INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust. Nonlinear Control 2012; **22**:1602–1630
Published online 1 August 2011 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.1773

A new small-gain theorem with an application to the stabilization of the chemostat

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SUMMARY

New small-gain results are obtained for nonlinear feedback systems under relaxed assumptions. Specifically, during a transient period, the solutions of the feedback system may not satisfy some key inequalities that previous small-gain results usually utilize to prove stability properties. The results allow the application of the small-gain perspective to various systems that satisfy less-demanding stability notions than the input-to-output stability property. The robust global feedback stabilization problem of an uncertain time-delayed chemostat model is solved by means of the new small-gain results. Copyright © 2011 John Wiley & Sons, Ltd.

Received 23 February 2010; Revised 11 February 2011; Accepted 16 May 2011

KEY WORDS: input-to-output stability; feedback systems; small-gain theorem; chemostat models

1. INTRODUCTION

Small-gain results are important tools for robustness analysis and robust controller design in mathematical control theory. In [1], a nonlinear, generalized small-gain theorem, based on the notion of input-to-state stability (ISS) originally introduced by Sontag [2], was developed. Recently, nonlinear small-gain results were developed for monotone systems, an important class of nonlinear systems in mathematical biology [3, 4]. Further extensions of the small-gain perspective to the cases of nonuniform-in-time stability, discrete-time systems, and Lyapunov characterizations are pursued by several authors independently; see, for instance, [5–18]. A general vector small-gain result, which can be applied to a wide class of control systems, was developed in [19].

One of the most important obstacles in applying nonlinear small-gain results is the representation of the original composite system as the feedback interconnection of subsystems that satisfy the input-to-output stability (IOS) property. More specifically, sometimes, the subsystems do not satisfy the IOS property: there is a transient period after which the solution enters a certain region of the state space. Within this region of the state space, the subsystems satisfy the small-gain requirements. In other words, the essential inequalities, which small-gain results utilize in order to prove stability properties, do not hold for all times: this feature excludes all available small-gain results from possible application. Particularly, this feature is important in systems arising from mathematical biology and population dynamics. Indeed, the idea of developing stability results that utilize certain Lyapunov-like conditions after an initial transient period was used in [20, 21] with primary motivation from addressing robust feedback stabilization problems for certain chemostat models.

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In this work, we present small-gain results that can allow a transient period during which the solutions do not satisfy the IOS inequalities (Theorems 2.5 and 2.6). The obtained results are direct extensions of the recent vector small-gain result in [19], and if the initial transient vanishes, then the results coincide with Theorem 3.1 in [19]. The significance of the obtained results is twofold:

- it allows the application of the small-gain perspective to various systems that satisfy less-demanding stability notions than ISS, and
- it allows the study of systems in mathematical biology and population dynamics.

During the time that the paper was submitted for publication, we became aware of the recent results in [22], where the same idea of combining small-gain results with an initial transient period is utilized and is used as an interpretation of the results in [9, 23]. However, we must point out some crucial differences between our results and the results in [22].

- Our results can be applied to a wide class of time-varying systems with disturbances, whereas the results in [22] can be applied to autonomous systems described by ordinary differential equations (ODEs). The general case of IOS is considered in our case, whereas the results demand (integral [i]) ISS in [22]. Nonuniform stability phenomena can be taken into account by Theorems 2.5 and 2.6.
- Our results guarantee IOS for the overall system, whereas the results in [22] guarantee iISS for the overall system. Moreover, in our case, the gain function of the overall system can be estimated by means of the functions involved in the key estimates of the hypotheses (Theorem 2.5).
- Finally, our results can be applied to large-scale systems, because they can take into account multiple interconnections, whereas the results in [22] only deal with an interconnection of two subsystems.

It should be noted that Theorems 2.5 and 2.6 can be formulated in terms of vector Lyapunov functionals for the particular cases of systems described by ODEs (Theorem 2.7). The theory of vector Lyapunov functionals is a rich theory that has recently attracted attention (see, e.g., [24, 25] and the recent works [19, 26, 27]).

To emphasize the fact that the obtained small-gain results can be applied to biological systems, we show how the obtained small-gain results can be used for the feedback stabilization of uncertain chemostat models. Chemostat models are often adequately represented by a simple dynamic model involving two state variables, the microbial biomass concentration X and the limiting nutrient concentration S [28]. The common delay-free model for microbial growth on a limiting substrate in a chemostat is of the form

$$\begin{aligned} &\dot{X}(t) = (\mu(S(t)) - D(t))\,X(t), \\ &\dot{S}(t) = D(t)(S_i - S(t)) - K\mu(S(t))X(t), \\ &X(t) \in (0, +\infty)\,,\ S(t) \in (0, S_i)\,,\ D(t) \geqslant 0, \end{aligned} \tag{1.1}$$

where S_i is the feed substrate concentration, D is the dilution rate (which is used as the control input), $\mu(S)$ is the specific growth rate and $K > 0$ is a biomass yield factor. The literature on control studies of chemostat models of the form (1.1) is extensive. In [29], feedback control of the chemostat by manipulating the dilution rate was studied for the promotion of coexistence. Other interesting control studies of the chemostat can be found in [26, 30–35]. The stability and robustness of periodic solutions of the chemostat was studied in [36, 37]. The problem of the stabilization of a nontrivial steady state (X_s, S_s) of the chemostat model (1.1) was considered in [33], where it was shown that the simple feedback law $D = \mu(S)X/X_s$ is a globally stabilizing feedback. See also the recent work [26] for the study of the robustness properties of the closed-loop system (1.1) with $D = \mu(S)X/X_s$ for time-varying inlet substrate concentration S_i. The recent work [20] studied the sampled-data stabilization of the nontrivial steady state (X_s, S_s) of the chemostat model (1.1), whereas [21] considered uncertain chemostat models.

DOI: 10.1002/rnc

In this work, we consider the robust global feedback stabilization problem for the chemostat model with delays:

$$\begin{aligned} \dot{X}(t) &= (p\,(T_r(t)S) - D(t) - b)\,X(t), \\ \dot{S}(t) &= D(t)(S_i - S(t)) - K(S(t))\mu(S(t))X(t), \\ X(t) &\in (0, +\infty)\,,\ S(t) \in (0, S_i)\,,\ D(t) \geqslant 0, \end{aligned} \tag{1.2}$$

where $T_r(t)S : [-r, 0] \to (0, S_i)$ denotes the r-history of S defined by $(T_r(t)S)\,(\theta) = S(t+\theta)$ for $\theta \in [-r, 0]$, $b \geqslant 0$ is the cell mortality rate, $r \geqslant 0$ is the maximum delay, $K(S) > 0$ is a possibly variable yield coefficient, and $p : C^0([-r, 0]; (0, S_i)) \to (0, +\infty)$ is a continuous functional that satisfies

$$\min_{t-r\leqslant\tau\leqslant t} \mu(S(\tau)) \leqslant p\,(T_r(t)S) \leqslant \max_{t-r\leqslant\tau\leqslant t} \mu(S(\tau)). \tag{1.3}$$

The functions $\mu : [0, S_i] \to [0, \mu_{\max}]$, $K : [0, S_i] \to (0, +\infty)$ with $\mu(0) = 0$, $\mu(S) > 0$ for all $S \in (0, S_i]$ are assumed to be locally Lipschitz functions. The chemostat model (1.2) under (1.3) is very general, because we may have

- $p\,(T_r(t)S) = \mu(S(t))$, which gives the standard chemostat model with no delays,
- $p\,(T_r(t)S) = \mu(S(t-r))$, which gives the time-delayed chemostat model studied in [28], and
- $p\,(T_r(t)S) = \lambda \sum_{i=0}^{n} w_i \mu(S(t-r_i)) + (1-\lambda) \int_{t-r}^{t} h(\tau + r - t)\mu(S(\tau))\,\mathrm{d}\tau$, where $\lambda \in [0, 1]$, $h \in C^0([0, r]; [0, +\infty))$ with $\int_0^r h(s)\,\mathrm{d}s = 1$, $w_i \geqslant 0$, $r_i \in [0, r]$ $(i = 0, \ldots, n)$ with $\sum_{i=0}^{n} w_i = 1$.

Moreover, it should be noted that the case of variable yield coefficients has been studied recently [38, 39] and has been proposed for the justification of experimental results. The reader should notice that chemostat models with time delays were considered in [40, 41]. We assume the existence of a nontrivial equilibrium point for (1.2), that is, the existence of $(S_s, X_s) \in (0, S_i) \times (0, +\infty)$ such that

$$\mu(S_s) = D_s + b, \qquad X_s = \frac{D_s(S_i - S_s)}{K(S_s)\,(D_s + b)}, \tag{1.4}$$

where $D_s > 0$ is the equilibrium value for the dilution rate. The stabilization problem for the equilibrium point $(S_s, X_s) \in (0, S_i) \times (0, +\infty)$ is crucial. In [28], it is shown that the equilibrium point is unstable even if $\mu : (0, S_i) \to (0, \mu_{\max}]$ is monotone (e.g., the Monod specific growth rate). Moreover, as remarked in [28], the chemostat model (1.2) under (1.3) allows the expression of the effect of the time difference between consumption of nutrient and growth of the cells [28, pp. 238–240]. We solve the feedback stabilization problem for the chemostat by providing a *delay-free* feedback that achieves global stabilization (Theorem 4.1). The proof of the theorem relies on the small-gain results of the paper. No knowledge of the maximum delay $r \geqslant 0$ is assumed.

The structure of the present work is as follows. Section 2 contains the statements of the new small-gain results (Theorems 2.5 and 2.6). Moreover, a vector Lyapunov formulation of the small-gain results for systems described by ODEs is given in Theorem 2.7. Section 3 provides illustrative examples of the applicability of the obtained results to systems that satisfy less-demanding stability notions than ISS. Section 4 is devoted to the development of the solution of the feedback stabilization problem for the uncertain chemostat (1.2). The conclusions are provided in Section 5. The proofs of the small-gain results are given in Appendix A. Finally, for readers' convenience, the definitions of the system-theoretic notions used in this work are given in Appendix B.

Notations

Throughout this paper, we adopt the following notations.

- $\Re_+ := \{x \in \Re : x \geqslant 0\}$ denotes the set of nonnegative real numbers.
- We say that a function $\rho : \Re_+ \to \Re_+$ is positive definite if $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$. We say that a function $V : \Re^n \to \Re$ is positive definite if $V(0) = 0$ and $V(x) > 0$ for

DOI: 10.1002/rnc

all $x \neq 0$. A continuous function $V : \Re^n \to \Re_+$ is called radially unbounded if the level sets $\{ x \in \Re^n : V(x) \leqslant M \}$ are compact for all $M \geqslant 0$.

- We denote by K^+ the class of all continuous functions $\varphi : \Re_+ \to (0, +\infty)$. By K, we denote the set of positive-definite, increasing, and continuous functions. We say that a positive-definite, increasing, and continuous function $\rho : \Re_+ \to \Re_+$ is of class K_∞ if $\lim_{s \to +\infty} \rho(s) = +\infty$. By KL, we denote the set of all continuous functions $\sigma : \Re^+ \times \Re^+ \to \Re^+$ with the following properties: (i) for each $t \geqslant 0$, the mapping $\sigma(\cdot, t)$ is of class K; and (ii) for each $s \geqslant 0$, the mapping $\sigma(s, \cdot)$ is non-increasing with $\lim_{t \to +\infty} \sigma(s, t) = 0$.
- By $\| \|_{\mathsf{X}}$, we denote the norm of the normed linear space X. By $| |$, we denote the Euclidean norm of $\Re^n$. Let $U \subseteq \mathsf{X}$ with $0 \in U$. By $B_U[0, r] := \{ u \in U \ ; \ \|u\|_{\mathsf{X}} \leqslant r \}$, we denote the intersection of $U \subseteq \mathsf{X}$ with the closed ball of radius $r \geqslant 0$, centered at $0 \in U$. If $U \subseteq \Re^n$, then $\mathrm{int}(U)$ denotes the interior of the set $U \subseteq \Re^n$.
- x' denotes the transpose of x.
- $\Re^n_+ := (\Re_+)^n = \left\{ (x_1, \ldots, x_n)' \in \Re^n : x_1 \geqslant 0, \ldots, x_n \geqslant 0 \right\}$. $\{e_i\}_{i=1}^n$ denotes the standard basis of $\Re^n$. Z_+ denotes the set of nonnegative integers.
- Let $x, y \in \Re^n$. We say that $x \leqslant y$ if and only if $(y - x) \in \Re^n_+$. We say that a function $\rho : \Re^n_+ \to \Re_+$ is of class N_n if ρ is continuous with $\rho(0) = 0$ and such that $\rho(x) \leqslant \rho(y)$ for all $x, y \in \Re^n_+$ with $x \leqslant y$. We say that $\Gamma : \Re^n_+ \to \Re^m_+$ is nondecreasing if $\Gamma(x) \leqslant \Gamma(y)$ for all $x, y \in \Re^n_+$ with $x \leqslant y$. For an integer $k \geqslant 1$, we define

$$\Gamma^{(k)}(x) = \underbrace{\Gamma \circ \Gamma \circ \ldots \circ \Gamma}_{k \ \text{times}}(x)$$

  when $m = n$.
- For $t \geqslant t_0 \geqslant 0$, let $V : [t_0, t] \to \Re^n$ be a bounded map with $V(\tau) = (V_1(\tau), \ldots, V_n(\tau))' \in \Re^n$ for $\tau \in [t_0, t]$. We define $[V]_{[t_0,t]} := \left(\sup_{\tau \in [t_0,t]} V_1(\tau), \ldots, \sup_{\tau \in [t_0,t]} V_n(\tau) \right)$. For a measurable and essentially bounded function $x : [a, b] \to \Re^n$, $\mathrm{ess\,sup}_{t \in [a,b]} |x(t)|$ denotes the essential supremum of $|x(\cdot)|$. Given a function $x : [a - r, b) \to \Re^n$, where $r > 0$, $a < b$, we define $T_r(t)x : [-r, 0] \to \Re^n$, for $t \in [a, b)$, to be the r-history of x, defined by $(T_r(t)x)(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.
- We define $\mathbf{1} = (1, 1, \ldots, 1)' \in \Re^n$. If $u, v \in \Re$ and $u \leqslant v$, then $\mathbf{1}u \leqslant \mathbf{1}v$.
- Let U be a subset of a normed linear space U, with $0 \in U$. By $M(U)$, we denote the set of all locally bounded functions $u : \Re_+ \to U$. By u_0, we denote the identically zero input, that is, the input that satisfies $u_0(t) = 0 \in U$ for all $t \geqslant 0$. If $U \subseteq \Re^n$, then M_U or $L^\infty_{\mathrm{loc}}(\Re_+; U)$ denote the space of measurable, locally bounded functions $u : \Re_+ \to U$.
- Let $A \subseteq \mathsf{X}$ and $B \subseteq \mathsf{Y}$, where X and Y are normed linear spaces. We denote by $C^0(A; B)$ the class of continuous mappings $f : A \to B$. For $x \in C^0([-r, 0]; \Re^n)$, we define $\|x\|_r := \max_{\theta \in [-r,0]} |x(\theta)|$.

2. NEW SMALL-GAIN THEOREMS

In this section, we state the main results of the present work. The proofs of the main results (Theorems 2.5 and 2.6) are provided in Appendix A. For the statement of the main result, one needs to know the abstract system theoretic framework introduced in [16, 42, 43] and used in [19]. For the convenience of the reader, all definitions of the basic notions are provided in Appendix B.
The following technical definitions were used in [19] and are needed here.

