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A CONVERSE LYAPUNOV THEOREM FOR NONUNIFORM IN TIME GLOBAL ASYMPTOTIC STABILITY AND ITS APPLICATION TO FEEDBACK STABILIZATION*

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Abstract. Lyapunov-like characterizations for the concepts of nonuniform in time robust global asymptotic stability and input-to-state stability for time-varying systems are established. The main result of our work enables us to derive (1) necessary and sufficient conditions for feedback stabilization for affine in the control systems and (2) sufficient conditions for the propagation of the input-to-state stability property through integrators.

Key words. nonuniform in time asymptotic stability, input-to-state stability, Lyapunov functions, feedback stabilization

AMS subject classifications. 93D20, 93D30, 37B55, 93D15

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1. Introduction.

The notion of nonuniform in time *robust global asymptotic stability* (RGAS) is basically motivated by the problem of feedback stabilization for a class of nonlinear systems that, although fail to be stabilized at a specific equilibrium by continuous static time-invariant feedback, a *time-varying* feedback controller can be constructed in such a way that the equilibrium for the resulting closed-loop time-varying system is asymptotically stable, in general being nonuniform with respect to the initial values of time. The notion of RGAS—without uniformity with respect to time—is also motivated by problems related to feedback stabilization, such as

- stabilization of systems with uncertainties,
- stabilization of systems at a reference trajectory.

In the problems mentioned above, the analysis is reduced to studying asymptotic stability at a specific equilibrium of a time-varying system, whose dynamics are in general unbounded with respect to time. Particularly, in [40, 41] it is shown that for a class of triangular systems whose dynamics contain time-varying unknown parameters, it is possible to find, by applying a backstepping design procedure, a smooth time-varying feedback controller in such a way that the equilibrium of the resulting closed-loop system is RGAS, in general nonuniform with respect to initial values of time. Further progress has been obtained in [12, 13, 14, 15, 16, 17] for a large class of nonlinear systems that in general fail to be uniformly asymptotically stabilized by smooth static time-invariant feedback at a specific equilibrium. It is worthwhile to note that among other things in the works [12, 14], by employing the concept of nonuniform in time RGAS and its Lyapunov characterizations, we derive sufficient conditions for the solvability of the state feedback tracking control problem for a class of nonholonomic systems that includes the nonholonomic case in chained form. The corresponding results generalize those obtained in the literature for the same problem, since they are based on much weaker hypotheses. We finally mention the

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recent work [16], where various equivalent descriptions of nonuniform in time input-to-state stability are proposed and a generalization of the well-known "small-gain theorem" of Jiang, Teel, and Praly in [11] is established for time-varying composite systems.

The main purpose of the present paper is to establish a Lyapunov characterization for the notion of nonuniform in time RGAS. Lyapunov functions play an important role to synthesis and design in control theory, and several important results have been recently established concerning Lyapunov-like descriptions of *robust uniform global asymptotic stability* (RUGAS) and *input-to-state stability* (ISS) (see [2, 4, 5, 6, 8, 9, 19, 20, 21, 24, 25, 33, 34, 43]), *forward completeness* [1], and *asymptotic controllability* (see, for instance, [23, 30]). Our goal is to establish converse Lyapunov theorems for the concepts of *nonuniform in time RGAS* and *nonuniform in time ISS* and give some applications to feedback stabilization. In [42] a converse Lyapunov theorem is established for the particular case of nonuniform in time exponential robust stability and exp-ISS. In the present paper, by extending the approach employed in [20, 34], we establish a Lyapunov characterization for the general concept of RGAS for time-varying systems:

$$\begin{aligned} &\dot{x} = f(t,x,d) \\ &x \in \Re^n, \quad d \in D, \quad t \geq 0. \end{aligned} \tag{1.1}$$

We assume that $D \subset \Re^m$ is a nonempty compact set and $f : \Re^+ \times \Re^n \times D \to \Re^n$ is mapping with $f(t,0,d) = 0$ for all $(t,d) \in \Re^+ \times D$ that satisfies the following hypotheses:

H1. The function $f(t,x,d)$ is measurable in t for all $(x,d) \in \Re^n \times D$.

H2. The function $f(t,x,d)$ is continuous in d for all $(t,x) \in \Re^+ \times \Re^n$.

H3. The function $f(t,x,d)$ is locally Lipschitz with respect to x, uniformly in $d \in D$, in the sense that for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^n$, there exists a constant $L \geq 0$ such that

$$\begin{aligned} &|f(t,x,d) - f(t,y,d)| \leq L|x-y| \\ &\forall t \in I, \quad (x,y) \in S \times S, \quad d \in D. \end{aligned}$$

It turns out from H3 that there exists a positive C^0 function $L : \Re^+ \times \Re^+ \to \Re^+$ such that for each fixed $s \geq 0$ the mappings $L(\cdot, s)$ and $L(s, \cdot)$ are nondecreasing and the following holds:

$$\begin{gathered} |f(t,x,d) - f(t,y,d)| \leq L(t, |x|+|y|)|x-y| \\ \forall (t,x,y,d) \in \Re^+ \times \Re^n \times \Re^n \times D. \end{gathered} \tag{1.2}$$

In section 2, we provide some equivalent characterizations for the concept of RGAS for systems (1.1) (Proposition 2.2), and in section 3, we establish its Lyapunov characterization (Theorem 3.1). Section 4 is devoted to various equivalent descriptions of the nonuniform in time ISS property based on the results obtained for RGAS. The results of section 4 are applicable to the ISS feedback stabilization problem. In section 5.1 we derive a necessary and sufficient Lyapunov-based condition for ISS feedback stabilization for systems of the form

$$\begin{gathered} \dot{x} = f(t,x,v) + g(t,x)u, \\ x \in \Re^n, \quad v \in \Re^l, \quad u \in \Re^m, \quad t \geq 0, \end{gathered} \tag{1.3}$$

where the dynamics $f(\cdot)$ and $g(\cdot) = (g_1(\cdot), g_2(\cdot), \ldots, g_m(\cdot))$ are both C^0 and locally Lipschitz with respect to (x, v) with $f(\cdot, 0, 0) = 0$. (Throughout this paper, given a map $F : \Re^+ \times \Re^{l_1} \to \Re^{l_2}$, we say that it is locally Lipschitz with respect to $x \in \Re^{l_1}$ if for every bounded interval $I \subset \Re^+$ and for every compact subset S of $\Re^{l_1}$, there exists a constant $L \geq 0$ such that $|F(t, x) - F(t, y)| \leq L|x - y|$ for every $(t, x, y) \in I \times S \times S$.) The main results of section 5.1 (Theorem 5.1 and Proposition 5.2) constitute extensions of the well-known Artstein–Sontag theorem [3, 27, 35] for autonomous systems and guarantee existence of a C^∞ mapping $u = k(t, x)$ in such a way that the resulting system

$$\dot{x} = f(t, x, v) + g(t, x)k(t, x) \tag{1.4}$$

satisfies the nonuniform in time ISS property with v as input. An explicit formula for a time-varying feedback stabilizer is proposed in Proposition 5.2. We also prove that, even for autonomous systems for which uniform in time asymptotic stabilization is not feasible, it is possible to exhibit nonuniform in time asymptotic stabilization by means of a time-varying feedback. In section 5.2 we establish an extension of a well-known result concerning the autonomous case (see [11, 36]) for systems of the following form:

$$\dot{x} = f(t, x, y), \tag{1.5a}$$

$$\begin{gathered} \dot{y} = g(t, x, y) + h(t, x, y)u, \\ x \in \Re^n, \quad y \in \Re, \quad u \in \Re, \quad t \geq 0, \end{gathered} \tag{1.5b}$$

where $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are C^0 and locally Lipschitz with respect to (x, y), with $f(\cdot, 0, 0) = 0$ and $g(\cdot, 0, 0) = 0$. Particularly, we show that, under the presence of the (nonuniform in time) ISS for the subsystem (1.5a) with y as input, there exists a feedback law exhibiting ISS stabilization for (1.5) (Proposition 5.6). This result enables us to examine the partial-state feedback stabilization problem for triangular systems. Particularly, by exploiting a Lyapunov function based approach we re-establish the main result in [40] for a special class of triangular systems whose dynamics are time-dependent.

Notations. Throughout this paper we adopt the following notations:

* By M_D we denote the set of all measurable functions from $\Re^+ := [0, +\infty)$ to D, where D is any given compact subset of $\Re^m$.
* For any $x \in \Re^n$, x^T denotes its transpose and $|x|$ its usual Euclidean norm.
* K^+denotes the class of positive nondecreasing C^∞ functions $\phi : \Re^+ \to (0, +\infty)$, and $\mathbf{E}$ denotes the class of nonnegative C^0 functions $\mu : \Re^+ \to \Re^+$, for which $\int_0^{+\infty} \mu(t)dt < +\infty$ and $\lim_{t\to+\infty} \mu(t) = 0$ hold.
* $\mathbf{L}^\infty_{loc}$ denotes the set of all measurable functions $u : \Re^+ \to \Re^m$ that are essentially bounded on any nonempty compact subset of $\Re^+$, and $\mathbf{L}^\infty$ denotes the set of all measurable functions $u : \Re^+ \to \Re^m$ that are essentially bounded on $\Re^+$.
* By $B[x, r]$, where $x \in \Re^n$ and $r > 0$, we denote the closed sphere in $\Re^n$ of radius r centered at x.
* By $x(t) = x(t, t_0, x_0; d)$ we denote the solution of (1.1) at time t that corresponds to some input $d \in M_D$ initiated from x_0 at time t_0. For convenience, in certain parts of the text we prefer the notation $\phi(\cdot)$ instead of $x(\cdot)$.
* For definitions of classes K, K_∞, KL, see [18, 20].
* By Π we denote the subclass of K_∞ consisting of all functions $r : \Re^+ \to \Re^+$, for which $r(s) = \sum_{i=1}^m a_i s^i$ with $a_i \geq 0$ for $i = 1, \ldots, m$, $a_1 > 0$ for some positive integer m.

2. The notion of RGAS. In this section we provide a general concept of global asymptotic stability (GAS) and establish some facts that will be used in proofs of main results in sections 3 and 5.

DEFINITION 2.1. *We say that zero $0 \in \Re^n$ is RGAS for* (1.1) *if for every $t_0 \geq 0$, $d \in M_D$, and $x_0 \in \Re^n$, the corresponding solution $x(\cdot)$ of* (1.1) *exists for all $t \geq t_0$ and satisfies the following properties:*

P1 (stability). *For every $\varepsilon > 0$, $T \geq 0$, it holds that*

$$\sup\{|x(t)| : d \in M_D, t \geq t_0, |x_0| \leq \varepsilon, t_0 \in [0,T]\} < +\infty \text{ (Lagrange stability)} \tag{2.1a}$$

and there exists a $\delta := \delta(\varepsilon, T) > 0$ such that

$$|x_0| \leq \delta, \quad t_0 \in [0,T] \Rightarrow |x(t)| \leq \varepsilon \quad \forall t \geq t_0, \quad d \in M_D \text{ (Lyapunov stability).} \tag{2.1b}$$

P2 (attractivity). *For every $\varepsilon > 0$, $T \geq 0$, and $R \geq 0$, there exists a $\tau := \tau(\varepsilon, T, R) \geq 0$ such that*

$$|x_0| \leq R, \quad t_0 \in [0,T] \Rightarrow |x(t)| \leq \varepsilon \quad \forall t \geq t_0 + \tau, \quad d \in M_D. \tag{2.1c}$$

As in the case of uniform in time RUGAS (see [20]) we have the following proposition.

PROPOSITION 2.2. *The origin $0 \in \Re^n$ is RGAS for* (1.1) *if and only if there exist a pair of functions a_1, a_2 of class K_∞, a_1 being locally Lipschitz on $(0, +\infty)$, and a function β of class K^+ such that for every $d \in M_D$, $t_0 \geq 0$, and $x_0 \in \Re^n$ the following holds:*

$$a_1(|x(t)|) \leq \exp(-t + t_0)\beta(t_0)a_2(|x_0|) \quad \forall t \geq t_0. \tag{2.2}$$

The proof of Proposition 2.2 requires the following technical result.

LEMMA 2.3. *Let $a : \Re^+ \times \Re^+ \to \Re^+$ be a function with $a(\cdot, 0) = 0$ that satisfies the following properties:*

(1) *For each fixed $t \geq 0$, the mapping $a(t, \cdot)$ is nondecreasing.*

(2) *For each fixed $s \geq 0$, the mapping $a(\cdot, s)$ is nondecreasing.*

(3) $\lim_{s \to 0^+} a(t, s) = 0$ *for all $t \geq 0$.*

Then there exists a pair of functions $\zeta \in K_\infty$ and $\gamma \in K^+$ such that

$$a(t, s) \leq \zeta(\gamma(t)s) \qquad \forall (t, s) \in (\Re^+)^2. \tag{2.3}$$

Proof of Lemma 2.3. Without loss of generality we may assume that a is $C^0(\Re^+ \times \Re^+)$. Indeed, otherwise we may consider the function

$$\hat{a}(t,s) := \begin{cases} \dfrac{1}{s} \displaystyle\int_s^{2s} \int_t^{t+1} a(\tau, \xi) d\tau \, d\xi & \text{for } s > 0, \\ 0 & \text{for } s = 0, \end{cases}$$

which by virtue of the inequality $a(t, s) \leq \hat{a}(t, s) \leq a(t + 1, 2s)$ is $C^0(\Re^+ \times \Re^+)$ and satisfies $\hat{a}(\cdot, 0) = 0$. Notice that $\hat{a}$ has the same properties (1)–(3) of our statement with a. By invoking property (3), there exists a C^0 strictly decreasing function $\eta : \Re^+ \to (0, +\infty)$ with $\lim_{t \to +\infty} \eta(t) = 0$ such that

$$s \leq \eta(t) \Rightarrow a(t, s) \leq \frac{1}{t + 1}. \tag{2.4a}$$

Let μ be the inverse function of η defined on $(0, \eta(0)]$ being nonnegative, continuous, strictly decreasing with $\lim_{t\to 0^+} \mu(t) = +\infty$. Define

$$\tilde{\mu}(s) := \begin{cases} \mu(s) & \text{if} \quad s \in (0, \eta(0)], \\ 0 & \text{if} \quad s > \eta(0). \end{cases} \tag{2.4b}$$

It turns out that $\tilde{\mu} : (0, +\infty) \to \Re^+$ is nonincreasing, continuous, and nonnegative and satisfies $\lim_{t\to 0^+} \tilde{\mu}(t) = +\infty$. Additionally, define

$$\beta(s) := s + \begin{cases} 0 & \text{if} \quad s = 0, \\ \sup\limits_{0<\tau\le s} a(\tilde{\mu}(\tau), \tau) & \text{if} \quad s > 0. \end{cases} \tag{2.5}$$

