Omega(t_0 + T_i + kh_i, h_i, z_k; v) \text{ for } k = 0, \ldots, N_i - 1, \text{ where}$$
$$z_0 = \left(\tan(q(t_0 + T_i)) - \tan(q_d(t_0 + T_i)), \frac{\dot{q}(t_0 + T_i)}{\cos^2(q(t_0 + T_i))} - \frac{\dot{q}_d(t_0 + T_i)}{\cos^2(q_d(t_0 + T_i))}\right)', \tag{10}$$

and where $N_i \geqslant 1$ is a sufficiently large integer (whose lower bound we specify in the succeeding text) and $h_i = \frac{\tau}{N_i}$. The preceding computations can be performed because they only require the values of v on the interval $[t_0 + T_i - \tau, t_0 + T_i)$. The control action $v(t)$ for $t \in [t_0 + T_i, t_0 + T_{i+1})$ is described by

$$v(t) = \frac{g_2(\zeta_d(t+\tau))v_d(t) - g_1(\zeta_d(t+\tau) + \xi(t)) + g_1(\zeta_d(t+\tau)) - (1+\mu^2)\xi_1(t) - 2\mu\xi_2(t)}{g_2(\zeta_d(t+\tau) + \xi(t))}, \tag{11}$$

where $\mu > 0$ is a constant and $\xi(t) \in \mathbb{R}^2$ is given by

$$\begin{aligned}
\xi_1(t) &= e^{-\mu(t-T_i-t_0)}\left((\xi_2(T_i + t_0) + \mu\xi_1(T_i + t_0))\sin(t - T_i - t_0)\right. \\
&\qquad \left. + \xi_1(T_i + t_0)\cos(t - T_i - t_0)\right) \\
\xi_2(t) &= e^{-\mu(t-T_i-t_0)}\left(-\left(\mu\xi_2(T_i + t_0) + (1+\mu^2)\xi_1(T_i + t_0)\right)\sin(t - T_i - t_0)\right. \\
&\qquad \left. + \xi_2(T_i + t_0)\cos(t - T_i - t_0)\right)
\end{aligned} \tag{12}$$

and

$$\xi(t_0 + T_i) = z_{N_i}. \tag{13}$$

The control scheme described by (10)–(13) is a combination of the following:

1. a numerical prediction of the error variables $x_1 = \tan(q) - \tan(q_d)$ and $x_2 = \dot{q}/\cos^2(q) - \dot{q}_d/\cos^2(q_d)$ at time $t_0 + T_i + \tau$ based on the knowledge of the state variables $(q(t_0 + T_i), \dot{q}(t_0 + T_i)) \in (-\pi/2, \pi/2) \times \mathbb{R}$, where the prediction is given by (13);
2. an intersample prediction of the error variables $x_1 = \tan(q) - \tan(q_d)$ and $x_2 = \dot{q}/\cos^2(q) - \dot{q}_d/\cos^2(q_d)$ for the time interval between two consecutive measurements, where the prediction is given by (12); and
3. the application of a nominal controller with the state variables replaced by their corresponding predicted values (predictor feedback), where the control action is given by (11).

See Section 5 for more details. The fact that $q(t_0 + T_i)$ in (10) stays in $(-\pi/2, \pi/2)$ will follow from our diffeomorphic mapping of the state space $(-\pi/2, \pi/2) \times \mathbb{R}$ for $(q, \dot{q})$ onto $\mathbb{R}^2$ and the fact that the system on $\mathbb{R}^2$ under this diffeomophism is forward complete. Our results are summarized in

Theorem 1

For all positive constants τ, r, and μ and for every signal $q_d : [0, +\infty) \to (-\pi/2, \pi/2)$ satisfying (4)–(5) for some reference input v_d, there exist a locally bounded mapping $N : [0, +\infty) \to \{1, 2, 3, \ldots\}$, a constant $\omega \in \left(0, \frac{\mu}{2}\right)$ and a locally Lipschitz, non-decreasing function $C : [0, +\infty) \to [0, +\infty)$ satisfying $C(0) = 0$ for which the following is true: for each partition $\{T_i\}$ of $[0, +\infty)$ such that $\sup_{i \geqslant 0}(T_{i+1} - T_i) \leqslant r$, each $t_0 \geqslant 0$, each $(q_0, \dot{q}_0) \in (-\pi/2, \pi/2) \times \mathbb{R}$, and each $v_0 \in L^\infty([-\tau, 0); \mathbb{R})$, the solution $(q(t), \dot{q}(t), v(t)) \in (-\pi/2, \pi/2) \times \mathbb{R}^2$ of the closed-loop system given by (2) and (10)–(13) with the choice

$$N_i = N\left(\left|\left(\tan(q(t_0 + T_i)) - \tan(q_d(t_0 + T_i)), \frac{\dot{q}(t_0 + T_i)}{\cos^2(q(t_0 + T_i))} - \frac{\dot{q}_d(t_0 + T_i)}{\cos^2(q_d(t_0 + T_i))}\right)\right|\right.$$
$$\left. + \sup\nolimits_{t_0 + T_i - \tau \leqslant s < t_0 + T_i} |v(s) - v_d(s)|\right) \tag{14}$$

and initial condition $(q(t_0), \dot{q}(t_0)) = (q_0, \dot{q}_0) \in (-\pi/2, \pi/2) \times \mathbb{R}$ and $v(t_0 + s) = v_0(s)$ for $s \in [-\tau, 0)$ exists for all $t \geqslant t_0$ and satisfies the inequality

$$\begin{aligned} &|q(t) - q_d(t)| + |\dot{q}(t) - \dot{q}_d(t)| + \sup_{t-\tau \leqslant s < t} |v(s) - v_d(s)| \\ &\leqslant \exp(-\omega(t - t_0)) C \left(\frac{|q_0 - q_d(t_0)| + |\dot{q}_0 - \dot{q}_d(t_0)|}{\cos^2(q_0)} + \sup_{-\tau \leqslant s < 0} |v_0(s) - v_d(t_0 + s)| \right) \end{aligned} \tag{15}$$

for all $t \geqslant t_0$.

Theorem 1 ensures robustness to perturbations of the sampling schedule, because (15) holds for all sampling schedules $\{t_0 + T_i : i = 1, 2, \ldots\}$ with $\sup_{i \geqslant 0}(T_{i+1} - T_i) \leqslant r$. Our proof is based on a general result for time-varying nonlinear systems, which is of independent interest. We turn to this general result next.

4. NUMERICAL APPROXIMATION OF SOLUTIONS OF TIME-VARYING FORWARD COMPLETE SYSTEMS

Our NMES control design is based on numerical predictions of solutions of the corresponding time-varying tracking dynamics, after a change of coordinates that transforms the state space to all of Euclidean space. The prediction is for a value of the NMES tracking error τ units into the future, where τ is the constant input delay. The work [24] gives a prediction method for solutions of time-invariant forward complete nonlinear systems, by introducing a special type of energy-like function W that only depends on the state. However, there is no obvious analog of the numerical approximation argument in [24] for time-varying systems, because [24] uses the time invariance of W in an essential way. In this section, we overcome this obstacle for time-varying systems. We study a large class of time-varying systems of the form

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{16}$$

where x is valued in $\mathbb{R}^n$ and u is valued in $\mathbb{R}^m$ for any dimensions m and n, under the following assumptions, which agree with the assumptions from [24] for time-invariant systems:

Assumption 1
The function $f : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, $f(t, 0, 0) = 0$ for all $t \geqslant 0$, and there is a continuous non-decreasing function $L : [0, +\infty) \to [1, +\infty)$ such that

$$|f(t, x, u) - f(t, y, u)| \leqslant L(|x| + |y| + |u|)\,|x - y| \tag{17}$$

$$\text{and } |f(t, x, u)| \leqslant (|x| + |u|) L(|x| + |u|) \tag{18}$$

hold for all $t \geqslant 0, x \in \mathbb{R}^n, y \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. □

Assumption 2
There are $W \in C^2([0, +\infty) \times \mathbb{R}^n; [1, +\infty))$ and $p \in \mathcal{K}_\infty$ and a constant $c > 0$ such that

$$\frac{\partial W}{\partial t}(t, x) + \frac{\partial W}{\partial x}(t, x) f(t, x, u) \;\leqslant\; c W(t, x) + p(|u|) \tag{19}$$

for all $t \geqslant 0, x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$. Also, there is a non-decreasing function $P \in C^0([0, +\infty); [0, +\infty))$ such that

$$\begin{aligned} P(s) \geqslant 1 + \sup \Bigg\{ &\left| \frac{\partial^2 W}{\partial t^2}(t, \xi) \right| + 2sL(s) \left| \frac{\partial^2 W}{\partial t \partial x}(t, \xi) \right| \\ &+ s^2 L^2(s) \left| \frac{\partial^2 W}{\partial x^2}(t, \xi) \right| : |\xi| \leqslant s(1 + \tau L(s)), t \geqslant 0 \Bigg\} \end{aligned} \tag{20}$$

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holds for all $s \geqslant 0$,

$$\left|\frac{\partial W}{\partial x}(t,x)\right| \leqslant \sqrt{P(|x|)} \tag{21}$$

holds for all $t \geqslant 0$ and $x \in \mathbb{R}^n$, and

$$|f(s,x,u) - f(t,x,u)| \leqslant (s-t)\sqrt{P(|x|+|u|)} \tag{22}$$

holds for all $t \geqslant 0, x \in \mathbb{R}^n, u \in \mathbb{R}^m$, and $s \geqslant t$. Also, for each constant $w \geqslant 0$, there is a non-decreasing, continuous function Q_w such that

$$\begin{aligned} Q_w(s) \geqslant 1 + \sup\Bigg\{&|x| : W(t+h,x) \leqslant \exp(2cw)\max_{|y|\leqslant s} W(t,y) \\ &+ \frac{1}{2c}\exp(2cw)p(s) \text{ holds for some } h \in [0,w] \text{ and some } t \geqslant 0\Bigg\} \end{aligned} \tag{23}$$

for all $s \geqslant 0$. □

Inequality (23) guarantees that for each $t \geqslant 0$, the mapping $x \mapsto W(t,x)$ is radially unbounded, because for each constant $\bar{M} \geqslant 0$, the set $\left\{x \in \mathbb{R}^n : W(t,x) \leqslant \bar{M}\right\}$ is empty or contained in a ball in $\mathbb{R}^n$ centered at zero with radius $Q_t(s)$, where $s = p^{-1}\left(2c\exp(-2ct)\bar{M}\right)$. To state our general result for (16), we use the following:

Lemma 1
For each constant $\tau > 0$, there is a function $a_\tau \in \mathcal{K}_\infty$ such that for each $t_0 \geqslant 0$, initial condition $x(t_0) = x_0$, and measurable bounded input $u : [t_0, t_0+\tau) \to \mathbb{R}^m$, the unique solution of (16) satisfies

$$|x(t)| \leqslant a_\tau(|x_0| + |u|_{[t_0,t_0+\tau)}) \text{ for all } t \in [t_0, t_0+\tau]. \tag{24}$$

Also, for each constant $\tau > 0$, there is a constant $M_\tau > 0$ such that $a_\tau(s) \leqslant M_\tau s$ for all $s \in [0,1]$.□

Proof
Choose any trajectory and input satisfying the assumptions, and set $||u|| = |u|_{[t_0,t_0+\tau)}$. By applying variation of parameters to (19), it follows that for each $t \in [t_0, t_0+\tau]$ where the solution exists, we get $W(t,x(t)) \leqslant \exp(2c\tau)W(t_0,x_0) + (2c)^{-1}\exp(2c\tau)p(\|u\|)$, so (23) implies that $|x(t)| \leqslant Q_\tau(|x_0|+\|u\|)$ and therefore also $L(|x(t)|+||u||) \leqslant L(Q_\tau(|x_0|+\|u\|)+||u||)$ for all $t \in [t_0, t_0+\tau]$ for which the solution exists. A standard contradiction argument (based on the existence of limits from the left for maximal solutions) shows that the solution exists for all $t \in [t_0, t_0+\tau]$. Therefore, (18) gives the following for all $t \in [t_0, t_0+\tau]$:

$$\begin{aligned} |x(t)| \leqslant |x_0| + L\left(Q_\tau(|x_0|+\|u\|)+\|u\|\right)\int_{t_0}^{t}|x(s)|\mathrm{d}s \\ + \|u\| L\left(Q_\tau(|x_0|+\|u\|)+\|u\|\right)(t-t_0) \end{aligned} \tag{25}$$

Then, the Gronwall–Bellman inequality implies that $a_\tau(s) = s(1+L(Q_\tau(s)+s)\tau)\exp(\tau L(Q_\tau(s)+s))$ satisfies our requirements. This readily gives the constant M_τ. □

For all $s \geqslant 0$, we can therefore define

$$A(s) = L\left(Q_\tau(s) + a_\tau(s) + s\right) \text{ and } B(s) = A(s)(a_\tau(s)+s)L(a_\tau(s)+s). \tag{26}$$

Consider the following numerical scheme, which is an extension of the explicit Euler method to time-varying systems with inputs. We select a positive integer N and define

$$x_{i+1} = x_i + \int_{t_0+ih}^{t_0+(i+1)h} f(s,x_i,u(s))\mathrm{d}s \text{ for } i = 0,1,\ldots,N-1, \tag{27}$$

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where $h = \frac{\tau}{N}$, and $x_0 \in \mathbb{R}^n$ and $u : [t_0, t_0 + \tau) \to \mathbb{R}^m$ are given. We still set $||u|| = |u|_{[t_0, t_0+\tau)}$ for brevity.

Theorem 2
Consider the system (16) under Assumptions 1–2. Let τ be any positive constant, and choose any $x_0 \in \mathbb{R}^n$ and $t_0 \geqslant 0$ and any measurable bounded input $u : [t_0, t_0 + \tau) \to \mathbb{R}^m$. If

$$N \geqslant \tau \frac{P\left(Q_\tau(|x_0| + \|u\|) + \|u\|\right)}{c}, \tag{28}$$

then

$$|x(t_0 + \tau) - x_N| \leqslant \frac{\tau B(|x_0| + \|u\|)}{2NA(|x_0| + \|u\|)} \left(\exp(\tau A(|x_0| + \|u\|)) - 1\right) \tag{29}$$

$$\text{and } |x_i| \leqslant Q_\tau(|x_0| + \|u\|) \tag{30}$$

hold for all $i = 0, 1, \ldots, N$, where $x(t)$ is the solution of (16) with initial condition $x(t_0) = x_0$ corresponding to input $u : [t_0, t_0 + \tau) \to \mathbb{R}^m$ at time t, and the constant $c > 0$ is from (19). □

We next give three technical lemmas, which we then use to prove Theorem 2.