Definition 2.1
Let $x = (x_1, \ldots, x_n)' \in \Re^n$, $y = (y_1, \ldots, y_n)' \in \Re^n$. We define $z = \mathrm{MAX}\{x, y\}$, where $z = (z_1, \ldots, z_n)' \in \Re^n$ satisfies $z_i = \max\{x_i, y_i\}$ for $i = 1, \ldots, n$. Similarly for $u_1, \ldots, u_m \in \Re^n$, we have $z = \mathrm{MAX}\{u_1, \ldots, u_m\}$, a vector $z = (z_1, \ldots, z_n)' \in \Re^n$ with $z_i = \max\{u_{1i}, \ldots, u_{mi}\}$, $i = 1, \ldots, n$.

Definition 2.2
We say that $\Gamma : \Re^n_+ \to \Re^n_+$ is MAX-preserving if $\Gamma : \Re^n_+ \to \Re^n_+$ is nondecreasing, and for every $x, y \in \Re^n_+$, the following equality holds:

$$\Gamma\left(\mathrm{MAX}\{x, y\}\right) = \mathrm{MAX}\{\Gamma(x), \Gamma(y)\}. \tag{2.1}$$

The defined MAX-preserving maps enjoy the following important property [19].

Proposition 2.3
$\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ is MAX-preserving if and only if there exist nondecreasing functions $\gamma_{i,j} : \Re_+ \to \Re_+$, $i, j = 1, \ldots, n$ with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$ for all $x \in \Re^n_+$, $i = 1, \ldots, n$.

The following class of MAX-preserving mappings plays an important role in what follows.

Definition 2.4
Let $\Gamma : \Re^n_+ \to \Re^n_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ be a MAX-preserving mapping for which there exist functions $\gamma_{i,j} \in \mathsf{N}_1$, $i, j = 1, \ldots, n$ with $\Gamma_i(x) = \max_{j=1,\ldots,n} \gamma_{i,j}(x_j)$ for all $x \in \Re^n_+$, $i = 1, \ldots, n$. We say that $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions if the following inequalities hold:

$$\gamma_{i,i}(s) < s, \quad \forall s > 0, \quad i = 1, \ldots, n, \tag{2.2}$$

and if $n > 1$, then for each $r = 2, \ldots, n$, it holds that

$$\left(\gamma_{i_1,i_2}, \gamma_{i_2,i_3}, \ldots, \gamma_{i_r,i_1}\right)(s) < s, \quad \forall s > 0, \tag{2.3}$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$.

Proposition 2.7 in [19] shows that the MAX-preserving continuous mapping $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions if and only if $0 \in \Re^n$ is globally asymptotically stable for the discrete-time $x(k+1) = \Gamma(x(k))$, where $x(k) \in \Re^n_+$, $k \in Z^+$. For every MAX-preserving continuous mapping $\Gamma : \Re^n_+ \to \Re^n_+$ we define the mapping $Q(x) := \mathrm{MAX}\left\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\right\}$ for all $x \in \Re^n_+$. The following facts are consequences of the related results in [7, 19, 44, 45] and the Definitions 2.1, 2.2, and 2.4 and provide important properties for MAX-preserving continuous mappings $\Gamma : \Re^n_+ \to \Re^n_+$, which will be used repeatedly in the proofs of the main results of the present section.

Fact I
If $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions, then $\lim_{k\to+\infty} \Gamma^{(k)}(x) = 0$ for all $x \in \Re^n_+$ and $\Gamma^{(k)}(x) \leqslant Q(x)$ for all $k \geqslant 1$ and $x \in \Re^n_+$.

Fact II
The mapping $Q(x)$ is a MAX-preserving continuous mapping.

Fact III
If $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions, then $\Gamma(Q(x)) \leqslant Q(x)$ and $Q(x) \geqslant x$ for all $x \in \Re^n_+$.

Fact IV
If $p \in \mathsf{N}_n$ and $R : \Re^n_+ \to \Re^n_+$ is a nondecreasing mapping, then the following inequality holds for all $s, r \in \Re_+ : p\left(\mathrm{MAX}\left\{R\left(\mathbf{1}s\right), R\left(\mathbf{1}r\right)\right\}\right) = \max\left(p\left(R\left(\mathbf{1}s\right)\right), p\left(R\left(\mathbf{1}r\right)\right)\right)$.

Fact V
If $\Gamma : \Re^n_+ \to \Re^n_+$ satisfies the cyclic small-gain conditions and $x, y \in \Re^n_+$ satisfy $x \leqslant \mathrm{MAX}\{y, \Gamma(x)\}$, then $x \leqslant Q(y)$.

We consider an abstract control system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ with the boundedness-implies-continuation (BIC) property for which $0 \in \mathsf{X}$ is a robust equilibrium point from the input $u \in M_U$ (see Appendix B for the notions of an abstract control system, the BIC property, and the notion of a robust equilibrium point). The reader should notice that for each $(t_0, x_0, u, d) \in \Re_+ \times X \times M_U \times M_D$, $x(t) = \phi(t, t_0, x_0, u, d) \in \mathsf{X}$ denotes the transition map (the value of the state) at time $t \geqslant t_0$ with initial condition $x(t_0) = \phi(t_0, t_0, x_0, u, d) = x_0$ and corresponding to external inputs $(u, d) \in M_U \times M_D$ (Appendix B). We suppose that there exists a set-valued map $S : \Re_+ \to 2^{\mathsf{X}}$ with $0 \in S(t)$ for all $t \geqslant 0$, mappings $V_i : \bigcup_{t \geqslant 0} \{\{t\} \times S(t) \times U\} \to \Re_+ (i = 1, \ldots, n)$, $L : \bigcup_{t \geqslant 0} \{\{t\} \times S(t)\} \to \Re_+$ with $L(t, 0) = 0$, $V_i(t, 0, 0) = 0$ for all $t \geqslant 0 (i = 1, \ldots, n)$, and a MAX-preserving (and therefore nondecreasing) continuous map $\Gamma : \Re_+^n \to \Re_+^n$ with $\Gamma(0) = 0$ such that the following hypotheses hold.

Hypothesis (H1) (The 'input-to-output-stability–like' inequalities)
There exist functions $\sigma \in KL$, $\zeta \in \mathsf{N}_1$, such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$ with $\phi(t, t_0, x_0, u, d) \in S(t)$ for all $t \in [t_0, t_{\max})$, the mappings $t \to V(t) = (V_1(t, \phi(t, t_0, x_0, u, d), u(t)), \ldots, V_n(t, \phi(t, t_0, x_0, u, d), u(t)))'$ and $t \to L(t) = L(t, \phi(t, t_0, x_0, u, d))$ are locally bounded on $[t_0, t_{\max})$, and the following estimates hold:

$$V(t) \leqslant \mathrm{MAX}\left\{ \mathbf{1}\sigma\left(L(t_0), t - t_0\right), \Gamma\left([V]_{[t_0,t]}\right), \mathbf{1}\zeta\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right) \right\}, \text{ for all } t \in [t_0, t_{\max}), \tag{2.4}$$

where $t_{\max}$ is the maximal existence time of the transition map of Σ.

Hypothesis (H2) (Estimates during and after the transient period)
For every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$, there exists $\xi \in \pi(t_0, x_0, u, d)$ such that $\phi(t, t_0, x_0, u, d) \in S(t)$ for all $t \in [\xi, t_{\max})$. Moreover, there exist functions $\nu, c, \tilde{c} \in K^+$, $a, \eta, \tilde{\eta}, p^u, g^u \in \mathsf{N}_1$, $p \in \mathsf{N}_n$, such that the following inequalities hold for every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$:

$$L(t) \leqslant \max\left\{ \nu(t - t_0), c(t_0), a\left(\|x_0\|_{\mathsf{X}}\right), p\left([V]_{[\xi,t]}\right), p^u\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right) \right\}, \text{ for all } t \in [\xi, t_{\max}); \tag{2.5}$$

$$\|\phi(t, t_0, x_0, u, d)\|_{\mathsf{X}} \leqslant \max\left\{ \nu(t - t_0), \tilde{c}(t_0), a\left(\|x_0\|_{\mathsf{X}}\right), \tilde{\eta}\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right) \right\}, \text{ for all } t \in [t_0, \xi]; \tag{2.6}$$

$$\xi \leqslant t_0 + a\left(\|x_0\|_{\mathsf{X}}\right) + c(t_0); \tag{2.7}$$

$$\|H(t, \phi(t, t_0, x_0, u, d), u(t))\|_{\mathsf{Y}} \leqslant \max\left\{ a\left(c(t_0)\ \|x_0\|_{\mathsf{X}}\right), \eta\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right) \right\}, \text{ for all } t \in [t_0, \xi]; \tag{2.8}$$

$$L\left(\xi, \phi(\xi, t_0, x_0, u, d)\right) \leqslant \max\left\{ a\left(c(t_0)\ \|x_0\|_{\mathsf{X}}\right), g^u\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,\xi]}\right) \right\}. \tag{2.9}$$

Hypothesis (H3) (Bounds for the norm of the state and the norm of the output)
There exist functions $b \in \mathsf{N}_1$, $q, g \in \mathsf{N}_n$, $\mu, \kappa \in K^+$ such that the following inequalities hold:

$$\mu(t)\,\|x\|_{\mathsf{X}} \leqslant b\left(L(t, x) + g(V(t, x, u)) + \kappa(t)\right), \text{ for all } (t, x, u) \in \bigcup_{t \geqslant 0} \{\{t\} \times S(t) \times U\}, \tag{2.10}$$

$$\|H(t, x, u)\|_{\mathsf{Y}} \leqslant q\left(V(t, x, u)\right), \text{ for all } (t, x, u) \in \bigcup_{t \geqslant 0} \{\{t\} \times S(t) \times U\}, \tag{2.11}$$

where $V(t, x, u) = (V_1(t, x, u), \ldots, V_n(t, x, u))'$.

Discussion of Hypotheses (H1), (H2), (H3)
By combining Hypotheses (H1) and (H2), we can conclude that for each trajectory, there exists a

time $\xi \in \pi(t_0, x_0, u, d)$ after which inequalities (2.4) and (2.5) hold. On the other hand, in order to be able to conclude IOS for the system, we have to assume additional inequalities that hold for the transient period $t \in [t_0, \xi]$, that is, inequalities (2.6), (2.7), (2.8), and (2.9) are required to hold. We next discuss each hypothesis in detail.

- Hypothesis (H1) is the hypothesis made in every small-gain result. It deals with the 'IOS-like' inequalities, which are to be used and combined, in order to prove the desired estimates. Notice that because we are using a family of n functionals, the 'IOS-like' inequalities are given for each functional separately. This is why (2.4) expresses n 'IOS-like' inequalities (in vector notation). The difference between Hypothesis (H1) and similar hypotheses involved in other small-gain results is that we do not assume that the 'IOS-like' inequalities (2.4) hold for every initial condition and every input; instead, we assume that (2.4) holds *only* for those initial conditions and inputs for which the state $\phi(t, t_0, x_0, u, d)$ is in the set $S(t)$ for all times $t \geqslant t_0$ for which the state exists.
- Hypothesis (H2) is the key hypothesis that guarantees that the state will necessarily enter the set $S(t)$. The time needed to in order enter the set $S(t)$ is denoted by $\xi \in \pi(t_0, x_0, u, d)$. Estimates (2.6), (2.8), and (2.9) are estimates for the evolution of the state and the output during the transient period $t \in [t_0, \xi]$, because during the transient period the 'IOS-like' inequalities (2.4) do not hold. Estimate (2.7) is an upper bound for the time needed in order to enter the set $S(t)$. Clearly, such an estimate is needed because we have to guarantee that the transient period (for which the state can behave erratically) is not 'too long'. Finally, estimate (2.5) is a key estimate for the functional $L : \bigcup_{t \geqslant 0} \{\{t\} \times S(t)\} \to \Re_+$ that appears in the right-hand side of inequalities (2.4). Inequality (2.5) holds for all times after the time needed in order to enter the set $S(t)$ (after the transient).
- Hypothesis (H3) is a hypothesis made in every small-gain result (explicitly or implicitly). It provides the bound that allows us to guarantee that the state does not 'blow up' and the bound that allows to conclude that the norm of the output is related to the functionals $V_i : \bigcup_{t \geqslant 0} \{\{t\} \times S(t) \times U\} \to \Re_+ (i = 1, \ldots, n)$. The reader should notice that for every small-gain result, such a hypothesis holds.
- Hypotheses (H1) and (H2) hold automatically when Hypotheses (H1–3) of Theorem 3.1 in [19] hold (Hypotheses (H1-3) in [19] correspond to the special case $S(t) \equiv \mathsf{X}$ and $\xi \equiv t_0$). Consequently, Hypotheses (H1) and (H2) are less-restrictive hypotheses. Indeed, inequalities (2.4) and (2.5) are not assumed to hold for all times $t \in [t_0, t_{\max})$ but only after the solution map $\phi(t, t_0, x_0, u, d)$ has entered the set $S(t) \subseteq \mathsf{X}$.
- Finally, it should be noted that the set-valued map $S(t) \subseteq \mathsf{X}$ is not assumed to be positively invariant. Instead, the state may enter and leave this set during the transient period $t \in [t_0, \xi]$. However, after the initial transient period, the state never leaves the set $S(t) \subseteq \mathsf{X}$. The set-valued map $S(t) \subseteq \mathsf{X}$ reminds the notion of the 'nonautonomous set' introduced in [46].

We are now ready to state the main results.

Theorem 2.5 (Trajectory-based small-gain result for input-to-output stability)
Consider system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ under the aforementioned hypotheses. Assume that the MAX-preserving continuous map $\Gamma : \Re_+^n \to \Re_+^n$ with $\Gamma(0) = 0$ satisfies the cyclic small-gain conditions. Then system Σ satisfies the IOS property from the input $u \in M_U$ with gain $\gamma(s) := \max\{\, \eta(s)\ ,\ q(G(s))\,\}$, where $G(s) = (G_1(s), \ldots, G_n(s))'$ is defined by

$$G(s) = Q\left(\mathbf{1}\max\left\{\sigma\left(p^u(s), 0\right), \sigma\left(g^u(s), 0\right), \sigma\left(p\left(Q\left(\mathbf{1}\sigma\left(g^u(s), 0\right)\right)\right), 0\right),\right.\right.$$
$$\left.\left.\sigma\left(p\left(Q\left(\mathbf{1}\zeta(s)\right)\right), 0\right), \zeta(s)\right\}\right) \tag{2.12}$$

with $Q(x) = \mathrm{MAX}\left\{x, \Gamma(x), \Gamma^{(2)}(x), \ldots, \Gamma^{(n-1)}(x)\right\}$ for all $x \in \Re_+^n$. Moreover, if $c \in K^+$ is bounded, then system Σ satisfies the uniform IOS property from the input $u \in M_U$ with gain $\gamma(s) := \max\{\eta(s), q(G(s))\}$.

We consider next an abstract control system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ with $U = \{0\}$ and the BIC property for which $0 \in \mathsf{X}$ is a robust equilibrium point from the input $u \in M_U$. Suppose that there exists a set-valued map $S : \Re_+ \to 2^{\mathsf{X}}$ with $0 \in S(t)$ for all $t \geqslant 0$, maps $V_i : \bigcup_{t \geqslant 0} \{\{t\} \times S(t)\} \to \Re_+ (i = 1, \ldots, n)$, $L : \bigcup_{t \geqslant 0} \{\{t\} \times S(t)\} \to \Re_+$ with $L(t, 0) = 0$, $V_i(t, 0) = 0$ for all $t \geqslant 0 (i = 1, \ldots, n)$, and a MAX-preserving continuous map $\Gamma : \Re_+^n \to \Re_+^n$ with $\Gamma(0) = 0$ such that the following hypothesis holds.