We show that $\beta \in K_\infty$. Indeed, by definition (2.5) it follows that $\beta(0) = 0$ and β is strictly increasing with $\lim_{s\to +\infty} \beta(s) = +\infty$. Continuity of β on $(0, +\infty)$ follows from the fact that both a and $\tilde{\mu}$ are C^0 on $(0, +\infty)$. Furthermore, notice that (2.4a) and (2.4b) imply

$$a(\tilde{\mu}(\tau), \tau) \le \frac{1}{\tilde{\mu}(\tau) + 1} \le \frac{1}{\tilde{\mu}(s) + 1} \qquad \forall \tau \in (0, s] \text{ and } s \le \eta(0). \tag{2.6}$$

Since $\lim_{s\to 0^+} \tilde{\mu}(s) = +\infty$ it follows from (2.6) that $\lim_{s\to 0^+} \beta(s) = 0$, and this establishes continuity of β at zero. Let $\zeta(s) := a(s, s) + \beta(s)$. Obviously, $\zeta(\cdot)$ is of class K_∞. Moreover, when $s \ge t$, by virtue of property (2) it holds that $a(t, s) \le a(s, s) \le \zeta(s)$, which implies

$$\sup_{s\ge t>0} \frac{\zeta^{-1}(a(t, s))}{s} \le 1. \tag{2.7}$$

Also, when $0 < s \le \eta(t)$, it follows from (2.4b) that $\tilde{\mu}(s) \ge t$; hence, by virtue of property (2) and (2.5), $a(t, s) \le a(\tilde{\mu}(s), s) \le \zeta(s)$. The latter implies that

$$\sup_{0<s\le \eta(t)} \frac{\zeta^{-1}(a(t, s))}{s} \le 1. \tag{2.8}$$

Using property (1), (2.7), and (2.8) we get

$$\sup_{s>0} \frac{\zeta^{-1}(a(t, s))}{s} \le 1 + \sup_{\eta(t)\le s\le t} \frac{\zeta^{-1}(a(t, s))}{s} \le 1 + \frac{\zeta^{-1}(a(t, t))}{\eta(t)}. \tag{2.9}$$

Finally let γ be any function of class K^+ which satisfies

$$\gamma(t) \ge \frac{\zeta^{-1}(a(t, t))}{\eta(t)} + 1 \qquad \forall t \ge 0. \tag{2.10}$$

The desired (2.3) is a consequence of (2.9) and (2.10). $\square$

We are in a position to establish Proposition 2.2. Its proof is based on Lemma 2.3 and is inspired by the analysis made in [32].

Proof of Proposition 2.2. ($\Rightarrow$) Suppose that $0 \in \Re^n$ is RGAS for (1.1). Let $\xi, T, s \ge 0$ and define

$$a(T, s) := \sup\{|x(t)| : d \in M_D, t \ge t_0, |x_0| \le s, 0 \le t_0 \le T\}, \tag{2.11a}$$

$$M(\xi, T, s) := \sup\{|x(t_0 + \xi)| : d \in M_D, |x_0| \le s, 0 \le t_0 \le T\}. \tag{2.11b}$$

Obviously, our hypothesis that $0 \in \Re^n$ is RGAS guarantees that both $a(\cdot)$ and $M(\cdot)$ are well defined. Moreover, $a(\cdot)$ satisfies all hypotheses of the Lemma 2.3; namely, for each fixed $s \geq 0$, $a(\cdot, s)$ is nondecreasing, and for each fixed $T \geq 0$, $a(T, \cdot)$ is nondecreasing and satisfies $a(\cdot, 0) = 0$. Furthermore, stability of zero asserts that, for every $T \geq 0$, $\lim_{s \to 0^+} a(T, s) = 0$. It turns out from Lemma 2.3 that there exist functions $\zeta_1 \in K_\infty$ and $\gamma_1 \in K^+$ such that

$$a(T, s) \leq \zeta_1(\gamma_1(T)s) \qquad \forall (T, s) \in (\Re^+)^2. \tag{2.12}$$

The previous inequality in conjunction with (2.11a) and (2.11b) implies

$$M(\xi, T, s) \leq \zeta_1(\gamma_1(T)s) \qquad \forall (\xi, T, s) \in (\Re^+)^3. \tag{2.13}$$

Moreover, attractivity of zero guarantees that for every $\varepsilon > 0$, $T \geq 0$, and $R \geq 0$, there exists a $\tau = \tau(\varepsilon, T, R) \geq 0$ such that

$$M(\xi, T, s) \leq \varepsilon \qquad \forall \xi \geq \tau(\varepsilon, T, R) \text{ and } 0 \leq s \leq R. \tag{2.14}$$

Let

$$g(s) := \sqrt{s} + s^2 \tag{2.15a}$$

and let p be a function of class K^+ with $p(0) = 1$ and

$$\lim_{t \to +\infty} p(t) = +\infty. \tag{2.15b}$$

Define

$$\mu(\xi) := \sup \left\{ \frac{M(\xi, T, s)}{p(T) g(\zeta_1(\gamma_1(T)s))}, T \geq 0, s > 0 \right\}. \tag{2.16}$$

Obviously, by (2.12) and (2.15a), the function $\mu : \Re^+ \to \Re^+$ is well defined and satisfies $\mu(\cdot) \leq 1$. We show that $\lim_{\xi \to +\infty} \mu(\xi) = 0$; equivalently, we establish that for any given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) \geq 0$ such that

$$\mu(\xi) \leq \varepsilon \text{ for } \xi \geq \delta(\varepsilon). \tag{2.17}$$

Notice first that for any given $\varepsilon > 0$ there exist constants $a := a(\varepsilon)$ and $b := b(\varepsilon)$ with $0 < a < b$ such that

$$x \notin (a, b) \Rightarrow \frac{x}{\sqrt{x} + x^2} \leq \varepsilon. \tag{2.18}$$

We next recall (2.15b), which asserts that, for the above ε for which (2.18) holds, there exists a $c := c(\varepsilon) \geq 0$ such that $p(T) \geq \frac{1}{\varepsilon}$ for all $T \geq c$. This by virtue of (2.13) and (2.15a) yields

$$\frac{M(\xi, T, s)}{p(T) g(\zeta_1(\gamma_1(T)s))} \leq \varepsilon \qquad \forall \xi \geq 0 \tag{2.19a}$$

$$\text{when either } T \geq c \text{ or } \zeta_1(\gamma_1(T)s) \notin (a, b). \tag{2.19b}$$

Hence, in order to establish (2.17), it remains to consider the case

$$a \leq \zeta_1(\gamma_1(T)s) \leq b \quad \text{and} \quad 0 \leq T \leq c. \tag{2.20}$$

Since, for each fixed $(\xi, s) \in (\Re^+)^2$, the mappings $M(\xi, \cdot, s)$, $M(\xi, T, \cdot)$, $\gamma_1(\cdot)$, and $p(\cdot)$ are nondecreasing, we have that

$$\frac{M(\xi, T, s)}{p(T)g(\zeta_1(\gamma_1(T)s))} \leq \frac{M\left(\xi, c, \frac{\zeta_1^{-1}(b)}{\gamma_1(0)}\right)}{g(a)} \tag{2.21}$$

provided that (2.20) holds. By using (2.14) and (2.21) with

$$\varepsilon := \varepsilon g(a), \quad T := c, \quad R := \frac{\zeta_1^{-1}(b)}{\gamma_1(0)},$$

it follows that

$$M\left(\xi, c, \frac{\zeta_1^{-1}(b)}{\gamma_1(0)}\right) \leq \varepsilon g(a) \quad \text{for } \xi \geq \delta(\varepsilon) := \tau\left(\varepsilon g(a), c, \frac{\zeta_1^{-1}(b)}{\gamma_1(0)}\right). \tag{2.22}$$

By taking into account (2.19), (2.20), (2.21), (2.22), and definition (2.16) of $\mu(\cdot)$, it follows that (2.17) holds with $\delta = \delta(\varepsilon)$ as selected in (2.22). Since $\varepsilon > 0$ was arbitrary we conclude that $\lim_{\xi \to +\infty} \mu(\xi) = 0$. Consequently, there exists a continuous strictly decreasing function $\bar{\mu} : \Re^+ \to (0, +\infty)$ such that $\bar{\mu}(\xi) \geq \mu(\xi)$ for all $\xi \geq 0$ and $\lim_{\xi \to +\infty} \bar{\mu}(\xi) = 0$. Thus, by recalling definition (2.16) we obtain

$$M(\xi, T, s) \leq \bar{\mu}(\xi)\theta(T, s) \qquad \forall (T, s) \in (\Re^+)^2, \quad \forall \xi \geq 0, \tag{2.23}$$

where $\theta(T, s) := p(T)g(\zeta_1(\gamma_1(T)s))$. Clearly, θ satisfies all hypotheses of Lemma 2.3 and therefore there exist $\zeta_2 \in K_\infty$ and $\gamma_2 \in K^+$ such that

$$\theta(T, s) \leq \zeta_2(\gamma_2(T)s) \qquad \forall (T, s) \in (\Re^+)^2. \tag{2.24}$$

Moreover, by recalling Proposition 7 in [32] there exist functions a_1, ρ of class K_∞, a_1, being locally Lipschitz on $(0, +\infty)$, such that the KL function $\mu(t)\zeta_2(s)$ is dominated by $a_1^{-1}(\exp(-t)\rho(s))$. Thus, by taking into account (2.11b), (2.23), and (2.24) we have

$$|x(t)| \leq a_1^{-1}(\exp(-t + t_0)\rho(\gamma_2(t_0)|x_0|)) \quad \forall t \geq t_0 \geq 0,\ x_0 \in \Re^n,\ d \in M_D. \tag{2.25}$$

By Corollary 10 in [32] a pair of functions a_2, $\tilde{\beta}$ of class K_∞ can be found such that

$$\rho(rs) \leq \tilde{\beta}(r)a_2(s) \qquad \forall r, s \geq 0, \tag{2.26}$$

and finally, let β be a function of class K^+ with

$$\tilde{\beta}(\gamma_2(t)) \leq \beta(t), \qquad t \geq 0. \tag{2.27}$$

The desired (2.2) is a consequence of (2.25), (2.26), and (2.27).

($\Leftarrow$) Conversely, assume that (2.2) holds. Existence of $x(\cdot)$ for all $t \geq t_0$ as well as (2.1a) are both immediate consequences of (2.2). Let $\varepsilon > 0$ and $T \geq 0$ be arbitrary constants. By selecting $\delta(\varepsilon, T) := a_2^{-1}(\frac{a_1(\varepsilon)}{\beta(T)})$ the desired (2.1b) is fulfilled; thus property P1 holds (stability). Moreover, for any arbitrary positive constants R, ε, T, we may select $\tau = \tau(\varepsilon, T, R) := -\log(\frac{a_1(\varepsilon)}{\beta(T)a_2(R)})$, and by using (2.2) it follows that (2.1c) is fulfilled, and this establishes property P2 (attractivity). $\square$

Remark 2.4.

* The notion of RGAS above is an extension of the well-known Sontag's robust uniform GAS (RUGAS) for autonomous systems, namely, when the solution $x(\cdot)$ satisfies $|x(t)| \leq G(|x_0|, t-t_0)$ for certain G of class KL (see, for instance, [18, 20]). To justify this, we may recall Proposition 7 in [32], which asserts that for any $G \in KL$ there exist functions a_1 and a_2 of class K_∞ with $G(s,t) \leq a_1^{-1}(\exp(-t)a_2(s))$. It turns out that RUGAS is characterized by the inequality $a_1(|x(t)|) \leq \exp(-t+t_0)a_2(|x_0|)$, which obviously is a special case of (2.2).
* It is also straightforward to see that, if (2.2) holds with β being bounded over $\Re^+$, then zero is RUGAS and thus it turns out that for this case RGAS is equivalent to RUGAS.

Finally, we provide the following proposition, which generalizes the well-known fact that for autonomous differential equations equi-attractivity implies stability (see [10]). The result of this proposition will be used in sections 3 and 5.

PROPOSITION 2.5. *The origin* $0 \in \Re^n$ *is RGAS for* (1.1) *if for every* $t_0 \geq 0$, $d \in M_D$, *and* $x_0 \in \Re^n$, *the corresponding solution* $x(\cdot)$ *of* (1.1) *exists for all* $t \geq t_0$ *and satisfies property* P2 *(attractivity) of Definition* 2.1 *and* (1.1) *is Lagrange stable; namely, for every* $\varepsilon > 0$ *and* $T \geq 0$, (2.1a) *holds. It turns out that, if there exist a constant* $M \geq 0$, *functions* $a_2 \in K_\infty$, $\sigma \in KL$, *and* $\beta \in K^+$ *such that the estimate*

$$|x(t)| \leq \sigma(a_2(\beta(t_0)|x_0|) + M, t-t_0) \qquad \forall t \geq t_0,\ (t_0, x_0) \in \Re^+ \times \Re^n,\ d(\cdot) \in M_D, \tag{2.28}$$

holds for the solution $x(\cdot)$ *of* (1.1)*, then* $0 \in \Re^n$ *is RGAS for* (1.1).

Proof. It suffices to show that for every $\varepsilon > 0$, $T \geq 0$, there exists a $\delta := \delta(\varepsilon, T) > 0$ such that (2.1b) holds. Let $\varepsilon > 0$, $T \geq 0$ be arbitrary. Define

$$R(\varepsilon, T) := \sup\{|x(t)| : d \in M_D, t \geq t_0, |x_0| \leq \varepsilon, t_0 \in [0,T]\}. \tag{2.29}$$

By taking into account (1.2), (2.29), completeness of solutions, and our assumption that zero $0 \in \Re^n$ is an equilibrium for (1.1), it follows by use of Gronwall's inequality that

$$|x(t)| \leq \exp\left(\int_{t_0}^{t} L(s, R(\varepsilon, T))ds\right)|x_0| \quad \forall t \geq t_0,\ d(\cdot) \in M_D,\ |x_0| \leq \varepsilon,\ t_0 \in [0,T]. \tag{2.30}$$

Moreover, property P2 of Definition 2.1 implies that for every $\varepsilon > 0$, $T \geq 0$, there exists a $\tau := \tau(\varepsilon, T) \geq 0$ such that

$$|x_0| \leq \varepsilon,\ t_0 \in [0,T] \Rightarrow |x(t)| \leq \varepsilon \quad \forall t \geq t_0 + \tau,\ d \in M_D. \tag{2.31}$$

Define

$$\delta(\varepsilon, T) := \varepsilon \exp\left(-\int_0^{T+\tau(\varepsilon,T)} L(s, R(\varepsilon, T))ds\right) \leq \varepsilon \tag{2.32}$$

and notice that estimate (2.30) and definition (2.32) guarantee the following implication:

$$|x_0| \leq \delta(\varepsilon, T),\ t_0 \in [0,T] \Rightarrow |x(t)| \leq \varepsilon \quad \forall t \in [t_0, t_0 + \tau(\varepsilon, T)],\ d(\cdot) \in M_D. \tag{2.33}$$

The desired implication (2.1b) is an immediate consequence of (2.31) and (2.33).