Lemma 2
Consider (16) under the assumptions of Theorem 2. Then, the following holds for each $i \in \{0, 1, \ldots, N - 1\}$: If $h \leqslant cW(t_0 + ih, x_i)/P(|x_i| + \|u\|)$, where P is the function satisfying (20)–(22), then

$$W(t_0+(i+1)h, x_{i+1}) \leqslant \exp(2ch)W(t_0+ih, x_i)+\int_{t_0+ih}^{t_0+(i+1)h} \exp(2c(t_0+(i+1)h-s))\,p(|u(s)|)\mathrm{d}s \tag{31}$$

holds. □

Proof
Given $i \in \{0, 1, \ldots, N - 1\}$, we define the function $g(\lambda) = W\left(t_0 + ih + \lambda h, x_i + \lambda(x_{i+1} - x_i)\right)$ for all $\lambda \in [0, 1]$. The following equalities hold for all $\lambda \in [0, 1]$:

$$\begin{aligned}
\frac{dg}{d\lambda}(\lambda) &= h\frac{\partial W}{\partial t}\left(t_0 + ih + \lambda h, x_i + \lambda(x_{i+1} - x_i)\right) \\
&\quad + \frac{\partial W}{\partial x}\left(t_0 + ih + \lambda h, x_i + \lambda(x_{i+1} - x_i)\right)(x_{i+1} - x_i) \\
\frac{d^2 g}{d\lambda^2}(\lambda) &= 2h\frac{\partial^2 W}{\partial t \partial x}\left(t_0 + ih + \lambda h, x_i + \lambda(x_{i+1} - x_i)\right)(x_{i+1} - x_i) \\
&\quad + h^2\frac{\partial^2 W}{\partial t^2}\left(t_0 + ih + \lambda h, x_i + \lambda(x_{i+1} - x_i)\right) \\
&\quad + (x_{i+1} - x_i)'\frac{\partial^2 W}{\partial x^2}\left(t_0 + ih + \lambda h, x_i + \lambda(x_{i+1} - x_i)\right)(x_{i+1} - x_i)
\end{aligned} \tag{32}$$

Moreover, notice that (18), (27), and the fact $h \leqslant \tau$ imply that $|x_{i+1} - x_i| \leqslant h(|x_i| + \|u\|)L(|x_i| + \|u\|)$ and $|x_i + \lambda(x_{i+1} - x_i)| \leqslant (|x_i| + \|u\|)(1 + \tau L(|x_i| + \|u\|))$. The previous inequalities, (20), and (32) give

$$\left|\frac{d^2 g}{d\lambda^2}(\lambda)\right| \leqslant h^2 P(|x_i| + \|u\|) \text{ for all } \lambda \in [0, 1]. \tag{33}$$

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Also, inequality (19) in conjunction with (27), (21), (22), and (32) gives

$$\begin{aligned}\frac{dg}{d\lambda}(0) &= h\frac{\partial W}{\partial t}(t_0+ih,x_i)+\int_{t_0+ih}^{t_0+(i+1)h}\frac{\partial W}{\partial x}(t_0+ih,x_i)f(s,x_i,u(s))\mathrm{d}s\\ &\leqslant chW(t_0+ih,x_i)+\int_{t_0+ih}^{t_0+(i+1)h}p(|u(s)|)\mathrm{d}s\\ &\quad+\int_{t_0+ih}^{t_0+(i+1)h}\frac{\partial W}{\partial x}(t_0+ih,x_i)\left(f(s,x_i,u(s))-f(t_0+ih,x_i,u(s))\right)\mathrm{d}s\\ &\leqslant chW(t_0+ih,x_i)+\int_{t_0+ih}^{t_0+(i+1)h}p(|u(s)|)\mathrm{d}s+\frac{h^2}{2}P(|x_i|+\|u\|)\end{aligned} \tag{34}$$

Combining (33)–(34) gives

$$\begin{aligned}W(t_0+(i+1)h,x_{i+1})=g(1)\;=\;g(0)+\int_0^1\left[\frac{dg}{d\lambda}(s)-\frac{dg}{d\lambda}(0)\right]\mathrm{d}s+\frac{dg}{d\lambda}(0)\\ \leqslant(1+ch)W(t_0+ih,x_i)+\int_{t_0+ih}^{t_0+(i+1)h}p(|u(s)|)\mathrm{d}s\\ +h^2P(|x_i|+\|u\|),\end{aligned} \tag{35}$$

by applying the mean value theorem to $\frac{dg}{d\lambda}$ on $[0,s]$ for all $s\in[0,1]$. Inequality (35) and the fact that

$$\begin{aligned}&(1+ch)W(t_0+ih,x_i)+\int_{t_0+ih}^{t_0+(i+1)h}p(|u(s)|)\mathrm{d}s+h^2P(|x_i|+\|u\|)\\ &\leqslant\exp(2ch)W(t_0+ih,x_i)+\int_{t_0+ih}^{t_0+(i+1)h}\exp(2c(t_0+(i+1)h-s))p(|u(s)|)\mathrm{d}s\end{aligned} \tag{36}$$

holds for all $h\leqslant\frac{cW(t_0+ih,x_i)}{P(|x_i|+\|u\|)}$ imply that (31) holds. The proof is complete. □

The proof of the preceding lemma differs significantly from the time-invariant result from [24] because it uses our more complicated function P. See Section A for the P needed for our NMES control. The next two lemmas are closer to the time-invariant analogs from [24], but we include them for completeness.

Lemma 3
Let the assumptions of Theorem 2 hold. If $h\leqslant c/P(Q_\tau(|x_0|+\|u\|)+\|u\|)$, then

$$W(t_0+ih,x_i)\leqslant\exp(2cih)W(t_0,x_0)+\int_{t_0}^{t_0+ih}\exp(2c(t_0+ih-s))p(|u(s)|)\mathrm{d}s \tag{37}$$

holds for all $i=0,\dots,N$, where $Q_\tau:[0,+\infty)\to[0,+\infty)$ is the function involved in (23) for $w=\tau$. □

Proof
We prove (37) by induction. First notice that (37) holds for $i=0$. Suppose (37) holds for some $i\in\{0,\dots,N-1\}$. Then, $W(t_0+ih,x_i)\leqslant\exp(2c\tau)W(t_0,x_0)+\exp(2c\tau)p(\|u\|)/(2c)$, because $h=\tau/N$. The previous inequality in conjunction with (23) gives

$$|x_i|\leqslant Q_\tau(|x_0|+\|u\|). \tag{38}$$

Hence, because P is non-decreasing and $W(t_0+ih,x_i)\geqslant 1$, we get $h\leqslant cW(t_0+ih,x_i)/P(|x_i|+\|u\|)$. Therefore, (31) holds. Substituting $W(t_0+ih,x_i)$ in the right side of (31) by the right side of (37), and then collecting terms, we conclude that (37) holds with i replaced by $i+1$. This proves the lemma. □

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Lemma 4
Consider system (16) under the assumptions of Theorem 2. Given any initial condition $x(t_0) = x_0$ and any input $u : [t_0, t_0 + \tau) \to \mathbb{R}^m$, define $e_i = x_i - x(t_0 + ih)$ for $i \in \{0, \dots, N\}$, where $x(t)$ is the corresponding solution of (2). Assume that $h \leqslant c/P(Q_\tau(|x_0| + \|u\|) + \|u\|)$. Then,

$$|e_i| \leqslant \frac{h^2}{2} B(|x_0| + \|u\|) \frac{\exp(ihA(|x_0| + \|u\|)) - 1}{\exp(hA(|x_0| + \|u\|)) - 1} \tag{39}$$

for all $i \in \{1, \dots, N\}$, where A and B are from (26). □

Proof
The following equation holds for all $i \in \{0, \dots, N-1\}$, as a direct consequence of (27):

$$e_{i+1} = e_i + \int_{t_0+ih}^{t_0+(i+1)h} (f(s, x_i, u(s)) - f(s, x(s), u(s)))\, \mathrm{d}s \tag{40}$$

Using the definition $e_i = x_i - x(t_0 + ih)$ and inequalities (18) and (24), and then noting that $|x_i - x(s)| \leqslant |e_i| + |x(s) - x(t_0 + ih)|$ for all $i \in \{0, \dots, N-1\}$ and all $s \in [t_0 + ih, t_0 + (i+1)h]$, we get

$$|x_i - x(s)| \leqslant |e_i| + (s - t_0 - ih)(a_\tau(|x_0| + \|u\|) + \|u\|)L(a_\tau(|x_0| + \|u\|) + \|u\|) \tag{41}$$

Notice that all hypotheses of Lemma 3 hold. Therefore, inequality (37) holds for all $i = 0, \dots, N$. Recall that (37) implies that (38) holds for all $i = 0, \dots, N$. Because Lemma 1 gives $|x(t)| \leqslant a_\tau(|x_0| + ||u||)$ for all $t \in [t_0, t_0 + \tau]$, we can therefore conclude from (40)–(41) that the following holds for all $i \in \{0, \dots, N-1\}$:

$$\begin{aligned}
|e_{i+1}| &\leqslant |e_i| + hL\left(Q_\tau(|x_0| + \|u\|) + a_\tau(|x_0| + \|u\|) + \|u\|\right)|e_i| \\
&\quad + \frac{h^2}{2} L\left(Q_\tau(|x_0| + \|u\|) + a_\tau(|x_0| + \|u\|) + \|u\|\right) \\
&\quad \times (a_\tau(|x_0| + \|u\|) + \|u\|)L\left(a_\tau(|x_0| + \|u\|) + \|u\|\right) \\
&\leqslant |e_i| + hA(|x_0| + \|u\|)|e_i| + \frac{h^2}{2} B(|x_0| + \|u\|),
\end{aligned}$$

by the definitions (26) of A and B, where we also used (17). This gives the recursive relation

$$|e_{i+1}| \leqslant \exp(hA(|x_0| + \|u\|))|e_i| + \frac{h^2}{2} B(|x_0| + \|u\|) \tag{42}$$

for all $i \in \{0, \dots, N-1\}$. Using (42) and the fact $e_0 = 0$ gives (39). The proof is complete. □

We can now prove Theorem 2. All assumptions of Lemmas 3–4 hold. Consequently, (37) and (39) hold. Inequality (29) follows from using the fact $\exp(hA(|x_0| + \|u\|)) - 1 \geqslant hA(|x_0| + \|u\|)$ and definition $h = \frac{\tau}{N}$ in conjunction with (39) for $i = N$. Moreover, inequality (37) implies (38), which is (30). This completes the proof of Theorem 2.

Theorem 2 allows us to construct mappings that approximate values of the solutions of (16) τ time units ahead with guaranteed accuracy as follows. Let $R : [0, +\infty) \to [0, +\infty)$ be a continuous positive definite function with $\liminf_{s \to 0^+} R(s)/s > 0$. Define the mapping $\Phi_{t_0} : \mathbb{R}^n \times L^\infty([t_0, t_0 + \tau); \mathbb{R}^m) \to \mathbb{R}^n$ by

$$\Phi_{t_0}(x_0, u) = x_N, \tag{43}$$

where x_i for $i = 1, \dots, N$ is defined by (27) with $h = \frac{\tau}{N}$ and N is as follows. We set

$$N(s) = \left[\tau \max\left\{\frac{a_\tau(s) + s}{2R(s)} L(a_\tau(s) + s)(\exp(\tau A(s)) - 1)\,,\, \frac{P(Q_\tau(s) + s)}{c}\right\}\right] + 1. \tag{44}$$

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for all $s > 0$ and $N(0) = 1$. Inequality (29) implies that the mapping Φ_{t_0} from (43) satisfies

$$\left|\Phi_{t_0}(x_0, u) - x(t_0 + \tau)\right| \leqslant R(|x_0| + \|u\|) \tag{45}$$

for all choices of $x_0 = x(t_0), u$, and t_0. Also, inequalities (29)–(30) in conjunction with (45) and (24) give

$$\left|\Phi_{t_0}(x_0, u)\right| \leqslant \min\{Q_\tau(|x_0| + \|u\|), R(|x_0| + \|u\|) + a_\tau(|x_0| + \|u\|)\}. \tag{46}$$

Notice that the mapping N in (44) is locally bounded. Indeed, there exists a constant $M_\tau > 0$ such that $a_\tau(s) \leqslant M_\tau s$ for all $s \in [0, 1]$. Therefore, continuity of all functions involved in (44) in conjunction with the fact that $\liminf_{s\to 0^+} R(s)/s > 0$ gives the local boundedness. We conclude as follows:

Corollary 1

Consider system (16) under the assumptions of Theorem 2. For every positive definite function $R \in C^0([0, +\infty); [0, +\infty))$ satisfying $\liminf_{s\to 0^+} R(s)/s > 0$ and for every constant $\tau > 0$, consider the mapping $\Phi_{t_0} : \mathbb{R}^n \times L^\infty([t_0, t_0 + \tau); \mathbb{R}^m) \to \mathbb{R}^n$ defined by (43), where x_i for $i = 1, \ldots, N$ is defined by the numerical scheme (27) with $h = \tau/N(|x_0| + ||u||)$ and N is defined in (44). Then, (45)–(46) hold for all $t_0 \geqslant 0$ and $(x_0, u) \in \mathbb{R}^n \times L^\infty([t_0, t_0 + \tau); \mathbb{R}^m)$, where $x(t)$ is the solution of (16) with initial condition $x(t_0) = x_0$ corresponding to input $u : [t_0, t_0 + \tau) \to \mathbb{R}^m$ and $\|u\| = |u|_{[t_0,t_0+\tau)}$. Also, N is locally bounded. □

5. PROOF OF THEOREM 1

We now use Theorem 2 and Corollary 1 to prove Theorem 1. The proof of Theorem 1 is constructive and formulas will be given for the locally bounded mapping $N : [0, +\infty) \to \{1, 2, 3, \ldots\}$ involved in the hybrid dynamic feedback law defined by (10)–(13). We first perform the following change of coordinates, which is beneficial because it maps the restricted state space $(-\pi/2, \pi/2) \times \mathbb{R}$ onto all of $\mathbb{R}^2$:

$$\zeta_1 = \tan(q) \text{ and } \zeta_2 = \frac{\dot{q}}{\cos^2(q)} \tag{47}$$

Then, $\cos^2(q) = \frac{1}{1+\zeta_1^2}$ and $\dot{q} = \frac{\zeta_2}{1+\zeta_1^2}$. Hence, (47) and (6) give

$$\begin{aligned} \dot{\zeta}_1(t) &= \zeta_2(t) \\ \dot{\zeta}_2(t) &= g_1(\zeta(t)) + g_2(\zeta(t))v(t - \tau) \end{aligned} \tag{48}$$

where $\zeta(t) = (\zeta_1(t), \zeta_2(t)) \in \mathbb{R}^2$ and $v(t) \in \mathbb{R}$, and

$$\begin{aligned} \dot{\zeta}_{1,d}(t) &= \zeta_{2,d}(t) \\ \dot{\zeta}_{2,d}(t) &= g_1(\zeta_d(t)) + g_2(\zeta_d(t))v_d(t - \tau), \end{aligned} \tag{49}$$

where

$$\begin{aligned} g_1(\zeta) &= -\left(1 + \zeta_1^2\right)\frac{dF}{dq}\left(\tan^{-1}(\zeta_1)\right) + \frac{2\zeta_1}{1 + \zeta_1^2}\zeta_2^2 - \left(1 + \zeta_1^2\right)H\left(\frac{\zeta_2}{1 + \zeta_1^2}\right) \text{ and} \\ g_2(\zeta) &= \left(1 + \zeta_1^2\right)G\left(\tan^{-1}(\zeta_1), \frac{\zeta_2}{1 + \zeta_1^2}\right). \end{aligned} \tag{50}$$

