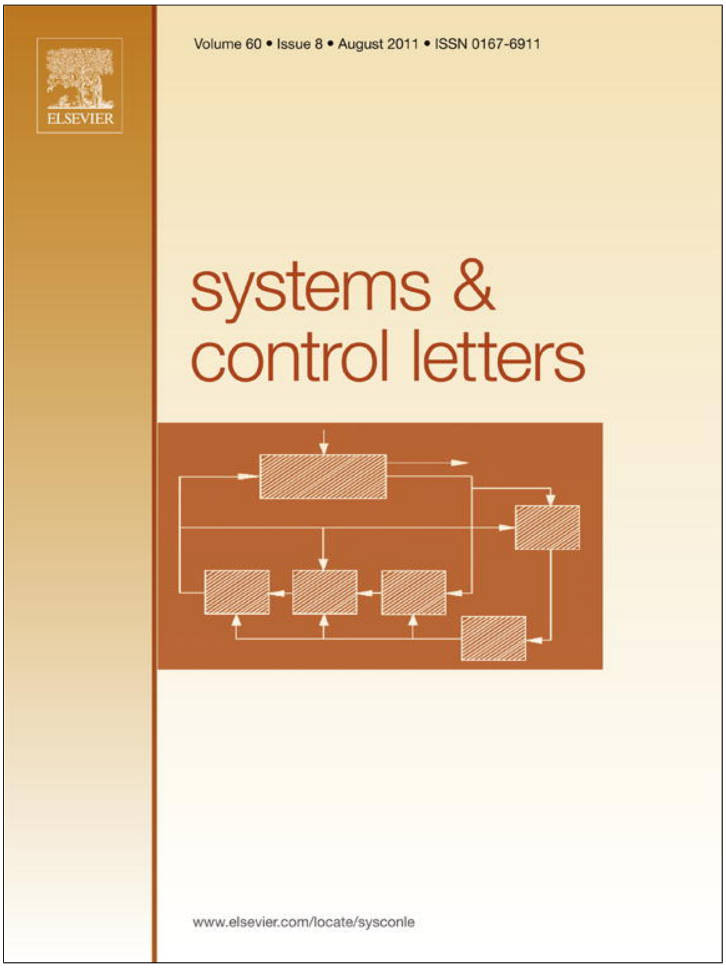
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Hybrid dead-beat observers for a class of nonlinear systems

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ABSTRACT

This paper studies both the full order and the reduced order dead-beat observer problem for a class of nonlinear systems, linear in the unmeasured states. A novel hybrid observer design strategy is proposed, with the help of the notion of strong observability in finite time. The proposed methodology is applied for the estimation of the frequency of a sinusoidal signal. The results show that accurate estimation can be provided even if the signal is corrupted by high frequency noise. A brief discussion of the robustness properties of the proposed observer with respect to measurement errors is also provided.

1. Introduction

The observer problem occupies an important place in mathematical control theory. It is concerned with the estimation of unmeasured states of a dynamic control system using the information of inputs and outputs. There has been a large amount of literature on the problems of existence and design of observers (see for instance [1–17] and the references therein). In this work, we focus on nonlinear systems of the form:

$$\begin{aligned}
&\dot{x}(t) = A(y(t), u(t))x(t) + b(y(t), u(t)) \\
&\dot{y}_i(t) = f_i(y(t), u(t)) + \sum_{j=1}^{n} c_{i,j}(y(t))x_j(t), \quad i = 1, \ldots, k \\
&x(t) = (x_1(t), \ldots, x_n(t))' \in \Re^n, \\
&y(t) = (y_1(t), \ldots, y_k(t))' \in \Re^k \\
&(x(t), y(t)) \in O \subseteq \Re^n \times \Re^k, \quad u(t) \in U \subseteq \Re^m
\end{aligned} \tag{1.1}$$

where $O \subseteq \Re^{n+k}$ is an open set, $U \subseteq \Re^m$ is a non-empty closed set, $A(y, u) = \{a_{i,j}(y, u), i, j = 1, \ldots, n\}$ and all mappings $a_{i,j} : \Omega \times U \to \Re$ $(i, j = 1, \ldots, n)$, $b : \Omega \times U \to \Re^n$, $c_{i,j} : \Omega \to \Re$ $(i = 1, \ldots, k, j = 1, \ldots, n)$ and $f_i : \Omega \times U \to \Re$ $(i = 1, \ldots, k)$ are locally Lipschitz, where $\Omega = \{y \in \Re^k : \exists x \text{ such that } (x, y) \in O\}$. It is assumed that the component of the state vector y, also known as the output, is available for the feedback design and that the remaining state component x is unmeasured and is to be estimated.