Finally, notice that when estimate (2.28) holds, then property P2 holds and (1.1) is Lagrange stable; hence zero is RGAS. $\square$

3. A converse Lyapunov theorem for RGAS. We next establish a Lyapunov characterization of the notion of RGAS, which constitutes generalization of the main result in [20] for the RUGAS case. Its proof is inspired from the analysis employed in [6, 20, 34].

THEOREM 3.1. *For the system* (1.1) *suppose that* H1, H2, H3 *are fulfilled and further* $f \in C^0(\Re^+ \times \Re^n \times D; \Re^n)$. *Then the following statements are equivalent:*

(i) *Zero* $0 \in \Re^n$ *is RGAS.*

(ii) *There exist a* C^∞ *function* $V : \Re^+ \times \Re^n \to \Re^+$, *functions* $\bar{a}_1$, $\bar{a}_2$ *of class* K_∞, $\bar{\beta}$ *of class* K^+ *such that for all* $(t, x, d) \in \Re^+ \times \Re^n \times D$ *it holds that*

$$\bar{a}_1(|x|) \le V(t,x) \le \bar{a}_2(\bar{\beta}(t)|x|), \tag{3.1a}$$

$$\dot{V}(t,x,d)\big|_{(1.1)} := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x,d) \le -V(t,x). \tag{3.1b}$$

(iii) *There exist a* C^1 *function* $V : \Re^+ \times \Re^n \to \Re^+$, *functions* $\bar{a}_1$, $\bar{a}_2$ *of class* K_∞, $\bar{\beta}$ *of class* K^+, μ *of class* $\mathbf{E}$ *(see notations for the definition of class* $\mathbf{E}$*), and a* C^0 *positive definite function* $\rho : \Re^+ \to \Re^+$ *such that for all* $(t, x, d) \in \Re^+ \times \Re^n \times D$ *it holds that*

$$\bar{a}_1(|x|) \le V(t,x) \le \bar{a}_2(\bar{\beta}(t)|x|), \tag{3.2a}$$

$$\dot{V}(t,x,d)\big|_{(1.1)} := \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) f(t,x,d) \le -\rho(V(t,x)) + \mu(t). \tag{3.2b}$$

For the proof of Theorem 3.1 we need a pair of technical lemmas. The first constitutes an extension of [20, Lemma 4.4] and was inspired by the main result in [22].

LEMMA 3.2. *Let* $y_d : \Re^+ \to \Re^+$ *be a family of absolutely continuous functions parameterized by* $d \in A$ *that satisfies the following differential inequality for almost all* $t \ge t_0$:

$$\dot{y}_d(t) \le -\rho(y_d(t)) + \mu(t), \tag{3.3}$$

where $\rho : \Re^+ \to \Re^+$ *is a* C^0 *positive definite function and* μ *is of class* $\mathbf{E}$. *Then there exists a KL function* $\sigma : (\Re^+)^2 \to \Re^+$ *such that for all* $y_d(t_0) = y_0 \ge 0$ *and* $d \in A$ *it holds that*

$$y_d(t) \le \sigma\left(y_0 + \int_{\Re^+} \mu(t)dt, t - t_0\right) \qquad \forall t \ge t_0. \tag{3.4}$$

Proof. Without loss of generality we may assume that $\int_{\Re^+} \mu(t)dt > 0$ (otherwise $\mu(t) = 0$ for all $t \ge 0$ and this is exactly the case of [20, Lemma 4.4]). First, notice that (3.3) yields

$$y_d(t) \le y_0 + M \qquad \forall t \ge t_0,\ d \in A, \tag{3.5}$$

$$M := \int_{\Re^+} \mu(t)dt, \tag{3.6}$$

and this shows that $y_d(t)$ is bounded. Let $R \ge 0$ and $0 < \varepsilon \le R + M$. Since $\lim_{t\to+\infty} \mu(t) = 0$ for any constants $r, \varepsilon > 0$ there exists a time $\tau := \tau(\varepsilon, r) \ge 0$ such that

$$t \ge \tau \Rightarrow \mu(t) \le \min\left\{\frac{1}{2}\rho(s); \frac{\varepsilon}{2} \le s \le r\right\}. \tag{3.7}$$

We now show that the region

$$L_{\varepsilon,R} := \{(t,y) \in \Re^+ \times \Re^+ : y \leq \varepsilon, t \geq \tau(\varepsilon, R+M)\} \tag{3.8}$$

is positively invariant. To see this, notice that, when $R + M \geq y_d(t) \geq \frac{\varepsilon}{2}$ and $t \geq \tau(\varepsilon, R+M)$ for some $d \in A$, then by (3.3) and (3.7) we have

$$\dot{y}_d(t) \leq -\rho(y_d(t)) + \mu(t) \leq -\frac{1}{2}\rho(y_d(t)) < 0 \tag{3.9}$$

and this establishes positive invariance of $L_{\varepsilon,R}$. We next establish that, if we define

$$T(\varepsilon, r) := \tau(\varepsilon, r) + \frac{2r}{\min_{\varepsilon \leq s \leq r} \rho(s)}, \tag{3.10}$$

then the following is fulfilled:

(3.11) For every $t \geq t_0 + T(\varepsilon, R+M)$, $\quad d \in A$ and $y_0 \leq R \Rightarrow (t, y_d(t)) \in L_{\varepsilon,R}$.

Indeed, otherwise, by positive invariance of $L_{\varepsilon,R}$ there would exist $d \in A$ and $y_0 \leq R$ such that

$$(t, y_d(t)) \notin L_{\varepsilon,R} \qquad \forall t \in [t_0 + \tau(\varepsilon, R+M), t_0 + T(\varepsilon, R+M)],$$

and since $t \geq \tau(\varepsilon, R, M)$, we would have

$$y_d(t) > \varepsilon \qquad \forall t \in [t_0 + \tau, t_0 + T]. \tag{3.12}$$

On the other hand, by (3.3), (3.5), (3.7) and (3.12), it follows that

$$\dot{y}_d(t) \leq -\frac{1}{2} \min_{\varepsilon \leq s \leq R+M} \rho(s) \qquad \forall t \in [t_0 + \tau, t_0 + T]. \tag{3.13}$$

It turns out from (3.12) and (3.13) that

$$\varepsilon < y_d(t) \leq R + M - \frac{1}{2}(t - t_0 - \tau) \min_{\varepsilon \leq s \leq R+M} \rho(s) \qquad \forall t \in [t_0 + \tau, t_0 + T]. \tag{3.14}$$

Using (3.14) and taking into account definition (3.10) of $T(\cdot)$ we get $\varepsilon < y_d(t_0+T) \leq 0$, which is a contradiction. This establishes (3.11).

Positive invariance of $L_{\varepsilon,R}$ and property (3.11) guarantee that the following attractivity property holds:

$$\begin{aligned} &\text{For all } (\varepsilon, R, t_0, d) \in (0, +\infty) \times \Re^+ \times \Re^+ \times A \text{ and} \\ &t \geq t_0 + T(\varepsilon, R+M), y_0 \leq R \Rightarrow 0 \leq y_d(t) \leq \varepsilon. \end{aligned} \tag{3.15}$$

In order to establish inequality (3.4), we exploit (3.15) and apply an approach similar to that used in Proposition 2.2. We proceed as follows. Define

$$g(s) := \sqrt{s} + s^2, \tag{3.16a}$$

$$v(t) := \sup\left\{\frac{y_d(t_0+\xi)}{g(y_0+M)}; d \in A, y_0 \geq 0, t_0 \geq 0, \xi \geq t\right\}, \tag{3.16b}$$

where $M > 0$ is defined by (3.6). Since $M > 0$, the denominator in (3.16b) is strictly positive and (3.5), (3.16a) imply that $v(t) \leq 1$ for all $t \geq 0$. We show that

$\lim_{t \to +\infty} v(t) = 0$. Let $\varepsilon > 0$ and let $a := a(\varepsilon)$, $b := b(\varepsilon)$ be a pair of constants with $0 < a < b$ and being defined in such a way that $x \notin [a, b] \Rightarrow x/(\sqrt{x} + x^2) < \varepsilon$. Then by (3.5) it follows that

$$\frac{y_d(t_0 + \xi)}{g(y_0 + M)} < \varepsilon \quad \forall \xi \geq 0,\ d \in A,\ \text{and } t_0 \geq 0,$$
$$\text{provided that either } y_0 + M < a \text{ or } y_0 + M > b. \tag{3.17a}$$

It remains to consider the case $a \leq y_0 + M \leq b$. By (3.15) we get

$$\frac{y_d(t_0 + \xi)}{g(y_0 + M)} \leq \frac{y_d(t_0 + \xi)}{g(a)} \leq \varepsilon \quad \forall \xi \geq T(\varepsilon g(a), b),\ \forall (t_0, d) \in \Re^+ \times A. \tag{3.17b}$$

It turns out from (3.16b), (3.17a), and (3.17b) that

$$v(t) \leq \varepsilon \qquad \forall t \geq T(\varepsilon g(a), b). \tag{3.18}$$

Since $\varepsilon > 0$ was arbitrary, (3.18) asserts that $\lim_{t \to +\infty} v(t) = 0$. Finally, let $\bar{v} : \Re^+ \to (0, +\infty)$ be a C^0, strictly decreasing function, with $v(t) \leq \bar{v}(t)$ for all $t \geq 0$ and in such a way that $\lim_{t \to +\infty} \bar{v}(t) = 0$. Then, obviously (3.4) is fulfilled with $\sigma(s, t) := g(s)\bar{v}(t)$. ∎

The second technical lemma provides a Lyapunov characterization of RGAS for (1.1) when its dynamics $f(\cdot)$ satisfy hypotheses H1, H2, and H3.

LEMMA 3.3. *Consider system* (1.1) *where its dynamics satisfy hypotheses* H1*,* H2 *and* H3 *and assume that* $0 \in \Re^n$ *is RGAS for* (1.1)*; particularly, there exists a pair of functions* a_1*,* a_2 *of class* K_∞*,* a_1 *being locally Lipschitz on* $(0, +\infty)$*, and a function* β *of class* K^+ *in such a way that* (2.2) *is satisfied. Then there exists a* C^0 *function* $U : \Re^+ \times \Re^n \to \Re^+$*, which is locally Lipschitz on* $\Re^+ \times (\Re^n \setminus \{0\})$*, such that for all* $(t, x) \in \Re^+ \times \Re^n$*,* $h \geq 0$*, and* $d(\cdot) \in M_D$ *it holds that*

$$a_1(|x|) \leq U(t, x) \leq \beta(t) a_2(|x|), \tag{3.19a}$$

$$U(t + h, \phi(t + h, t, x; d)) \leq \exp\left(-\frac{h}{2}\right) U(t, x)$$
$$\forall (t, x) \in \Re^+ \times \Re^n,\ h \geq 0,\ d(\cdot) \in M_D, \tag{3.19b}$$

where for convenience we adopt the notation $\phi(\cdot, t, x; d)$ *to denote the solution of* (1.1) *that corresponds to the input* $d(\cdot) \in M_D$*, with* $\phi(t, t, x; d) = x$*.*

Proof. For the proof we need the following elementary properties for the solution of (1.1), which are immediate consequences of (1.2) and (2.2):

$$|\phi(t, t_0, x; d) - \phi(t, t_0, y; d)| \leq \exp\left(\int_{t_0}^{t} \tilde{L}(s, |x| + |y|) ds\right) |x - y|, \tag{3.20}$$

$$|\phi(t, t_0, x; d) - x| \leq \left(\exp\left(\int_{t_0}^{t} \tilde{L}(s, |x|) ds\right) - 1\right) |x|, \tag{3.21}$$

$$|\phi(t, t_0, x; d)| \geq \exp\left(-\int_{t_0}^{t} \tilde{L}(s, |x|) ds\right) |x|, \tag{3.22}$$

$$|\phi(t, t_1, x; d) - \phi(t, t_2, x; d)| \leq \exp\left(\int_{\min(t_1, t_2)}^{t} \tilde{L}(s, |x|) ds\right) \tilde{L}(\max(t_1, t_2), |x|) |x| |t_1 - t_2| \tag{3.23}$$

$$\forall t \geq t_0 \text{ and } (t_0, x, y; d) \in \Re^+ \times \Re^n \times \Re^n \times M_D,$$

where

$$\tilde{L}(t,s) := L(t, 2a_1^{-1}(\beta(t)a_2(s))).$$

We define

$$U(t,x) := \sup\left\{a_1(|\phi(\tau,t,x;d)|)\exp\left(\frac{1}{2}(\tau-t)\right) : \tau \geq t, d \in M_D\right\}. \tag{3.24}$$

The desired properties (3.19a) and (3.19b) are then immediate consequences of (2.2) and definition (3.24). Inequality (3.19a) asserts that $U : \Re^+ \times \Re^n \to \Re^+$ is continuous at $x = 0$ with $U(t,0) = 0$ for all $t \geq 0$. We next establish that $U(\cdot)$ is locally Lipschitz on $\Re^+ \times (\Re^n \setminus \{0\})$. By (2.2) and (3.24) it follows that for any $T > 0$ the following holds:

$$\begin{aligned} U(t,x) &= \max\left\{\sup\left\{a_1(|\phi(\tau,t,x;d)|)\exp\left(\frac{1}{2}(\tau-t)\right) : t \leq \tau \leq t+T, d \in M_D\right\},\right.\\ &\qquad \left.\sup\left\{a_1(|\phi(\tau,t,x;d)|)\exp\left(\frac{1}{2}(\tau-t)\right) : \tau \geq t+T, d \in M_D\right\}\right\}\\ &\leq \max\left\{\sup\left\{a_1(|\phi(\tau,t,x;d)|)\exp\left(\frac{1}{2}(\tau-t)\right) : t \leq \tau \leq t+T, d \in M_D\right\},\right.\\ &\qquad \left.\beta(t)a_2(|x|)\exp\left(-\frac{1}{2}T\right)\right\}. \end{aligned} \tag{3.25}$$

Let $T_i : \Re^+ \times (0,+\infty) \to (0,+\infty)$, $i = 1,2$, be a pair of positive, C^0 functions, defined as

$$T_1(t,s) := 2\log\left(\frac{2\beta(t)a_2(s)}{a_1(s)}\right), \qquad T_2(t,s) := 2\log\left(\frac{2\beta(t)a_2(2s)}{a_1\left(\frac{s}{2}\right)}\right). \tag{3.26}$$

Notice that for every $s > 0$ each $T_i(\cdot,s)$ is nondecreasing and the following holds:

$$T_2(t,|x|) \geq \sup\left\{T_1(t,|y|) : y \in B\left[x, \frac{1}{2}|x|\right]\right\}, \qquad x \neq 0. \tag{3.27}$$