Next, we set

$$x(t) = \zeta(t) - \zeta_d(t) \text{ and } u(t) = v(t) - v_d(t). \tag{51}$$

Then, (48)–(49) give

$$\dot{x}(t) = f(t, x(t), u(t - \tau)), \tag{52}$$

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where $x(t) = (x_1(t), x_2(t))$ is valued in $\mathbb{R}^2$ and $u(t)$ is valued in $\mathbb{R}$,

$$\begin{aligned} f(t,x,u) &= \begin{pmatrix} x_2 \\ \tilde{f}(t,x) + \tilde{g}(t,x)u \end{pmatrix}, \\ \tilde{f}(t,x) &= g_1(\zeta_d(t)+x) - g_1(\zeta_d(t)) + (g_2(\zeta_d(t)+x) - g_2(\zeta_d(t)))v_d(t-\tau), \text{ and} \\ \tilde{g}(t,x) &= g_2(\zeta_d(t)+x). \end{aligned} \tag{53}$$

Notice that $f(t,0,0) = 0$ for all $t \geqslant 0$. To simplify the procedure of the proof, we break the proof up into three steps. In the first step, we show that the time-varying system (52) satisfies Assumptions 1–2 of Section 4. The second step is the construction of $N : [0,+\infty) \to \{1,2,3,\ldots\}$. The third step is the rest of the proof.

5.1. First step of proof of Theorem 1

The fact that (52) satisfies Assumption 1 is a direct consequence of the definitions (53) and our bounds (5) for the reference trajectory and reference input. To check Assumption 2, define the function

$$W(t,x) = 1 + \frac{1}{2}\left(\frac{\zeta_{2,d}(t)+x_2}{1+(\zeta_{1,d}(t)+x_1)^2}\right)^2 + F\left(\tan^{-1}(\zeta_{1,d}(t)+x_1)\right) \tag{54}$$

Because $F : (-\pi/2, \pi/2) \to [0,+\infty)$ is C^2, it follows that $W : [0,+\infty) \times \mathbb{R}^2 \to [1,+\infty)$ is a C^2 function. The formulas (53) for $\tilde{f}$ and $\tilde{g}$ give the following for all $t \geqslant 0, x \in \mathbb{R}^2$, and $u \in \mathbb{R}$:

$$\begin{aligned} &\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x_1}(t,x)x_2 + \frac{\partial W}{\partial x_2}(t,x)\left(\tilde{f}(t,x)+\tilde{g}(t,x)u\right) \\ &= -\frac{\zeta_{2,d}(t)+x_2}{1+(\zeta_{1,d}(t)+x_1)^2} H\left(\frac{\zeta_{2,d}(t)+x_2}{1+(\zeta_{1,d}(t)+x_1)^2}\right) \\ &\quad + \frac{\zeta_{2,d}(t)+x_2}{1+(\zeta_{1,d}(t)+x_1)^2} G\left(\tan^{-1}(\zeta_{1,d}(t)+x_1), \frac{\zeta_{2,d}(t)+x_2}{1+(\zeta_{1,d}(t)+x_1)^2}\right)(v_d(t-\tau)+u) \end{aligned} \tag{55}$$

Equation (55) follows because $W(t,x) = 1 + \frac{1}{2}\dot{q}^2 + F(q)$. Using the fact that $\inf_{x\in\mathbb{R}} xH(x) \geqslant 0$, the fact that $G : (-\pi/2,\pi/2) \times \mathbb{R} \to [0,+\infty)$ is bounded and the fact $W(t,x) \geqslant 1$ for all $(t,x) \in [0,+\infty) \times \mathbb{R}^2$, we get the following from Equation (55) for all $t \geqslant 0, x \in \mathbb{R}^2$ and $u \in \mathbb{R}$:

$$\begin{aligned} &\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x_1}(t,x)x_2 + \frac{\partial W}{\partial x_2}(t,x)\left(\tilde{f}(t,x)+\tilde{g}(t,x)u\right) \\ &\leqslant \frac{1}{2}\left(\frac{\zeta_{2,d}(t)+x_2}{1+(\zeta_{1,d}(t)+x_1)^2}\right)^2 + \frac{1}{2}\tilde{G}^2(v_d(t-\tau)+u)^2 \\ &\leqslant \frac{1}{2}\left(\frac{\zeta_{2,d}(t)+x_2}{1+(\zeta_{1,d}(t)+x_1)^2}\right)^2 + \tilde{G}^2 v_d^2(t-\tau) + \tilde{G}^2 u^2 \\ &\leqslant \frac{1}{2}\left(\frac{\zeta_{2,d}(t)+x_2}{1+(\zeta_{1,d}(t)+x_1)^2}\right)^2 + \tilde{G}^2 v_d^2(t-\tau)W(t,x) + \tilde{G}^2 u^2 \end{aligned} \tag{56}$$

where $\tilde{G} = \sup\{G(q,\dot{q}) : (q,\dot{q}) \in (-\pi/2,\pi/2) \times \mathbb{R}\}$. The aforementioned inequality in conjunction with the definition (54) of W implies that inequality (19) holds with

$$c = 1 + \tilde{G}^2|v_d|_\infty^2 \text{ and } p(s) = \tilde{G}^2 s^2. \tag{57}$$

The fact that there exists a continuous, non-decreasing function $P : [0,+\infty) \to [0,+\infty)$ such that inequalities (20), (21), and (22) hold is shown in Appendix A.

We next turn to satisfying (23). Because $F : (-\pi/2,\pi/2) \to [0,+\infty)$ is a C^2, non-negative function that satisfies $\lim_{q\to\pm\pi/2} F(q) = +\infty$, it follows that the function $\tilde{W} : \mathbb{R}^2 \to \mathbb{R}$ defined by

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$$\tilde{W}(x) = 1 + \frac{1}{2}\left(\frac{x_2}{1+x_1^2}\right)^2 + F\left(\tan^{-1}(x_1)\right) \tag{58}$$

is a C^1, positive valued, radially unbounded function. Consequently, there are $\mathcal{K}_\infty$ functions θ_i for $i = 1, 2$ and a constant $R_2 \geqslant 0$ such that

$$\theta_1(|x|) \leqslant \tilde{W}(x) \leqslant R_2 + \theta_2(|x|) \text{ for all } x \in \mathbb{R}^2. \tag{59}$$

Notice that $W(t,x) = \tilde{W}(\zeta_d(t) + x)$ holds for all $(t,x) \in [0,+\infty) \times \mathbb{R}^2$. Hence, (59) gives

$$|x| \leqslant \theta_1^{-1}(W(t,x)) + |\zeta_d|_\infty \text{ and } W(t,x) \leqslant R_2 + \theta_2(|\zeta_d|_\infty + |x|) \tag{60}$$

for all $(t,x) \in [0,+\infty) \times \mathbb{R}^2$. Inequalities (60) imply that (23) holds with

$$Q_w(s) = 1 + \theta_1^{-1}\left(\exp(2cw)(R_2 + \theta_2(s + |\zeta_d|_\infty)) + (2c)^{-1}\exp(2cw)p(s)\right) + |\zeta_d|_\infty \tag{61}$$

To see why, notice that for each $t \geqslant 0$ and $s \geqslant 0$, (60) gives $\max_{|y| \leqslant s} W(t,y) \leqslant R_2 + \theta_2(|\zeta_d|_\infty + s)$ and consequently, the condition

$$W(t+h,x) \leqslant \exp(2cw)\max_{|yt| \leqslant s} W(t,y) + (2c)^{-1}\exp(2cw)p(s) \tag{62}$$

implies that $W(t+h,x) \leqslant \exp(2cw)(R_2 + \theta_2(s + |\zeta_d|_\infty)) + (2c)^{-1}\exp(2cw)p(s)$ for all $h \in [0,w]$. The choice of Q_w therefore follows from the first inequality in (60).

5.2. *Second step of proof of Theorem 1*

For all $(t,x) \in [0,+\infty) \times \mathbb{R}^2$, we set

$$V(x) = \frac{2}{\mu^2 + 2 - \mu\sqrt{\mu^2+4}}\left(x_1^2 + (x_2 + \mu x_1)^2\right) \tag{63}$$

and

$$k(t,x) = -\frac{(1+\mu^2)x_1 + 2\mu x_2 + \tilde{f}(t,x)}{\tilde{g}(t,x)}, \tag{64}$$

which we later use to define our control in our new coordinate system (in (79) in the succeeding text), where the constant $\mu > 0$ is from the control action (11). Then, the following relations hold:

$$\frac{\partial V}{\partial x_1}(x)x_2 + \frac{\partial V}{\partial x_2}(x)\left(\tilde{f}(t,x) + \tilde{g}(t,x)k(t,x)\right) = -2\mu V(x) \text{ for all } (t,x) \in [0,+\infty) \times \mathbb{R}^2, \tag{65}$$

$$|x|^2 \leqslant V(x) \leqslant K|x|^2 \text{ for all } x \in \mathbb{R}^2, \tag{66}$$

and

$$|\nabla V(x)| \leqslant 2K|x| \text{ for all } x \in \mathbb{R}^2, \text{ where } K = \frac{\mu^2 + 2 + \mu\sqrt{\mu^2+4}}{\mu^2 + 2 - \mu\sqrt{\mu^2+4}}. \tag{67}$$

This is because the eigenvalues of the matrix for the quadratic form $x_1^2 + (x_2 + \mu x_1)^2$ are $\frac{1}{2}\left(\mu^2 + 2 \pm \mu\sqrt{\mu^2+4}\right)$.

Definitions (53) and (64) provide a function $\tilde{a} \in \mathcal{K}_\infty$, a continuous, non-decreasing function $M : [0,+\infty) \to [1,+\infty)$ and positive constants $\tilde{k}$ and ε that satisfy the following:

$$|k(t,x)| \leqslant \tilde{a}(|x|) \text{ for all } (t,x) \in [0,+\infty) \times \mathbb{R}^2, \tag{68}$$

$$\tilde{a}(s) = \tilde{k}s \text{ for all } s \in [0,\varepsilon], \text{ and} \tag{69}$$

$$\tilde{g}(t,x)|k(t,x) - k(t,\xi)| \leqslant M(|x| + |\xi|)|\xi - x| \text{ for all } t \geqslant 0,\ \zeta \in \mathbb{R}^n \text{ and } x \in \mathbb{R}^n. \tag{70}$$

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The existence of $\tilde{a} \in \mathcal{K}_\infty$ follows from the bounds for the reference trajectory; see (5). See Appendix A for a construction of a function $M : [0, +\infty) \to [1, +\infty)$ satisfying the preceding requirements. We also set

$$D_r(s) = 2K(a_r(s) + s)\, M(a_r(s) + s) \exp(rL(a_r(s) + s)) \text{ and } \beta(s) = \tilde{a}\left(s\sqrt{K}\right) + s\sqrt{K} \quad (71)$$

for all $s \geqslant 0$, where $a_r \in \mathcal{K}_\infty$ is the function in (24) from Lemma 1 for system (52) with τ replaced by $r > 0$, and $L : [0, +\infty) \to [1, +\infty)$ is the function from Assumption 1 for the choice of (53) of $f(t, x, u)$.

Next, we select a constant $\delta > 0$, such that

$$2\sqrt{K\delta} \leqslant \varepsilon. \quad (72)$$

Having selected $\delta > 0$, we can select a constant $\gamma > 0$ such that

$$\gamma \leqslant \min\left\{\sqrt{\delta}, \frac{\mu\delta}{2}\right\}. \quad (73)$$

We next set

$$\phi = 2KM\left(2\sqrt{K\delta} + \sqrt{\delta}\right)\exp\left(r\tilde{L}\right), \text{ where } \tilde{L} = L\left(\left(1 + \tilde{k}\right)2\sqrt{K\delta} + \sqrt{\delta}\right), \quad (74)$$

and we select a constant $\tilde{R} > 0$ such that

$$\tilde{R}\tilde{k}\sqrt{K} < 1 \text{ and } \frac{\phi\tilde{R}}{\mu\sqrt{2}}\left(1 + \frac{\tilde{k}\sqrt{K}\left(\tilde{R} + 1\right)}{1 - \tilde{R}\tilde{k}\sqrt{K}}\right) < 1. \quad (75)$$

Finally, we define

$$R(s) = \min\left\{\frac{\gamma}{\max\left\{1, D_r(a_\tau(s) + \beta(Q_\tau(s)))\right\}}, \tilde{R}s, \frac{1}{2\sqrt{K}}\tilde{a}^{-1}\left(\frac{s}{2}\right)\right\}. \quad (76)$$

Equation (69), definition (76) of R, and the fact that $Q_\tau(s) \geqslant 1$ for all $s \geqslant 0$ imply that

$$\liminf_{s \to 0^+} \frac{R(s)}{s} = \min\left\{\tilde{R}, \frac{1}{4\tilde{k}\sqrt{K}}\right\} > 0. \quad (77)$$

Therefore, Corollary 1 guarantees that the mapping $N : [0, +\infty) \to \{1, 2, 3, \ldots\}$ defined by (44) and $N(0) = 1$ for system $\dot{x} = f(t, x, u)$ (i.e., the delay-free version of (52)) is locally bounded and the mapping $\Phi_{t_0} : \mathbb{R}^2 \times L^\infty([t_0, t_0 + \tau); \mathbb{R}) \to \mathbb{R}^2$ defined by (43) satisfies inequalities (45)–(46) for all $(t_0, x_0) \in [0, +\infty) \times \mathbb{R}^2$ and $u \in L^\infty([t_0, t_0 + \tau); \mathbb{R})$, where $x(t)$ denotes the solution of $\dot{x} = f(t, x, u)$ for the initial condition $x(t_0) = x_0$ corresponding to input $u : [t_0, t_0 + \tau) \to \mathbb{R}$ and $\|u\| = \operatorname{ess\,sup}_{t \in [t_0, t_0+\tau)} |u(t)|$.