Hypothesis (H4) (The 'input-to-output-stability–like' inequalities)
There exists a function $\sigma \in KL$ such that for every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$ with $\phi(t, t_0, x_0, u_0, d) \in S(t)$ for all $t \in [t_0, t_{\max})$, the mappings $t \to V(t) = (V_1(t, \phi(t, t_0, x_0, u_0, d)), \ldots, V_n(t, \phi(t, t_0, x_0, u_0, d)))'$ and $t \to L(t) = L(t, \phi(t, t_0, x_0, u_0, d))$ are locally bounded on $[t_0, t_{\max})$, and the following estimates hold:

$$V(t) \leqslant \mathrm{MAX}\left\{ \mathbf{1}\sigma\left(L(t_0), t - t_0 \right), \Gamma\left([V]_{[t_0, t]} \right) \right\}, \text{ for all } t \in [t_0, t_{\max}), \tag{2.13}$$

where $t_{\max}$ is the maximal existence time of the transition map of Σ.

Hypothesis (H5) (Estimates during and after the transient period)
For every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$ there exists $\xi \in \pi(t_0, x_0, u_0, d)$ such that $\phi(t, t_0, x_0, u_0, d) \in S(t)$ for all $t \in [\xi, t_{\max})$. Moreover, there exist functions $\nu, c \in K^+$, $a \in \mathsf{N}_1$, $p \in \mathsf{N}_n$, such that for every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$ the following inequalities hold:

$$L(t) \leqslant \max\left\{ \nu(t - t_0), c(t_0), a\left(\|x_0\|_{\mathsf{X}} \right), p\left([V]_{[\xi, t]} \right) \right\}, \text{ for all } t \in [\xi, t_{\max}); \tag{2.14}$$

$$\|\phi(t, t_0, x_0, u_0, d)\|_{\mathsf{X}} \leqslant \max\left\{ \nu(t - t_0), c(t_0), a\left(\|x_0\|_{\mathsf{X}} \right) \right\}, \text{ for all } t \in [t_0, \xi]; \tag{2.15}$$

$$\xi \leqslant t_0 + a\left(\|x_0\|_{\mathsf{X}} \right) + c(t_0); \tag{2.16}$$

$$L(\xi) \leqslant a\left(\|x_0\|_{\mathsf{X}} \right) + c(t_0). \tag{2.17}$$

Discussion of Hypothesis (H5)
Hypothesis (H5) is almost the same with Hypothesis (H2) applied to the case $U = \{0\}$. Nonetheless, notice the difference that the estimate for $L(\xi)$ in inequality (2.17) is less tight than the estimate needed in inequality (2.9) of Hypothesis (H2). Indeed, when $x_0 = 0$, estimate (2.17) does not yield $L(\xi) = 0$, contrary to the estimate (2.9), which gives $L(\xi) = 0$. Finally, the analogue of inequality (2.8) for $U = \{0\}$ (estimation of the norm of the output during the transient period) is not needed in Hypothesis (H5).

Theorem 2.6 (Trajectory-based small-gain theorem for robust global asymptotic output stability)
Consider system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ with $U = \{0\}$ under hypotheses (H3–5). Assume that the MAX-preserving continuous map $\Gamma : \Re_+^n \to \Re_+^n$ with $\Gamma(0) = 0$ satisfies the cyclic small-gain conditions. Then system Σ is robustly globally asymptotically output stable (RGAOS). Moreover, if $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic for a certain $T > 0$, then system Σ is uniformly RGAOS (URGAOS).

Remarks on Theorems 2.5 and 2.6

(i) It is clear that Hypotheses (H4) and (H5) are less demanding than Hypotheses (H1) and (H2). On the other hand, the conclusion of Theorem 2.6 is weaker than the conclusion of Theorem 2.5. Theorem 2.6 guarantees RGAOS, whereas Theorem 2.5 guarantees IOS. The proofs of Theorems 2.5 and 2.6 are provided in Appendix A and are similar in spirit to the proof of Theorem 3.1 in [19].

(ii) For the case $H(t, x, u) := x$, Theorems 2.5 and 2.6 allow us to conclude ISS and robust global asymptotic stability (RGAS), respectively. In this case, some inequalities in Hypotheses (H1)–(H5) become redundant. For example, inequalities (2.6) and (2.10) are not needed (because inequalities (2.8) and (2.11) with $H(t,x,u) := x$ guarantee that (2.6) and (2.10) hold). Further simplifications are possible if the functional $L : \bigcup_{t \geqslant 0} \{\{t\} \times S(t)\} \to \Re_+$ is defined to be $L(t,x) := \|x\|_{\mathsf{X}}$. In this case, inequality (2.5) is a direct consequence of inequalities (2.4) and (2.11), and inequality (2.9) is a direct consequence of (2.8). Similarly, inequalities (2.14) and (2.17) are redundant for the case $H(t,x,u) := x$ and $L(t,x) := \|x\|_{\mathsf{X}}$.

It should be noted that Theorems 2.5 and 2.6 can be expressed in terms of vector Lyapunov functionals for the particular cases of systems described by ODEs, systems described by retarded functional differential equations, and sampled-data systems exactly as in [19]. We next give sufficient conditions for uniform RGAS (URGAS) in terms of a vector Lyapunov function for systems described by ODEs, which are based on Theorems 2.5 and 2.6.

Consider the following nonlinear system described by ODEs of the form

$$\dot{x} = f(x,d,u), \quad x \in \Re^n,\ d \in D,\ u \in U, \tag{2.18}$$

where $D \subseteq \Re^l$, $U \subseteq \Re^m$ with $0 \in U$ and $f : \Re^n \times D \times U \to \Re^n$ is a continuous mapping with $f(0,d,0) = 0$ for all $d \in D$ that satisfies the following hypotheses.

Hypothesis (A1) (Existence and uniqueness)
There exists $a \in K_\infty$ such that $|f(x,d,u)| \leqslant a(|x| + |u|)$ for all $(x,d,u) \in \Re^n \times D \times U$. Moreover, there exists a symmetric positive-definite matrix $P \in \Re^{n \times n}$ such that for every bounded $O \subset \Re^n \times U$, there exists a constant $L \geqslant 0$ satisfying the following inequality:

$$(x-y)'\,P\,(f(x,d,u) - f(y,d,u)) \leqslant L\,|x-y|^2 \quad \forall (x,u,y,u) \in O \times O\,,\ \forall d \in D.$$

Hypothesis (A2) (The vector Lyapunov formulation of the small-gain hypotheses)
There exist functions $h \in C^1(\Re^n;\Re)$ with $h(0) \leqslant 0$, $V_i \in C^1(\Re^n;\Re_+)(i = 1,\ldots,k)$, $W \in C^1(\Re^n;\Re_+)$ being radially unbounded, a function $\delta \in C^0(\Re_+;(0,+\infty))$, a nondecreasing function $K \in C^0(\Re_+;\Re_+)$, $a_1, a_2 \in K_\infty$, $\zeta \in \mathsf{N}_1$, $\gamma_{i,j} \in \mathsf{N}_1$, $i,j = 1,\ldots,k$, and a family of positive-definite functions $\rho_i \in C^0(\Re_+;\Re_+)(i = 1,\ldots,k)$ such that the following inequalities hold:

$$a_1(|x|) \leqslant \max_{i=1,\ldots,k} V_i(x) \leqslant a_2(|x|)\,, \text{ for all } x \in \Re^n \text{ with } h(x) \leqslant 0; \tag{2.19}$$

$$\sup_{d \in D} \nabla h(x) f(x,d,u) \leqslant -\delta(h(x)), \text{ for all } (x,u) \in \Re^n \times U \text{ with } h(x) \geqslant 0; \tag{2.20}$$

$$\sup_{d \in D} \nabla W(x) f(x,d,u) \leqslant K(h(x))W(x) + K(h(x))\,\zeta(|u|)\,, \text{ for all } (x,u) \in \Re^n \times U \text{ with } h(x) \geqslant 0. \tag{2.21}$$

Moreover, for every $i = 1,\ldots,k$ and $x \in \Re^n$ with $h(x) \leqslant 0$, the following implication holds:

$$\text{'If } \max\left\{\zeta(|u|)\,,\ \max_{j=1,\ldots,k} \gamma_{i,j}\left(V_j(x)\right)\right\} \leqslant V_i(x), \text{ then } \sup_{d \in D} \nabla V_i(x) f(x,d,u) \leqslant -\rho_i\left(V_i(x)\right)\text{'}. \tag{2.22}$$

Discussion of Hypotheses (A1) and (A2)
Hypothesis (A1) is a usual hypothesis that guarantees local existence and uniqueness of solutions for system (2.18) for every measurable and locally essentially bounded inputs $d : \Re_+ \to D$ and $u : \Re_+ \to U$. Hypothesis (A2) is the 'translation' of Hypotheses (H1)–(H3) or Hypotheses (H3)–(H5) in terms of vector Lyapunov functions. In other words, if Hypothesis (A2) holds, then Hypotheses (H1)–(H3) or Hypotheses (H3)–(H5) also hold for system (2.18) with identity output mapping. More specifically, the proof of Theorem 2.7 illustrates that

- inequality (2.21) guarantees that the solution does not 'blow up' when it evolves out of the set $S := \{\, x \in \Re^n : h(x) \leqslant 0 \,\}$, that is, it provides sufficient conditions for inequalities (2.6), (2.8), and (2.9) (or (2.15) and (2.17));

DOI: 10.1002/rnc

- inequality (2.20) guarantees that the solution enters the set $S := \{ x \in \Re^n : h(x) \leqslant 0 \}$ in finite time and that the set $S := \{ x \in \Re^n : h(x) \leqslant 0 \}$ is positively invariant, that is, it provides sufficient conditions for inequality (2.7) (or (2.16));
- implications (2.22) guarantee the IOS-like inequalities when the solution evolves in the set $S := \{ x \in \Re^n : h(x) \leqslant 0 \}$; and
- inequality (2.19) provides the necessary bounds for the norm of the state.

The following result guarantees that Hypotheses (H1)–(H3) or Hypotheses (H3)–(H5) hold for system (2.18). Its proof can be found at Appendix A.

Theorem 2.7
Consider system (2.18) under Hypotheses (A1) and (A2). Suppose that the MAX-preserving mapping $\Gamma : \Re^k_+ \to \Re^k_+$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_k(x))'$, $\Gamma_i(x) = \max_{j=1,\ldots,k} \gamma_{i,j}(x_j)$ for all $x \in \Re^k_+$, $i = 1, \ldots, k$ satisfies the cyclic small-gain conditions. Then the following statements hold:

- If $W \in C^1(\Re^n; \Re_+)$ is positive definite, then system (2.18) satisfies the uniform ISS (UISS) property from the input $u \in U$.
- If $U = \{0\}$, then system (2.18) is URGAS.

3. EXAMPLES AND DISCUSSIONS

The first example indicates that the trajectory-based small-gain results of the previous section can be used to study the feedback interconnection of systems that do not necessarily satisfy the IOS property.

Example 3.1
Consider the system

$$\begin{aligned} &\dot{x} = f(d,x,y,u), \\ &\dot{y} = g(d,x,y), \\ &x \in \Re^n,\ y \in \Re^k,\ d \in D \subset \Re^l,\ u \in \Re^m, \end{aligned} \tag{3.1}$$

where $D \subset \Re^l$ is a nonempty compact set, $f : D \times \Re^n \times \Re^k \times \Re^m \to \Re^n$, $g : D \times \Re^n \times \Re^k \to \Re^k$ are locally Lipschitz mappings with $f(d,0,0,0) = 0$, $g(d,0,0) = 0$ for all $d \in D$. Suppose that there exist positive-definite, continuously differentiable, and radially unbounded functions $V_1 : \Re^n \to \Re_+$, $V_2 : \Re^k \to \Re_+$, a constant $a \in [0,1]$, and a function $k \in K_\infty$ satisfying the following inequalities for all $(x,y,u) \in \Re^n \times \Re^k \times \Re^m$:

$$\max_{d\in D} \nabla V_1(x) f(d,x,y,u) \leqslant -(2+a)\frac{V_1(x)}{1+V_1(x)} + (1-a)\frac{V_2(y)}{(1+V_1(x))(1+V_2(y))} + a\frac{k(|u|)}{1+k(|u|)}, \tag{3.2}$$

$$\max_{d\in D} \nabla V_2(y) g(d,x,y) \leqslant -2\frac{V_2(y)}{1+V_2(y)} + V_1(x). \tag{3.3}$$

It is clear that the subsystem $\dot{y} = g(d,x,y)$ does not satisfy necessarily the ISS property from the input $x \in \Re^n$. Consequently, the classical small-gain theorem in [1] cannot be applied because the y-subsystem in (3.1) is not ISS but iISS with $x \in \Re^n$ as input [47,48]. Recent small-gain approaches have been used for system (3.1), where it is shown that $0 \in \Re^n \times \Re^k$ is globally asymptotically stable [5,49] for the disturbance-free case with $a = 0$. From (3.2) and (3.3), the small-gain approaches in [9,23] directly establish the UISS property as well as the iISS property for system (3.1) with respect to the input u through an explicit construction of a Lyapunov function. Here, we will show, by making use of Theorem 2.7, that system (3.1) satisfies the UISS property from the input $u \in \Re^m$.

DOI: 10.1002/rnc

Let $\varepsilon \in (0,1)$ arbitrary and define

$$h(x,y) := V_1(x) - \frac{1+\varepsilon}{2-\varepsilon}, \quad W(x,y) := V_1(x) + V_2(y). \tag{3.4}$$

Inequalities (3.2) and (3.3) guarantee that if $h(x,y) \geqslant 0$, then inequalities (2.20) and (2.21) hold with $\delta(s) \equiv \varepsilon$, $K(s) \equiv 1$, and $\zeta(s) \equiv 0$. Using (3.2), (3.3), (3.4), and the inequality $s + w \leqslant \max\left\{(1+\mu)s\,,\,(1+\mu^{-1})w\right\}$, which holds for all $\mu > 0$, $s, w \geqslant 0$, we can prove that for every $\lambda \in (0,1)$, $\mu > 0$ with $\mu \geqslant (a/(2-\lambda))$ and $\varepsilon \in (0,1)$ with $\varepsilon < ((3-2\lambda)/(3-\lambda))$, the following implications hold:

$$V_1(x) \geqslant \max\left\{\frac{(1-a)(1+\mu)}{2+a-\lambda}\frac{V_2(y)}{1+V_2(y)}\,,\,k(|u|)\right\} \Rightarrow \max_{d\in D} \nabla V_1(x) f(d,x,y,u) \leqslant -\rho(V_1(x)), \tag{3.5}$$

$$V_2(y) \geqslant \frac{V_1(x)}{2-\lambda-V_1(x)} \text{ and } h(x,y) \leqslant 0 \quad \Rightarrow \max_{d\in D} \nabla V_2(y) g(d,x,y) \leqslant -\rho(V_2(y)), \tag{3.6}$$

where $\rho(s) := (\lambda s/(1+s))$. Therefore, implications (2.22) hold with $\zeta(s) := k(|u|)$, $\gamma_{1,1}(s) = \gamma_{2,2}(s) \equiv 0$, $\gamma_{1,2}(s) := \frac{(1-a)(1+\mu)}{2+a-\lambda}\frac{s}{1+s}$, $\gamma_{2,1}(s) := \frac{s}{2-\lambda-s}$ for $s \in \left[0, \frac{1+\varepsilon}{2-\varepsilon}\right]$, and $\gamma_{2,1}(s) := \frac{1+\varepsilon}{3-3\varepsilon-2\lambda+\varepsilon\lambda}$ for $s > \frac{1+\varepsilon}{2-\varepsilon}$. Finally, because $V_1 : \Re^n \to \Re_+$, $V_2 : \Re^k \to \Re_+$ are radially unbounded, positive-definite functions, it follows that inequality (2.19) holds for appropriate functions $a_1, a_2 \in K_\infty$.