It turns out from (3.25) and (3.26) that

$$\begin{aligned} U(t,x) \leq \max&\left\{\sup\left\{a_1(|\phi(\tau,t,x;d)|)\exp\left(\frac{1}{2}(\tau-t)\right) : t \leq \tau \leq t+\xi, d \in M_D\right\},\right.\\ &\left.\frac{1}{2}a_1(|x|)\right\} \qquad \text{for } \xi \geq T_1(t,|x|),\ x \neq 0, \end{aligned} \tag{3.28}$$

which by virtue of (3.19a) gives

$$\begin{aligned} U(t,x) = \sup&\left\{a_1(|\phi(\tau,t,x;d)|)\exp\left(\frac{1}{2}(\tau-t)\right) : t \leq \tau \leq t+\xi, d \in M_D\right\}\\ &\text{for } \xi \geq T_1(t,|x|),\ x \neq 0. \end{aligned} \tag{3.29}$$

It follows by taking into account (2.2), (3.22), (3.27) and (3.29) that

(3.30)
$$|U(t,y)-U(t,x)| = \left|\sup\left\{a_1(|\phi(\tau,t,y;d)|)\exp\left(\frac{1}{2}(\tau-t)\right): t \le \tau \le t+T_2(t,|x|), d \in M_D\right\}\right.$$
$$\left. - \sup\left\{a_1(|\phi(\tau,t,x;d)|)\exp\left(\frac{1}{2}(\tau-t)\right): t \le \tau \le t+T_2(t,|x|), d \in M_D\right\}\right|$$
$$\le \sup\left\{\exp\left(\frac{1}{2}(\tau-t)\right)|a_1(|\phi(\tau,t,y;d)|)-a_1(|\phi(\tau,t,x;d)|)| : t \le \tau \le t+T_2(t,|x|), d \in M_D\right\}$$
$$\le M_I \sup\left\{\exp\left(\frac{1}{2}(\tau-t)\right)|\phi(\tau,t,y;d)-\phi(\tau,t,x;d)| : t \le \tau \le t+T_2(t,|x|), d \in M_D\right\}$$
$$\forall y \in B\left[x, \frac{1}{2}|x|\right],\ x \neq 0,$$

where M_I is any Lipschitz constant for $a_1(\cdot)$ on the interval

$$I := \left[\frac{1}{2}\exp\left\{-\int_t^{t+T_2(t,|x|)} \tilde{L}\left(s, \frac{3}{2}|x|\right) ds\right\}|x|, a_1^{-1}\left(\beta(t) a_2\left(\frac{3}{2}|x|\right)\right)\right],$$

namely, $|a_1(s_1)-a_1(s_2)| \le M_I|s_1-s_2|$ for every $s_1, s_2 \in I$. From (3.20) and (3.30) we deduce

$$\text{(3.31a)} \quad |U(t,y)-U(t,x)| \le G_1(t,|x|)|y-x|, \qquad \forall y \in B\left[x, \frac{1}{2}|x|\right],\ x \neq 0,$$

$$\text{(3.31b)} \qquad G_1(t,|x|) := M_I \exp\left(\frac{1}{2}T_2(t,|x|) + \int_t^{t+T_2(t,|x|)} \tilde{L}\left(s, \frac{5}{2}|x|\right) ds\right).$$

This establishes that, for each $t \ge 0$, $U(t,\cdot)$ is locally Lipschitz on $\Re^n \setminus \{0\}$.

Likewise, we may establish that for each fixed nonzero $x \in \Re^n$, the map $U(\cdot,x)$ is locally Lipschitz on $\Re^+$. Indeed, consider a compact interval $I \subset \Re^+$ and let $t_1, t_2 \in I$. Then, according to (3.29), for any $\varepsilon > 0$, there exists a $d_\varepsilon \in M_D$ and time τ with $t_2 \le \tau \le t_2 + T_1(t_2,|x|)$ such that

$$\text{(3.32)} \qquad U(t_2,x) - \varepsilon \le a_1(|\phi(\tau,t_2,x;d_\varepsilon)|)\exp\left(\frac{1}{2}(\tau-t_2)\right) \le U(t_2,x).$$

We distinguish three cases. The first is

$$\text{(3.33)} \qquad t_1 < t_2 \le \tau.$$

It then follows by virtue of definition (3.24) that

$$\text{(3.34)} \qquad a_1(|\phi(\tau,t_1,x;d_\varepsilon)|)\exp\left(\frac{1}{2}(\tau-t_1)\right) \le U(t_1,x);$$

hence, by (3.32) and (3.34) we get

(3.35)
$$U(t_2,x)-U(t_1,x) \le \exp\left(\frac{1}{2}(\tau-t_2)\right)|a_1(|\phi(\tau,t_2,x;d_\varepsilon)|)-a_1(|\phi(\tau,t_1,x;d_\varepsilon)|)| + \varepsilon.$$

Using (3.22) and (3.23) and exploiting the fact that $a_1(\cdot)$ is locally Lipschitz on $(0,+\infty)$, we deduce from (3.35) that for any compact $\Delta \subset \Re^n \setminus \{0\}$ a constant $L_1 > 0$ (being independent of ε and τ) can be found such that

$$\begin{gathered} U(t_2,x) - U(t_1,x) \le L_1|t_2 - t_1| + \varepsilon \\ \forall t_2 > t_1, t_1,\ t_2 \in I,\ x \in \Delta. \end{gathered} \tag{3.36}$$

The second case is

$$t_2 \le t_1 \le \tau. \tag{3.37}$$

We may recall again (3.32), (3.34) and estimate

$$\begin{aligned} U(t_2,x) - U(t_1,x) &\le a_1(|\phi(\tau,t_2,x;d_\varepsilon)|)\exp\left(\frac{1}{2}(\tau - t_2)\right) \\ &\quad - a_1(|\phi(\tau,t_1,x;d_\varepsilon)|)\exp\left(\frac{1}{2}(\tau - t_1)\right) + \varepsilon \\ &= \exp\left(\frac{1}{2}(\tau - t_2)\right)\left(a_1(|\phi(\tau,t_2,x;d_\varepsilon)|) - a_1(|\phi(\tau,t_1,x;d_\varepsilon)|)\right) \\ &\quad + a_1(|\phi(\tau,t_1,x;d_\varepsilon)|)\left(\exp\left(\frac{1}{2}(\tau - t_2)\right) - \exp\left(\frac{1}{2}(\tau - t_1)\right)\right) + \varepsilon, \end{aligned} \tag{3.38}$$

and, as previously, it follows by (3.23) and (3.38) that there exists a constant $L_2 > 0$ (being independent of ε and τ) such that

$$\begin{gathered} U(t_2,x) - U(t_1,x) \le L_2|t_2 - t_1| + \varepsilon \\ \forall x \in \Delta, \quad t_1, t_2 \in I,\ \text{provided that (3.37) holds.} \end{gathered} \tag{3.39}$$

Finally, consider the case

$$t_2 \le \tau < t_1 \tag{3.40}$$

for certain τ and d_ε such that (3.32) holds. We now invoke the left-hand-side inequality of (3.19a):

$$a_1(|x|) \le U(t_1,x). \tag{3.41}$$

It follows by virtue of (3.32), (3.40), and (3.41) that

$$\begin{aligned} U(t_2,x) - U(t_1,x) &\le a_1(|\phi(\tau,t_2,x;d_\varepsilon)|)\exp\left(\frac{1}{2}(\tau - t_2)\right) - a_1(|x|) \\ &\le \exp\left(\frac{1}{2}(\tau - t_2)\right)\left(a_1(|\phi(\tau,t_2,x;d_\varepsilon)|) - a_1(|x|)\right) \\ &\quad + a_1(|x|)\left(\exp\left(\frac{1}{2}(t_1 - t_2)\right) - 1\right). \end{aligned} \tag{3.42}$$

Using (3.21) and (3.22) and the fact that $a_1(\cdot)$ is locally Lipschitz on $(0,+\infty)$, we deduce from (3.42) that for any compact $\Delta \subset \Re^n \setminus \{0\}$ a constant $L_3 > 0$ (being independent of ε and τ) can be found such that

$$\begin{gathered} U(t_2,x) - U(t_1,x) \le L_3|t_2 - t_1| + \varepsilon \\ \forall x \in \Delta, \quad t_1, t_2 \in I,\ \text{provided that (3.40) holds.} \end{gathered} \tag{3.43}$$

From (3.37), (3.39), and (3.43) it follows that $U(t_2,x)-U(t_1,x)\leq \max(L_1,L_2,L_3)|t_2-t_1|+\varepsilon$ for all $t_1,t_2\in I$, $\varepsilon>0$, and $x\in\Delta$. Similarly, we handle the case $U(t_1,x)-U(t_2,x)$ and conclude that for any compact sets $I\subset\Re^+$ and $\Delta\subset\Re^n\setminus\{0\}$, there is a constant $C>0$ such that

$$|U(t_2,x)-U(t_1,x)|\leq C|t_2-t_1|+\varepsilon\quad \forall t_1,t_2\in I,\ x\in\Delta. \tag{3.44}$$

Since $\varepsilon>0$ is arbitrary, inequalities (3.31) and (3.44) establish that $U(\cdot)$ is locally Lipschitz. The proof is complete. $\square$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. (i) $\Rightarrow$ (ii) For convenience we still adopt here the notation $\phi(t,t_0,x_0;d)$ to denote the solution of (1.1) that corresponds to the input $d\in M_D$, initiated from $x_0\in\Re^n$ at time $t_0\geq 0$. Suppose first that $0\in\Re^n$ is RGAS and establish existence of $V(\cdot)$ satisfying (3.1). Since $0\in\Re^n$ is RGAS for (1.1), it follows by Lemma 3.3 that there exists a C^0 function $U:\Re^+\times\Re^n\to\Re^+$, which is locally Lipschitz on $\Re^n\times(\Re^n\setminus\{0\})$; a pair of functions a_1, a_2 of class K_∞, a_1 being locally Lipschitz on $(0,+\infty)$; and a function β of class K^+, such that for all $(t,x)\in\Re^+\times\Re^n$, $h\geq 0$, and $d(\cdot)\in M_D$ both (3.19a), (3.19b) hold. The proof is divided into two parts. In Part I we construct a function $W:\Re^+\times\Re^n\to\Re^+$ of class $C^0(\Re^+\times\Re^n)\cap C^\infty(\Re^+\times(\Re^n\setminus\{0\}))$, which satisfies

$$\frac{1}{2}a_1(|x|)\leq W(t,x)\leq\frac{3}{2}\beta(t)a_2(|x|)\qquad\forall (t,x)\in\Re^+\times\Re^n, \tag{3.45a}$$

$$\begin{gathered}\frac{\partial W}{\partial t}(t,x)+\frac{\partial W}{\partial x}(t,x)f(t,x,d)\leq -\frac{1}{4}W(t,x)\\ \forall (t,x,d)\in\Re^+\times(\Re^n\setminus\{0\})\times D,\end{gathered} \tag{3.45b}$$

where a_1, $a_2\in K_\infty$ and $\beta\in K^+$ are the functions defined in (3.19a), (3.19b).

In Part II, by exploiting (3.45), we build the desired Lyapunov function $V:\Re^+\times\Re^n\to\Re^+$ that satisfies (3.1) for appropriate functions $\bar a_1$, $\bar a_2$ of class K_∞ and $\bar\beta$ of class K^+.

Part I. We proceed to the construction of an "almost smooth" W satisfying (3.45a), (3.45b). If the dynamics $f(\cdot)$ were Lipschitzian in both t and x, then the smoothing approach of [20] applied to the time-extended system $\dot x=f(t,x,d)$, $\dot t=1$, would lead to the existence of a function W satisfying both (3.45a) and (3.45b). However, we have assumed that $f(\cdot)$ is continuous in t, so we need to make a modification of the approach in [20]. We proceed as follows. Let $\psi_1:\Re^n\to\Re^+$, $\psi_2:\Re\to\Re^+$ be a pair of C^∞ functions with $\psi_1(\xi)=0$ and $\psi_2(\tau)=0$ when $|\xi|\geq 1$ and $\tau\notin(0,1)$, respectively, in such a way that

$$\int_{\Re^n}\psi_1(\xi)d\xi=\int_{\Re}\psi_2(\tau)d\tau=1.$$

Let S be a compact subset of $\Re^+\times(\Re^n\setminus\{0\})$. We consider the following family of functions:

$$W_\sigma(t,x):=\int_{\Re}\int_{\Re^n}U(t+\sigma\tau,x+\sigma\xi)\psi_1(\xi)\psi_2(\tau)d\xi d\tau,\qquad \sigma>0, \tag{3.46}$$

where $U(\cdot)$ is the function provided by Lemma 3.3. Let

$$\begin{gathered} r:=\min_{(t,x)\in S}|x|>0,\\ \tilde S:=\left\{(t+c\tau,x+c\xi)\in\Re^+\times\Re^n:(t,x)\in S,c\in\left[0,\frac{1}{2}r\right],\xi\in B[0,1],\tau\in[0,1]\right\}.\end{gathered} \tag{3.47}$$

Obviously, $S \subseteq \tilde{S} \subseteq \Re^+ \times (\Re^n \setminus \{0\})$, $\tilde{S}$ is compact, and let C be a Lipschitz constant for U on $\tilde{S}$. It follows by virtue of (3.46) and (3.47) that for $\sigma < \frac{1}{2}r$, W_σ is well defined and C^∞ on S and satisfies

$$|W_\sigma(t,x) - U(t,x)| \leq C\sigma \quad \forall (t,x) \in S, \quad \sigma < \frac{1}{2}r. \tag{3.48}$$

We also obtain the following by recalling (3.21) and (3.47):

$$(t+h+\sigma\tau, \phi(t+h,t,x;d)+\sigma\xi) \in \tilde{S}, \tag{3.49a}$$

$$\begin{gathered}(t+h+\sigma\tau, \phi(t+h+\sigma\tau, t+\sigma\tau, x+\sigma\xi; d)) \in \tilde{S}\\ \forall (t,x) \in S,\ d \in M_D,\ (\tau,\xi) \in [0,1] \times B[0,1],\\ \sigma \leq \frac{1}{4}r,\ h > 0,\ \text{sufficiently small.}\end{gathered} \tag{3.49b}$$

Then by using (3.19b), (3.46), (3.49a) and (3.49b) we get

$$\begin{aligned}
W_\sigma(t+h, \phi(t+h,t,x;d)) - W_\sigma(t,x) &\leq \left(\exp\left(-\frac{h}{2}\right) - 1\right) W_\sigma(t,x)\\
&+ \int_\Re \int_{\Re^n} (U(t+h+\sigma\tau, \phi(t+h,t,x;d)+\sigma\xi)\\
&\qquad - U(t+h+\sigma\tau, \phi(t+h+\sigma\tau, t+\sigma\tau, x+\sigma\xi; d)))\psi_1(\xi)\psi_2(\tau)d\xi d\tau\\
&\leq \left(\exp\left(-\frac{h}{2}\right) - 1\right) W_\sigma(t,x)\\
&+ C\int_\Re \int_{\Re^n} |\phi(t+h,t,x;d)+\sigma\xi - \phi(t+h+\sigma\tau, t+\sigma\tau, x+\sigma\xi; d)|\psi_1(\xi)\psi_2(\tau)d\xi d\tau\\
&\qquad \forall (t,x) \in S,\ d \in M_D,\ h > 0 \text{ sufficiently small.}
\end{aligned} \tag{3.50}$$