In the new coordinate system defined by (47) and (51), the closed-loop system given by (10)–(13) with the choices (14) of the N_i's is described by the equations

$$\dot{x}(t) = f(t, x(t), u(t - \tau)) \quad (78)$$

with x valued in $\mathbb{R}^2$ and u valued in $\mathbb{R}$, and with $v(t) = v_d(t) + u(t)$ and

$$\begin{aligned} &\dot{\xi}(t) = f(t + \tau, \xi(t), k(t + \tau, \xi(t))), \\ &u(t) = k(t + \tau, \xi(t)) \text{ for all } t \in [t_0 + T_i, t_0 + T_{i+1}) \text{ and all } i, \text{ where } \xi(t_0 + T_i) = z_{N_i} \end{aligned} \quad (79)$$

and where $N_i = N\left(|x(t_0 + T_i)| + \sup_{t_0+T_i-\tau \leqslant s < t_0+T_i} |u(s)|\right), h_i = \frac{\tau}{N_i}$ and

$$z_{j+1} = z_j + \int_{t_0+T_i+jh_i}^{t_0+T_i+(j+1)h_i} f(s, z_j, u(s - \tau))\mathrm{d}s \text{ for } j = 0, \ldots, N_i - 1 \text{ and } z_0 = x(t_0 + T_i). \quad (80)$$

To verify that (12) and (79) agree, notice that our choice (64) of k and our formula for f from (53) imply that the ξ dynamics from (79) can be written as

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$$\begin{aligned} \dot{\xi}_1(t) &= \xi_2(t) \\ \dot{\xi}_2(t) &= -(1+\mu^2)\xi_1(t) - 2\mu\xi_2(t) \text{ for all } t \in [t_0+T_i, t_0+T_{i+1}). \end{aligned} \tag{81}$$

The solution of (81) satisfying $\xi(t_0+T_i) = z_{N_i}$ is given by (12)–(13).

5.3. *Third step of proof of Theorem 1*

For every $t_0 \geqslant 0, (q_0, \dot{q}_0) \in (-\pi/2, \pi/2) \times \mathbb{R}$ and $v_0 \in L^\infty([-\tau, 0); \mathbb{R})$, the solution $(q(t), \dot{q}(t), v(t)) \in (-\pi/2, \pi/2) \times \mathbb{R}^2$ of the closed-loop system (2) and (10)–(13) with

$$\begin{aligned} N_i = N\Bigg(&\left|\left(\tan(q(t_0+T_i)) - \tan(q_d(t_0+T_i)), \frac{\dot{q}(t_0+T_i)}{\cos^2(q(t_0+T_i))} - \frac{\dot{q}_d(t_0+T_i)}{\cos^2(q_d(t_0+T_i))}\right)\right| \\ &+ \sup_{t_0+T_i-\tau \leqslant s < t_0+T_i} |v(s) - v_d(s)|\Bigg) \end{aligned} \tag{82}$$

and initial condition $(q(t_0), \dot{q}(t_0)) = (q_0, \dot{q}_0) \in (-\pi/2, \pi/2) \times \mathbb{R}$ and $v(t_0+s) = v_0(s)$ for $s \in [-\tau, 0)$ is related to the solution $(x(t), u(t)) \in \mathbb{R}^3$ of the closed-loop system (78)–(80) with initial condition

$$x(t_0) = \left(\tan(q(t_0)) - \tan(q_d(t_0)), \frac{\dot{q}(t_0)}{\cos^2(q(t_0))} - \frac{\dot{q}_d(t_0)}{\cos^2(q_d(t_0))}\right) \text{ and } u(t_0+s) = v_0(s) - v_d(t_0+s) \tag{83}$$

by

$$x(t) = \left(\tan(q(t)) - \tan(q_d(t)), \frac{\dot{q}(t)}{\cos^2(q(t))} - \frac{\dot{q}_d(t)}{\cos^2(q_d(t))}\right),\ u(t-\tau) = v(t-\tau) - v_d(t-\tau) \tag{84}$$

and

$$\begin{aligned} q(t) &= \tan^{-1}(x_1(t) + \tan(q_d(t))) \text{ and} \\ \dot{q}(t) &= \frac{1}{1 + (x_1(t) + \tan(q_d(t)))^2}\left(x_2(t) + \frac{\dot{q}_d(t)}{\cos^2(q_d(t))}\right), \end{aligned} \tag{85}$$

which hold for all $t \geqslant t_0$ for which the solutions exist. The global relations $|q(t) - q_d(t)| \leqslant |x_1(t)|, |\dot{q}(t) - \dot{q}_d(t)| \leqslant |x_2(t)| + M_1|x_1(t)|$,

$$|x_1(t_0)| \leqslant M_2\frac{|q(t_0) - q_d(t_0)|}{|\cos(q(t_0))|}, \text{ and } |x_2(t_0)| \leqslant \frac{|\dot{q}(t_0) - \dot{q}_d(t_0)| + M_2^2 M_3|q(t_0) - q_d(t_0)|}{\cos^2(q(t_0))},$$

where $M_1 = 2\sup_{t\geqslant 0}|\zeta_{2,d}(t)|, M_2 = \sup_{t\geqslant 0}(2/|\cos(q_d(t))|)$ and $M_3 = |\dot{q}_d|_\infty$ (which are direct consequences of (84)–(85) and the mean value theorem), allow us to conclude that in order to prove Theorem 1, it suffices to show that there exists a locally Lipschitz, non-decreasing function $\hat{C} : [0, +\infty) \to [0, +\infty)$ with $\hat{C}(0) = 0$ and a constant $\omega \in (0, \mu/2)$ such that for every partition $\{T_i\}_{i=0}^\infty$ of $[0, +\infty)$ with $\sup_{i\geqslant 0}(T_{i+1} - T_i) \leqslant r$, and every $t_0 \geqslant 0, x_0 \in \mathbb{R}^2$ and $u_0 \in L^\infty([-\tau, 0); \mathbb{R})$, the solution $(x(t), u(t))$ of the closed-loop system given by (78)–(80) with initial condition $x(t_0) = x_0$ and $u(t_0+s) = u_0(s)$ for $s \in [-\tau, 0)$ exists for all $t \geqslant t_0$ and satisfies the following inequality for all $t \geqslant t_0$:

$$|x(t)| + \sup_{t-\tau \leqslant s < t}|u(s)| \leqslant \exp(-\omega(t-t_0))\hat{C}(|x_0| + |u_0|_\infty) \tag{86}$$

Therefore, the rest of proof is devoted to the proof of (86) for the closed-loop system (78)–(80). To prove (86), we first prove some basic results for (78)–(80). The claims we are about to give are analogous to the claims in [24], but [24] is limited to time-invariant systems and cannot be applied to our NMES tracking dynamics. The following claim shows that practical stabilization is achieved. Its proof is in Appendix A.

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Claim 1
There exists a function $\sigma \in \mathcal{KL}$ such that for every partition $\{T_i\}$ of $[0,+\infty)$ with $T_{i+1}-T_i \leqslant r$ for all $i \geqslant 0$ and all initial conditions $x(t_0)=x_0$ and $\check{T}_\tau(t_0)u = u_0$ for every $(t_0,x_0) \in [0,+\infty)\times\mathbb{R}^2$ and $u_0 \in L^\infty([-\tau,0);\mathbb{R})$, the solution of (78)–(80) satisfies

$$V(x(t)) \leqslant \max\left\{\sigma\left(|x_0| + \|u_0\|_\tau, t-t_0\right), \mu^{-1}\gamma\right\} \tag{87}$$

for all $t \geqslant t_0$, where $\gamma > 0$ is the constant involved in (73) and V is from (63).

The following claim shows local exponential stabilization. It is also proved in Appendix A.

Claim 2
There are positive constants $\overline{S}_1, \overline{S}_2$, and $\omega \in (0,\mu/2)$ such that for each partition $\{T_i\}$ of $[0,+\infty)$ with $\sup_{i\geqslant 0}\{T_{i+1}-T_i\} \leqslant r$, each pair $(t_0,x_0) \in [0,+\infty)\times\mathbb{R}^2$, and each initial function $u_0 \in L^\infty([-\tau,0);\mathbb{R})$ for the control, the solution of (78)–(80) with initial condition $x(t_0)=x_0$ satisfies

$$|u(t)|\exp(\omega(t-t_0-T_j)) \leqslant \overline{S}_1\left(\sup_{t_0+T_j\leqslant w\leqslant t_0+T_j+\tau}|x(w)| + \left\|\check{T}_\tau(t_0+T_j)u\right\|_\tau\right) \ \forall t \geqslant t_0+T_j \tag{88}$$

and

$$\begin{aligned}&|x(t)|\exp(\omega(t-t_0-T_j-\tau))\\ &\leqslant \overline{S}_2\left(\sup_{t_0+T_j\leqslant w\leqslant t_0+T_j+\tau}|x(w)| + \left\|\check{T}_\tau(t_0+T_j)u\right\|_\tau\right) \ \forall t \geqslant t_0+T_j+\tau,\end{aligned} \tag{89}$$

where j is the smallest index such that $V(x(t_0+T_j+\tau)) \leqslant \delta$ and $\delta > 0$ is from (72). □

Our final claim guarantees that u is bounded. It too is proven in Appendix A.

Claim 3
There exists a non-decreasing function $S:[0,+\infty)\to[0,+\infty)$ such that for each partition $\{T_i\}_{i=0}^\infty$ of $[0,+\infty)$ satisfying $T_{i+1}-T_i \leqslant r$ for all $i \geqslant 0$, each $(t_0,x_0) \in [0,+\infty)\times\mathbb{R}^2$, and each $u_0 \in L^\infty([-\tau,0);\mathbb{R})$, the solution of (78)–(80) with initial condition $x(t_0)=x_0$ and $\check{T}_\tau(t_0)u=u_0$ satisfies

$$|x(t)| + \left\|\check{T}_\tau(t)u\right\|_\tau \leqslant S(|x_0| + ||u_0||_\tau) \tag{90}$$

for all $t \geqslant t_0$. □

Finally, we prove (86). Let $\{T_i\}_{i=0}^\infty$ be any partition of $[0,+\infty)$ such that $\sup\{T_{i+1}-T_i\} \leqslant r$, and $(t_0,x_0) \in [0,+\infty)\times\mathbb{R}^2$ and $u_0 \in L^\infty([-\tau,0);\mathbb{R})$ be given. Consider the solution of (78)–(80) with the initial condition $x(t_0)=x_0$ and the initial input $\check{T}_\tau(t_0)u=u_0$.

Inequalities (66) and (24) imply that the smallest sampling time t_0+T_j for which $V(x(t_0+T_j+\tau)) \leqslant \delta$ holds gives $T_0 = 0$ if $K(a_\tau(|x_0|+||u_0||_\tau))^2 \leqslant \delta$. Moreover, because there is a constant $M_\tau > 0$ such that $a_\tau(s) \leqslant M_\tau s$ for all $s \in [0,1]$, we can use inequalities (24) and (88)–(89) to find a constant $\tilde{\Omega} > 0$ such that

$$|x(t)| + \left\|\check{T}_\tau(t)u\right\|_\tau \leqslant \tilde{\Omega}\exp(-\omega(t-t_0))(|x_0|+||u_0||_\tau) \tag{91}$$

for all $t \geqslant t_0$, provided that

$$|x_0| + ||u_0||_\tau \leqslant \min\left\{1, \frac{1}{M_\tau}\sqrt{\frac{\delta}{K}}\right\}, \tag{92}$$

because (92) gives $Ka_\tau^2(|x_0|+||u_0||_\tau) \leqslant \delta$.

The $\mathcal{KL}$ lemma from [26] provides class $\mathcal{K}_\infty$ functions β_1 and β_2 such that $\sigma(s,t) \leqslant \beta_1(\exp(-t)\beta_2(s))$ holds for all $s \geqslant 0$ and $t \geqslant 0$. In conjunction with (87), (73), and the fact that $T_{i+1}-T_i \leqslant r$ for all $i \geqslant 0$, this guarantees the existence of a non-decreasing function $\tilde{T}$: $[0,+\infty)\to[0,+\infty)$ such that the smallest sampling time t_0+T_j for which $V(x(t_0+T_j+\tau)) \leqslant \delta$

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holds satisfies $T_j \leqslant \tilde{T}(|x_0| + ||u_0||_\tau)$ for all $(t_0, x_0) \in [0, +\infty) \times \mathbb{R}^2$ and $u_0 \in L^\infty([-\tau, 0); \mathbb{R})$. Combining (88), (89), and (90) with the previous inequality provides a continuous non-decreasing function $\tilde{G} : [0, +\infty) \to [0, +\infty)$ such that

$$|x(t)| + \left\| \breve{T}_\tau(t)u \right\|_\tau \leqslant \exp(-\omega(t - t_0))\tilde{G}(|x_0| + ||u_0||_\tau) \text{ for all } t \geqslant t_0. \tag{93}$$

Consequently, (86) holds with the choice

$$\hat{C}(s) = \begin{cases} \frac{1}{s}\int_s^{2s} \tilde{C}(w)\mathrm{d}w, & s > 0 \\ 0, & s = 0 \end{cases}, \text{ where } \tilde{C}(s) = \begin{cases} \max\left\{1, \frac{\tilde{G}(l)}{\tilde{\Omega} l}\right\} \tilde{\Omega} s, & s \in [0, l] \\ \max\left\{\tilde{\Omega} s, \tilde{G}(s)\right\}, & s > l \end{cases} \tag{94}$$

and $l = \min\{1, (1/M_\tau)\sqrt{\delta/K}\}$. This completes the proof of Theorem 1.

6. SIMULATIONS

We simulated the NMES model (1) in closed-loop with the hybrid control (10)–(13) with a time delay of $\tau = 0.07$s. We set the parameters in (1) to the following values:

$$\begin{aligned} &J = 0.39\text{kg}-\text{m}^2/\text{rad},\ b_1 = 0.6\text{kg}-\text{m}^2/(\text{rad}-\text{s}),\ b_2 = 0.1\text{kg}-\text{m}^2/(\text{rad}-\text{s}), \\ &b_3 = 50\text{s}/\text{rad},\ k_1 = 7.9\text{kg}-\text{m}^2/(\text{rad}-\text{s}^2),\ k_2 = 1.681/\text{rad}, \\ &k_3 = 1.17\text{kg}-\text{m}^2/(\text{rad}-\text{s}^2),\ m = 4.38\text{kg},\ l = 0.248\text{m}. \end{aligned} \tag{95}$$

We chose the positive valued moment arm [27, Appendix 1]

$$\zeta(q) = Ae^{-2q^2}\sin(q) + B \tag{96}$$

with $A = 0.058$ m and $B = 0.0284$ m. Because an analytical model for the muscle recruitment function $\eta(q, \dot{q})$ is not available in the literature ‡, we assumed the control input to be the muscle force rather than the electrode voltage, that is, $U = \zeta(q)\mathcal{F}$ where $\mathcal{F}$ is the force applied by the quadriceps muscles. This is equivalent to setting $\eta(q, \dot{q}) = 1$ and $v = \mathcal{F}$ in Section 3, but our results apply for a broad class of possible functions ζ and η.