Therefore, Hypothesis (A2) of Theorem 2.7 holds. Hypothesis (A1) of Theorem 2.7 holds as well, because the mappings $f : D \times \Re^n \times \Re^k \times \Re^m \to \Re^n$, $g : D \times \Re^n \times \Re^k \to \Re^k$ are locally Lipschitz mappings with $f(d,0,0,0) = 0$, $g(d,0,0) = 0$ for all $d \in D$, and $D \subset \Re^l$ is compact. It follows from Theorem 2.7 that system (3.1) satisfies the UISS property from the input $u \in \Re^m$ provided that the small-gain inequalities hold. In this case, the small-gain inequalities are equivalent to the following inequality:

$$(1-a)(1+\mu) < (2-\lambda)\,(2+a-\lambda)\,.$$

Because $a \in [0,1]$, the aforementioned inequality as well as the inequality $\mu \geqslant (a/(2-\lambda))$ hold for $\mu = 2/3$ and $\lambda = 1/2$.

The following example deals with the robust global sampled-data stabilization of a nonlinear planar system.

Example 3.2

Consider the planar system

$$\begin{aligned} &\dot{x} = -\left(1+y^2\right)x + y, \\ &\dot{y} = f(x) + g(x)y + u, \\ &(x,y) \in \Re^2\,,\; u \in \Re, \end{aligned} \tag{3.7}$$

where $f, g : \Re \to \Re$ are locally Lipschitz functions with $f(0) = 0$. We will show that there exist constants $M > 0$ sufficiently large and $r > 0$ sufficiently small so that system (3.7) in closed loop with the feedback law $u = -My$ applied with zero order hold, that is, the closed-loop system

$$\begin{aligned} &\dot{x}(t) = -\left(1+y^2(t)\right)x(t) + y(t), \\ &\dot{y}(t) = -My(\tau_i) + f(x(t)) + g(x(t))y(t))\,,\; t \in [\tau_i, \tau_{i+1}), \\ &\tau_{i+1} = \tau_i + \exp\left(-w(\tau_i)\right) r, \quad w(t) \in \Re^+ \end{aligned} \tag{3.8}$$

satisfies the UISS property with zero gain when w is considered as input.

DOI: 10.1002/rnc

First, notice that there exists a function $\sigma \in KL$ such that for all $(x_0, y) \in \Re^n \times L^\infty_{\text{loc}}(\Re^+;\Re)$ the solution of $\dot{x} = -\left(1+y^2\right)x + y$ with initial condition $x(0) = x_0$ corresponding to inputs $y \in L^\infty_{\text{loc}}(\Re^+;\Re)$ satisfies the following estimate for all $t \geqslant 0$:

$$|x(t)| \leqslant \max\left\{\sigma\left(|x_0|\ ,\ t\right)\ ,\ \sup_{0\leqslant\tau\leqslant t}\ \gamma\left(|y(\tau)|\right)\right\} \tag{3.9}$$

with $\gamma(s) := \left(\sqrt{2}\,s/\left(\sqrt{1+4s^2}\right)\right)$. Indeed, inequality (3.9) can be verified by using the Lyapunov function $V(x) = x^2$ that satisfies the following implication:

$$\text{if } V(x) = x^2 \geqslant \frac{2y^2}{1+4y^2} \text{ then } \dot{V} \leqslant -\frac{1}{4}V(x).$$

The aforementioned implication in conjunction with Lemma 3.5 in [27] guarantees that (3.9) holds for appropriate $\sigma \in KL$. Next, we show the following claim.

Claim 1
For every $\varepsilon, a > 0$, there exist $\sigma \in KL$, $M > 0$ sufficiently large and $r > 0$ sufficiently small such that for every $(y_0, x, w) \in \Re \times L^\infty_{\text{loc}}(\Re^+; B[0,a]) \times L^\infty_{\text{loc}}(\Re^+;\Re^+)$, the solution of

$$\begin{aligned} &\dot{y}(t) = -My(\tau_i) + f(x(t)) + g(x(t))y(t)\ ,\ t \in [\tau_i, \tau_{i+1}) \\ &\tau_{i+1} = \tau_i + \exp\left(-w(\tau_i)\right) r, \quad w(t) \in \Re^+ \end{aligned} \tag{3.10}$$

with initial condition $y(0) = y_0$ corresponding to inputs $(x, w) \in L^\infty_{\text{loc}}(\Re^+; B[0,a]) \times L^\infty_{\text{loc}}(\Re^+;\Re^+)$ satisfies the following inequality:

$$|y(t)| \leqslant \max\left\{\sigma\left(|y_0|\ ,\ t\right),\ \varepsilon \sup_{0\leqslant\tau\leqslant t}\ |x(\tau)|\right\}. \tag{3.11}$$

Proof of Claim 1
Let $\varepsilon, a > 0$ be arbitrary. Because $f, g : \Re \to \Re$ are locally Lipschitz functions with $f(0) = 0$, there exist constants $P, Q > 0$ such that

$$|f(x)| \leqslant P\,|x| \text{ and } g(x) \leqslant Q, \text{ for all } x \in B[0,a]. \tag{3.12}$$

Let $M > 0$ and $r > 0$ be chosen so that

$$M \geqslant 2 + 2Q + \frac{9P^2}{2\varepsilon^2} \text{ and } 3(M+Q)r\exp(Qr) \leqslant 1. \tag{3.13}$$

Consider a solution $y(t)$ of (3.10) corresponding to arbitrary $(x, w) \in L^\infty_{\text{loc}}(\Re^+; B[0,a]) \times L^\infty_{\text{loc}}(\Re^+;\Re^+)$ with initial condition $y(0) = y_0 \in \Re$. By virtue of Proposition 2.5 in [42], there exists a maximal existence time for the solution denoted by $t_{\max} \leqslant +\infty$. Moreover, let $\pi := \{\tau_0, \tau_1, \ldots\}$ the set of sampling times (which may be finite if $t_{\max} < +\infty$) and $mp(t) := \max\{\tau \in \pi\ :\ \tau \leqslant t\}$. Let $\|x\| := \sup_{0\leqslant s\leqslant t}|x(s)|$ and $\tau = mp(t)$. Inequalities (3.12) and (3.13) and the fact that $t - \tau \leqslant r$ in conjunction with the Gronwall–Bellman inequality implies

$$|y(t) - y(\tau)| \leqslant \frac{(M+Q)r\exp(Qr)}{1-(M+Q)r\exp(Qr)}\,|y(t)| + \frac{P\,r}{1-(M+Q)r\exp(Qr)}\exp(Qr)\,\|x\|\,. \tag{3.14}$$

Define $V(t) = y^2(t)$ on $[0, t_{\max})$. Let $I \subset [0, t_{\max})$ be the zero Lebesgue measure set where $y(t)$ is not differentiable or where $\dot{y}(t) \neq -My(\tau_i) + f(x(t)) + g(x(t))y(t)$). Using (3.12), (3.13), and (3.14), we obtain

$$\dot{V} \leqslant -2V(t) + \frac{\varepsilon^2}{2}\,\|x\|^2\,,\ \text{for all } t \in [0, t_{\max})\backslash I\,. \tag{3.15}$$

Direct integration of the differential inequality (3.15) and the fact that $V(t) = y^2(t)$ implies that

$$|y(t)| \leqslant \max\left\{ \sqrt{2}\exp(-t)\,|y_0|\ ,\ \varepsilon\,\|x\| \right\},\ \text{ for all } t \in [0, t_{\max}). \tag{3.16}$$

Clearly, inequality (3.16) implies that as long as the solution of (3.10) exists, $y(t)$ is bounded. A standard contradiction argument in conjunction with the 'BIC' property for (3.10) [42, Proposition 2.5] implies that $t_{\max} = +\infty$. Inequality (3.11) is a direct consequence of inequality (3.16). The proof is complete. □

We select $M > 0$ sufficiently large and $r > 0$ sufficiently small such that inequality (3.11) holds with $\varepsilon < 1/\sqrt{2}$ and $a = 1 + \sqrt{2}/2$. The solution of the closed-loop system (3.8) exists for all $t \geqslant 0$. The existence of the solution is guaranteed by the following claim.

Claim 2

For every $M > 0, r > 0$, and $(y_0, x_0, w) \in \Re \times \Re^n \times L^\infty_{\text{loc}}(\Re^+; \Re^+)$, the solution of (3.8) with initial condition $(x(0), y(0)) = (x_0, y_0)$ corresponding to input $w \in L^\infty_{\text{loc}}(\Re^+; \Re^+)$ exists for all $t \geqslant 0$. Moreover, for $M > 0$ sufficiently large and $r > 0$ sufficiently small, there exist $g \in K_\infty$ and $\xi \in \pi$ such that

$$|x(t)| + |y(t)| \leqslant g\left(|(x_0, y_0)|\right),\ \text{ for all } t \in [0, \xi], \tag{3.17}$$

$$|x(t)| \leqslant a,\ \text{ for all } t \geqslant \xi, \tag{3.18}$$

$$\xi \leqslant 1 + r + g\left(|x_0|\right), \tag{3.19}$$

where $a = 1 + \sqrt{2}/2$.

Proof of Claim 2

Let $M > 0$, $r > 0$, and $(y_0, x_0, w) \in \Re \times \Re^n \times L^\infty_{\text{loc}}(\Re^+; \Re^+)$ be arbitrary. Consider a solution $(x(t), y(t))$ of (3.8) corresponding to arbitrary $w \in L^\infty_{\text{loc}}(\Re^+; \Re^+)$ with initial condition $(x(0), y(0)) = (x_0, y_0)$. By virtue of Proposition 2.5 in [42], there exists a maximal existence time for the solution denoted by $t_{\max} \leqslant +\infty$. Moreover, let $\pi := \{\tau_0, \tau_1, \ldots\}$ be the set of sampling times (which may be finite if $t_{\max} < +\infty$) and $mp(t) := \max\{\tau \in \pi\ :\ \tau \leqslant t\}$. By virtue of (3.9), we have for all $t \in [0, t_{\max})$

$$|x(t)| \leqslant \max\left\{\sigma\left(|x_0|\ ,\ 0\right)\ ,\ a\right\}. \tag{3.20}$$

Define

$$P := \max\left\{|f(x)| : |x| \leqslant \max\left\{\sigma\left(|x_0|, 0\right), a\right\}\right\} \text{ and } Q := \max\left\{|g(x)| : |x| \leqslant \max\left\{\sigma\left(|x_0|, 0\right), a\right\}\right\}. \tag{3.21}$$

Using (3.10) and (3.21) in conjunction with Gronwall–Bellman's lemma, we obtain the following inequality for all $t \in [0, t_{\max})$:

$$|y(t)| \leqslant |y(\tau)|\exp((M + 2Q)(t - \tau)) + P(t - \tau)\exp(Q(t - \tau)), \tag{3.22}$$

where $\tau = mp(t)$. Using (3.22) and by induction, we can show the following inequality for all $\tau_i \in \pi$:

$$|y(\tau_i)| \leqslant |y_0|\exp((M + 2Q)\tau_i) + P\tau_i\exp(Qr)\exp((M + 2Q)\tau_i), \tag{3.23}$$

where we have used the fact that $\tau_{i+1} - \tau_i \leqslant r$. Estimate (3.22) in conjunction with (3.32) gives for all $t \in [0, t_{\max})$

$$|y(t)| \leqslant [|y_0| + Pt\exp(Qr)]\exp((M + 2Q)t). \tag{3.24}$$

A standard contradiction argument in conjunction with the 'BIC' property for (3.8) [42, Proposition 2.5] implies that $t_{\max} = +\infty$.

DOI: 10.1002/rnc

The existence of $\xi \in \pi$ such that (3.18) holds is a direct consequence of (3.9) and definitions $\gamma(s) := (\sqrt{2}s/(\sqrt{1+4s^2}))$, $a = 1 + \sqrt{2}/2$. By virtue of (3.9) and Proposition 7 in [48], there exists $\beta \in K_\infty$ such that

$$\xi \leqslant 1 + r + \beta(|x_0|). \tag{3.25}$$

Finally, let $M > 0$ sufficiently large and $r > 0$ sufficiently small so that (3.11) holds for $a = 1 + \sqrt{2}/2$ and $\varepsilon < 1/\sqrt{2}$. For $x_0 \in \Re^n$ with $\sigma(|x_0|, 0) \leqslant a$, we obtain from (3.11) and (3.9) for all $t \geqslant 0$

$$|x(t)| + |y(t)| \leqslant \left(1 + \varepsilon + \sqrt{2}\right)(\sigma(|x_0|, 0) + \sigma(|y_0|, 0)). \tag{3.26}$$

Using (3.20), (3.21), (3.24), (3.25), and (3.26), we guarantee the existence of $\tilde{\beta} \in K_\infty$ such that

$$|x(t)| + |y(t)| \leqslant \tilde{\beta}(|(x_0, y_0)|), \text{ for all } t \in [0, \xi]. \tag{3.27}$$

The existence of $g \in K_\infty$ satisfying (3.17) and (3.19) is a direct consequence of (3.25) and (3.27). The proof is complete. □

The fact that the robust global stabilization problem for (3.8) with sampled-data feedback applied with zero order hold is solvable with $M > 0$ being sufficiently large and $r > 0$ being sufficiently small is a consequence from all that were discussed and Theorem 2.5. Indeed, we apply Theorem 2.5 with $n = 2$, $V_1 = |x|$, $V_2 = |y|$, $L = |x| + |y|$, $H = (x, y)$, $S(t) := \{(x, y) \in \Re \times \Re : |x| \leqslant a\}$, $\gamma_{1,2}(s) := \sqrt{2}s$, $\gamma_{2,1}(s) := \varepsilon s$, $\gamma_{1,1} \equiv 0$, $\gamma_{2,2} \equiv 0$, $\zeta \equiv 0$, $g^u \equiv 0$, $\eta \equiv 0$, $\tilde{\eta} \equiv 0$, $p^u \equiv 0$, $c(t) = \tilde{c}(t) = \nu(t) = \mu(t) = \kappa(t) \equiv 1 + r$, $g \equiv 0$, $p(s, w) := s + w$, for appropriate $a, b \in K_\infty$, $\sigma \in KL$, and $q \in \mathsf{N}_2$. All Hypotheses (H1)–(H3) are satisfied by using the aforementioned definitions and previous results. Therefore, we say that the closed-loop system (3.8) with $M > 0$ being sufficiently large and $r > 0$ being sufficiently small satisfies the UISS property from the input w with zero gain.

The reader should notice that alternative sampled-data feedback designs for system (3.7) applied with zero order hold, and positive sampling rate can be obtained by using the results [50, 51] that, however, achieve semiglobal and practical stabilization. It should be emphasized that the feedback design obtained by using the trajectory-based small-gain results of the present work guarantee global and asymptotic stabilization. Moreover, robustness to perturbations of the sampling schedule is guaranteed (that is the reason for introducing the input w in the closed-loop system (3.13)).

The following example is a large-scale system and shows how efficiently the small-gain results of the present work can be applied to large-scale systems.

Example 3.3
Consider the following system described by ODEs:

$$\dot{x}_i(t) = -a_i x_i(t) + g_i(d(t), y(t), x(t)), \quad i = 1, \dots, n, \tag{3.28}$$

$$\dot{y}(t) = -(\omega + P(x(t)))\, y(t) + q(x(t)), \tag{3.29}$$

where $x(t) = (x_1(t), \dots, x_n(t)) \in \Re^n$, $y(t) \in \Re$, $d(t) \in D \subseteq \Re^m$, $D \subseteq \Re^m$ is compact, $a_i > 0$ $(i = 1, \dots, n)$, $\omega > 0$ and $g_i : D \times \Re \times \Re^n \to \Re$ $(i = 1, \dots, n)$, $P : \Re^n \to \Re_+$, $q : \Re^n \to \Re$ are locally Lipschitz mappings with $q(0) = 0$ for which there exist constants $\lambda \in (0, 1)$, $c_{i,j} \geqslant 0$ $(i, j = 1, \dots, n)$ and $R > 0$ such that

$$\frac{|q(x)|}{\lambda\omega + P(x)} \leqslant R, \text{ for all } x \in \Re^n, \tag{3.30}$$

$$\sup_{d \in D} |g_i(d, y, x)| \leqslant \max_{j=1,\dots,n} c_{i,j}\left|x_j\right|, \text{ for all } x \in \Re^n, \ y \in \Re \text{ with } |y| \leqslant R. \tag{3.31}$$

DOI: 10.1002/rnc

Moreover, we assume that there exists a nondecreasing function $L : \Re_+ \to \Re_+$ such that

$$\sup_{d \in D} \left[x_i g_i\left(d, y, x\right)\right] \leqslant L(|y|) \left(1 + |x|^2\right), \quad i = 1, \ldots, n, \tag{3.32}$$

for all $x \in \Re^n$, $y \in \Re$.