Since f is C^0 and therefore uniformly continuous on compact sets, there exists a function $\delta_1 : \Re^+ \to \Re^+$ of class K such that

$$\sup\{|f(t+\sigma\tau, x, d) - f(t,x,d)| : (t,x) \in \tilde{S}, \tau \in [0,1], d \in D\} \leq \delta_1(\sigma). \tag{3.51}$$

Using (1.2) and (3.51) and applying Gronwall's inequality, a function $\delta_2 : \Re^+ \to \Re^+$ of class K can be found such that

$$\begin{gathered}|\phi(t+h,t,x;d) + \sigma\xi - \phi(t+h+\sigma\tau, t+\sigma\tau, x+\sigma\xi; d)| \leq \delta_2(\sigma)h\\ \forall (t,x) \in S,\ d \in M_D,\ (\tau,\xi) \in [0,1] \times B[0,1],\ h > 0 \text{ sufficiently small.}\end{gathered} \tag{3.52}$$

Specifically, in order to establish (3.52), define $p(s) := |\phi(t+s,t,x;d) + \sigma\xi - \phi(t+s+\sigma\tau, t+\sigma\tau, x+\sigma\xi; d)|$ and let L be a Lipschitz constant for f on $\tilde{S} \times D$, namely, $|f(t,x,d) - f(t,y,d)| \leq L|x-y|$ for all $(t,x,d) \in \tilde{S} \times D$ and $(t,y,d) \in \tilde{S} \times D$. We then obtain by (3.51)

$$\begin{aligned}
p(h) &\leq \int_0^h |f(t+s, \phi(t+s,t,x;d), d) - f(t+\sigma\tau+s, \phi(t+\sigma\tau+s, t+\sigma\tau, x+\sigma\xi; d), d)|\,ds\\
&\leq \delta_1(\sigma)h + \int_0^h |f(t+\sigma\tau+s, \phi(t+s,t,x;d), d)\\
&\qquad - f(t+\sigma\tau+s, \phi(t+\sigma\tau+s, t+\sigma\tau, x+\sigma\xi; d), d)|\,ds\\
&\leq \delta_1(\sigma)h + L\int_0^h |\phi(t+s,t,x;d) - \phi(t+\sigma\tau+s, t+\sigma\tau, x+\sigma\xi; d)|\,ds\\
&\leq \delta_1(\sigma)h + L\int_0^h p(s)ds + \sigma Lh.
\end{aligned}$$

The desired (3.52) is then a straightforward consequence of the previous inequality and Gronwall's lemma.

From (3.50) and (3.52) it follows that

$$\lim_{h\to 0^+} \frac{W_\sigma(t+h,\phi(t+h,t,x;d)) - W_\sigma(t,x)}{h} = \frac{\partial W_\sigma}{\partial t}(t,x) + \frac{\partial W_\sigma}{\partial x}(t,x)f(t,x,d) \le -\frac{1}{2}W_\sigma(t,x) + C\delta_2(\sigma). \tag{3.53}$$

By (3.48) and (3.53) we conclude that for any compact $S \subseteq \Re^+ \times (\Re^n \setminus \{0\})$ and $\varepsilon > 0$, there exists a constant $\sigma_0 > 0$ such that for every $\sigma < \sigma_0$ the function W_σ is well defined and C^∞ on S and satisfies for all $(t,x,d) \in S \times D$

$$|W_\sigma(t,x) - U(t,x)| \le \varepsilon, \tag{3.54a}$$

$$\frac{\partial W_\sigma}{\partial t}(t,x) + \frac{\partial W_\sigma}{\partial x}(t,x)f(t,x,d) \le -\frac{1}{2}W_\sigma(t,x) + \varepsilon. \tag{3.54b}$$

We may use (3.19a), (3.54a) and (3.54b) and apply partition of unity, as in the proof of [20, Theorem B.1], to build a function $W : \Re^+ \times \Re^n \to \Re^+$ of class $C^0(\Re^+ \times \Re^n) \cap C^\infty(\Re^+ \times (\Re^n \setminus \{0\}))$ that satisfies both (3.45a) and (3.45b).

Part II. We finally proceed to the construction of an everywhere C^∞ function V satisfying (3.1a), (3.1b). This part of proof is based on [34, Lemma 17], which in conjunction with (3.45a) and (3.45b) guarantees the existence of a function $\eta : \Re^+ \to \Re^+$ of class K_∞ with $\eta(s) \le \frac{d\eta}{ds}(s)s$, such that the map

$$V(t,x) := (\eta(W(t,x)))^4 \tag{3.55}$$

is everywhere C^∞ and satisfies (3.1b). Furthermore, by Lemma 2.3 there exist functions $\tilde{a}_2 \in K_\infty$, $\bar{\beta} \in K^+$ such that

$$\frac{3}{2}\beta(t)a_2(s) \le \tilde{a}_2(\bar{\beta}(t)s) \qquad \forall t,s \ge 0. \tag{3.56a}$$

Define

$$\bar{a}_1(s) := \left(\eta\left(\frac{1}{2}a_1(s)\right)\right)^4, \qquad \bar{a}_2(s) := \left(\eta\left(\tilde{a}_2(s)\right)\right)^4. \tag{3.56b}$$

By using (3.56a), (3.56b) and invoking (3.45a), (3.45b), it follows that the function V as defined by (3.55) satisfies the desired inequalities (3.1a), (3.1b).

(ii) $\Rightarrow$ (iii) The implication is obvious since (3.1a), (3.1b) implies (3.2a), (3.2b) with $\rho(s) = s$, $\mu(t) \equiv 0 \in \mathbf{E}$, and some $\bar{a}_1 \in K_\infty$, $\bar{a}_2 \in K_\infty$, and $\bar{\beta} \in K^+$.

(iii) $\Rightarrow$ (i) We finally establish the converse part of our theorem, namely, that $0 \in \Re^n$ is RGAS with respect to (1.1) when both (3.2a) and (3.2b) are fulfilled. Define $A := M_D$ and let us again denote the solution of (1.1) by $\phi(t,t_0,x_0;d)$. Then using (3.2a), (3.2b) and applying the result of Lemma 3.2 with $y_d(t) := V(t,\phi(t,t_0,x_0;d))$, it follows that (3.3) holds; thus there exists a KL function σ and a constant $M = \int_0^{+\infty}\mu(t)dt \ge 0$ such that

$$|\phi(t,t_0,x_0;d)| \le \bar{a}_1^{-1}(\sigma(\bar{a}_2(\bar{\beta}(t_0)|x_0|) + M, t - t_0)) \quad \forall t \ge t_0,\ d(\cdot) \in M_D$$

The latter estimate in conjunction with the result of Proposition 2.5 implies that $0 \in \Re^n$ is RGAS with respect to (1.1). The proof is complete. $\square$

4. The nonuniform in time ISS property for time-varying systems. The results of the previous section enable us to characterize the nonuniform in time notion of ISS in terms of Lyapunov functions. We first introduce the notion of (nonuniform in time) ISS, as an extension of the notion of uniform in time ISS as presented in [36, 37]. In [16] we establish further equivalent descriptions of nonuniform in time ISS that constitute extensions of Sontag's ISS.

DEFINITION 4.1. *Consider the system*

$$\begin{gathered}\dot{x} = f(t,x,u),\\ x \in \Re^n, \quad u \in \Re^m, \quad t \geq 0,\end{gathered} \tag{4.1}$$

where $f(t,x,u)$ is measurable in $t \geq 0$ for all $(x,u) \in \Re^n \times \Re^m$ and is locally Lipschitz with respect to (x,u) with $f(\cdot,0,0) = 0$; denote $x(t) = x(t,t_0,x_0;u)$ its solution at time t that corresponds to some input $u \in \mathbf{L}^\infty_{loc}$, initiated from x_0 at time t_0. Let $\gamma(t,s) : (\Re^+)^2 \to \Re^+$ be a C^0 function, which is locally Lipschitz in s, such that for each fixed $t \geq 0$ the map $\gamma(t,\cdot)$ is positive definite. We say that (4.1) *satisfies the weak (nonuniform in time) input-to-state stability property (wISS) from the input u with gain $\gamma(\cdot)$ if each solution of* (4.1) *exists for all $t \geq t_0$ and satisfies properties* P1 *and* P2 *of Definition* 2.1 *provided that*

$$|u(t)| \leq \gamma(t,|x(t)|) \qquad \text{a.e. for } t \geq t_0. \tag{4.2}$$

We say that (4.1) *satisfies the (nonuniform in time) ISS from the input u with gain $\gamma(\cdot)$ if it is wISS from the input u with gain $\gamma(\cdot)$ and in addition for each fixed $t \geq 0$ the map $\gamma(t,\cdot)$ is of class K_∞.*

As in the autonomous case (see [29, 37]) we can easily establish the following elementary fact.

Fact 4.2. System (4.1) satisfies the nonuniform in time wISS property from the input u with gain $\gamma(\cdot)$ if and only if $0 \in \Re^n$ is RGAS for the system

$$\begin{gathered}\dot{x} = f(t,x,\gamma(t,|x|)d),\\ x \in \Re^n, \quad d \in B[0,1] \subset \Re^m, \quad t \geq 0.\end{gathered} \tag{4.3}$$

The following theorem summarizes some useful equivalent descriptions of nonuniform in time wISS. Its proof is a direct consequence of Proposition 2.2, Lemma 3.2, Theorem 3.1, and Fact 4.2.

PROPOSITION 4.3. *Consider the system* (4.1) *whose dynamics satisfy the regularity assumptions of Definition* 4.1, *and let $\gamma(t,s) : (\Re^+)^2 \to \Re^+$ be a C^0 function, which is locally Lipschitz in s, such that for each fixed $t \geq 0$ the map $\gamma(t,\cdot)$ is positive definite. Then the following statements are equivalent:*

(i) *System* (4.1) *satisfies the nonuniform in time wISS property from the input with gain $\gamma(\cdot)$.*
(ii) *There exists a pair of functions a_1, a_2 of class K_∞, a_1 being locally Lipschitz on $(0,+\infty)$, and a function β of class K^+ such that the following property holds:*

$$\begin{gathered}|u(t)| \leq \gamma(t,|x(t)|) \text{ a.e. for } t \geq t_0 \Rightarrow a_1(|x(t)|) \leq \exp(-t+t_0)\beta(t_0)a_2(|x_0|),\\ \forall t \geq t_0,\ x_0 \in \Re^n.\end{gathered} \tag{4.4}$$

(iii) *There exists a C^0 function $U : \Re^+ \times \Re^n \to \Re^+$, which is locally Lipschitz on $\Re^+ \times (\Re^n \setminus \{0\})$ and satisfies*

$$a_1(|x|) \le U(t,x) \le \beta(t) a_2(|x|) \qquad \forall (t,x) \in \Re^+ \times \Re^n, \tag{4.5a}$$

$$|u(t)| \le \gamma(t,|x(t)|) \text{ a.e. for } t \ge t_0 \Rightarrow U(t,x(t)) \le \exp\left(-\frac{1}{2}(t-t_0)\right) U(t_0,x_0) \quad \forall (t_0,x_0) \in \Re^+ \times \Re^n \text{ and } t \ge t_0 \tag{4.5b}$$

with the some a_1, a_2, and β as defined in (4.4).

If in addition $f(\cdot) \in C^0(\Re^+ \times \Re^n; \Re^n)$, then the following are equivalent to the previous statements:

(iv) *There exist a C^∞ function $V : \Re^+ \times \Re^n \to \Re^+$ and functions $\bar{a}_1$, $\bar{a}_2 \in K_\infty$, $\bar{\beta} \in K^+$ such that the following hold for all $(t,x,u) \in \Re^+ \times \Re^n \times \Re^m$:*

$$\bar{a}_1(|x|) \le V(t,x) \le \bar{a}_2(\bar{\beta}(t)|x|), \tag{4.6a}$$

$$|u| \le \gamma(t,|x|) \Rightarrow \dot{V}(t,x,u)|_{(4.1)} \le -V(t,x). \tag{4.6b}$$

(v) *There exist a C^∞ function $V : \Re^+ \times \Re^n \to \Re^+$ and functions $\bar{a}_1$, $\bar{a}_2 \in K_\infty$, $\bar{\beta} \in K^+$, $\mu \in \mathbf{E}$ and a C^0 positive definite function $\rho : \Re^+ \to \Re^+$ such that the following hold for all $(t,x,u) \in \Re^+ \times \Re^n \times \Re^m$:*

$$\bar{a}_1(|x|) \le V(t,x) \le \bar{a}_2(\bar{\beta}(t)|x|), \tag{4.7a}$$

$$|u| \le \gamma(t,|x|) \Rightarrow \dot{V}(t,x,u)|_{(4.1)} \le -\rho(V(t,x)) + \mu(t). \tag{4.7b}$$

5. Applications to feedback stabilization. In this section we apply the converse Lyapunov Theorem 3.1 in order to derive necessary and sufficient conditions for ISS-feedback stabilization for affine in the control time-varying systems. For the general case (1.3) we extend the Artstein–Sontag theorem by introducing the concept of time-varying control Lyapunov function (Theorem 5.1). Among other things we establish that, even for a class of autonomous systems, it is possible to achieve nonuniform in time ISS stabilization by smooth time-varying feedback, although an everywhere smooth time-independent feedback exhibiting uniform in time stabilization does not exist (Corollary 5.4).

For the special case (1.5) an extension of a well-known result concerning autonomous systems (see [11, 36]) is established (Proposition 5.6). This result generalizes [40, Lemma 2.3] since is based on weaker hypotheses. Its Lyapunov function based establishment extremely simplifies the analysis made in [40].

5.1. A necessary and sufficient condition for ISS-feedback stabilization. The following theorem is an extension of the Artstein–Sontag theorem (see, for instance, [3, 27, 35]). We consider here the time-varying case (1.3) and in what follows assume that the dynamics f, g are C^0 and locally Lipschitz with respect to $(x,v) \in \Re^n \times \Re^l$, with $f(\cdot,0,0) = 0$.

THEOREM 5.1. *Consider the system* (1.3) *and let $\gamma(t,s) : (\Re^+)^2 \to \Re^+$ be a function, which is C^0, locally Lipschitz in s, and in such a way that for each $t \ge 0$ the mapping $\gamma(t,\cdot)$ is positive definite. Then the following statements are equivalent:*

(i) *There exists a C^∞ function $k : \Re^+ \times \Re^n \to \Re^m$ with $k(t,0) = 0$ for all $t \ge 0$, in such a way that the resulting system* (1.3) *with $u = k(t,x)$, namely, system* (1.4), *satisfies the nonuniform in time wISS property with gain $\gamma(\cdot)$ from the input v. It turns out that* (1.4) *satisfies the nonuniform in time ISS property with gain $\gamma(\cdot)$ from the input v, provided that for each fixed $t \ge 0$ the map $\gamma(t,\cdot)$ is of class K_∞.*

(ii) *There exists a* C^0 *function* $k : \Re^+ \times \Re^n \to \Re^m$ *with* $k(t,0) = 0$ *for all* $t \geq 0$*, being locally Lipschitz in* x*, in such a way that the resulting system* (1.4) *satisfies the same property as in statement* (i).