We ran simulations for two different types of reference trajectories. First, we chose the reference trajectory

$$q_d(t) = \frac{\pi}{3}(1 - \exp(-3t))\text{rad} \tag{97}$$

to simulate a smoothed step command for the lower leg limb. We used the initial conditions $q(0) = \pi/18$ rad, $\dot{q}(0) = 0$ rad/s, and $v(t) = 0$ on $[-0.07, 0)$. Instead of using a variable number of grid points, we used a constant number of grid points $N_i = N = 10$ for $i = 0, 1, 2, \ldots$, that is, we performed the numerical prediction in (10) with the constant discretization time step $h = \tau/N = 0.007$. We also set $t_0 = 0$ s and $T_{i+1} - T_i = 0.014$s for $i = 0, 1, 2, \ldots$. The control gain μ was tuned by trial and error. Figure 1 shows the plots of $q_d(t)$ versus $q(t)$, $\dot{q}_d(t)$ vs. $\dot{q}(t)$, and $v(t)$ when $\mu = 2$. We found that increasing μ resulted in faster convergence of the position tracking error to zero, at the expense of larger overshoots. The order of magnitude of the control (10^2N) is reasonable and within the expected range for the leg quadriceps muscle [27].

We then investigated the robustness of the proposed control scheme to parametric uncertainties by running the preceding simulation with the same initial conditions, but with a mismatch between the plant parameters in (95) and the corresponding parameter values used in the control. Specifically, we set

$$\begin{aligned} &J' = 1.25J,\ b_1' = 1.2b_1,\ b_2' = 0.9b_2,\ b_3' = 0.85b_3,\ k_1' = 1.1k_1, \\ &k_2' = 0.912k_2,\ k_3' = 0.9k_3,\ m' = 0.97m,\ l' = 1.013l,\ A' = 1.185A, \text{ and } B' = 0.98B \end{aligned} \tag{98}$$

‡Some voltage-level controllers have been designed to compensate for the unknown term $\eta(q, \dot{q})$ [1, 2, 15]. Because these controllers were tested experimentally, an analytical model for this term was not needed.

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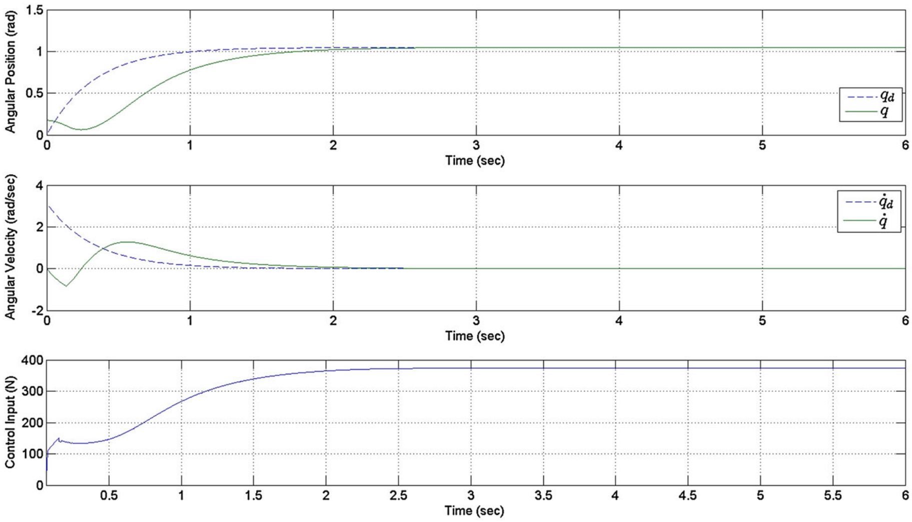

Figure 1. Simulation for smoothed step function with exact parameter knowledge and $\mu = 2$.

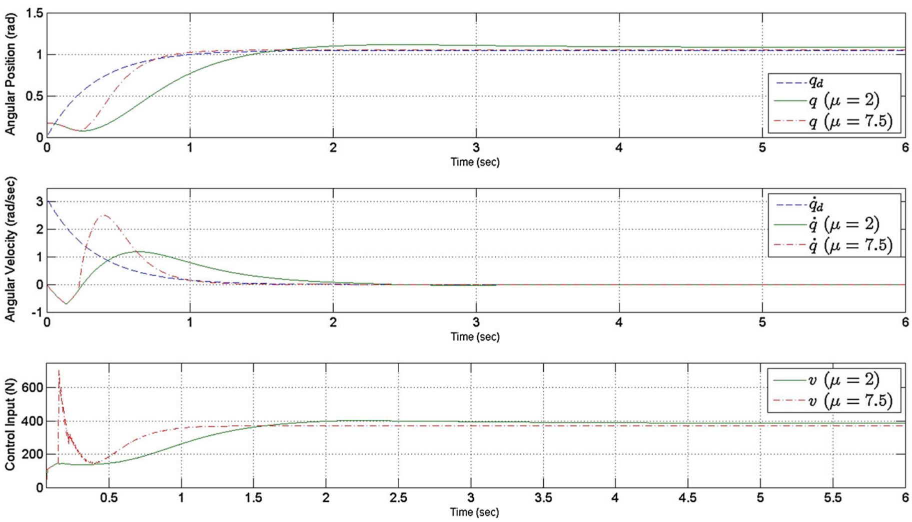

Figure 2. Simulation for smoothed step function with parametric uncertainty.

where the parameter values with primes were the ones used in the control, and where the ones without the primes were used in the model and were set to (95). The results with the control gain kept at $\mu = 2$ are shown in Figure 2, where expectedly there is a small steady state error in the position (of approximately 0.034 rad). We observed that the closed-loop system was most sensitive to uncertainties in the parameters k_2, m, l, A, and B. The sensitivity in m and l is not surprising, because it affects the gravitational torque, which in turn affects the steady-state (equilibrium) position of the closed-loop system. We then retuned the control gain to help reduce the steady-state error. The results for $\mu = 7.5$ are also given in Figure 2, showing that the steady-state error was virtually eliminated at the expense of higher transients in the control input.

Next, we set the reference trajectory to

$$q_d(t) = \frac{\pi}{8}\sin(t)(1 - \exp(-8t))\text{rad} \tag{99}$$

to simulate a sinusoidal command with a smooth start. We used the initial conditions $q(0) = 0.5$rad, $\dot{q}(0) = 0$ rad/s, and $v(t) = 0$ on $[-0.07, 0)$, and we kept N and $T_{i+1} - T_i$ and the parameters (95) at

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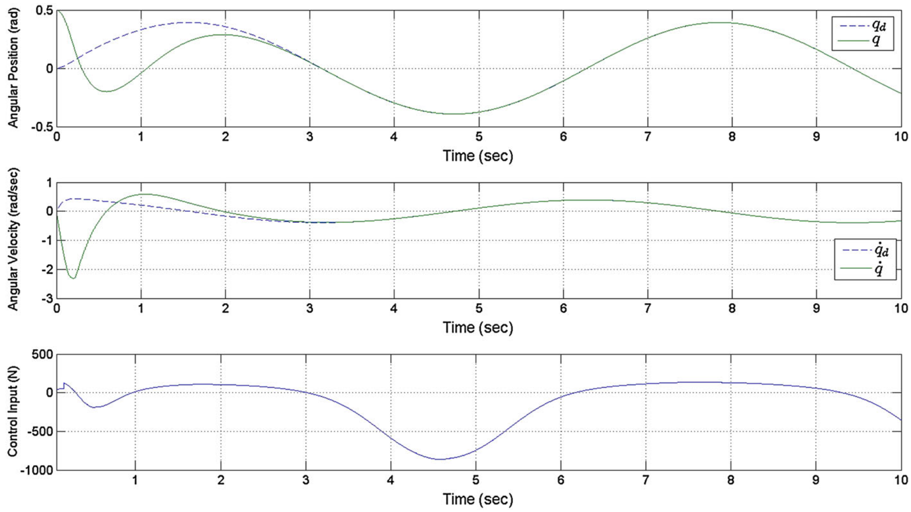

Figure 3. Simulation for sinusoidal trajectory with exact parameter knowledge and $\mu = 2$.

the same values we used previously. We assumed exact parameter knowledge and we chose $\mu = 2$. We plotted the result in Figure 3. We noticed that after a transient period of 3 s, the state $(q(t), \dot{q}(t))$ closely tracked $(q_d(t), \dot{q}_d(t))$. This illustrates the effectiveness of our control for tracking sinusoidal reference trajectories. The preceding examples also illustrate how the use of a constant number of grid points (i.e., explicit Euler steps) for the numerical prediction of the state variables is sufficient for global asymptotic tracking. Moreover, usually, a small number of grid points suffices, such as $N = 10$ for the preceding examples.

7. CONCLUDING REMARKS

Neuromuscular electrical stimulation is an important technique that can help restore movement in human limbs in patients with motor neuron disorders. However, it is not amenable to open loop control, and it is prone to input delays that can seriously degrade the performance of closed loop controls. This paper provided a new predictor controller that helps overcome these challenges and ensures exponentially stable tracking of a broad class of reference trajectories while respecting the state constraint imposed by the physical system. The advantages of our approach are that (a) it only requires sampled measurements of the state (instead of continuous measurements); (b) it allows perturbed sampling schedules; (c) it proves tracking of both the position and the velocity under the state constraint on the position; and (d) it does not impose any upper bound on the input delay. The control scheme only requires knowledge of the signal to be tracked, the functions in the NMES dynamics, the delay appearing in the NMES control, and the upper diameter of the sampling schedule; it is a model-based nonlinear hybrid predictor feedback. In Appendix A, we give tables of all of the formulas and constants needed to implement our control scheme. The formulas and constants in the tables are selected in such a way that all of the required inequalities and equalities in the previous sections are satisfied automatically. Our NMES control was based on a new general prediction theorem that is of independent interest. Moreover, our simulations demonstrate that our predictive NMES control performs well under uncertainty in the model parameters. In our future work, we aim to generalize our theorems to capture uncertainties in the model.

APPENDIX

A.1. Proof of Claims 1–3

A.1.1. Proof of Claim 1. First, we show that for each partition T_i of $[0, +\infty)$ such that $T_{i+1} - T_i \leqslant r$ for all $i \geqslant 0$ and each $(t_0, x_0) \in [0, +\infty) \times \mathbb{R}^2$ and $u_0 \in L^\infty([-\tau, 0); \mathbb{R})$, the solution of (78)–(80)

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with initial condition $x(t_0) = x_0$ and $\breve{T}_\tau(t_0)u = u_0$ is unique and exists for all $t \geqslant t_0$. The solution of (78)–(80) is determined as follows:

Initial step: given $x(t_0) = x_0$ and $\breve{T}_\tau(t_0)u = u_0$, we determine the solution $x(t)$ of (78) for $t \in [t_0, t_0 + \tau]$. The solution is unique, and it satisfies the estimate (24) from Lemma 1.

i-th Step: given $x(t)$ for $t \in [t_0, t_0 + T_i + \tau]$ and $u(t)$ for $t \in [-\tau, t_0 + T_i)$, we determine $x(t)$ for $t \in [t_0, t_0 + T_{i+1} + \tau]$ and $u(t)$ for $t \in [-\tau, t_0 + T_{i+1})$ as follows. The solution $\xi(t)$ from (79) for $t \in [t_0 + T_i, t_0 + T_{i+1})$ with initial condition $\xi(t_0 + T_i) = z_{N_i}$ is unique and is given by (12). Inequality (65) implies that

$$V(\xi(t)) \leqslant V(\xi(t_0 + T_i) \text{ for all } t \in [t_0 + T_i, t_0 + T_{i+1}). \tag{A.1}$$

We determine $u(t)$ for $t \in [t_0 + T_i, t_0 + T_{i+1})$ using the equation $u(t) = k(t + \tau, \xi(t))$. Notice that inequalities (66) and (68) in conjunction with (A.1) imply the following inequality for all $t \in [t_0 + T_i, t_0 + T_{i+1})$:

$$|u(t)| = |k(t + \tau, \xi(t))| \leqslant \tilde{a}\left(|\xi(t_0 + T_i)|\sqrt{K}\right) \tag{A.2}$$

This allows us to determine the unique solution $x(t)$ of (78) for $t \in [t_0, t_0 + T_{i+1} + \tau]$. The fact that $T_{i+1} - T_i \leqslant r$ in conjunction with (24) with τ replaced by $r > 0$ and (A.2) implies this estimate:

$$|x(t)| \leqslant a_r\left(|x(t_0 + T_i + \tau)| + \tilde{a}\left(|\xi(t_0 + T_i)|\sqrt{K}\right)\right) \text{ for all } t \in [t_0 + T_i + \tau, t_0 + T_{i+1} + \tau] \tag{A.3}$$

Next, we evaluate the difference $\xi(t) - x(t + \tau)$ for $t \in [t_0 + T_i, t_0 + T_{i+1})$. Exploiting (17), we get:

$$\begin{aligned}
|\xi(t) - x(t + \tau)| &= \Bigg|\xi(t_0 + T_i) - x(t_0 + T_i + \tau) + \int_{t_0+T_i}^{t} (f(s + \tau, \xi(s), k(s + \tau, \xi(s))) \\
&\quad - f(s + \tau, x(s + \tau), k(s + \tau, \xi(s))))\, \mathrm{d}s\Bigg| \\
&\leqslant |\xi(t_0 + T_i) - x(t_0 + T_i + \tau)| + \int_{t_0+T_i}^{t} L(|\xi(s)| + |x(s + \tau)| \\
&\quad + |k(s + \tau, \xi(s))|)|\xi(s) - x(s + \tau)|\mathrm{d}s
\end{aligned} \tag{A.4}$$

Using the quadratic upper and lower bounds on V from (66) and (A.1)–(A.4), we obtain:

$$\begin{aligned}
|\xi(t) - x(t + \tau)| \leqslant |\xi(t_0 + T_i) - x(t_0 + T_i + \tau)| + L\Big(&|\xi(t_0 + T_i)|\sqrt{K} + \tilde{a}(|\xi(t_0 + T_i)|\sqrt{K}) \\
&+ a_r\left(|x(t_0 + T_i + \tau)| + \tilde{a}(|\xi(t_0 + T_i)|\sqrt{K})\right)\Big) \int_{t_0+T_i}^{t} |\xi(s) - x(s + \tau)|\mathrm{d}s
\end{aligned}$$