We will next show that systems (3.28) and (3.29) are URGAS, if $c_{i,i} < a_i$ for all $i = 1, \ldots, n$, and the following small-gain conditions hold for each $r = 2, \ldots, n$:

$$c_{i_1,i_2} c_{i_2,i_3} \ldots c_{i_r,i_1} < a_{i_1} a_{i_2} \ldots a_{i_r} \tag{3.33}$$

for all $i_j \in \{1, \ldots, n\}$, $i_j \neq i_k$ if $j \neq k$.

Because the set $D \subseteq \Re^m$ is compact and the mappings $g_i : D \times \Re \times \Re^n \to \Re$ $(i = 1, \ldots, n)$, $P : \Re^n \to \Re_+$, $q : \Re^n \to \Re$ are locally Lipschitz mappings vanishing at zero, it follows that Hypothesis (A1) of Theorem 2.7 holds for systems (3.28) and (3.29). Define the family of functions $V_i(x, y) := (1/2)x_i^2$ $(i = 1, \ldots, n)$, $V_{n+1}(x, y) := (1/2)y^2$, $W(x, y) := 1 + (1/2)|x|^2 + (1/2)y^2$ and $h(x, y) := y^2 - R^2$ for $(x, y) \in \Re^n \times \Re$. The inequalities

$$\begin{aligned} y\dot{y} &\leqslant -\left(\omega + P(x)\right) y^2 + |q(x)|\, |y| \leqslant R\left(\lambda\,\omega + P(x)\right)|y| - \left(\lambda\,\omega + P(x)\right)|y|^2 - \omega(1-\lambda)\,|y|^2 \\ &= -\,|y|\left(\lambda\omega + P(x)\right)\left(|y| - R\right) - \omega(1-\lambda)\,|y|^2 \end{aligned}$$

show that inequality (2.20) holds $\delta(s) \equiv 2\omega(1-\lambda)R^2$. Moreover, by virtue of (3.32) and the previous inequality, we obtain

$$\nabla W(x, y) \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \leqslant L(|y|) \left(1 + |x|^2,\right)$$

and consequently inequality (2.21) holds with $K(s) := 2L\left(\sqrt{R^2 + s^2}\right)$. Inequality (2.19) holds with $a_1(s) := (s^2/2(n+1))$, $a_2(s) := s^2/2$. Finally, notice that for every $\mu \in (0, 1)$ and $i = 1, \ldots, n$, the following implications hold:

$$V_i(x) \geqslant \frac{1}{2\mu^2 a_i^2} \left|g_i(d, y, x)\right|^2, \text{ then } x_i\left(-a_i x_i + g_i(d, y, x)\right) \leqslant -(1-\mu) a_i x_i^2. \tag{3.34}$$

It follows from (3.34) and (3.31) that, implications (2.22) for $i = 1, \ldots, n$ hold with

$$\rho_i(s) := 2(1-\mu)a_i s, \quad \gamma_{i,j}(s) := \frac{c_{i,j}^2}{\mu^2 a_i^2} s \text{ and } \gamma_{i,n+1}(s) := 0, \text{ for } s \geqslant 0 \text{ and } i, j = 1, \ldots, n. \tag{3.35}$$

Because $q : \Re^n \to \Re$ is a continuous mapping with $q(0) = 0$, it follows that there exist a function $\gamma \in K_\infty$ such that the following inequality holds:

$$|q(x)| \leqslant \gamma\left(|x|\right), \text{ for all } x \in \Re^n. \tag{3.36}$$

Inequality (3.36) and the fact that $P(x) \geqslant 0$ for all $x \in \Re^n$ (notice that $P : \Re^n \to \Re_+$) imply that the following implication holds for every $\mu \in (0, 1)$:

$$V_{n+1}(y) \geqslant \frac{1}{2\mu^2\omega^2} \left(\gamma\left(|x|\right)\right)^2 \text{ then } y\left[-\left(\omega + P(x)\right) y + q(x)\right] \leqslant -(1-\mu)\omega\, y^2. \tag{3.37}$$

It follows from (3.37) that implication (2.22) holds for $i = n + 1$ with

$$\gamma_{n+1,j}(s) := \frac{1}{2\mu^2\omega^2} \left(\gamma\left(\sqrt{2ns}\right)\right)^2 \text{ and } \gamma_{n+1,n+1}(s) := 0, \text{ for } s \geqslant 0 \text{ and } j = 1, \ldots, n. \tag{3.38}$$

Definitions (3.35) and (3.38) in conjunction with (3.33) and the fact that $c_{i,i} < a_i$ for all $i = 1, \ldots, n$ guarantee that there exists $\mu \in (0, 1)$ (sufficiently close to 1) such that the MAX-preserving mapping $\Gamma : \Re_+^{n+1} \to \Re_+^{n+1}$ with $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_{n+1}(x))'$, $\Gamma_i(x) = \max_{j=1,\ldots,n+1} \gamma_{i,j}(x_j)$ for all $x \in \Re_+^{n+1}$, $i = 1, \ldots, n+1$ satisfies the cyclic small-gain conditions. It follows from Theorem 2.7 that systems (3.28) and (3.29) is URGAS.

DOI: 10.1002/rnc

4. A DELAYED CHEMOSTAT MODEL

In this section, we study the robust global feedback stabilization problem for system (1.2) under (1.3). More specifically, in order to emphasize the fact that the mapping $p : C^0([-r,0];(0,S_i)) \to (0,+\infty)$ is unknown, we will consider the stabilization problem of the equilibrium point $(X_s, S(\cdot)) \in (0,+\infty) \times C^0([-r,0];(0,S_i))$ with $S(\theta) = S_s$ for all $\theta \in [-r,0]$ satisfying (1.4) for the uncertain chemostat model:

$$\begin{aligned}
&\dot{X}(t) = \left(\min_{t-r\leqslant\tau\leqslant t} \mu(S(\tau)) + d(t) \left(\max_{t-r\leqslant\tau\leqslant t} \mu(S(\tau)) - \min_{t-r\leqslant\tau\leqslant t} \mu(S(\tau)) \right) - D(t) - b \right) X(t),\\
&\dot{S}(t) = D(t)(S_i - S(t)) - K(S(t))\mu(S(t))X(t),\\
&X(t) \in (0,+\infty)\,,\ S(t) \in (0,S_i)\,,\ D(t) \geqslant 0\,,\ d(t) \in [0,1],
\end{aligned} \tag{4.1}$$

where $d(t) \in [0,1]$ is the uncertainty. We will assume the following:

Hypothesis (H)
There exists $S^* < S_s$ such that $\mu(S) > b$ for all $S \in [S^*, S_i]$.

Hypothesis (H) is automatically satisfied for the case of a monotone-specific growth rate. Hypothesis (H) can be satisfied for nonmonotone-specific growth rates (e.g., Haldane or generalized Haldane growth expressions). By using the trajectory-based small-gain Theorem 2.6, we can prove the following theorem.

Theorem 4.1
Let $a > 0$ be a constant that satisfies

$$\min_{S^*\leqslant S\leqslant S_i} \mu(S) - b > a D_s \frac{S_s}{S_i}. \tag{4.2}$$

Then the locally Lipschitz delay-free feedback law

$$D(t) = \frac{K(S(t))\mu(S(t))X(t) + a D_s \left(S_s - \min\left(S(t)\,,\ S_s\right)\right)}{S_i - \min\left(S(t)\,,\ S_s\right)} \tag{4.3}$$

achieves robust global stabilization of the equilibrium point $(X_s, S(\cdot)) \in (0,+\infty) \times C^0([-r,0];(0,S_i))$ with $S(\theta) = S_s$ for all $\theta \in [-r,0]$ for the uncertain chemostat model (4.1) under Hypothesis (H).

It should be noted that the change of coordinates

$$X = X_s \exp(x), \qquad S = \frac{S_i \exp(y)}{G + \exp(y)}, \tag{4.4a}$$

where $G := (S_i/S_s) - 1$, and the input transformation

$$D = D_s \exp(u) \tag{4.4b}$$

maps the set $(0,S_i) \times (0,+\infty)$ onto $\Re^2$ and the equilibrium point $(X_s, S(\cdot)) \in (0,+\infty) \times C^0([-r,0];(0,S_i))$ with $S(\theta) = S_s$ for all $\theta \in [-r,0]$ of system (4.1) to the equilibrium point $0 \in \Re \times C^0([-r,0];\Re)$ of the transformed control system:

$$\begin{aligned}
&\dot{x}(t) = \min_{t-r\leqslant\tau\leqslant t} \tilde{\mu}(y(\tau)) + d(t) \left(\max_{t-r\leqslant\tau\leqslant t} \tilde{\mu}(y(\tau)) - \min_{t-r\leqslant\tau\leqslant t} \tilde{\mu}(y(\tau)) \right) - D_s \exp(u(t)) - b,\\
&\dot{y}(t) = D_s(G\exp(-y(t)) + 1)\left[\exp(u(t)) - (G + \exp(y(t)))g(y(t))\exp(x(t))\right],\\
&(x,y) \in \Re^2,\ u(t) \in \Re\,,\ d(t) \in [0,1],
\end{aligned} \tag{4.5}$$

where

$$\tilde{\mu}(y) := \mu\left(\frac{S_i \exp(y)}{G+\exp(y)}\right),$$
$$g(y) := \frac{X_s}{D_s S_i G} K\left(\frac{S_i \exp(y)}{G+\exp(y)}\right) \mu\left(\frac{S_i \exp(y)}{G+\exp(y)}\right). \tag{4.6}$$

In the new coordinates, the feedback law (4.3) takes the form

$$u(t) = \ln\left(g(y(t))\exp(x(t))\min(G+\exp(y(t)), G+1) + \frac{a}{G+1}\max\left(1-\exp(y(t))\,,\,0\right)\right). \tag{4.7}$$

The feedback law (4.7) (or (4.3)) is a delay-free feedback that achieves global stabilization of $0 \in \Re \times C^0([-r,0];\Re)$ for system (4.5) no matter how large the delay is. Furthermore, no knowledge of the maximum delay $r \geqslant 0$ is needed for the implementation of (4.7).

The proof of Theorem 4.1 is therefore equivalent to the proof of RGAS of the equilibrium point $0 \in \Re \times C^0([-r,0];\Re)$ for system (4.5).

Before we give the proof of Theorem 4.1, it is important to understand the intuition that leads to the construction of the feedback law (4.7) and the ideas behind the proof of Theorem 4.1. To explain the procedure, we follow the following arguments.

(i) For the stabilization of the equilibrium point $0 \in \Re \times C^0([-r,0];\Re)$, we first start with the stabilization of subsystem $\dot{y}(t) = D_s(G\exp(-y(t))+1)\,[\exp(u(t)) - (G+\exp(y(t)))\; g(y(t))\; \exp(x(t))]$ with x as input. Any feedback law that satisfies $u(t) = \ln\left(g(y(t))\exp(x(t))(G+1)\right)$ for $y(t) \geqslant 0$ and $u(t) > \ln\left(g(y(t))\exp(x(t))(G+\exp(y(t)))\right)$ for $y(t) < 0$ achieves ISS stabilization of the subsystem with x as input.

(ii) In order to prove URGAS for the composite system by means of small-gain arguments, one has to show the ISS property of the x-subsystem $\dot{x}(t) = \min_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau)) + d(t)\;(\max_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau)) - \min_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau))) - D_s\exp(u(t)) - b$ with y as input. Notice that the feedback selection from the previous step gives $\dot{x}(t) = \min_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau)) + d(t)\;(\max_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau)) - \min_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau))) - D_s g(y(t))\exp(x(t))(G+1) - b$ for $y(t) \geqslant 0$ and $\dot{x}(t) < \min_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau)) + d(t)\;(\max_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau)) - \min_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau))) - D_s g(y(t))\exp(x(t))(G+\exp(y(t))) - b$ for $y(t) < 0$. The estimation of the derivative $\dot{x}(t)$ shows that the ISS inequality for the $x-$subsystem does not hold, unless we have $\min_{t-r\leqslant\tau\leqslant t}\tilde{\mu}(y(\tau)) > b$ for all t being sufficiently large. By virtue of Hypothesis (H), there exists $y^* < 0$, such that the ISS inequality for the the x-subsystem holds if $\min_{t-r\leqslant\tau\leqslant t} y(\tau) \geqslant y^*$ holds for all t sufficiently being large.

(iii) The feedback law $u(t) > \ln\left(g(y(t))\exp(x(t))(G+\exp(y(t)))\right)$ for $y(t) < 0$ is selected such that the inequality $\min_{t-r\leqslant\tau\leqslant t} y(\tau) \geqslant y^*$ holds for all initial conditions after a transient period. Because the ISS inequalities will hold only after this transient period, the trajectory-based small-gain result Theorem 2.6 must be used for the proof of URGAS of the closed-loop system.

Proof of Theorem 4.1

Consider the solution $(x(t), y(t)) \in \Re^2$ of (4.5) with (4.7) with arbitrary initial condition $x(0) = x_0 \in \Re$, $T_r(0)y = y_0 \in C^0([-r,0];\Re)$ and corresponding to arbitrary input $d \in M_D$. The following equations hold for system (4.5) with (4.7):

$$\dot{y}(t) = aD_s\frac{G\exp(-y(t))+1}{G+1}\left(1-\exp(y(t))\right),\ \text{if } y(t) \leqslant 0;$$
$$\dot{y}(t) = D_s g(y(t))(G\exp(-y(t))+1)\exp(x(t))\left(1-\exp(y(t))\right),\ \text{if } y(t) > 0. \tag{4.8}$$

Equation (4.8) imply that the function $V(t) = y^2(t)$ is nonincreasing; consequently, we obtain

$$|y(t)| \leqslant \|y_0\|_r\,,\ \text{ for all } t \in [0, t_{\max}). \tag{4.9}$$

DOI: 10.1002/rnc

Using the fact that $\mu : (0, S_i) \to (0, \mu_{\max}]$ and Definition (4.6) of $\tilde{\mu}$, we obtain that $\tilde{\mu}(y) \leqslant \mu_{\max}$ for all $y \in \Re$. This implies the differential inequality

$$\dot{x}(t) \leqslant 2\mu_{\max} - b,$$

which, by direct integration, yields the estimate

$$x(t) \leqslant x_0 + (2\mu_{\max} - b)\ t, \text{ for all } t \in [0, t_{\max}). \tag{4.10}$$

Define $\kappa(s) := (G+1)D_s \max_{|y|\leqslant s} g(y)$. Inequalities (4.9) and (4.10) imply that the following differential inequality holds:

$$\dot{x}(t) \geqslant -b - \frac{aD_s}{G+1} - \kappa\left(\|y_0\|_r\right)\exp(x_0 + (2\mu_{\max} - b)\ t),$$

which, by direct integration, yields the estimate:

$$x(t) \geqslant x_0 - \left(b + \frac{aD_s}{G+1}\right) t - \kappa\left(\|y_0\|_r\right)\exp(x_0)\frac{\exp((2\mu_{\max} - b)\ t) - 1}{2\mu_{\max} - b}, \text{ for all } t \in [0, t_{\max}). \tag{4.11}$$

Inequalities (4.9), (4.10), and (4.11) and a standard contradiction argument show that system (4.5) with (4.7) is forward complete, that is, $t_{\max} = +\infty$. Therefore, inequalities (4.9), (4.10), and (4.11) hold for all $t \geqslant 0$, and because system (4.5) with (4.7) is autonomous, it follows that system (4.5) with (4.7) is robustly forward complete (RFC, see Appendix B).