(iii) *System* (1.3) *admits a "control Lyapunov function," namely, there exists a* C^1 *function* $V : \Re^+ \times \Re^n \to \Re^+$*, functions* $a_1, a_2 \in K_\infty$*,* $\beta \in K^+$*,* $\mu \in \mathbf{E}$*, and a* C^0 *positive definite function* $\rho : \Re^+ \to \Re^+$*, such that*

$$a_1(|x|) \leq V(t,x) \leq a_2(\beta(t)|x|), \tag{5.1a}$$

$$\frac{\partial V}{\partial x}(t,x)g(t,x) = 0,\ |v| \leq \gamma(t,|x|) \tag{5.1b}$$
$$\Rightarrow \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x,v) \leq -\rho(V(t,x)) + \mu(t).$$

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. We establish implication (ii) $\Rightarrow$ (iii). Suppose that there exists a map $k(\cdot)$, as in statement (ii) of the theorem, such that system (1.4) satisfies the wISS property with gain $\gamma(\cdot)$. By recalling (iv) of Proposition 4.3, there exists a C^∞ function $V : \Re^+ \times \Re^n \to \Re^+$ in such a way that (5.1a) holds and

$$|v| \leq \gamma(t,x) \Rightarrow \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)(f(t,x,v) + g(t,x)k(t,x)) \leq -V(t,x). \tag{5.2}$$

The latter implies (5.1b) with $\mu(t) \equiv 0 \in \mathbf{E}$ and $\rho(s) = s$. We next establish (iii) $\Rightarrow$ (i). Consider the functions a_1, a_2, β, V, and μ as defined in (5.1a), (5.1b) and without any loss of generality assume

$$\mu(t) > 0 \qquad \forall t \geq 0. \tag{5.3}$$

Notice, by virtue of (5.1a), that

$$\frac{\partial V}{\partial t}(t,0) = 0, \qquad \frac{\partial V}{\partial x}(t,0) = 0. \tag{5.4}$$

Condition (5.1b) in conjunction with (5.3) and (5.4) enables us to build by standard partition of unity arguments a C^∞ map $k : \Re^+ \times \Re^n \to \Re^m$ with $k(\cdot,0) = 0$ such that

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x,v) + \frac{\partial V}{\partial x}(t,x)g(t,x)k(t,x) \leq -\rho(V(t,x)) + \mu(t) \tag{5.5}$$
$$\forall (t,x) \in \Re^+ \times \Re^n, \quad |v| \leq \gamma(t,|x|).$$

Define $v = \gamma(t,|x|)d$, where $d(\cdot) \in A := M_{B[0,1]}$. Then using (5.5) it follows that for the solution $x(t,t_0,x_0;d)$ of the system $\dot{x} = f(t,x,\gamma(t,|x|)d) + g(t,x)k(t,x)$ it holds that $\dot{y}_d(t) \leq -\rho(y_d(t)) + \mu(t)$ for all $t \geq t_0$, where $y_d(t) := V(t,x(t,t_0,x_0;d))$. It turns out from Lemma 3.2 that there exists a KL function $\sigma : (\Re^+)^2 \to \Re^+$ such that

$$V(t,x(t,t_0,x_0;d)) \leq \sigma(M + V(t_0,x_0), t - t_0) \qquad \forall t \geq t_0, \quad d(\cdot) \in M_{B[0,1]},$$

where $M := \int_{\Re^+} \mu(t)dt$, and thus by virtue of (5.1a)

$$|x(t,t_0,x_0;d)| \leq a_1^{-1}(\sigma(M + a_2(\beta(t_0)|x_0|), t - t_0)) \qquad \forall t \geq t_0, \quad d(\cdot) \in M_{B[0,1]}, \tag{5.6}$$

for any initial (t_0,x_0). Inequality (5.6) in conjunction with Proposition 2.5 implies that $0 \in \Re^n$ is RGAS with respect to (1.4). The desired wISS property for system (1.4) is a consequence of Fact 4.2. $\square$

The next proposition establishes the existence of an *explicit* formula of a feedback law exhibiting ISS stabilization for system (1.3).

PROPOSITION 5.2. *Consider the system* (1.3) *and suppose that statement* (iii) *of Theorem* 5.1 *is fulfilled for some positive function* $\mu \in \mathbf{E}$*, certain* $V : \Re^+ \times \Re^n \to \Re^+$ *of class* $C^2(\Re^+ \times \Re^n)$*, and some positive definite, locally Lipschitz function* $\rho : \Re^+ \to \Re^+$*. Let* $\theta : \Re \to \Re^+$ *be any* C^∞ *nondecreasing map with*

$$\theta(s) \begin{cases} = 0, & s \leq 0, \\ < 1, & s < 1, \\ = 1, & s \geq 1, \end{cases} \tag{5.7}$$

and let

$$\zeta(t,x) := \left| \frac{\partial V}{\partial t}(t,x) + \max_{|v| \leq \gamma(t,|x|)} \frac{\partial V}{\partial x}(t,x) f(t,x,v) + \rho(V(t,x)) \right|, \tag{5.8a}$$

$$W(t,x) := \frac{\partial V}{\partial t}(t,x) + \max_{|v| \leq \gamma(t,|x|)} \frac{\partial V}{\partial x}(t,x) f(t,x,v) + \frac{1}{2}\rho(V(t,x)) - \mu(t). \tag{5.8b}$$

Then the feedback law

$$k(t,x) := -\left\{ \frac{\left(\frac{\partial V}{\partial x}(t,x) g(t,x)\right)^T}{1 - \theta\left(\frac{W(t,x)}{\mu(t)}\right) + \left|\frac{\partial V}{\partial x}(t,x) g(t,x)\right|^2} \right\} \zeta(t,x), \tag{5.9}$$

which is everywhere continuous and locally Lipschitz with respect to x *and satisfies* $k(\cdot, 0) = 0$*, exhibits wISS stabilization for* (1.4) *with gain* $\gamma(\cdot)$ *from the input* v*.*

Proof. From (5.1b) and definition (5.8b) of $W(\cdot)$ it follows that

$$\frac{\partial V}{\partial x}(t,x) g(t,x) = 0 \Rightarrow W(t,x) \leq 0, \tag{5.10a}$$

$$W(t,x) \leq \mu(t) \Rightarrow \frac{\partial V}{\partial t}(t,x) + \max_{|v| \leq \gamma(t,|x|)} \frac{\partial V}{\partial x}(t,x) f(t,x,v) \leq -\frac{1}{2}\rho(V(t,x)) + 2\mu(t). \tag{5.10b}$$

Notice that k is well defined for all (t,x), since the denominator in (5.9) is strictly positive for all $(t,x) \in \Re^+ \times \Re^n$, and is of class $C^0(\Re^+ \times \Re^n)$. Indeed, $\theta\left(\frac{W(t,x)}{\mu(t)}\right) \leq 1$ for all $(t,x) \in \Re^+ \times \Re^n$, and suppose that $\theta\left(\frac{W(t,x)}{\mu(t)}\right) = 1$ for certain $(t,x) \in \Re^+ \times \Re^n$. It then follows from (5.7) that $W(t,x) \geq \mu(t)$, and thus by virtue of (5.8a) $\frac{\partial V}{\partial x}(t,x) g(t,x)$ is nonzero. Furthermore, according to regularity assumptions made for $V(\cdot)$, $f(\cdot)$, $\gamma(\cdot)$, $g(\cdot)$, and $\rho(\cdot)$, the map $k(t,x)$ as defined by (5.9) is C^0 on $\Re^+ \times \Re^n$ and locally Lipschitz with respect to $x \in \Re^n$, with $k(t,0) = 0$ for all $t \geq 0$. We next estimate the derivative $\dot{V}(\cdot)$ of $V(\cdot)$ along the trajectories of the solutions of the closed-loop system (1.4). We find

$$\begin{aligned} \dot{V}(t,x) &:= \frac{\partial V}{\partial t}(t,x) + \max_{|v| \leq \gamma(t,|x|)} \frac{\partial V}{\partial x}(t,x) f(t,x,v) + \frac{\partial V}{\partial x}(t,x) g(t,x) k(t,x) \\ &\leq -\frac{1}{2}\rho(V(t,x)) + 2\mu(t). \end{aligned} \tag{5.11}$$

Indeed, for those t, x for which $W(t,x) \leq \mu(t)$, we have by taking into account (5.9) and (5.10b) that

$$\frac{\partial V}{\partial t}(t,x) + \max_{|v| \leq \gamma(t,|x|)} \frac{\partial V}{\partial x}(t,x) f(t,x,v) \leq -\frac{1}{2}\rho(V(t,x)) + 2\mu(t),$$
$$\frac{\partial V}{\partial x}(t,x) g(t,x) k(t,x) \leq 0,$$

which implies (5.11). On the other hand, for those t, x for which $W(t,x) \geq \mu(t)$, it follows from (5.7), (5.9), and (5.10a) that

$$\frac{\partial V}{\partial x}(t,x)g(t,x) \neq 0,$$
$$\frac{\partial V}{\partial x}(t,x)g(t,x)k(t,x) = -\zeta(t,x),$$

and thus by taking into account definition (5.8a) of $\zeta(\cdot)$ it follows that

$$\dot{V}(t,x) \leq -\rho(V(t,x)) \leq -\frac{1}{2}\rho(V(t,x)) + 2\mu(t).$$

This establishes (5.11). We complete the proof by applying Lemma 3.2 as exactly done in the proof of Theorem 5.1. □

We next specialize the result of Theorem 5.1 to the following case of time-varying systems:

$$\begin{aligned} &\dot{x} = f(t,x) + g(t,x)u, \\ &x \in \Re^n, \quad u \in \Re^m, \quad t \geq 0, \end{aligned} \tag{5.12}$$

where the mappings f, g are C^0 and locally Lipschitz with respect to x with $f(t,0) = 0$ for all $t \geq 0$.

Corollary 5.3. *The following statements are equivalent:*

(i) *There exist a C^1 function $V : \Re^+ \times \Re^n \to \Re^+$, functions $a_1, a_2 \in K_\infty$, $\beta \in K^+$, $\mu \in \mathbf{E}$, and a C^0 positive definite map $\rho : \Re^+ \to \Re^+$, such that for all $(t,x) \in \Re^+ \times \Re^n$*

$$a_1(|x|) \leq V(t,x) \leq a_2(\beta(t)|x|), \tag{5.13a}$$

$$\frac{\partial V}{\partial x}(t,x)g(t,x) = 0 \Rightarrow \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \leq -\rho(V(t,x)) + \mu(t). \tag{5.13b}$$

(ii) *There exists a C^∞ function $k : \Re^+ \times \Re^n \to \Re^m$ with $k(t,0) = 0$ for all $t \geq 0$, such that $0 \in \Re^n$ is GAS for the system*

$$\dot{x} = f(t,x) + g(t,x)k(t,x).$$

(iii) *For every C^0 function $\gamma(t,s) : (\Re^+)^2 \to \Re^+$, being locally Lipschitz in s and such that, for each $t \geq 0$, $\gamma(t,\cdot)$ is positive definite, there exists a C^∞ function $\tilde{k} : \Re^+ \times \Re^n \to \Re^m$ with $\tilde{k}(t,0) = 0$ for all $t \geq 0$, in such a way that the system*

$$\dot{x} = f(t,x) + g(t,x)\left(\tilde{k}(t,x) + v\right)$$

satisfies the wISS property with gain $\gamma(\cdot)$ from the input $v \in \Re^m$.

Proof. Equivalence between (i) and (ii) is an immediate consequence of Theorem 5.1. In order to establish (i) $\Leftrightarrow$ (iii) consider the system

$$\dot{x} = \tilde{f}(t,x,v) + g(t,x)u, \tag{5.14}$$

where $\tilde{f}(t,x,v) := f(t,x) + g(t,x)v$, which has the form (1.3). The equivalence between (i) and (iii) follows directly from Theorem 5.1 and the obvious consequence of (5.14):

$$\frac{\partial V}{\partial x}(t,x)g(t,x) = 0, |v| \leq \gamma(t,|x|) \Rightarrow \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)\tilde{f}(t,x,v) \leq -\rho(V(t,x)) + \mu(t).$$

The rest part of proof is straightforward and is left to the reader. □

COROLLARY 5.4. *Consider the system*

$$\begin{aligned} &\dot{x} = f(x) + g(x)u, \\ &x \in \Re^n, \quad u \in \Re, \end{aligned} \tag{5.15}$$

where f and g are locally Lipschitz with $f(0) = 0$, and suppose that (5.15) *is globally uniformly asymptotically stabilized at the origin by means of a C^0 static feedback $u = k(x)$ with $k(0) = 0$. Then for every C^0 function $\gamma(t, s) : (\Re^+)^2 \to \Re^+$, being locally Lipschitz in s and such that, for each $t \geq 0$, $\gamma(t, \cdot)$ is positive definite, there exists a C^∞ time-varying feedback law $u = k(t, x)$ with $k(\cdot, 0) = 0$ such that the system*

$$\dot{x} = f(x) + g(x)(k(t, x) + u)$$

satisfies the wISS property with gain $\gamma(\cdot)$ from the input $u \in \Re$.