Define $\varphi(s) = a_r(s) + s$. Using the Gronwall–Bellman lemma, the aforementioned inequality, formula $\beta(s) = \tilde{a}\left(s\sqrt{K}\right) + s\sqrt{K}$ from (71) and the fact that $T_{i+1} - T_i \leqslant r$, we get the following for all $t \in [t_0 + T_i, t_0 + T_{i+1})$:

$$|\xi(t) - x(t+\tau)| \leqslant |\xi(t_0 + T_i) - x(t_0 + T_i + \tau)| \exp\left(rL\left(\varphi\left(|x(t_0 + T_i + \tau)| + \beta(|\xi(t_0 + T_i)|)\right)\right)\right) \tag{A.5}$$

Next, we evaluate the quantity $\nabla V(x(t + \tau))f(t + \tau, x(t + \tau), k(t + \tau, \xi(t)))$ for $t \in [t_0 + T_i, t_0 + T_{i+1})$. Using the decay estimate (65) on V and our choice (53) of the dynamics f, we get

$$\begin{aligned}
&\nabla V(x(t + \tau))f(t + \tau, x(t + \tau), k(t + \tau, \xi(t))) \leqslant -2\mu V(x(t + \tau)) \\
&\quad + \nabla V(x(t + \tau))\left(f(t + \tau, x(t + \tau), k(t + \tau, \xi(t))) - f(t+\tau, x(t+\tau), k(t+\tau, x(t+\tau)))\right)
\end{aligned}$$

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Using (67), (70), (A.1), (A.3), and definitions $\beta(s) = \tilde{a}\left(s\sqrt{K}\right) + s\sqrt{K}$ and $\varphi(s) = a_r(s) + s$ gives

$$\begin{aligned}&\nabla V(x(t+\tau)) f(t+\tau, x(t+\tau), k(t+\tau, \xi(t))) \\ &\quad \leqslant -2\mu V(x(t+\tau)) + 2K\left(\varphi\left(|x(t_0+T_i+\tau)| + \beta(|\xi(t_0+T_i)|)\right)\right) \\ &\qquad \times M\left(\varphi\left(|x(t_0+T_i+\tau)| + \beta(|\xi(t_0+T_i)|)\right)\right) |x(t+\tau) - \xi(t)|\end{aligned}$$

Combining the preceding inequality with (A.5) and the definition (71) of D_r, we obtain the following for all $t \in [t_0+T_i, t_0+T_{i+1})$:

$$\begin{aligned}&\nabla V(x(t+\tau)) f(t+\tau, x(t+\tau), k(t+\tau, \xi(t))) \\ &\quad \leqslant -2\mu V(x(t+\tau)) + D_r\left(|x(t_0+T_i+\tau)| + \beta(|\xi(t_0+T_i)|)\right) |x(t_0+T_i+\tau) - \xi(t_0+T_i)|\end{aligned} \tag{A.6}$$

Because $\xi(t_0+T_i) = z_{N_i}$, we can apply (45)–(46) (with initial time t_0+T_i) to get the following for all $i = 0, 1, 2, \ldots$:

$$|\xi(t_0+T_i) - x(t_0+T_i+\tau)| \leqslant R\left(|x(t_0+T_i)| + \left\|\breve{T}_\tau(t_0+T_i)u\right\|_\tau\right) \tag{A.7}$$

$$|\xi(t_0+T_i)| \leqslant Q_\tau\left(|x(t_0+T_i)| + \left\|\breve{T}_\tau(t_0+T_i)u\right\|_\tau\right) \tag{A.8}$$

Because (24) gives

$$|x(t_0+T_i+\tau)| \leqslant a_\tau\left(|x(t_0+T_i)| + \left\|\breve{T}_\tau(t_0+T_i)u\right\|_\tau\right), \tag{A.9}$$

we can use (A.6), (A.7), (A.8), and the definition (76) of R to get $\frac{d}{dt}V(x(t+\tau)) \leqslant -2\mu V(x(t+\tau)) + \gamma$ for all $t \in [t_0+T_i, t_0+T_{i+1})$. Then, we can integrate to get the following for all $t \geqslant t_0$:

$$\begin{aligned}V(x(t+\tau)) &\leqslant \exp(-2\mu(t-t_0))V(x(t_0+\tau)) + \frac{\gamma}{2\mu} \\ &\leqslant 2\max\left\{\exp(-2\mu(t-t_0))V(x(t_0+\tau)), \frac{\gamma}{2\mu}\right\}\end{aligned} \tag{A.10}$$

Combining the quadratic lower and upper bounds for V from (66) with (24) and (A.10), we get (87) with $\sigma(s,t) = 2Ka_\tau^2(s)\exp(-2\mu(t-\tau))$ for all $t > \tau$ and $\sigma(s,t) = 2Ka_\tau^2(s)$ for all $t \in [0,\tau]$. This proves Claim 1.

A.1.2. Proof of Claim 2. Pick any partition $\{T_i\}_{i=0}^{\infty}$ of $[0, +\infty)$ with $\sup_{i \geqslant 0}(T_{i+1} - T_i) \leqslant r$ and any $t_0 \geqslant 0, x_0 \in \mathbb{R}^2$, and $u_0 \in L^\infty([-\tau, 0); \mathbb{R}^m)$, and consider the solution of (78)–(80) with (arbitrary) initial conditions $x(t_0) = x_0$ and $\breve{T}_\tau(t_0)u = u_0$. Inequality (87) guarantees that there exists a unique smallest sampling time $t_0 + T_j$ such that $V(x(t_0+T_j+\tau)) \leqslant \delta$, because (73) gives $\frac{\gamma}{\mu} < \frac{\delta}{2}$. Moreover, inequalities (73) and (66) give

$$|x(t)| \leqslant \sqrt{\delta} \text{ and } V(x(t)) \leqslant \delta \text{ for all } t \geqslant t_0 + T_j + \tau, \tag{A.11}$$

because $\frac{d}{dt}V(x(t+\tau)) \leqslant -2\mu V(x(t+\tau)) + \gamma < 0$ when $V(x(t+\tau)) \geqslant \delta$. Recall from (76) that $R(s) \leq \gamma$ for all $s \geqslant 0$. Hence, (A.7), (73), and (A.11) give

$$|\xi(t_0+T_i)| \leqslant |\xi(t_0+T_i) - x(t_0+T_i+\tau)| + |x(t_0+T_i+\tau)| \leqslant \gamma + \sqrt{\delta} \leqslant 2\sqrt{\delta} \text{ for all } i \geqslant j. \tag{A.12}$$

Using (A.1), (66), and (A.12), we get $|\xi(t)|^2 \leqslant V(\xi(t)) \leqslant V(\xi(t_0+T_i)) \leqslant 4\delta K$ for all $t \geqslant t_0 + T_j$, hence

$$|\xi(t)| \leqslant 2\sqrt{K\delta} \leqslant \varepsilon \tag{A.13}$$

for all $t \geqslant t_0 + T_j$, where the last inequality in (A.13) used (72).

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Next, we evaluate the difference $\xi(t) - x(t+\tau)$ for $t \geqslant t_0 + T_j$. Exploiting (17), our choice (53) of f, and inequalities (68), (69), (A.11), and (A.13), we get the following for all $i \geqslant j$ and $t \in [t_0 + T_i, t_0 + T_{i+1})$:

$$\begin{aligned}|\xi(t) - x(t+\tau)| &= \Big|\xi(t_0 + T_i) - x(t_0 + T_i + \tau) + \int_{t_0+T_i}^{t} (f(s+\tau, \xi(s), k(s+\tau, \xi(s))) \\ &\qquad - f(s+\tau, x(s+\tau), k(s+\tau, \xi(s))))\,\mathrm{d}s\Big| \\ &\leqslant |\xi(t_0+T_i) - x(t_0+T_i+\tau)| + \tilde{L}\int_{t_0+T_i}^{t} |\xi(s) - x(s+\tau)|\mathrm{d}s\end{aligned}$$

where $\tilde{L} = L\left((1+\tilde{k})2\sqrt{K\delta} + \sqrt{\delta}\right)$ is from (74). Using the Gronwall–Bellman lemma, the aforementioned inequality, and the fact that $T_{i+1} - T_i \leqslant r$, we conclude that for all $i \geqslant j$ and $t \in [t_0 + T_i, t_0 + T_{i+1})$, we have

$$|\xi(t) - x(t+\tau)| \leqslant |\xi(t_0+T_i) - x(t_0+T_i+\tau)| \exp\left(r\tilde{L}\right) \tag{A.14}$$

Next, we evaluate the quantity $\nabla V(x(t+\tau)) f(t+\tau, x(t+\tau), k(t+\tau, \xi(t)))$ for $t \in [t_0 + T_i, t_0 + T_{i+1})$. Using inequalities (65), (67), (A.11), (70), (A.13), and (A.14) and our definition $\phi = 2KM\left(2\sqrt{K\delta} + \sqrt{\delta}\right)\exp\left(r\tilde{L}\right)$ from (74), we get the following for all $i \geqslant j$ and $t \in [t_0 + T_i, t_0 + T_{i+1})$:

$$\begin{aligned}&\nabla V(x(t+\tau)) f(t+\tau, x(t+\tau), k(t+\tau, \xi(t))) \\ &\quad\leqslant -2\mu V(x(t+\tau)) + 2K|x(t+\tau)|\,|f(t+\tau, x(t+\tau), k(t+\tau, \xi(t))) \\ &\qquad - f(t+\tau, x(t+\tau), k(t+\tau, x(t+\tau)))| \\ &\quad\leqslant -2\mu V(x(t+\tau)) + 2K|x(t+\tau)||\tilde{g}(t, x(t+\tau))||k(t+\tau, \xi(t)) - k(t+\tau, x(t+\tau))| \\ &\quad\leqslant -2\mu V(x(t+\tau)) + 2K|x(t+\tau)|M(|x(t+\tau)| + |\xi(t)|)|\xi(t) - x(t+\tau)| \\ &\quad\leqslant -2\mu V(x(t+\tau)) + \phi|x(t+\tau)|\ |x(t_0+T_i+\tau) - \xi(t_0+T_i)|\end{aligned}$$

Therefore, because our quadratic lower bound on V from (66) gives $|x(t+\tau)| \leqslant \sqrt{V(x(t+\tau))}$ for all t, the triangle inequality gives the following for all $i \geqslant j$ and $t \in [t_0 + T_i, t_0 + T_{i+1})$:

$$\begin{aligned}\dot{V}(t+\tau) &\leqslant -\mu V(t+\tau) + \frac{\phi^2}{4\mu}|x(t_0+T_i+\tau) - \xi(t_0+T_i)|^2 \\ &\leqslant -\mu V(t+\tau) + \frac{\phi^2\tilde{R}^2}{2\mu}\,|x(t_0+T_i)|^2 + \frac{\phi^2\tilde{R}^2}{2\mu}\,\left\|\breve{T}_\tau(t_0+T_i)u\right\|_\tau^2\end{aligned} \tag{A.15}$$

where $V(t) = V(x(t))$ and where the second inequality used (A.7) and the fact that $R(s) \leqslant \tilde{R}s$ for all $s \geqslant 0$.

Using (75), we can find a constant $\omega \in \left(0, \frac{\mu}{2}\right)$ that is sufficiently small such that

$$\begin{aligned}&\tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau)) < 1 \text{ and} \\ &\frac{\phi\tilde{R}}{\sqrt{2\mu}}\frac{\exp(\omega(r+\tau))}{\sqrt{\mu - 2\omega}}\left(1 + \frac{\tilde{k}\exp(\omega r)\sqrt{K}\left(\tilde{R} + \exp(-\omega\tau)\right)}{1 - \tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau))}\right) < 1.\end{aligned} \tag{A.16}$$

Using (A.15) and the fact that $\sup_{i\geqslant 0}(T_{i+1} - T_i) \leqslant r$, we get this for all $i \geqslant j$ and $t \in [t_0 + T_i, t_0 + T_{i+1})$:

$$\begin{aligned}\dot{V}(t+\tau) \leqslant& -\mu V(t+\tau) + \frac{\phi^2\tilde{R}^2}{2\mu}\,\exp(-2\omega t)\exp(2\omega r)\ \sup_{t_0+T_i\leqslant s\leqslant t}\left(\exp(2\omega s)|x(s)|^2\right) \\ &+ \frac{\phi^2\tilde{R}^2}{2\mu}\exp(-2\omega t)\exp(2\omega(r+\tau))\ \sup_{t_0+T_i-\tau\leqslant s\leqslant t}\left(\exp(2\omega s)|u(s)|^2\right)\end{aligned} \tag{A.17}$$

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The differential inequality (A.17) allows us to conclude that the following holds for almost all $t \geqslant t_0 + T_j$:

$$\dot{V}(t+\tau) \leqslant -\mu V(t+\tau) + \frac{\phi^2 \tilde{R}^2}{2\mu} \exp(-2\omega t)\exp(2\omega r)\ \sup_{t_0+T_j \leqslant s \leqslant t}\left(\exp(2\omega s)|x(s)|^2\right)$$
$$+ \frac{\phi^2 \tilde{R}^2}{2\mu}\exp(-2\omega t)\exp(2\omega(r+\tau))\ \sup_{t_0+T_j-\tau \leqslant s \leqslant t}\left(\exp(2\omega s)|u(s)|^2\right) \quad \text{(A.18)}$$

Multiplying (A.18) through by $\exp(\mu(t+\tau))$ and then integrating the result over $[t_0 + T_j, t]$ for any $t \geqslant t_0 + T_j$ and using the fact that $2\omega < \mu$, we obtain the following for all $t \geqslant t_0 + T_j$:

$$\begin{aligned} V(t+\tau) \leqslant{} & \exp(-2\omega(t - t_0 - T_j))V(t_0 + T_j + \tau) \\ & + \frac{\phi^2\tilde{R}^2}{2\mu}\frac{\exp(-2\omega t)}{\mu - 2\omega}\exp(2\omega r)\ \sup_{t_0+T_j \leqslant s \leqslant t}\left(\exp(2\omega s)|x(s)|^2\right) \\ & + \frac{\phi^2\tilde{R}^2}{2\mu}\frac{\exp(-2\omega t)}{\mu - 2\omega}\exp(2\omega(r+\tau))\ \sup_{t_0+T_j-\tau \leqslant s \leqslant t}\left(\exp(2\omega s)|u(s)|^2\right) \end{aligned} \quad \text{(A.19)}$$