By considering (4.8) and the function

$$W(t) = \begin{cases} y^2(t) & \text{if } \ y(t) \leqslant 0, \\ 0 & \text{if } \ y(t) > 0, \end{cases}$$

we obtain the existence of a positive-definite function $\rho \in C^0(\Re_+; \Re_+)$ such that

$$\dot{W}(t) \leqslant -\rho\left(W(t)\right), \text{ for all } t \geqslant 0. \tag{4.12}$$

Lemma 3.5 in [27] implies the existence of $\sigma \in KL$ such that for every $x(0) = x_0 \in \Re$, $T_r(0)y = y_0 \in C^0([-r, 0]; \Re)$ and $d \in M_D$, it holds that

$$W(t) \leqslant \sigma\left(W(0), t\right), \text{ for all } t \geqslant 0. \tag{4.13}$$

Inequality (4.13), in conjunction with Proposition 7 in [48], shows the existence of $a \in K_\infty$ such that for every $x(0) = x_0 \in \Re$, $T_r(0)y = y_0 \in C^0([-r, 0]; \Re)$ and $d \in M_D$, there exists $\xi \geqslant r$ with $\xi \leqslant r + a\left(\|y_0\|_r\right)$ satisfying

$$y(t - r) \geqslant -\frac{c}{2}, \text{ for all } t \geqslant \xi, \tag{4.14}$$

where $c := \ln((S_i - S^*)/(S^* G)) > 0$, and $S^* < S_s$ is the constant involved in Hypothesis (H). Define $S := \Re \times C^0([-r, 0]; [-c/2, +\infty))$. Inequality (4.14) shows that $(x(t), T_r(t)y) \in S$ for all $t \geqslant \xi$ and that inequality (2.16) holds for appropriate $a \in K_\infty$ and $c(t) \equiv 1$.

Notice that for $(x(t), T_r(t)y) \in S$, the functionals

$$V_1(t) = \max_{\theta \in [-r, 0]} \exp(2\sigma\theta)\left|z(t+\theta)\right|^2, \quad V_2 = |x(t)|^2, \tag{4.15}$$

where $\sigma > 0$, and

$$y(t) = c\left(\exp(z(t)) - 1\right) \tag{4.16}$$

are well-defined. Moreover, by considering the differential equations

$$\dot{z}(t) = aD_s \frac{G\exp(c\ (1 - \exp(z(t)))) + 1}{c(G+1)}\exp(-z(t))\left(1 - \exp\left(c\ (\exp(z(t)) - 1)\right)\right) \quad \text{if } \ z(t) \leqslant 0,$$

$$\begin{aligned}\dot{z}(t) = {} & c^{-1} D_s g\left(c\ (\exp(z(t)) - 1)\right)\left(G\exp\left(c\ (1 - \exp(z(t)))\right) + 1\right)\exp(x(t) - z(t)) \\ & \times \left(1 - \exp\left(c\ (\exp(z(t)) - 1)\right)\right) \quad \text{if } \ z(t) > 0,\end{aligned}$$

we conclude from Lemma 3.5 in [26] that for every $\gamma_{1,2} \in K_\infty$, there exists $\sigma_1 \in KL$ such that

$$V_1(t) \leqslant \max\left\{ \sigma_1\left(V_1(t_0), t-t_0\right) \ , \ \sup_{t_0 \leqslant \tau \leqslant t} \gamma_{1,2}\left(V_2(\tau)\right) \right\}, \text{ for all } t \geqslant t_0 \geqslant 0. \tag{4.17}$$

Finally, using Hypothesis (H) and definitions (4.15), we guarantee that there exists a positive-definite function $\rho \in C^0(\Re_+;\Re_+)$ such that the following implication holds for every $\varepsilon > 0$:
'If

$$(1+\varepsilon)\ln\left(\frac{\left(G+\exp\left(c\left(\exp(\sqrt{V_1(t)})-1\right)\right)\right)D_s \max\limits_{|z|\leqslant\sqrt{V_1(t)}} g\left(c\left(\exp(z)-1\right)\right)}{\min\limits_{|z|\leqslant \exp(\sigma r)\sqrt{V_1(t)}} \tilde{\mu}\left(c\left(\exp(z)-1\right)\right) - b - \frac{a}{G+1}D_s\left(1-\exp\left(c\left(\exp(-\sqrt{V_1(t)})-1\right)\right)\right)}\right) \leqslant |x(t)|$$

and

$$(1+\varepsilon)\ln\left(\frac{\max\limits_{|z|\leqslant \exp(\sigma r)\sqrt{V_1(t)}} \tilde{\mu}\left(c\left(\exp(z)-1\right)\right) - b}{\left(G+\exp\left(c\left(\exp(-\sqrt{V_1(t)})-1\right)\right)\right)D_s \min\limits_{|z|\leqslant\sqrt{V_1(t)}} g\left(c\left(\exp(z)-1\right)\right)}\right) \leqslant |x(t)|,$$

then

$$2x(t)\dot{x}(t) \leqslant -\rho\left(x^2(t)\right)\text{'}.$$

Therefore, Lemma 3.5 in [26] implies that there exists $\sigma_2 \in KL$ such that

$$V_2(t) \leqslant \max\left\{ \sigma_2\left(V_2(t_0), t-t_0\right) \ , \ \sup_{t_0 \leqslant \tau \leqslant t} \gamma_{2,1}\left(V_1(\tau)\right) \right\}, \text{ for all } t \geqslant t_0 \geqslant 0, \tag{4.18}$$

where

$$\begin{aligned}
\gamma_{2,1}(s) &:= (1+\varepsilon)^2\left(\ln\left(\max\{g_1(s)\ ,\ g_2(s)\}\right)\right)^2, \\
g_1(s) &:= \frac{\left(G+\exp\left(c\left(\exp(\sqrt{s})-1\right)\right)\right)D_s \max\limits_{|z|\leqslant\sqrt{s}} g\left(c\left(\exp(z)-1\right)\right)}{\min\limits_{|z|\leqslant \exp(\sigma r)\sqrt{s}} \tilde{\mu}\left(c\left(\exp(z)-1\right)\right) - b - \frac{aD_s}{G+1}\left(1-\exp\left(c\left(\exp(-\sqrt{s})-1\right)\right)\right)}, \\
g_2(s) &:= \frac{\max\limits_{|z|\leqslant \exp(\sigma r)\sqrt{s}} \tilde{\mu}\left(c\left(\exp(z)-1\right)\right) - b}{\left(G+\exp\left(c\left(\exp(-\sqrt{s})-1\right)\right)\right)D_s \min\limits_{|z|\leqslant\sqrt{s}} g\left(c\left(\exp(z)-1\right)\right)}.
\end{aligned} \tag{4.19}$$

Inequalities (4.9), (4.10), (4.11), (4.17), and (4.18) guarantee that inequalities (2.10), (2.11), (2.13), (2.14), (2.15), and (2.17) hold for appropriate $\sigma \in KL, \nu \in K^+, a \in K_\infty$ with $c(t) \equiv 1$, $p \equiv 0$, $\gamma_{1,2}(s) := \gamma_{2,1}(s/2)$, $\gamma_{1,1}(s) = \gamma_{2,2}(s) \equiv 0$, $L := V_1 + V_2$ and $H(t,x,y) := \sqrt{x^2 + \|y\|_r^2}$. Finally, notice that the MAX-preserving mapping $\Gamma : \Re_+^2 \to \Re_+^2$ with $\Gamma_i(x) = \max_{j=1,2}\gamma_{i,j}(x_j)$ $(i = 1,2)$ satisfies the cyclic small-gain conditions.

By virtue of Theorem 2.6, we conclude that the autonomous system (4.5) with (4.7) is URGAS. $\square$

5. CONCLUSIONS

One of the most important obstacles in applying nonlinear small-gain results is the fact that the essential inequalities, which small-gain results utilize in order to prove stability properties, do not hold for all times. This feature excludes all available small-gain results from possible application. In

DOI: 10.1002/rnc

this work, novel small-gain results that can allow for a transient period during which the solutions do not satisfy the usual inequalities required by previous small-gain results (Theorems 2.5 and 2.6) were presented. The obtained results allow the application of the small-gain methodology to various classes of systems that satisfy less-demanding stability notions than the IOS property. Moreover, a vector Lyapunov function formulation of the small-gain results was presented in Theorem 2.7 and was shown by an illustrative example that it is applicable to large-scale systems.

The robust global feedback stabilization problem of an uncertain time-delay chemostat model is solved by means of the trajectory-based small-gain results. Future research will focus on the application of the trajectory-based small-gain results to Lotka–Volterra systems in mathematical biology ([52]).

ACKNOWLEDGEMENTS

The authors would like to thank Yu Jiang. This work was supported in part by the NSF grants DMS-0504462 and DMS-0906659.

APPENDIX A: PROOFS OF THEOREM 2.5, THEOREM 2.6, AND THEOREM 2.7

Proof of Theorem 2.5

The proof is similar to the proof of Theorem 3.1 in [19] and consists of four steps:

Step 1: We show that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$, the following inequality holds for all $t \in [\xi, t_{\max})$:

$$V(t) \leqslant \mathrm{MAX}\left\{ Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\ ,\ Q\left(\mathbf{1}\zeta\left(\left[\|u\|_{\mathsf{U}}\right]_{[\xi,t]}\right)\right)\right\}, \tag{A.1}$$

where $\xi \in \pi(t_0, x_0, u, d)$ is the time such that $\phi(t, t_0, x_0, u, d) \in S(t)$ for all $t \in [\xi, t_{\max})$ (recall Hypothesis (H2)).

This step is proved in exactly the same way as in the proof of Theorem 3.1 in [19] (using Fact V).

Step 2: We show that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$, it holds that $t_{\max} = +\infty$.

The proof of Step 2 is exactly the same with the proof of Theorem 3.1 in [19]. The only difference is the additional use of inequality (2.6), which guarantees that the transition map is bounded during the transient period $t \in [t_0, \xi]$.

Step 3: We show that Σ is RFC from the input $u \in M_U$.

Again, the proof of Step 3 is exactly the same with the proof of Theorem 3.1 in [19]. The only difference is the additional use of inequalities (2.6) and (2.9).

Step 4: We prove the following claim.

Claim

For every $\varepsilon > 0, k \in Z_+, R, T \geqslant 0$, there exists $\tau_k(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathsf{X}} \leqslant R$, the following inequality holds:

$$V(t) \leqslant \mathrm{MAX}\left\{ Q(\mathbf{1}\varepsilon)\, , \Gamma^{(k)}\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right)\, ,\, G\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right)\right\},\ \text{for all } t \geqslant \xi + \tau_k. \tag{A.2}$$

Moreover, if $c \in K^+$ is bounded, then for every $\varepsilon > 0, k \in Z_+, R \geqslant 0$, there exists $\tau_k(\varepsilon, R) \geqslant 0$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$ with $\|x_0\|_{\mathsf{X}} \leqslant R$, inequality (A.2) holds.

Proof of Step 4: The proof of the claim is made by an induction on $k \in Z_+$.

Inequality (A.2) for $k = 1$ is a direct consequence of inequalities (2.4) and (2.9) and definition (2.12).

DOI: 10.1002/rnc

We notice that inequality (2.5) in conjunction with inequality (A.1) and Fact IV imply, for all $t \geqslant \xi$, that

$$L(t) \leqslant \max\left\{ \nu(t-t_0)\,,\ c(t_0)\,,\ a\left(\|x_0\|_{\mathsf{X}}\right)\,,\ p\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right)\,, p\left(Q\left(\mathbf{1}\zeta\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right)\right)\right),\ p^u\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right)\right\}. \tag{A.3}$$

Next, suppose that for every $\varepsilon > 0$, $R, T \geqslant 0$, there exists $\tau_k(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathsf{X}} \leqslant R$, (A.2) holds for some $k \in Z_+$. Let arbitrary $\varepsilon > 0$, $R, T \geqslant 0$, $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathsf{X}} \leqslant R$ be given. Notice that the weak semigroup property implies that $\pi(t_0, x_0, u, d) \cap [\xi+\tau_k, \xi+\tau_k+r] \neq \emptyset$. Let $t_k \in \pi(t_0, x_0, u, d) \cap [\xi + \tau_k, \xi + \tau_k + r]$. Then, (2.4) implies

$$V(t) \leqslant \mathrm{MAX}\left\{ \mathbf{1}\sigma\left(L(t_k),\ t-t_k\right),\ \Gamma\left([V]_{[t_k,t]}\right),\ \mathbf{1}\zeta\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_k,t]}\right)\right\},\ \text{for all } t \geqslant t_k. \tag{A.4}$$

Using inequalities (A.2), (A.3), (A.4), and (2.9) and working in the same way as in the proof of Theorem 3.1 in [19], we can derive the following inequality for all $t \geqslant \xi + \tau_k + r$:

$$V(t) \leqslant \mathrm{MAX}\left\{ \mathbf{1}\sigma\left(L(t_k),\ t-\xi-\tau_k-r\right),\ Q(\mathbf{1}\varepsilon),\ \Gamma^{(k+1)}\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right),\ G\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right)\right\}. \tag{A.5}$$

Definition (2.12) in conjunction with (2.7), (2.9), inequality (A.3), and the facts that $t_k \leqslant \xi + \tau_k + r$, $t_0 \in [0, T]$, and $\|x_0\|_{\mathsf{X}} \leqslant R$ implies that, for all $t \geqslant \xi + \tau_k + r$,

$$\mathbf{1}\sigma\left(L(t_k),\ t-\xi-\tau_k-r\right) \leqslant \mathrm{MAX}\left\{ \mathbf{1}\sigma\left(f(\varepsilon,T,R),\ t-\xi-\tau_k-r\right),\ G\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right)\right\}, \tag{A.6}$$

where

$$f(\varepsilon,T,R) := \max\left\{ \max_{0\leqslant t\leqslant a(R)+C(T)+\tau_k(\varepsilon,R,T)+r} \nu(t),\ C(T),\ a(R),\ p\left(Q\left(\mathbf{1}\sigma\left(a(RC(T)),0\right)\right)\right)\right\} \tag{A.7}$$

and

$$C(T) := \max_{0\leqslant t\leqslant T} c(t). \tag{A.8}$$

The reader should notice that if $c \in K^+$ is bounded and τ_k is independent of T, then f can be chosen to be independent of T as well. The rest of the proof of the claim follows from a combination of inequalities (A.5) and (A.6) and from an appropriate selection of τ_{k+1} (set $\tau_{k+1}(\varepsilon, R, T) = \tau_k(\varepsilon, R, T) + r + \tau(\varepsilon, R, T)$, where $\tau(\varepsilon, R, T) \geqslant 0$ satisfies $\sigma\left(f(\varepsilon, T, R)\,,\ \tau\right) \leqslant \varepsilon$).