Proof. Using Kurzweil's converse Lyapunov theorem in [19] we may find a C^1 radially unbounded, positive definite function $V : \Re^n \to \Re^+$ that satisfies $\frac{\partial V}{\partial x}(x)(f(x) + g(x)k_0(x)) < 0$ for $x \neq 0$. It then follows that

$$\frac{\partial V}{\partial x}(x)g(x) = 0 \Rightarrow \frac{\partial V}{\partial x}(x)f(x) \leq -\rho(V(x)) + \mu(t) \tag{5.16}$$

for a certain C^0 positive definite function $\rho : \Re^+ \to \Re^+$ and for arbitrary $\mu \in \mathbf{E}$. The rest of the proof is straightforward consequence of (5.16) and Corollary 5.3 (implication (i) $\Rightarrow$ (iii)). $\square$

Example 5.5. Consider the affine in the control system

$$\begin{aligned} &\dot{x} = x + y^3, \\ &\dot{y} = u, \\ &(x, y) \in \Re^2, \quad u \in \Re. \end{aligned} \tag{5.17}$$

It is known that there is no C^1 static feedback exhibiting uniform in time asymptotic stabilization at the origin for (5.17). However, a C^0 static feedback law exhibiting global uniform in time asymptotic stability exists, and several approaches can be used to obtain such a feedback. Alternatively, we may apply Corollary 5.3 to establish existence of a locally Lipschitz time-varying feedback $k(t, x, y)$ that guarantees nonuniform in time ISS for any given gain function $\gamma(\cdot)$ for the resulting system:

$$\begin{aligned} &\dot{x} = x + y^3, \\ &\dot{y} = k(t, x, y) + u, \\ &(x, y) \in \Re^2, \quad u \in \Re, \end{aligned} \tag{5.18}$$

with u as input. We may also use Proposition 5.2 to determine an explicit formula for a stabilizing feedback. Indeed, let $f(t, x, y) = (x + y^3, 0)$, $g(t, x, y) = (0, 1)$ and define

$$V(t, x, y) := 2\exp(2t)x^2 + (y + \exp(t)x)^2 . \tag{5.19}$$

A simple calculation shows that

$$\frac{\partial V}{\partial y}(t, x, y) = 0 \Leftrightarrow y = -\exp(t)x. \tag{5.20}$$

For those (t,x,y) for which (5.20) holds we have $V(t,x,y)=2\exp(2t)x^2$ and thus

$$\begin{aligned}\frac{\partial V}{\partial t}(t,x,y)+\frac{\partial V}{\partial x}(t,x,y)(x+y^3)&=8\exp(2t)x^2-4\exp(5t)x^4\\&=4V(t,x,y)-\exp(t)V^2(t,x,y)\\&\le-\frac{1}{2}V^2(t,x,y)+4\exp(-t).\end{aligned}$$

Therefore, both (5.13a) and (5.13b) are satisfied with $\rho(s)=\frac{1}{2}s^2$, $\mu(t)=4\exp(-t)$, $a_1(s):=\frac{1}{2}s^2$, $a_2(s):=4s^2$, and $\beta(t):=\exp(t)$, and thus, according to Corollary 5.3, for any gain function $\gamma(t,|(x,y)|)$, there exists a C^∞ time-varying feedback $k(t,x,y)$ with $k(\cdot,0,0)=0$ such that the ISS property with gain $\gamma(\cdot)$ is fulfilled for (5.18). Finally, we may invoke Proposition 5.2 to find an explicit formula for a locally Lipschitz time-varying feedback. Indeed, by (5.7), (5.8), and (5.9) we find

$$k(t,x,y)=\frac{-(y+\exp(t)x)|16\exp(-t)D(t,x,y)+V^4(t,x,y)|}{2-2\theta(D(t,x,y)-1)+8(y+\exp(t)x)^2},$$

where $V(\cdot)$ is defined by (5.19) and

$$\begin{aligned}D(t,x,y):=3\exp(3t)x^2+\exp(2t)xy+\frac{3}{2}\exp(3t)xy^3+\frac{1}{4}\exp(2t)y^4\\+\frac{1}{2}\exp(t)|y+\exp(t)x|\gamma(t,|(x,y)|)+\frac{1}{16}\exp(t)V^4(t,x,y).\end{aligned}\qquad\square$$

5.2. Propagating the ISS property through integrators. In this section we apply Proposition 4.3 in order to derive sufficient conditions for ISS feedback stabilization for the particular class of systems (1.5), where $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are C^0 and locally Lipschitz with respect to (x,y) with $f(\cdot,0,0)=0$ and $g(\cdot,0,0)=0$. In addition to the regularity assumptions made for f,g,h, we further assume that there exists an everywhere strictly positive C^0 function $h_0:\Re^+\to(0,+\infty)$, such that

$$h(t,x,y)\ge h_0(t)\qquad\forall(t,x,y)\in\Re^+\times\Re^n\times\Re. \tag{5.21}$$

As in the time-invariant case (see, for instance, [11, 36]), we impose ISS for the subsystem (1.5a); particularly, we make the following assumptions:

(A1) There exists a C^∞ function $k:\Re^+\times\Re^n\to\Re$, with $k(\cdot,0)=0$, such that the system

$$\dot{x}=f(t,x,k(t,x)+y) \tag{5.22}$$

satisfies the nonuniform in time ISS property from the input y. Specifically, assume that there exist functions a_1, a_2 of class K_∞, with a_1 being a locally Lipschitz function; a function β of class K^+; and a C^0 function $\gamma(t,s):(\Re^+)^2\to\Re^+$, which is locally Lipschitz in s and for each fixed $t\ge0$ the map $\gamma(t,\cdot)$ is of class K_∞, in such a way that the following holds:

(5.23)
$$|y(t)|\le\gamma(t,|x(t)|)\text{ a.e. for }t\ge t_0\Rightarrow a_1(|x(t)|)\le\exp(-t+t_0)\beta(t_0)a_2(|x_0|),$$

where $x(t):=x(t,t_0,x_0;y)$ denotes the trajectory of (5.22) with input y.

(A2) For the function $k(\cdot)$ above we make the following additional hypothesis. There exists a function $E : (\Re^+)^2 \to \Re^+$, with $E(\cdot, 0) = 0$, being nondecreasing in s for each fixed $t \geq 0$ in such a way that

$$|k(t,x)| \leq E(t,|x|) \qquad \forall (t,x) \in \Re^+ \times \Re^n, \tag{5.24a}$$

$$\lim_{t\to+\infty} E\left(t, a_1^{-1}\left(c\exp\left(-\frac{1}{2}t\right)\right)\right) = 0 \qquad \forall c \geq 0. \tag{5.24b}$$

(A3) There exist constants $R > 0$, $m \geq 1$ and a C^0 function $M : \Re^+ \to (0, +\infty)$ such that

$$a_2(s) \leq Rs^{2m} \quad \text{for } s \text{ near zero}, \tag{5.25a}$$

$$M(t)s^m \leq \gamma(t,s) \quad \forall t \geq 0, \quad s \text{ near zero}. \tag{5.25b}$$

The following proposition generalizes a well-known result concerning ISS-feedback stabilization for autonomous systems under the presence of uniform in time ISS (see, for instance, [36]). It also constitutes an extension of the main result in [40] under the presence of "exponential," nonuniform in time ISS.

PROPOSITION 5.6. *Under* (A1), (A2), *and* (A3), *for any gain function* $\bar{\gamma}(t,s) : (\Re^+)^2 \to \Re^+$ *with the same properties as* γ *there exists an everywhere* C^∞ *function* $\bar{k} : \Re^+ \times \Re \to \Re$, *with* $\bar{k}(t,0) = 0$ *for all* $t \geq 0$, *such that system* (1.5) *with* $u := \bar{k}(t, y - k(t,x)) + u$ *satisfies the nonuniform in time ISS property with gain* $\bar{\gamma}$ *from the input* u.

Proof. The proof is based on the Lyapunov characterization of wISS (Proposition 4.3). The corresponding analysis is similar to that employed in [38, 39] and extremely simplifies the approach in [40], where ISS stabilization is exhibited under stricter assumptions. We proceed as follows. Our hypothesis (A1) guarantees, according to Proposition 4.3(iii), the existence of a C^0 function $U : \Re^+ \times \Re^n \to \Re^+$, which is locally Lipschitz on $\Re^+ \times (\Re^n \setminus \{0\})$, such that

$$a_1(|x|) \leq U(t,x) \leq \beta(t)a_2(|x|) \qquad \forall (t,x) \in \Re^+ \times \Re^n, \tag{5.26a}$$

$$\begin{aligned} &|y(t) - k(t,x(t))| \leq \gamma(t,|x(t)|) \text{ a.e. for } t \geq t_0 \\ &\Rightarrow U(t,x(t)) \leq \exp\left(-\frac{1}{2}(t-t_0)\right) U(t_0,x_0) \qquad \forall (t_0,x_0) \in \Re^+ \times \Re^n \text{ and } t \geq t_0, \end{aligned} \tag{5.26b}$$

where $(x(t), y(t))$ denotes the trajectory of the closed-loop system (1.5) with $u := \bar{k}(t, y - k(t,x)) + u$. Let us denote by $\gamma^{-1}(t,s)$ the inverse function of $\gamma(t,s)$ with respect to s; i.e., $\gamma^{-1}(\cdot)$ satisfies

$$\gamma\left(t, \gamma^{-1}(t,s)\right) = \gamma^{-1}(t, \gamma(t,s)) = s \qquad \forall (t,s) \in \Re^+ \times \Re^+.$$

Clearly, $\gamma^{-1}(t,s)$ is C^0 and for each fixed $t \geq 0$ the mapping $\gamma^{-1}(t,\cdot)$ is of class K_∞ as well. By Lemma 2.3, a pair of functions $a \in K_\infty \cap C^\infty((0,+\infty))$ and $\kappa \in K^+ \cap C^\infty(\Re^+)$ can be found in such a way that

$$\beta(t)a_2\left(\gamma^{-1}(t,s)\right) \leq a(\kappa(t)s) \qquad \forall (t,s) \in \Re^+ \times \Re^+. \tag{5.27}$$

We define

$$W(t,s) := a(\kappa(t)s). \tag{5.28}$$

Notice that, according to (A3), the function $W(\cdot)$ can be constructed in such a way that, in addition to (5.27), the following holds:

$$W(t,s) = \bar{M}(t)s^2, \quad t \geq 0,\ s \text{ near zero}, \tag{5.29}$$

for a certain function $\bar{M}(\cdot)$ of class $K^+ \cap C^\infty(\Re^+)$. Therefore without loss of generality we may assume that $W(\cdot)$ as defined by (5.28) is of class $C^\infty(\Re^+ \times \Re; \Re^+)$. It follows by (5.26a) and (5.27) that

$$W(t,|y-k(t,x)|) \leq U(t,x) \Rightarrow |y-k(t,x)| \leq \gamma(t,|x|), \tag{5.30a}$$

$$U(t,x) \leq W(t,|y-k(t,x)|) \Rightarrow |x| \leq a_1^{-1}(W(t,|y-k(t,x)|)). \tag{5.30b}$$

Next define

$$\begin{aligned} &S_1 := \{(t,x,y) \in \Re^+ \times \Re^n \times \Re : W(t,|y-k(t,x)|) \leq U(t,x)\}, \\ &S_2 := (\Re^+ \times \Re^n \times \Re) \setminus S_1, \end{aligned} \tag{5.31}$$

$$\Phi(t,x,y) := \begin{cases} U(t,x), & (t,x,y) \in S_1, \\ W(t,|y-k(t,x)|), & (t,x,y) \in S_2. \end{cases} \tag{5.32}$$

From (5.27), (5.28), (5.31), and definition (5.32) of Φ, it follows that Φ is C^0 and satisfies

$$\begin{aligned} \bar{a}_1(|(x,y-k(t,x))|) \leq \Phi(t,x,y) \leq \bar{\beta}(t)\bar{a}_2(|(x,y-k(t,x))|) \\ \forall (t,x,y) \in \Re^+ \times \Re^n \times \Re \end{aligned} \tag{5.33}$$

for certain $\bar{a}_1$, $\bar{a}_2 \in K_\infty$ and $\bar{\beta} \in K^+$. By taking into account (5.21) and (5.29) and applying standard partition of unity arguments, it follows that for every gain $\bar{\gamma}$ with the same properties as γ, a $C^\infty(\Re^+ \times \Re^n)$ function $\bar{k}(t,z)$ can be determined in such a way that $\bar{k}(t,0) = 0$ for all $t \geq 0$, and furthermore, for every $(t,z) \in \Re^+ \times (\Re \setminus \{0\})$, the following holds:

$$\begin{aligned} &\frac{\partial W}{\partial t}(t,|z|) + \frac{\partial W}{\partial s}(t,|z|)\mathrm{sgn}(z) \Big(g(t,x,k(t,x)+z) + h(t,x,k(t,x)+z) \left(\bar{k}(t,z)+u\right) \\ &\qquad - \frac{\partial k}{\partial t}(t,x) - \frac{\partial k}{\partial x}(t,x) f(t,x,k(t,x)+z) \Big) \\ &\leq -\frac{1}{2}W(t,|z|) + \exp(-t) \qquad \forall |u| \leq \bar{\gamma}(t,|(x,k(t,x)+z)|), |x| \leq a_1^{-1}(W(t,|z|)). \end{aligned} \tag{5.34}$$

We are now in a position to establish the ISS property for the resulting system

$$\begin{aligned} &\dot{x} = f(t,x,y), \\ &\dot{y} = g(t,x,y) + h(t,x,y)\bar{k}(t,y-k(t,x)) + h(t,x,y)u. \end{aligned} \tag{5.35}$$

Particularly, we show that, if $(x(t),y(t))$ denotes the trajectory of (5.35) initiated from (x_0,y_0) at time t_0 with input $v \in \mathbf{L}^\infty_{loc}$, then the following holds:

$$\begin{aligned} &|u(t)| \leq \bar{\gamma}(t,|(x(t),y(t))|) \text{ a.e. for } t \geq t_0 \\ &\quad \Rightarrow \Phi(t,x(t),y(t)) \leq \exp\left(-\frac{1}{2}(t-t_0)\right)(\Phi(t_0,x_0,y_0)+2). \end{aligned} \tag{5.36}$$

Indeed, by taking into account (5.26b), (5.30a), (5.31), and (5.32) it follows that

$$\begin{aligned}\Phi(t,x(t),y(t)) = U(t,x(t)) &\le \exp\left(-\frac{1}{2}(t-t_0)\right)U(t_0,x(t_0))\\ &\le \exp\left(-\frac{1}{2}(t-t_0)\right)\Phi(t_0,x(t_0),y(t_0))\\ &\text{for the case } (t,x(t),y(t))\in S_1,\ t\ge t_0,\end{aligned} \tag{5.37a}$$

whereas, by virtue of (5.30b), (5.31), (5.32), and (5.34) we obtain

$$\begin{aligned}\Phi(t,x(t),y(t)) &= W(t,|y(t)-k(t,x(t))|)\\ &\le \exp\left(-\frac{1}{2}(t-t_0)\right)\Bigg(W(t_0,|y(t_0)-k(t_0,x(t_0))|)\\ &\qquad + \int_{t_0}^{t}\exp\left(-\frac{1}{2}(\tau+t_0)\right)d\tau\Bigg)\\ &\le \exp\left(-\frac{1}{2}(t-t_0)\right)\left(\Phi(t_0,x(t_0),y(t_0)) + \int_{t_0}^{t}\exp\left(-\frac{1}{2}(\tau+t_0)\right)d\tau\right)\\ &\text{for the case } (t,x(t),y(t))\in S_2,\ t\ge t_0.\end{aligned} \tag{5.37b}$$

Combining both cases (5.37a), (5.37b) above and exploiting continuity of Φ, we get (5.36). It turns out by taking into account (5.24a), (5.26a), (5.28), (5.32), and (5.33) that

$$|u(t)| \le \bar{\gamma}(t,|(x(t),y(t))|) \text{ a.e. for } t\ge t_0 \Rightarrow a_1(|x(t)|) \le D(t,t_0,|(x_0,y_0)|), \tag{5.38}$$

$$|u(t)| \le \bar{\gamma}(t,|(x(t),y(t))|) \text{ a.e. for } t\ge t_0 \Rightarrow a(|y-k(t,x)|) \le D(t,t_0,|(x_0,y_0)|), \tag{5.39}$$

where $D(t,t_0,s) := \exp(-\frac{1}{2}(t-t_0))(1+\bar{\beta}(t_0)\bar{a}_2(s+E(t_0,s)))$. It follows from (5.24a), (5.38), and (5.39) that

$$\begin{aligned}|u(t)| &\le \bar{\gamma}(t,|(x(t),y(t))|) \text{ a.e. for } t\ge t_0\\ &\Rightarrow |y(t)| \le E(t,a_1^{-1}(D(t,t_0,|(x_0,y_0)|))) + a^{-1}(D(t,t_0,|(x_0,y_0)|)),\end{aligned}$$

which by virtue of (5.24b), (5.38), and (5.39) guarantee the ISS property for (5.35) with gain $\bar{\gamma}$ from the input u. ☐

Conditions (A1), (A2), and (A3) do not in general guarantee that the feedback stabilizer $\bar{k}(\cdot)$ satisfies the same property (A2) imposed for the original feedback $k(\cdot)$. This is a drawback for the achievement of ISS partial-state feedback stabilization for higher dimensional triangular time-varying systems by applying backstepping design. Therefore, some additional conditions should be imposed for the original subsystem (1.5a) and the map $k(\cdot)$ in order to propagate (A2) to the new feedback $\bar{k}(\cdot)$, like those imposed in [40]. For instance, in [40] it was assumed that (1.5a) satisfies an exponential type of ISS from the input y and the dynamics have polynomial structure with respect to (t,x). Further generalizations of Proposition 5.6, as well as conditions weaker than those imposed in [40], which enable us to construct a smooth feedback with the same properties as $k(\cdot)$, are presented in [15]. We limited ourselves instead, to the case examined in [40], by re-establishing ISS stabilization for (1.5) by means of a smooth feedback $\bar{k}(\cdot)$ for which (A2) holds. We next show that the main result in [40] is a straightforward consequence of Proposition 5.6.