Using our quadratic bounds on V from (66) and (A.19) then gives the following for all $t \geqslant t_0 + T_j$:

$$\begin{aligned} |x(t+\tau)|\exp(\omega(t+\tau)) \leqslant{} & \sqrt{K}\exp(\omega(t_0 + T_j + \tau))|x(t_0 + T_j + \tau)| \\ & + \frac{\phi\tilde{R}}{\sqrt{2\mu}}\frac{\exp(\omega(r+\tau))}{\sqrt{\mu - 2\omega}}\sup_{t_0+T_j \leqslant s \leqslant t}(\exp(\omega s)|x(s)|) \\ & + \frac{\phi\tilde{R}}{\sqrt{2\mu}}\frac{\exp(\omega(r+2\tau))}{\sqrt{\mu - 2\omega}}\sup_{t_0+T_j-\tau \leqslant s \leqslant t}(\exp(\omega s)|u(s)|) \end{aligned} \quad \text{(A.20)}$$

Recall from (A.13) that $|\xi(t)| \leqslant \varepsilon$ for all $t \geqslant t_0 + T_j$. Hence, using our upper bound (68) for $|k|$, (A.1), our quadratic bounds (66) on V, (A.7), and our formula (76) for R, we obtain $R(s) \leqslant \tilde{R}s$ for all $s \geqslant 0$, and therefore also the following for all $i \geqslant j$ and $t \in [t_0 + T_i, t_0 + T_{i+1})$:

$$\begin{aligned} |u(t)| = |k(t+\tau, \xi(t))| \ \leqslant\ & \tilde{k}|\xi(t)| \ \leqslant\ \tilde{k}\sqrt{K}|\xi(t_0 + T_i)| \\ \leqslant{} & \tilde{k}\sqrt{K}|\xi(t_0+T_i) - x(t_0 + T_i + \tau)| + \tilde{k}\sqrt{K}|x(t_0 + T_i + \tau)| \\ \leqslant{} & \tilde{R}\tilde{k}\sqrt{K}|x(t_0+T_i)| + \tilde{R}\tilde{k}\sqrt{K}\left\|\breve{T}_\tau(t_0+T_i)u\right\|_\tau + \tilde{k}\sqrt{K}|x(t_0+T_i+\tau)| \end{aligned} \quad \text{(A.21)}$$

Inequality (A.21) and the fact that $\sup_{i \geqslant 0}(T_{i+1} - T_i) \leqslant r$ implies the following for all $i \geqslant j$:

$$\begin{aligned} |u(t)|\exp(\omega t) \leqslant{} & \tilde{R}\tilde{k}\sqrt{K}\exp(\omega r)|x(t_0 + T_i)|\exp(\omega(t_0 + T_i)) \\ & + \tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau))\sup_{t_0+T_i-\tau \leqslant s < t_0+T_i}(\exp(\omega s)|u(s)|) \\ & + \tilde{k}\sqrt{K}\exp(\omega(r-\tau))|x(t_0+T_i+\tau)|\exp(\omega(t_0+T_i+\tau)) \end{aligned} \quad \text{(A.22)}$$

The preceding inequality gives the following for all $t \geq t_0 + T_j$:

$$\begin{aligned} |u(t)|\exp(\omega t) \leqslant{} & \tilde{k}\exp(\omega r)\sqrt{K}\left(\tilde{R} + \exp(-\omega\tau)\right)\sup_{t_0+T_j-\tau \leqslant s \leqslant t}(\exp(\omega(s+\tau))|x(s+\tau)|) \\ & + \tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau))\sup_{t_0+T_j-\tau \leqslant s \leqslant t}(\exp(\omega s)|u(s)|) \end{aligned} \quad \text{(A.23)}$$

Hence, by separately considering cases $\sup_{t_0+T_j-\tau \leqslant s \leqslant t}(\exp(\omega s)|u(s)|) = \sup_{t_0+T_j \leqslant s \leqslant t}(\exp(\omega s)|u(s)|)$ and $\sup_{t_0+T_j-\tau \leqslant s \leqslant t}(\exp(\omega s)|u(s)|) = \sup_{t_0+T_j-\tau \leqslant s < t_0+T_j}(\exp(\omega s)|u(s)|)$, and then recalling (A.16), we get

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$$\begin{aligned}\sup_{t_0+T_j\leqslant s\leqslant t}\exp(\omega s)|u(s)| \leqslant{}& \frac{\tilde{k}\exp(\omega r)\sqrt{K}\left(\tilde{R}+\exp(-\omega\tau)\right)}{1-\tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau))}\sup_{t_0+T_j-\tau\leqslant s\leqslant t}\exp(\omega(s+\tau))|x(s+\tau)| \\ &+\tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau))\sup_{t_0+T_j-\tau\leqslant s<t_0+T_j}\exp(\omega s)|u(s)|\end{aligned} \tag{A.24}$$

for all $t \geqslant t_0+T_j$. By again separately considering the preceding two cases, and then recalling from (A.16) that $\tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau)) < 1$ and combining (A.20) and (A.24), we get the following for all $t \geqslant t_0+T_j$:

$$\begin{aligned}&|x(t+\tau)|\exp(\omega(t+\tau)) \\ &\leqslant \sqrt{K}\exp(\omega(t_0+T_j+\tau))|x(t_0+T_j+\tau)| + \frac{\phi\tilde{R}}{\sqrt{2\mu}}\frac{\exp(\omega(r+2\tau))}{\sqrt{\mu-2\omega}} \\ &\times\left(1+\frac{\tilde{k}\exp(\omega r)\sqrt{K}\left(\tilde{R}+\exp(-\omega\tau)\right)}{1-\tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau))}\right)\sup_{t_0+T_j-\tau\leqslant s\leqslant t}(\exp(\omega(s+\tau))|x(s+\tau)|) \\ &+\frac{\phi\tilde{R}}{\sqrt{2\mu}}\frac{\exp(\omega(r+2\tau))}{\sqrt{\mu-2\omega}}\sup_{t_0+T_j-\tau\leqslant s<t_0+T_j}(\exp(\omega s)|u(s)|)\end{aligned} \tag{A.25}$$

By separately considering the cases

$$\begin{aligned}&\sup_{t_0+T_j-\tau\leqslant s\leqslant t}\exp(\omega(s+\tau))|x(s+\tau)| = \sup_{t_0+T_j-\tau\leqslant s\leqslant t_0+T_j}(\exp(\omega(s+\tau))|x(s+\tau)|) \text{ and} \\ &\sup_{t_0+T_j-\tau\leqslant s\leqslant t}\exp(\omega(s+\tau))|x(s+\tau)| = \sup_{t_0+T_j\leqslant s\leqslant t}(\exp(\omega(s+\tau))|x(s+\tau)|)\end{aligned} \tag{A.26}$$

and using (A.25), we obtain the following for all $t \geqslant t_0+T_j$:

$$\begin{aligned}&|x(t+\tau)|\exp(\omega(t+\tau)) \\ &\leqslant \frac{\sqrt{K}\exp(\omega(t_0+T_j+\tau))}{1-\lambda}|x(t_0+T_j+\tau)| + \frac{\phi\tilde{R}}{\sqrt{2\mu}}\frac{\exp(\omega(r+\tau))}{\sqrt{\mu-2\omega}} \\ &\times\left(1+\frac{\tilde{k}\exp(\omega r)\sqrt{K}\left(\tilde{R}+\exp(-\omega\tau)\right)}{1-\tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau))}\right)\sup_{t_0+T_j-\tau\leqslant s\leqslant t_0+T_j}(\exp(\omega(s+\tau))|x(s+\tau)|) \\ &+\frac{\phi\tilde{R}}{\sqrt{2\mu}}\frac{\exp(\omega(r+2\tau))}{(1-\lambda)\sqrt{\mu-2\omega}}\sup_{t_0+T_j-\tau\leqslant s<t_0+T_j}(\exp(\omega s)|u(s)|),\end{aligned} \tag{A.27}$$

where

$$\lambda = \frac{\phi\tilde{R}}{\sqrt{2\mu}}\frac{\exp(\omega(r+\tau))}{\sqrt{\mu-2\omega}}\left(1+\frac{\tilde{k}\exp(\omega r)\sqrt{K}\left(\tilde{R}+\exp(-\omega\tau)\right)}{1-\tilde{R}\tilde{k}\sqrt{K}\exp(\omega(r+\tau))}\right). \tag{A.28}$$

Inequalities (A.24) and (A.27) imply that there exist positive constants $\bar{S}_1$ and $\bar{S}_2$ such that (88)–(89) hold.

A.1.3. Proof of Claim 3. Choose an arbitrary partition $\{T_i\}_{i=0}^{\infty}$ of $[0,+\infty)$ satisfying $\sup_{i\geqslant 0}(T_{i+1}-T_i) \leqslant r$, and any $(t_0,x_0) \in [0,+\infty)\times\mathbb{R}^2$ and $u_0 \in L^{\infty}([-\tau,0);\mathbb{R})$, and consider the solution of (78)-(80) with (arbitrary) initial conditions $x(t_0)=x_0$ and $\breve{T}_\tau(t_0)u=u_0$. Set $b(s)=\tilde{a}\left(s\sqrt{K}\right)$ for all $s \geqslant 0$, where $\tilde{a} \in \mathcal{K}_\infty$ is from (68). Then $b \in \mathcal{K}_\infty$. Notice that our formula (76) for R implies that

$$R(s) \leqslant \frac{1}{2}b^{-1}\left(\frac{s}{2}\right) \text{ for all } s \geqslant 0. \tag{A.29}$$

Furthermore, inequality (A.2) implies the following for all $i \in \mathbb{Z}_+$ and $t \in [t_0+T_i, t_0+T_{i+1})$:

$$|u(t)| \leqslant b(|\xi(t_0+T_i)|). \tag{A.30}$$

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The quadratic lower bound on V from (66) and (87) give a non-decreasing function g such that

$$|x(t)| \leqslant g(|x_0| + \|u_0\|_\tau) \text{ for all } t \geqslant t_0. \tag{A.31}$$

Combined with (A.7) and (A.29), we get the following for all $i \in \mathbb{Z}_+$:

$$\begin{aligned}
|\xi(t_0+T_i) - x(t_0+T_i+\tau)| &\leqslant R(|x(t_0+T_i)| + \sup_{t_0+T_i-\tau\leqslant s<t_0+T_i} |u(s)|) \\
&\leqslant \frac{1}{2}b^{-1}\left(\frac{1}{2}|x(t_0+T_i)| + \frac{1}{2}\sup_{t_0+T_i-\tau\leqslant s<t_0+T_i} |u(s)|\right) \\
&\leqslant \max\left\{\frac{1}{2}b^{-1}(g(|x_0| + \|u_0\|_\tau)),\ \frac{1}{2}b^{-1}\left(\sup_{t_0+T_i-\tau\leqslant s<t_0+T_i} |u(s)|\right)\right\},
\end{aligned}$$

because $b^{-1}(r+s) \leqslant b^{-1}(2r) + b^{-1}(2s)$ for all $r \geqslant 0$ and $s \geqslant 0$. The aforementioned inequality in conjunction with (A.31) gives the following for all $i \in \mathbb{Z}_+$:

$$\begin{aligned}
&|\xi(t_0+T_i)| \\
&\leqslant |x(t_0+T_i+\tau)| + \max\left\{\frac{1}{2}b^{-1}(g(|x_0| + \|u_0\|_\tau)),\ \frac{1}{2}b^{-1}\left(\sup_{t_0+T_i-\tau\leqslant s<t_0+T_i} |u(s)|\right)\right\} \\
&\leqslant g(|x_0| + \|u_0\|_\tau) + \frac{1}{2}\max\left\{b^{-1}(g(|x_0| + \|u_0\|_\tau)),\ b^{-1}\left(\sup_{t_0+T_i-\tau\leqslant s<t_0+T_i} |u(s)|\right)\right\} \\
&\leqslant \max\left\{2g(|x_0| + \|u_0\|_\tau), b^{-1}(g(|x_0| + \|u_0\|_\tau)),\ b^{-1}\left(\sup_{t_0-\tau\leqslant s<t_0+T_i} |u(s)|\right)\right\}
\end{aligned}$$

where we used $a_1 + a_2 \leqslant \max\{2a_1, 2a_2\}$ and $\max\{\lambda a_1, \lambda a_2\} = \lambda \max\{a_1, a_2\}$, which hold for all $a_i \in [0, +\infty)(i = 1, 2)$ and $\lambda \geqslant 0$. Also, using (A.30) and the above inequality, we obtain the following for all $i \in \mathbb{Z}_+$:

$$\sup_{t_0+T_i\leqslant s<t_0+T_{i+1}} |u(s)| \leqslant \max\left\{\hat{g}(|x_0| + \|u_0\|_\tau),\ \sup_{t_0-\tau\leqslant s<t_0+T_i} |u(s)|\right\}, \tag{A.32}$$

where $\hat{g}(s) = \max\{g(s), b(2g(s))\}$. Define the sequence $F_i = \sup_{t_0-\tau\leqslant s<t_0+T_i} |u(s)|$. The fact that

$$F_{i+1} = \sup_{t_0-\tau\leqslant s<t_0+T_{i+1}} |u(s)| = \max\left\{\sup_{t_0+T_i\leqslant s<t_0+T_{i+1}} |u(s)|,\ \sup_{t_0-\tau\leqslant s<t_0+T_i} |u(s)|\right\} \tag{A.33}$$

in conjunction with (A.32) imply that $F_{i+1} \leqslant \max\{\hat{g}(|x_0| + \|u_0\|_\tau), F_i\}$ holds for all $i \in \mathbb{Z}_+$. This and the fact that $F_0 = \|u_0\|_\tau$ allows us to prove by induction that $F_i \leqslant \max\{\hat{g}(|x_0| + \|u_0\|_\tau), \|u_0\|_\tau\}$ holds for all $i \in \mathbb{Z}_+$. Therefore, (A.31) implies that (90) holds with $S(s) = g(s) + \max\{\hat{g}(s), s\}$. This proves Claim 3.