To finish the proof, let arbitrary $\varepsilon > 0$, $R, T \geqslant 0$, $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$ and denote $Y(t) = H\left(t, \phi(t, t_0, x_0, u, d), u(t)\right)$ for $t \geqslant t_0$. Using Fact IV, (2.11), and (A.1), we obtain for all $t \geqslant \xi$:

$$\|Y(t)\|_{\mathsf{Y}} \leqslant \max\left\{ q\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right)\ ,\ q\left(Q\left(\mathbf{1}\zeta\left(\left[\|u\|_{\mathsf{U}}\right]_{[\xi,t]}\right)\right)\right)\right\}.$$

The aforementioned inequality in conjunction with (2.9) implies that, for all $t \geqslant \xi$,

$$\|Y(t)\|_{\mathsf{Y}} \leqslant \max\left\{ q\left(Q\left(\mathbf{1}\sigma\left(a\left(c(t_0)\,\|x_0\|_{\mathsf{X}}\right),0\right)\right)\right),\ q\left(Q\left(\mathbf{1}\sigma\left(g^u\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right),0\right)\right)\right),\ q\left(Q\left(\mathbf{1}\zeta\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right)\right)\right)\right\}. \tag{A.9}$$

Using (2.8) and (A.9), we conclude that the following estimate holds for all $t \geqslant t_0$:

$$\|Y(t)\|_{\mathsf{Y}} \leqslant \max\left\{\begin{matrix} q\left(Q\left(\mathbf{1}\sigma\left(a\left(c(t_0)\,\|x_0\|_{\mathsf{X}}\right),0\right)\right)\right),\ a\left(c(t_0)\,\|x_0\|_{\mathsf{X}}\right), \\ \eta\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right),\ q\left(Q\left(\mathbf{1}\sigma\left(g^u\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right),0\right)\right)\right), q\left(Q\left(\mathbf{1}\zeta\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right)\right)\right)\end{matrix}\right\}. \tag{A.10}$$

Inequality (A.10) shows that properties P1 and P2 of Lemma 2.16 in [16] hold for system Σ with $V = \|H(t,x,u)\|_{\mathsf{Y}}$ and $\gamma(s) := \max\{\eta(s), q(G(s))\}$. Moreover, if $c \in K^+$ is bounded, then (A.10) implies that properties P1 and P2 of Lemma 2.17 in [16] hold for system Σ with $V = \|H(t,x,u)\|_{\mathsf{Y}}$ and $\gamma(s) := \max\{\eta(s), q(G(s))\}$.

Inequality (A.2) in conjunction with Fact III, (2.9), (A.8), and definition (2.12) guarantees that for every $\varepsilon > 0$, $k \in Z_+$, $R, T \geqslant 0$, there exists $\tau_k(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathsf{X}} \leqslant R$, the following inequality holds:

$$V(t) \leqslant \mathrm{MAX}\left\{ Q(\mathbf{1}\varepsilon), \Gamma^{(k)}\left(Q\left(\mathbf{1}\sigma\left(a\left(RC(T)\right),0\right)\right)\right), G\left(\left[\|u\|_{\mathsf{U}}\right]_{[t_0,t]}\right)\right\}, \text{ for all } t \geqslant \xi + \tau_k. \tag{A.11}$$

Notice that Fact I guarantees the existence of $k(\varepsilon, T, R) \in Z_+$ such that $Q(\mathbf{1}\varepsilon) \geqslant \Gamma^{(l)}\left(Q\left(\mathbf{1}\sigma\left(a\left(RC(T)\right),0\right)\right)\right)$ for all $l \geqslant k$. If $c \in K^+$ is bounded, then k is independent of T. Therefore, by virtue of (A.11) and (2.7), property P3 of Lemma 2.16 in [16] holds for system Σ with $V = \|H(t,x,u)\|_{\mathsf{Y}}$ and $\gamma(s) := \max\{\eta(s), q(G(s))\}$. Moreover, if $c \in K^+$ is bounded, then (A.11) and (2.7) imply that property P3 of Lemma 2.17 in [16] hold for system Σ with $V = \|H(t,x,u)\|_{\mathsf{Y}}$ and $\gamma(s) := \max\{\eta(s), q(G(s))\}$.

The proof of Theorem 2.4 is thus completed with the help of Lemma 2.16 (or Lemma 2.17) in [16]. □

Proof of Theorem 2.6

By virtue of Lemma 3.3 in [15], we have to show that Σ is RFC and satisfies the *robust output attractivity property*, that is, for every $\varepsilon > 0$, $T \geqslant 0$ and $R \geqslant 0$, there exists a $\tau := \tau(\varepsilon, T, R) \geqslant 0$, such that

$$\|x_0\|_{\mathsf{X}} \leqslant R,\ t_0 \in [0,T] \Rightarrow \|H(t, \phi(t, t_0, x_0, u_0, d), 0)\|_{\mathsf{Y}} \leqslant \varepsilon,\ \forall t \in [t_0 + \tau, +\infty), \quad \forall d \in M_D.$$

The reader should notice that Lemma 3.3 in [15] assumes the classical semigroup property; however, the semigroup property is not used in the proof of Lemma 3.3 in [15]. Consequently, Lemma 3.3 in [15] holds as well for systems satisfying the weak semigroup property.

Moreover, Lemma 3.5 in [42] guarantees that system Σ is uniformly RGAOS in case that $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ is T-periodic for certain $T > 0$.

Again, the proof consists of four steps:

Step 1: We show that for every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$, the following inequality holds for all $t \in [\xi, t_{\max})$:

$$V(t) \leqslant Q\left(\mathbf{1}\sigma\left(L(\xi), 0\right)\right), \tag{A.12}$$

where $\xi \in \pi(t_0, x_0, u_0, d)$ is the time such that $\phi(t, t_0, x_0, u_0, d) \in S(t)$ for all $t \in [\xi, t_{\max})$ (recall Hypothesis (H2)).

Step 2: We show that for every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$, $t_{\max} = +\infty$ holds.

Step 3: We show that Σ is RFC.

Step 4: We prove the following claim.

Claim

For every $\varepsilon > 0, k \in Z_+, R, T \geqslant 0$, there exists $\tau_k(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathsf{X}} \leqslant R$ the following inequality holds:

$$V(t) \leqslant \mathrm{MAX}\left\{ Q(\mathbf{1}\varepsilon), \Gamma^{(k)}\left(Q\left(\mathbf{1}\sigma\left(L(\xi),0\right)\right)\right)\right\}, \text{ for all } t \geqslant \xi + \tau_k. \tag{A.13}$$

The proofs of the previous steps are exactly the same with the proof of Theorem 2.5 and are omitted. The difference between inequalities (2.9) and (2.17) does not play any role. To finish the proof, let arbitrary $\varepsilon > 0$, $R, T \geqslant 0$, $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$ and denote $Y(t) = H\left(t, \phi(t, t_0, x_0, u_0, d), 0\right)$ for $t \geqslant t_0$.

DOI: 10.1002/rnc

Inequality (A.13) in conjunction with (2.17) and (A.8) guarantees that for every $\varepsilon > 0$, $k \in Z_+$, $R, T \geqslant 0$, there exists $\tau_k(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathsf{X}} \leqslant R$, the following inequality holds:

$$V(t) \leqslant \mathrm{MAX}\left\{ Q(\mathbf{1}\varepsilon)\,,\ \Gamma^{(k)}\left(Q\left(\mathbf{1}\sigma\left(a\left(R\right)+C(T),0\right)\right)\right)\right\},\ \text{for all } t \geqslant \xi + \tau_k. \tag{A.14}$$

Notice that Fact I guarantees the existence of $k(\varepsilon, T, R) \in Z_+$ such that $Q(\mathbf{1}\varepsilon) \geqslant \Gamma^{(l)}\left(Q\left(\mathbf{1}\sigma\left(a\left(R\right)+C(T),0\right)\right)\right)$ for all $l \geqslant k$. Therefore, (A.14) implies that for every $\varepsilon > 0$, $R, T \geqslant 0$, there exists $\tau(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathsf{X}} \leqslant R$, it holds that

$$V(t) \leqslant Q(\mathbf{1}\varepsilon),\ \text{for all } t \geqslant \xi + \tau. \tag{A.15}$$

It follows from inequalities (2.11) and (A.15) that for every $\varepsilon > 0$, $R, T \geqslant 0$, there exists $\tau(\varepsilon, R, T) \geqslant 0$ such that for every $(t_0, x_0, d) \in \Re_+ \times \mathsf{X} \times M_D$ with $t_0 \in [0, T]$ and $\|x_0\|_{\mathsf{X}} \leqslant R$, it holds that

$$\|Y(t)\|_{\mathsf{Y}} \leqslant q(Q(\mathbf{1}\varepsilon)),\ \text{for all } t \geqslant \xi + \tau. \tag{A.16}$$

Therefore, by virtue of (A.16) and (2.16), the *robust output attractivity property* holds for system Σ. The proof is complete. □

Proof of Theorem 2.7

Consider a solution $x(t)$ of (2.18) corresponding to arbitrary $(u, d) \in M_U \times M_D$ (here, M_U, M_D denote the classes of measurable and locally essentially bounded functions $u : \Re_+ \to U$ and $d : \Re_+ \to D$, respectively) with arbitrary initial condition $x(0) = x_0 \in \Re^n$. Clearly, there exists a maximal existence time for the solution denoted by $t_{\max} \leqslant +\infty$. Suppose that $h(x(t)) \leqslant 0$ for all $t \in [0, t_{\max})$ and let $V_i(t) = V_i(x(t))$, $i = 1, \ldots, k$, absolutely continuous functions on $[0, t_{\max})$. Moreover, let $I \subset [0, t_{\max})$ be the zero Lebesgue measure set where $x(t)$ is not differentiable or $\dot{x}(t) \neq f(x(t), d(t), u(t))$. By virtue of (2.22), it follows that the following implication holds for $t \in [0, t_{\max})\backslash I$ and $i = 1, \ldots, k$:

$$V_i(t) \geqslant \max\left\{\zeta\left(|u(t)|\right),\ \max_{j=1,\ldots,k} \gamma_{i,j}\left(V_j(t)\right)\right\} \Rightarrow \quad \dot{V}_i(t) \leqslant -\rho_i(V_i(t)). \tag{A.17}$$

Lemma 3.5 in [27] in conjunction with (A.17) implies that there exists a family of continuous functions σ_i $(i = 1, \ldots, k)$ of class KL, with $\sigma_i(s, 0) = s$ for all $s \geqslant 0$ such that for all $t \in [0, t_{\max})$ and $i = 1, \ldots, k$, we have

$$\begin{aligned} V_i(t) \leqslant \max\Bigg\{&\sigma_i\left(V_i(0),\, t\right),\ \sup_{0\leqslant\tau\leqslant t} \sigma_i\left(\max_{j=1,\ldots,k}\ \sup_{0\leqslant s\leqslant\tau} \gamma_{i,j}\left(V_j(s)\right),\, t-\tau\right), \\ &\sup_{0\leqslant\tau\leqslant t} \sigma_i\left(\zeta\left(\sup_{0\leqslant s\leqslant\tau} |u(s)|\right),\, t-\tau\right)\Bigg\}. \end{aligned} \tag{A.18}$$

Let $\sigma(s, t) := \max_{i=1,\ldots,k} \sigma_i(s, t)$, which is a function of class KL that satisfies $\sigma(s, 0) = s$ for all $s \geqslant 0$. It follows from (A.18) that, if the solution $x(t)$ of (2.18) satisfies $h(x(t)) \leqslant 0$ for all $t \in [0, t_{\max})$, then the following inequalities hold for all $t \in [0, t_{\max})$ and $i = 1, \ldots, k$:

$$V_i(t) \leqslant \max\left\{\sigma\left(\max_{i=1,\ldots,k} V_i(0),\, t\right),\ \max_{j=1,\ldots,k} \gamma_{i,j}\left(\sup_{0\leqslant s\leqslant t} V_j(s)\right),\ \zeta\left(\sup_{0\leqslant s\leqslant t} |u(s)|\right)\right\}. \tag{A.19}$$

Clearly, inequalities (A.19) in conjunction with the fact that system (2.18) is autonomous (and, consequently, we need to consider only the case that the initial time is 0) show that Hypothesis (H1) holds with $\Gamma : \Re_+^k \to \Re_+^k$, $\Gamma(x) = (\Gamma_1(x), \ldots, \Gamma_n(x))'$ with $\Gamma_i(x) = \max_{j=1,\ldots,k} \gamma_{i,j}(x_j)$ for all $i = 1, \ldots, k$, $x \in \Re_+^n$ and

$$\begin{aligned} L(t, x) := \max_{i=1,\ldots,k} V_i(x),\ \text{for all } (t, x) \in \Re_+ \times \Re^n,\ \text{and } S(t) := S = \{\, x \in \Re^n : h(x) \leqslant 0 \,\}, \\ \text{for all } t \geqslant 0. \end{aligned} \tag{A.20}$$

Furthermore, if $U = \{0\}$, then inequalities (A.19) and definitions (A.20) imply that Hypothesis (H4) holds. Definitions (A.20) in conjunction with (2.19) show that Hypothesis (H3) holds as well with $q(x) := a_1^{-1}\left(\max_{i=1,\ldots,k} x_i\right)$ for all $x \in \Re_+^k$, $H(t,x) := x$ for all $(t,x) \in \Re_+ \times \Re^n$, and $g \equiv 0$, $\kappa(t) = \mu(t) \equiv 1$, and $b(s) := a_1^{-1}(s)$ for all $s \geqslant 0$.

It should be noticed that inequality (2.20) guarantees that the set $S = \{ x \in \Re^n : h(x) \leqslant 0 \}$ is positively invariant for system (2.18) and for every applied input $(u,d) \in M_U \times M_D$.

We next consider the solution $x(t)$ of (2.18) corresponding to arbitrary $(u,d) \in M_U \times M_D$ with arbitrary initial condition $x(0) = x_0 \notin S$. Define

$$\xi := \sup\{ t \in [0, t_{\max}) : h(x(t)) > 0 \}. \tag{A.21}$$

The continuity of h and the fact that $x(0) = x_0 \notin S$ imply that $\xi > 0$. Definition (A.21) and positive invariance of the set $S = \{ x \in \Re^n : h(x) \leqslant 0 \}$ imply that

(i) $h(x(t)) > 0$ for all $t \in [0,\xi)$, and
(ii) either $\xi = t_{\max}$ or $\xi < t_{\max}$ and $h(x(\xi)) = 0$.

Therefore, inequalities (2.20) and (2.21) imply that the following differential inequalities hold:

$$\dot{h}(t) \leqslant -\delta(h(t)), \text{ for almost all } t \in [0,\xi), \tag{A.22}$$

$$\dot{W}(t) \leqslant K(h(t))W(t) + K(h(t))\,\zeta\left(|u(t)|\right), \text{ for almost all } t \in [0,\xi), \tag{A.23}$$

where $h(t) := h(x(t))$ and $W(t) := W(x(t))$. Inequality (A.22) implies that the mapping $t \to h(t)$ is nonincreasing on $[0,\xi]$. Using the fact that K is nondecreasing, we obtain from (A.22) and (A.23) the following inequalities:

$$0 < h(t) \leqslant h(x_0) - \tilde{\delta}\, t, \text{for all } t \in [0,\xi), \tag{A.24}$$

$$W(t) \leqslant \exp\left(K(h(x_0))\,t\right)\left[W(x_0) + \sup_{0 \leqslant \tau \leqslant t} \zeta\left(|u(\tau)|\right)\right], \text{ for all } t \in [0,\xi), \tag{A.25}$$

where $\tilde{\delta} := \min_{0 \leqslant s \leqslant h(x_0)} \delta(s) > 0$.

We next show by contradiction that the case $\xi = t_{\max}$ cannot happen. Suppose that $\xi = t_{\max}$.

- If $t_{\max} < +\infty$, then standard theory implies that $\limsup_{t \to t_{\max}^-} |x(t)| = +\infty$. Because W is radially unbounded, we must have $\limsup_{t \to t_{\max}^-} W(x(t)) = +\infty$. On the other hand, inequality (A.25) shows that there exists a finite constant $A > 0$ such that $W(x(t)) \leqslant A$ for all $t \in [0, t_{\max})$, a contradiction.
- If $\xi = t_{\max} = +\infty$, then inequality (A.24) shows that $0 < h(t) \leqslant h(x_0) - \tilde{\delta}\, t$, for all $t \geqslant 0$, a contradiction.

Therefore, we can conclude that $\xi < t_{\max}$ and $h(x(\xi)) = 0$. Inequality (A.24) implies that $\xi \leqslant \frac{h(x_0)}{\tilde{\delta}}$.