PROPOSITION 5.7. *Consider the system* (1.5) *with* $h(t,x,y)\equiv 1$, *and in addition to the regularity properties for* f, g, k, γ *imposed in Proposition* 5.6, *we assume that* *there exists a function r of class Π (see "Notations" for the definition of class Π) and constants a, $K > 0$ such that*

$$|f(t,x,y)| + |g(t,x,y)| \leq (1+t)^a r(|(x,y)|), \tag{5.40a}$$

$$|k(t,x)| + \left|\frac{\partial k}{\partial t}(t,x)\right| \leq (1+t)^a r(|x|), \tag{5.40b}$$

$$\left|\frac{\partial k}{\partial x}(t,x)\right| \leq (1+t)^a (1 + r(|x|)), \tag{5.40c}$$

$$\frac{1}{K(1+t)^a} s \leq \gamma(t,s) \leq (1+t)^a r(s). \tag{5.40d}$$

Moreover, assume that subsystem (1.5a) *satisfies assumption* (A1) *with* $a_1(s) = a_2(s) = s^2$ *and* $\beta(t) = M(1+t)^a$ *for some constant* $M > 0$*. Then for any* $\Gamma(\cdot) \in \Pi$ *there exist a function* $\bar{r} \in \Pi$*, constants* $\bar{a} \geq a$ *and* $\bar{M} \geq M$*, and a feedback* $\bar{k}(\cdot)$ *as in statement of Proposition* 5.6 *such that property* (A1) *holds for* (1.5) *with* $u := u + \bar{k}(t, y - k(t,x))$*,* $\gamma := \Gamma$*, and some* $a_1(\cdot)$*,* $a_2(\cdot)$*, and* $\bar{\beta}(t) = \bar{M}(1+t)^{\bar{a}}$*, as well as inequalities* (5.40a), (5.40b), (5.40c), (5.40d)*, are fulfilled with* $\bar{k}(t, y - k(t,x))$*,* Γ*,* (x,y)*,* $\bar{a}$*, and* $\bar{r}$ *instead of* $k(t,x)$*,* γ*,* x*,* a*, and* r*, respectively.*

Proof (outline). It can be easily verified that all hypotheses (A1), (A2), and (A3) of Proposition 5.6 are fulfilled for (1.5). Particularly, (A2) holds as a consequence of (5.40b) and the fact that $r \in \Pi$. In order to establish our statement we proceed exactly as in the proof of Proposition 5.6. In our case we may use

$$W(t,s) = C(1+t)^{3a} s^2 \tag{5.41}$$

for some constant $C > 0$ (the constant a is defined in (5.40)) and we can find a polynomial $R \in \Pi$ of the form $R(s) = R_0(s + s^l)$ for $R_0 > 0$ and l being an odd positive integer and a constant $\theta \geq a$ such that (5.34) is fulfilled with

$$\bar{k}(t,y) := -(1+t)^\theta R(y) \tag{5.42}$$

and with $W(\cdot)$ as given by (5.41). The rest of the proof is the same as that given in proof of Proposition 5.6. Finally, it is immediate to see that, according to definition (5.42), the feedback $\bar{k}(\cdot)$ satisfies the same properties as those imposed for $k(\cdot)$; hence, it turns out that (A2) holds for the map $\bar{k}(\cdot)$. ∎

We may use the result of Proposition 5.7 and apply the induction procedure in order to re-establish Theorem 2.4 in [40], concerning partial-state feedback stabilization for a class of triangular systems.

COROLLARY 5.8. *Consider the system*

$$\dot{x} = f(t,x,y), \tag{5.43a}$$

$$\begin{gathered} \dot{y}_i = g_i(t,x,y_1,\ldots,y_i) + y_{i+1}, \qquad i = 1,\ldots,m, \\ x \in \Re^n, \quad y = (y_1,\ldots,y_m)^T \in \Re^m, \quad t \geq 0, \quad u = y_{m+1} \in \Re, \end{gathered} \tag{5.43b}$$

where f*,* g_i *are* C^0 *everywhere and locally Lipschitz with respect to* (x,y) *with* $f(t,0,0) = 0$*,* $g_i(t,0,\ldots,0) = 0$ *for* $i = 1,\ldots,m$ *and for all* $t \geq 0$*. Suppose that there exists a function* r *of class* Π *and a constant* $a > 0$ *such that*

$$|f(t,x,y)| \leq (1+t)^a r(|(x,y)|), \tag{5.44a}$$

$$|g_i(t,x,y_1,\ldots,y_i)| \leq (1+t)^a r(|(x,y_1,\ldots,y_i)|). \tag{5.44b}$$

Moreover, assume that subsystem (5.43a) *satisfies assumption* (A1) *with* $k \equiv 0$, $a_1(s) = a_2(s) = s^2$, *and* $\beta(t) = M(1+t)^a$ *for some constant* $M > 0$ *and gain* $\gamma(t,s)$, *which is* C^0 *on* $\Re^+ \times \Re^+$ *and locally Lipschitz with respect to* $s \geq 0$ *and satisfies* $\gamma(t,\cdot) \in K_\infty$ *for all* $t \geq 0$, *in such a way that the following holds for some constant* $K > 0$*:*

$$\frac{1}{K(1+t)^a} s \leq \gamma(t,s) \leq (1+t)^a r(s). \tag{5.44c}$$

Then for any $\Gamma(\cdot) \in \Pi$ *there exists a* C^∞ *feedback law* $u = \bar{k}(t, y_1, \ldots, y_m)$ *such that system* (5.43) *with* $u := \bar{k}(t,y) + u$ *satisfies the ISS property with gain* Γ *from the input* u.

6. Conclusions. We have provided equivalent characterizations for the concept of robust global asymptotic stability (RGAS) for time-varying systems. Lyapunov characterizations for this concept as well as for the concept of nonuniform in time input-to-state stability (ISS) are given. Moreover, we have provided necessary and sufficient conditions for nonuniform in time ISS stabilization of affine in the control systems by means of a smooth time-varying feedback. An explicit formula for the time-varying feedback stabilizer is also presented. The problem of partial-state nonuniform in time ISS-feedback stabilization for triangular systems is considered.

REFERENCES

[1] D. Angeli and E.D. Sontag, *Forward completeness, unbounded observability and their Lyapunov characterizations*, Systems Control Lett., 38 (1999), pp. 209–217.

[2] D. Angeli, E.D. Sontag, and Y. Wang, *A characterization of integral input-to-state stability*, IEEE Trans. Automat. Control, 45 (2000), pp. 1082–1096.

[3] Z. Artstein, *Stabilization with relaxed controls*, Nonlinear Anal., 7 (1983), pp. 1163–1173.

[4] A. Bacciotti and L. Rosier, *Lyapunov stability and Lagrange stability: Inverse theorems for discontinuous systems*, Math. Control Signals Systems, 11 (1998), pp. 101–125.

[5] A. Bacciotti and L. Rosier, *On the converse of first Lyapunov theorem: The regularity issue*, Systems Control Lett., 41 (2000), pp. 265–270.

[6] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory*, Lecture Notes in Control and Inform. Sci. 267, Springer–Verlag, London, 2001.

[7] J.M. Coron and L. Praly, *Adding an integrator for the stabilization problem*, Systems Control Lett., 17 (1991), pp. 89–104.

[8] W.P. Dayawansa and C.F. Martin, *A converse Lyapunov theorem for a class of dynamical systems which undergo switching*, IEEE Trans. Automat. Control, 44 (1999), pp. 751–760.

[9] L. Grune, *Input-to-state dynamical stability and its Lyapunov function characterization*, to appear.

[10] W. Hahn, *Stability of Motion*, Springer–Verlag, Berlin, 1967.

[11] Z.P. Jiang, A. Teel, and L. Praly, *Small-gain theorem for ISS systems and applications*, Math. Control Signals Systems, 7 (1994), pp. 95–120.

[12] I. Karafyllis and J. Tsinias, *Global stabilization and asymptotic tracking for a class of nonlinear systems by means of time-varying feedback*, Int. J. Robust Nonlinear Control, to appear.

[13] I. Karafyllis and J. Tsinias, *Converse Lyapunov theorems for non-uniform in time global asymptotic stability and stabilization by means of time-varying feedback*, in Nonlinear Control Systems 2001, Elsevier, New York, 2002, pp. 801–805.

[14] I. Karafyllis and J. Tsinias, *Non-uniform in time stabilization for linear systems and tracking control for nonholonomic systems in chained form*, Internat. J. Control, submitted.

[15] I. Karafyllis, *Non-uniform stabilization of control systems*, IMA J. Math. Control Inform., 19 (2002), pp. 419–444.

[16] I. Karafyllis and J. Tsinias, *Characterizations of the non-uniform in time ISS property and applications*, in Proceedings of the Fifteenth International Symposium on Mathematical Theory of Networks and Systems, Notre Dame, IN, 2002.

[17] I. Karafyllis, *Necessary and sufficient conditions for the existence of stabilizing feedback for control systems*, IMA J. Math. Control Inform., 20 (2003), pp. 37–64.
[18] H.K. Khalil, *Nonlinear Systems*, 2nd ed., Prentice–Hall, Englewood Cliffs, NJ, 1996.
[19] J. Kurzweil, *On the inversion of Lyapunov's second theorem on stability of motion*, Amer. Math. Soc. Transl. Ser. 2, 24 (1956), pp. 19–77.
[20] Y. Lin, E.D. Sontag, and Y. Wang, *A smooth converse Lyapunov theorem for robust stability*, SIAM J. Control Optim., 34 (1996), pp. 124–160.
[21] J.L. Mancilla-Aguilar and R.A. Garcia, *On converse Lyapunov theorems for ISS and iISS switched nonlinear systems*, Systems Control Lett., 42 (2001), pp. 47–53.
[22] E. Panteley and A. Loria, *On global uniform asymptotic stability of nonlinear time-varying systems in cascade*, Systems Control Lett., 33 (1998), pp. 131–138.
[23] L. Rifford, *Existence of Lipschitz and semiconcave control-Lyapunov functions*, SIAM J. Control Optim., 39 (2000), pp. 1043–1064.
[24] L. Rifford, *On the existence of non-smooth control Lyapunov functions in the sense of generalized gradients*, ESAIM Control Optim. Calc. Var., 6 (2001), pp. 593–611.
[25] L. Rosier, *Smooth Lyapunov functions for discontinuous stable systems*, Set-Valued Anal., 7 (1999), pp. 375–405.
[26] E.D. Sontag, *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Control Optim., 21 (1983), pp. 462–471.
[27] E.D. Sontag, *A "universal" construction of Artstein's theorem on nonlinear stabilization*, Systems Control Lett., 13 (1989), pp. 117–123.
[28] E.D. Sontag, *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Control, 34 (1989), pp. 435–443.
[29] E.D. Sontag and Y. Wang, *On characterizations of the input-to-state stability property*, Systems Control Lett., 24 (1995), pp. 351–359.
[30] E.D. Sontag and H.J. Sussmann, *Nonsmooth control-Lyapunov functions*, in Proceedings of the 34th IEEE Conference on Decision and Control, Vol. 3, IEEE Press, Piscataway, NJ, 1995, pp. 2799–2805.
[31] E.D. Sontag. and Y. Wang, *New characterizations of the input-to-state stability*, IEEE Trans. Automat. Control, 41 (1996), pp. 1283–1294.
[32] E.D. Sontag, *Comments on integral variants of ISS*, Systems Control Lett., 34 (1998), pp. 93–100.
[33] E.D. Sontag and Y. Wang, *Lyapunov characterizations of input to output stability*, SIAM J. Control Optim., 39 (2000), pp. 226–249.
[34] A.R. Teel and L. Praly, *A smooth Lyapunov function from a class-KL estimate involving two positive semidefinite functions*, ESAIM Control Optim. Calc. Var., 5 (2000), pp. 313–367.
[35] J. Tsinias, *Sufficient Lyapunov like conditions for stabilization*, Math. Control Signals Systems, 2 (1989), pp. 343-357.
[36] J. Tsinias, *Versions of Sontag's input to state stability condition and output feedback stabilization*, J. Math. Systems Estim. Control, 6 (1996), pp. 113–126.
[37] J. Tsinias, *Input to state stability properties of nonlinear systems and applications to bounded feedback stabilization using saturation*, ESAIM Control Optim. Calc. Var., 2 (1997), pp. 57–85.
[38] J. Tsinias, *Stochastic input-to-state stability and applications to global feedback stabilization*, Internat. J. Control, 71 (1998), pp. 907–931.
[39] J. Tsinias, *The concept of "exponential ISS" for stochastic systems and applications to feedback stabilization*, Systems Control Lett., 36 (1999), pp. 221–229.
[40] J. Tsinias and I. Karafyllis, *ISS property for time-varying systems and application to partial-static feedback stabilization and asymptotic tracking*, IEEE Trans. Automat. Control, 44 (1999), pp. 2179–2185.
[41] J. Tsinias, *Backstepping design for time-varying nonlinear systems with unknown parameters*, Systems Control Lett., 39 (2000), pp. 219–227.
[42] J. Tsinias, *A converse Lyapunov theorem for non-uniform in time, global exponential robust stability*, Systems Control Lett., 44 (2001), pp. 373–384.
[43] J. Tsinias and J. Spiliotis, *Notions of exponential robust stability, ISS and their Lyapunov characterizations*, Int. J. Robust Nonlinear Control, to appear.