A.2. Construction of a Function P Satisfying (20)-(22)

We construct a non-decreasing continuous function P satisfying (20)–(22) for the first step of the proof of Theorem 1. Our choice (53) of f implies that the following holds for all $(s, t, x, u) \in [0, +\infty) \times [0, +\infty) \times \mathbb{R}^2 \times \mathbb{R}$:

$$|f(s, x, u) - f(t, x, u)| \leqslant |s - t| \sup_{l\geqslant 0} \left|\frac{\partial}{\partial l}\tilde{f}(l, x)\right| + |s - t||u| \sup_{l\geqslant 0} \left|\frac{\partial}{\partial l}\tilde{g}(l, x)\right| \tag{A.34}$$

Let $\psi_i(s)$ be a continuous non-decreasing function such that $\max\{|\nabla g_i(\zeta)| : |\zeta| \leqslant |\zeta_d|_\infty + s\} \leqslant \psi_i(s)$ for all $s \geqslant 0$ for $i = 1, 2$. Inequality (A.34), in conjunction with the previous inequalities and the definitions of $\tilde{f}$ and $\tilde{g}$ from (53), gives the following inequality for all $(s, t, x, u) \in [0, +\infty) \times [0, +\infty) \times \mathbb{R}^2 \times \mathbb{R}$:

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$$|f(s,x,u)-f(t,x,u)| \leqslant |s-t||u|\psi_2(|x|)\left|\dot{\zeta}_d\right|_\infty + |s-t| \times \left(2\psi_1(|x|)\left|\dot{\zeta}_d\right|_\infty + 2\psi_2(|x|)|v_d|_\infty\left|\dot{\zeta}_d\right|_\infty + |x|\psi_2(|x|)\,|\dot{v}_d|_\infty\right) \tag{A.35}$$

Therefore, (A.35) implies that (22) holds provided that the following inequality holds for all $s \geqslant 0$:

$$\left(2|\dot{\zeta}_d|_\infty\psi_1(s) + 2|v_d|_\infty|\dot{\zeta}_d|_\infty\psi_2(s) + \left(|\dot{v}_d|_\infty + |\dot{\zeta}_d|_\infty\right)s\psi_2(s)\right)^2 \leqslant P(s). \tag{A.36}$$

Define $\tilde{W}$ by $\tilde{W}(x) = 1 + \frac{1}{2}\left(x_2/\left(1+x_1^2\right)\right)^2 + F(\tan^{-1}(x_1))$ for all $x \in \mathbb{R}^2$. Let $\psi_i(s)$ for $i = 3, 4$ be continuous, non-decreasing functions that satisfy

$$\max\left\{\left|\nabla\tilde{W}(x)\right| : |x| \leqslant |\zeta_d|_\infty + s\right\} \leqslant \psi_3(s) \text{ and } \max\left\{\left|\nabla^2\tilde{W}(x)\right| : |x| \leqslant |\zeta_d|_\infty + s\right\} \leqslant \psi_4(s) \tag{A.37}$$

for all $s \geqslant 0$. Because our definition (54) of W gives $W(t,x) = \tilde{W}(\zeta_d(t)+x)$, the previous inequalities give $|(\partial W/\partial x)(t,x)| \leqslant \psi_3(|x|)$ for all $(t,x) \in [0,+\infty)\times\mathbb{R}^2$. Therefore, (21) holds provided that the following holds for all $s \geqslant 0$:

$$\psi_3^2(s) \leqslant P(s) \tag{A.38}$$

Also, if L satisfies the requirements from Assumption 1, then because $W(t,x) = \tilde{W}(\zeta_d(t)+x)$, we can then use the subadditivity of the sup operator and (A.37) to conclude that the following holds for all $s \geqslant 0$:

$$\begin{aligned}
&1 + \sup\left\{\left|\frac{\partial^2 W}{\partial t^2}(t,x)\right| + 2sL(s)\left|\frac{\partial^2 W}{\partial t\,\partial x}(t,x)\right| + s^2L^2(s)\left|\frac{\partial^2 W}{\partial x^2}(t,x)\right| : |x| \leqslant s(1+\tau L(s)), t \geqslant 0\right\} \\
&\leqslant 1 + \psi_4(s(1+\tau L(s)))\left|\dot{\zeta}_d\right|_\infty^2 + \psi_3(s(1+\tau L(s)))\left|\ddot{\zeta}_d\right|_\infty \\
&\quad + 2sL(s)\psi_4(s(1+\tau L(s)))\left|\dot{\zeta}_d\right|_\infty + s^2L^2(s)\psi_4(s(1+\tau L(s)))
\end{aligned} \tag{A.39}$$

Therefore, (A.39) implies that inequality (20) holds provided that

$$1 + \psi_4(s(1+\tau L(s)))\left(|\dot{\zeta}_d|_\infty + sL(s)\right)^2 + \left|\ddot{\zeta}_d\right|_\infty \psi_3(s(1+\tau L(s))) \leqslant P(s) \tag{A.40}$$

holds for all $s \geqslant 0$. We then have our formula for P by adding the left sides of (A.36), (A.38), and (A.40).

A.3. Construction of $M : [0,+\infty) \to [1,+\infty)$ *Satisfying (70)*

Using (64), we obtain the following inequality for all $(t,x,\xi) \in [0,+\infty)\times\mathbb{R}^2\times\mathbb{R}^2$:

$$\begin{aligned}
\tilde{g}(t,x)|k(t,x)-k(t,\xi)| \leqslant \Big|&(1+\mu^2)(x_1-\xi_1) + 2\mu(x_2-\xi_2) + \tilde{f}(t,x) - \tilde{f}(t,\xi) \\
&+ \frac{\tilde{g}(t,\xi)-\tilde{g}(t,x)}{\tilde{g}(t,\xi)}\left((1+\mu^2)\xi_1 + 2\mu\xi_2 + \tilde{f}(t,\xi)\right)\Big|
\end{aligned}$$

Hence, using the facts that $|x_i - \xi_i| \leqslant |x-\xi|$ (for $i = 1, 2$) and the triangle inequality, we obtain the following inequality for all $(t,x,\xi) \in [0,+\infty)\times\mathbb{R}^2\times\mathbb{R}^2$:

$$\begin{aligned}
\tilde{g}(t,x)|k(t,x)-k(t,\xi)| \leqslant (1+\mu)^2|\xi-x| + \left|\tilde{f}(t,x)-\tilde{f}(t,\xi)\right| \\
+ \left|(1+\mu^2)\xi_1 + 2\mu\xi_2 + \tilde{f}(t,\xi)\right|\frac{|\tilde{g}(t,\xi)-\tilde{g}(t,x)|}{\tilde{g}(t,\xi)}
\end{aligned} \tag{A.41}$$

Using Assumption 1 (with $u = 0$), the formula for $\tilde{f}$ from (53), (A.41), the facts that $|\xi_i| \leqslant |\xi|(i = 1, 2)$, and the triangle inequality, we get the following for all $(t,x,\xi) \in [0,+\infty)\times\mathbb{R}^2\times\mathbb{R}^2$:

$$\tilde{g}(t,x)|k(t,x)-k(t,\xi)| \leqslant (1+\mu)^2|\xi-x| + L(|x|+|\xi|)|\xi-x| + ((1+\mu)^2+L(|\xi|))|\xi|\frac{|\tilde{g}(t,\xi)-\tilde{g}(t,x)|}{\tilde{g}(t,\xi)} \tag{A.42}$$

Let $\psi_2(s)$ be a continuous, non-decreasing function that satisfies $\max\{|\nabla g_2(\zeta)| : |\zeta| \leqslant |\zeta_d|_\infty + s\} \leqslant \psi_2(s)$ for all $s \geqslant 0$. Then, (A.42) gives the following for all $(t,x,\xi) \in [0,+\infty) \times \mathbb{R}^2 \times \mathbb{R}^2$:

$$\tilde{g}(t,x)|k(t,x)-k(t,\xi)| \leqslant (1+\mu)^2|\xi-x| + L(|x|+|\xi|)|\xi-x| + ((1+\mu)^2+L(|x|))|\xi|\frac{\psi_2(|x|+|\xi|)}{\min\{g_2(\zeta) : |\zeta| \leqslant |\zeta_d|_\infty + |\xi|\}}|x-\xi| \tag{A.43}$$

Finally, using the facts that $L(|\xi|) \leqslant L(|\xi|+|x|)$, $|\xi| \leqslant |\xi|+|x|$, and $\min\{g_2(\zeta) : |\zeta| \leqslant |\zeta_d|_\infty + |\xi|+|x|\} \leqslant \min\{g_2(\zeta) : |\zeta| \leqslant |\zeta_d|_\infty + |\xi|\}$, in conjunction with (A.43), we conclude that inequality (70) holds with

$$M(s) = ((1+\mu)^2 + L(s))\left(1 + \frac{s\psi_2(s)}{\min\{g_2(\zeta) : |\zeta| \leqslant |\zeta_d|_\infty + s\}}\right) \text{ for all } s \geqslant 0. \tag{A.44}$$

This concludes the construction.

A.4. Tables of Formulas and Constants

The following table shows all constants involved in the feedback (10)-(13). Here, $\psi_i(s)$ is any continuous non-decreasing function that satisfies $\max\{|\nabla g_i(\zeta)| : |\zeta| \leqslant |\zeta_d|_\infty + s\} \leqslant \psi_i(s)$ for all $s \geqslant 0$ for $i = 1, 2$. See pp. 10–12 for some details on their derivations.

c	$1 + \tilde{G}^2 |v_d|_\infty^2$
$\tilde{G}$	$\sup\{G(q,\dot{q}) : (q,\dot{q}) \in (-\pi/2, \pi/2) \times \mathbb{R}\}$
K	$\frac{\mu^2+2+\mu\sqrt{\mu^2+4}}{\mu^2+2-\mu\sqrt{\mu^2+4}}$
γ	$\min\left\{\frac{\varepsilon}{2\sqrt{K}}, \frac{\mu\varepsilon^2}{8K}\right\}$
ϕ	$2KM\left(\varepsilon + \frac{\varepsilon}{2\sqrt{K}}\right)\exp(r\tilde{L})$
$\tilde{L}$	$L\left(\left(1+\tilde{k}\right)\varepsilon + \frac{\varepsilon}{2\sqrt{K}}\right)$
$\tilde{R}$	$\min\left\{\frac{\mu\sqrt{2}}{2\phi\left(1+4\tilde{k}\sqrt{K}\right)}, \frac{1}{2\tilde{k}\sqrt{K}}, \frac{1}{2}\right\}$
$\tilde{k}$	$\frac{(1+\mu)^2+\psi_1(\varepsilon)+|v_d|_\infty\psi_2(\varepsilon)}{\min\{g_2(\zeta):|\zeta|\leqslant|\zeta_d|_\infty+\varepsilon\}}$

The next table gives the functions involved in the feedback (10)–(13). The functions ψ_1 and ψ_2 are as in the first table, and $\psi_i(s)$ for $i = 3, 4$ are any continuous, non-decreasing functions that satisfy $\max\left\{\left|\nabla\tilde{W}(x)\right| : |x| \leqslant |\zeta_d|_\infty + s\right\} \leqslant \psi_3(s)$ and $\max\left\{\left|\nabla^2\tilde{W}(x)\right| : |x| \leqslant |\zeta_d|_\infty + s\right\} \leqslant \psi_4(s)$ for all $s \geqslant 0$. The functions $\theta_i \in \mathcal{K}_\infty$ for $i = 1, 2$ and the constant $R_2 \geqslant 0$ are chosen so that $\theta_1(|x|) \leqslant \tilde{W}(x) \leqslant R_2 + \theta_2(|x|)$ holds for all $x \in \mathbb{R}^2$.

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$N(s)$	$\begin{cases} \left[\tau \max\left\{\frac{a_\tau(s)+s}{2R(s)} L(a_\tau(s)+s)(\exp(\tau A(s))-1), c^{-1}P(Q_\tau(s)+s)\right\}\right]+1, \text{ if } s>0 \\ 1, \text{ if } s=0 \end{cases}$
$L(s)$	$1+\psi_1(s)+(1+s+\lvert v_d\rvert_\infty)\psi_2(s)+\tilde{G}(1+2\lvert\zeta_d\rvert_\infty^2+2s^2)$
$g_1(\zeta)$	$-\left(1+\zeta_1^2\right)\frac{dF}{dq}(\tan^{-1}(\zeta_1))+\frac{2\zeta_1}{1+\zeta_1^2}\zeta_2^2-\left(1+\zeta_1^2\right)H\left(\frac{\zeta_2}{1+\zeta_1^2}\right)$
$g_2(\zeta)$	$\left(1+\zeta_1^2\right)G\left(\tan^{-1}(\zeta_1), \frac{\zeta_2}{1+\zeta_1^2}\right)$
$A(s)$	$L(Q_\tau(s)+a_\tau(s)+s)$
$\tilde{W}(x)$	$1+\frac{1}{2}\left(\frac{x_2}{1+x_1^2}\right)^2+F(\tan^{-1}(x_1))$
$a_\tau(s)$	$s(1+L(Q_\tau(s)+s)\tau)\exp(\tau L(Q_\tau(s)+s))$
$Q_\tau(s)$	$1+\theta_1^{-1}\left(\exp(2c\tau)(R_2+\theta_2(s+\lvert\zeta_d\rvert_\infty))+(2c)^{-1}\exp(2c\tau)\tilde{G}^2s^2\right)+\lvert\zeta_d\rvert_\infty$
$R(s)$	$\min\left\{\frac{\gamma}{\max\{1, D_r(a_\tau(s)+\beta(Q_\tau(s)))\}}, \tilde{R}s, \frac{1}{2\sqrt{K}}\tilde{a}^{-1}\left(\frac{s}{2}\right)\right\}$
$D_r(s)$	$2K(a_r(s)+s)\,M(a_r(s)+s)\exp(rL(a_r(s)+s))$
$M(s)$	$((1+\mu)^2+L(s))\left(1+\frac{s\psi_2(s)}{\min\{g_2(\zeta):\lvert\zeta\rvert\leqslant\lvert\zeta_d\rvert_\infty+s\}}\right)$
$\beta(s)$	$\tilde{a}\left(s\sqrt{K}\right)+s\sqrt{K}$
$\tilde{a}(s)$	$\begin{cases} \frac{(1+\mu)^2+\psi_1(s)+\lvert v_d\rvert_\infty\psi_2(s)}{\min\{g_2(\zeta):\lvert\zeta\rvert\leqslant\lvert\zeta_d\rvert_\infty+s\}}s, \text{ if } s\geqslant\varepsilon \\ \tilde{k}s, \text{ if } s\in[0,\varepsilon) \end{cases}$
$P(s)$	$\left(2\left\lvert\dot{\zeta}_d\right\rvert_\infty\psi_1(s)+2\lvert v_d\rvert_\infty\left\lvert\dot{\zeta}_d\right\rvert_\infty\psi_2(s)+\left(\lvert\dot{v}_d\rvert_\infty+\left\lvert\dot{\zeta}_d\right\rvert_\infty\right)s\psi_2(s)\right)^2+\psi_3^2(s)$ $+1+\psi_4(s(1+\tau L(s)))\left(\left\lvert\dot{\zeta}_d\right\rvert_\infty+sL(s)\right)^2+\left\lvert\ddot{\zeta}_d\right\rvert_\infty\psi_3(s(1+\tau L(s)))$
$\zeta_d(t)$	$\left(\tan(q_d(t)), \frac{\dot{q}_d(t)}{\cos^2(q_d(t))}\right)$
$v_d(t-\tau)$	$\frac{\ddot{q}_d(t)+\frac{dF}{dq}(q_d(t))+H(\dot{q}_d(t))}{G(q_d(t),\dot{q}_d(t))}$

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