Positive invariance of the set $S = \{ x \in \Re^n : h(x) \leqslant 0 \}$ implies that for every $x(0) = x_0 \in \Re^n$, $(u,d) \in M_U \times M_D$, there exists $\xi \in [0, t_{\max})$ with $\xi \leqslant \Xi(x_0)$ satisfying $x(t) \in S$, for all $t \in [\xi, t_{\max})$, where

$$\Xi(x) := \frac{\max\{0, h(x)\}}{\min\limits_{0 \leqslant s \leqslant \max\{0,h(x)\}} \delta(s)}. \tag{A.26}$$

It should be noticed that the function $\Xi : \Re^n \to \Re_+$ defined by (A.26) is a continuous mapping with $\Xi(0) = 0$. Therefore, there exists $A \in K_\infty$ such that $\Xi(x) \leqslant A\left(|x|\right)$ for all $x \in \Re^n$. It follows from those discussed that inequalities (2.7) and (2.16) hold with $c(t) \equiv 1$. Moreover, definition (A.20) implies that inequalities (2.5) and (2.14) hold with $p(x) := \max_{i=1,\ldots,k} x_i$ for all $x \in \Re_+^k$, $v(t) = c(t) \equiv 1$, $p^u \equiv 0$, and arbitrary $a \in K_\infty$.

In order to finish the proof we have to show the following.

- There exist $a, \eta \in \mathsf{N}_1$ such that (2.8) holds with $H(t,x) := x$, $c(t) \equiv 1$, for the case that $W \in C^1(\Re^n;\Re_+)$ is positive definite. Notice that inequalities (2.6) and (2.9) are direct consequences of (2.8) with $H(t,x) := x$, $c(t) \equiv 1$ and (2.19) and (A.20) for appropriate $a, \tilde{\eta}, g^u \in K_\infty$ and $c(t) \equiv 1$. In this case, the conclusion of the theorem follows from Theorem 2.5.
- There exists $a \in K_\infty$ and $R \geqslant 0$ such that (2.15) holds with $\nu(t) = c(t) \equiv 2(1+R)$, for the case $U = \{0\}$. Notice that inequality (2.17) is a direct consequence of (2.15) with $\nu(t) = c(t) \equiv 2(1+R)$ and (2.19) and (A.20) for appropriate $a \in K_\infty$, $c \in K^+$. In this case, the conclusion of the theorem follows from Theorem 2.6.

If $W \in C^1(\Re^n;\Re_+)$ is positive definite and radially unbounded, there exist $b_1, b_2 \in K_\infty$ such that $b_1(|x|) \leqslant W(x) \leqslant b_2(|x|)$, for all $x \in \Re^n$. Moreover, using the fact that $\xi \leqslant \Xi(x_0)$, we obtain from (A.25) for all $x(0) = x_0 \in \Re^n$, $(u,d) \in M_U \times M_D$ and $t \in [0,\xi]$

$$\begin{aligned}
b_1(|x(t)|) \leqslant & W(t) \leqslant \exp\left(K(\max\{0,h(x_0)\})\,t\right)\left[W(x_0) + \sup_{0\leqslant\tau\leqslant t}\zeta(|u(\tau)|)\right] \\
& \leqslant \exp\left(K(\max\{0,h(x_0)\})\,\Xi(x_0)\right)\left[W(x_0) + \sup_{0\leqslant\tau\leqslant t}\zeta(|u(\tau)|)\right] \\
& = \exp\left(K(\max\{0,h(x_0)\})\,\Xi(x_0)\right)W(x_0) \\
& \quad + \left(\exp\left(K(\max\{0,h(x_0)\})\,\Xi(x_0)\right)-1\right)\sup_{0\leqslant\tau\leqslant t}\zeta(|u(\tau)|) + \sup_{0\leqslant\tau\leqslant t}\zeta(|u(\tau)|) \\
& \leqslant \exp\left(K(\max\{0,h(x_0)\})\,\Xi(x_0)\right)b_2(|x_0|) + \frac{1}{2}\left(\exp\left(K(\max\{0,h(x_0)\})\,\Xi(x_0)\right)-1\right)^2 \\
& \quad + \frac{1}{2}\sup_{0\leqslant\tau\leqslant t}\zeta^2(|u(\tau)|) + \sup_{0\leqslant\tau\leqslant t}\zeta(|u(\tau)|) \\
& \leqslant \max\left\{2\exp\left(K(\max\{0,h(x_0)\})\,\Xi(x_0)\right)b_2(|x_0|)\right. \\
& \quad \left. + \frac{1}{2}\left(\exp\left(K(\max\{0,h(x_0)\})\,\Xi(x_0)\right)-1\right)^2 \,,\ \sup_{0\leqslant\tau\leqslant t}\eta(|u(\tau)|)\right\},
\end{aligned}$$

where $\eta(s) := \zeta^2(s) + 2\zeta(s)$. Because the mapping

$$\Re^n \ni x \to B(x) := 2\exp\left(K(\max\{0,h(x)\})\,\Xi(x)\right)b_2(|x|) + \frac{1}{2}\left(\exp\left(K(\max\{0,h(x)\})\,\Xi(x)\right)-1\right)^2$$

is nonnegative, continuous, and vanishing at zero, there exists $b_1 \in K_\infty$ such that $B(x) \leqslant b_1(|x|)$, for all $x \in \Re^n$. The aforementioned inequalities show that (2.8) holds with $H(t,x) := x$, $c(t) \equiv 1$, $\eta(s) := \zeta^2(s) + 2\zeta(s)$ and $a(s) := b_1^{-1}(b_3(s))$.

Finally, if $U = \{0\}$, then we can define the function $P : \Re^n \to \Re_+$:

$$P(x) := \max\left\{|y| \,:\, y \in \Re^n,\ W(y) \leqslant \exp\left(K(\max\{0,\ h(x)\})\,\Xi(x)\right)W(x)\right\}. \tag{A.27}$$

Notice that because W is continuous, nonnegative, and radially unbounded, the functions $P : \Re^n \to \Re_+$ is locally bounded. Because $P : \Re^n \to \Re_+$ is locally bounded, there exists $b_4 \in K_\infty$ and $R \geqslant 0$ such that $P(x) \leqslant R + b_4(|x|)$, for all $x \in \Re^n$. It follows from (A.25) and (A.27) and the fact that $P(x) \leqslant R + b_4(|x|)$ for all $x \in \Re^n$ that the following inequality holds for all $x(0) = x_0 \in \Re^n$, $(u,d) \in M_U \times M_D$, and $t \in [0,\xi]$:

$$|x(t)| \leqslant \max\{2b_4(|x_0|)\,,\ 2R\}. \tag{A.28}$$

Therefore, inequality (2.15) holds with $\nu(t) = c(t) \equiv 2(1+R)$ and $a(s) := 2b_4(s)$. The proof is complete. □

DOI: 10.1002/rnc

APPENDIX B: BASIC NOTIONS

To make our work self-contained, we introduce some notions that are essential to the system theoretic framework presented in [16, 42, 43]. The abstract system theoretic framework used in [16, 42, 43] is utilized in the present work. The difference of the basic notions used here and the classical systems notions in [53–55] is that the classical semigroup property does not hold, and no continuity assumptions are made for the transition map.

The notion of a control system—Definition 2.1 in [16]
A control system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ with outputs consists of

(i) a set U (control set), which is a subset of a normed linear space U with $0 \in U$, and a set $M_U \subseteq \mathsf{M}(U)$ (allowable control inputs), which contains at least the identically zero input u_0;
(ii) a set D (disturbance set) and a set $M_D \subseteq \mathsf{M}(D)$, which is called the 'set of allowable disturbances';
(iii) a pair of normed linear spaces X and Y called the 'state space' and the 'output space', respectively;
(iv) a continuous map $H : \Re_+ \times \mathsf{X} \times U \to \mathsf{Y}$ that maps bounded sets of $\Re_+ \times \mathsf{X} \times U$ into bounded sets of Y, called the 'output map';
(v) a set-valued map $\Re_+ \times \mathsf{X} \times M_U \times M_D \ni (t_0, x_0, u, d) \to \pi\,(t_0, x_0, u, d) \subseteq [t_0, +\infty)$, with $t_0 \in \pi(t_0, x_0, u, d)$ for all $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$, called the set of 'sampling times'; and
(vi) the map $\phi : A_\phi \to \mathsf{X}$ where $A_\phi \subseteq \Re_+ \times \Re_+ \times \mathsf{X} \times M_U \times M_D$, called the 'transition map' that has the following properties:
   (1) Existence: For each $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$, there exists $t > t_0$ such that $[t_0, t] \times (t_0, x_0, u, d) \subseteq A_\phi$.
   (2) Identity property: For each $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$, it holds that $\phi(t_0, t_0, x_0, u, d) = x_0$.
   (3) Causality: For each $(t, t_0, x_0, u, d) \in A_\phi$ with $t > t_0$ and for each $(\tilde{u}, \tilde{d}) \in M_U \times M_D$ with $(\tilde{u}(\tau), \tilde{d}(\tau)) = (u(\tau), d(\tau))$ for all $\tau \in [t_0, t]$, it holds that $(t, t_0, x_0, \tilde{u}, \tilde{d}) \in A_\phi$ with $\phi(t, t_0, x_0, u, d) = \phi(t, t_0, x_0, \tilde{u}, \tilde{d})$.
   (4) Weak semigroup property: There exists a constant $r > 0$, such that for each $t \geqslant t_0$ with $(t, t_0, x_0, u, d) \in A_\phi$
      (a) $(\tau, t_0, x_0, u, d) \in A_\phi$ for all $\tau \in [t_0, t]$;
      (b) $\phi(t, \tau, \phi(\tau, t_0, x_0, u, d), u, d) = \phi(t, t_0, x_0, u, d)$ for all $\tau \in [t_0, t] \cap \pi(t_0, x_0, u, d)$;
      (c) if $(t + r, t_0, x_0, u, d) \in A_\phi$, then it holds that $\pi(t_0, x_0, u, d) \cap [t, t + r] \neq \emptyset$; and
      (d) for all $\tau \in \pi(t_0, x_0, u, d)$ with $(\tau, t_0, x_0, u, d) \in A_\phi$, we have $\pi(\tau, \phi(\tau, t_0, x_0, u, d), u, d) = \pi(t_0, x_0, u, d) \cap [\tau, +\infty)$.

The boundedness-implies-continuation and robust forward completeness (RFC) properties—Definition 2.4 in [16]
Consider a control system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ with outputs. We say the following for system Σ.

(i) System Σ has the 'BIC' property if for each $(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D$, there exists a maximal existence time, that is, there exists $t_{\max} := t_{\max}(t_0, x_0, u, d) \in (t_0, +\infty]$, such that

$$A_\phi = \bigcup_{(t_0, x_0, u, d) \in \Re_+ \times \mathsf{X} \times M_U \times M_D} [t_0, t_{\max}) \times \{(t_0, x_0, u, d)\}.$$

In addition, if $t_{\max} < +\infty$, then for every $M > 0$, there exists $t \in [t_0, t_{\max})$ with $\|\phi(t, t_0, x_0, u, d)\|_{\mathsf{X}} > M$.

(ii) System Σ is *RFC from the input* $u \in M_U$ if it has the BIC property and for every $r \geqslant 0, T \geqslant 0$, it holds that

$$\sup\{\|\phi(t_0+s,t_0,x_0,u,d)\|_{\mathsf{X}}\,;\ u \in \mathsf{M}(B_U[0,r]) \cap\ M_U,\ s \in [0,T],\ \|x_0\|_{\mathsf{X}} \leqslant r,\ t_0 \in [0,T],\ d \in M_D\} < +\infty.$$

The notion of a robust equilibrium point—Definition 2.5 in [16]
Consider a control system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ and suppose that $H(t,0,0) = 0$ for all $t \geqslant 0$. We say that $0 \in \mathsf{X}$ is a *robust equilibrium point from the input* $u \in M_U$ for Σ if

(i) for every $(t,t_0,d) \in \Re_+ \times \Re_+ \times M_D$ with $t \geqslant t_0$ it holds that $\phi(t,t_0,0,u_0,d) = 0$; and
(ii) for every $\varepsilon > 0, T, h \in \Re_+$, there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that for all $(t_0,x,u) \in [0,T] \times \mathsf{X} \times M_U, \tau \in [t_0, t_0+h]$ with $\|x\|_{\mathsf{X}} + \sup_{t \geqslant 0}\|u(t)\|_{\mathsf{U}} < \delta$, it holds that $(\tau,t_0,x,u,d) \in A_\phi$ for all $d \in M_D$ and

$$\sup\{\,\|\phi(\tau,t_0,x,u,d)\|_{\mathsf{X}}\ ;\ d \in M_D\ ,\ \tau \in [t_0,t_0+h]\ ,\ t_0 \in [0,T]\,\} < \varepsilon.$$

Let $T > 0$. A deterministic control system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ with outputs is called T-periodic, if

(i) $H(t+T,x,u) = H(t,x,u)$ for all $(t,x,u) \in \Re^+ \times \mathsf{X} \times U$;
(ii) for every $(u,d) \in M_U \times M_D$ and integer k, there exist inputs $P_{kT}u \in M_U$, $P_{kT}d \in M_D$ with $(P_{kT}u)(t) = u(t+kT)$ and $(P_{kT}d)(t) = d(t+kT)$ for all $t + kT \geqslant 0$; and
(iii) for each $(t,t_0,x_0,u,d) \in A_\phi$ with $t \geqslant t_0$ and for each integer k with $t_0 - kT \geqslant 0$, it follows that $(t-kT, t_0-kT, x_0, P_{kT}u, P_{kT}d) \in A_\phi$ and $\pi(t_0-kT, x_0, P_{kT}u, P_{kT}d) = \cup_{\tau \in \pi(t_0,x_0,u,d)}\{\tau - kT\}$ with $\phi(t,t_0,x_0,u,d) = \phi(t-kT, t_0-kT, x_0, P_{kT}u, P_{kT}d)$.

A deterministic control system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ with outputs is called time-invariant, or autonomous, if it is T-periodic for all $T > 0$.

Next, we present the IOS property for the class of systems described previously (see also [1,56] for finite-dimensional, time-invariant dynamic systems).

The notion of input-to-output stability, uniform input-to-output stability, input-to-state stability, and uniform input-to-state stability—Definition 2.5 in [16]
Consider a control system $\Sigma := (\mathsf{X}, \mathsf{Y}, M_U, M_D, \phi, \pi, H)$ with outputs and the BIC property and for which $0 \in \mathsf{X}$ is a robust equilibrium point from the input $u \in M_U$. Suppose that Σ is RFC from the input $u \in M_U$. If there exist functions $\sigma \in KL, \beta \in K^+, \gamma \in \mathsf{N}_1$ such that the following estimate holds for all $u \in M_U, (t_0,x_0,d) \in \Re_+ \times \mathsf{X} \times M_D$ and $t \geqslant t_0$:

$$\|H(t,\phi(t,t_0,x_0,u,d),u(t))\|_{\mathsf{Y}} \quad \leqslant\ \sigma\left(\beta(t_0)\ \|x_0\|_{\mathsf{X}}\ ,\ t-t_0\right)\ +\ \sup_{t_0 \leqslant \tau \leqslant t} \gamma\left(\|u(\tau)\|_{\mathsf{U}}\right),$$

then we say that Σ satisfies the IOS property from the input $u \in M_U$ with gain $\gamma \in \mathsf{N}$. Moreover, if $\beta \in K^+$ may be chosen as $\beta(t) \equiv 1$, then we say that Σ satisfies the uniform IOS property from the input $u \in M_U$ with gain $\gamma \in \mathsf{N}_1$.

For the special case of the identity output mapping, that is, $H(t,x,u) := x$, the (uniform) IOS property from the input $u \in M_U$ is called (uniform) ISS property from the input $u \in M_U$. When $U = \{0\}$ (the no-input case) and Σ satisfies the (U)IOS property, then we say that Σ satisfies the (uniform) RGAOS property. When $U = \{0\}$ (the no-input case) and Σ satisfies the (uniform) ISS property, then we say that Σ satisfies the (uniform) RGAS property.

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DOI: 10.1002/rnc