Systems of the form (1.1) are termed as "systems linear in the unmeasured state components" in the literature (see [18–23]). The dynamic output feedback stabilization problem has been studied extensively in the past for this class of systems in [18–23]. Exponential observers for systems linear in the unmeasured state components were provided in [3], under a persistency of excitation condition. It should be noted that systems of the form (1.1) are related to systems with output dependent incremental rate. For systems with output dependent incremental rate the dynamic high-gain approach was exploited in [24] for the solution of the output feedback stabilization problem.

The purpose of the present work is to study the observability properties of systems linear in the unmeasured state components and to propose a novel observer design procedure that guarantees features which cannot be provided by conventional observers: we propose hybrid observers which provide exact estimation of the unmeasured state components in finite time (dead-beat observers). Moreover, we consider the general case where the system evolves in an open set O and not in $\Re^{n+k}$. It should be noted that hybrid observers were recently proposed in [4] as well. Moreover, dead-beat observers have been proposed in the literature for linear systems:

- by means of sliding modes (see [8,12,25]),
- by means of delays (see [6]).

The approach of using delays for the observer design was exploited in [13] for a special class of nonlinear systems (with nonlinear output injection terms) and was extended to globally Lipschitz nonlinear systems in [11]. High-gain techniques were utilized in

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[5,15] for the design of semi-global finite-time observers for a class of nonlinear systems.

In Section 2 of the present work, we study the observability properties of system (1.1). The notion of strong observability is introduced for general nonlinear systems. Conditions for the construction of inputs which do not distinguish between events in finite time (see [26]) for system (1.1) are proposed. In Section 3 it is shown that under strong observability in time $r > 0$ for system (1.1), it is possible to give explicit formulas for a deterministic system (Σ) with states $z(t) \in O$ and inputs $y : \Re^+ \to \Omega$, $u : \Re^+ \to U$ so that

$$z(t) = (x(t), y(t)), \quad \text{for all } t \geq r. \tag{1.2}$$

The proposed observer (Σ) is a hybrid system which uses delays: the history of the output is utilized in order to estimate the state component x of system (1.1). For the overall system (1.1) with (Σ) the classical semigroup property does not hold: however, the weak semigroup property holds (see [27–29]). Also, the overall system (1.1) with (Σ) is autonomous in the sense described in [27–29]. The proposed hybrid observer relies on the minimization of an appropriate L^2 norm and is methodologically close to the procedure used for optimization-based observers (see [1] and references therein).

In Section 4 of the present work, we present an application dealing with the estimation of the frequency of a sinusoidal signal. The problem was recently studied in [30] (see also [31,32]). It is shown that the proposed hybrid, dead-beat observer provides robust estimation of the frequency of the measured signal. Indeed, the observer is tested in the presence of high frequency noise and exactly the same test of robustness with the one in [30] is performed. The results show that the sensitivity to measurement noise decreases as the time horizon of the minimized L^2 norm increases, i.e., as the length of the history of the output which is utilized for the state estimation increases. This feature is expected and it is common to optimization-based observers (see [1] and references therein).

The robustness properties of the proposed observer with respect to measurement errors are studied in detail in Section 5. Proposition 5.1 implies that the difference of the state of system (1.1) and the observer state satisfies the Bounded-Input Bounded-Output (BIBO) property (statement (a) of Proposition 5.1) and the Converging-Input Converging-Output (CICO) property (statement (c) of Proposition 5.1) with the measurement error as input, under certain hypotheses. The result is important, because the topic of the robustness properties of observers for nonlinear systems is rarely studied both numerically and theoretically.

Notations. Throughout this paper, we adopt the following notations:

* Let $I \subseteq \Re_+ := [0, +\infty)$ be an interval. By $L^\infty(I; U)$ ($L^\infty_{loc}(I; U)$), we denote the space of measurable and (locally) essentially bounded functions $u(\cdot)$ defined on I and taking values in $U \subseteq \Re^m$.
* By $C^0(A; \Omega)$, we denote the class of continuous functions on A, which take values in Ω.
* For a vector $x \in \Re^n$, we denote by x' its transpose and by $|x|$ its Euclidean norm. The determinant of a square matrix $A \in \Re^{n\times n}$ is denoted by $\det(A)$. $A' \in \Re^{n\times m}$ denotes the transpose of the matrix $A \in \Re^{m\times n}$.
* By $A = \mathrm{diag}(l_1, l_2, \ldots, l_n)$ we mean a diagonal matrix with $l_1, l_2, \ldots, l_n$ on its diagonal.

2. Strong observability

Consider an autonomous system described by ordinary differential equations of the form:

$$\begin{aligned} &\dot{x}(t) = F(x(t), u(t)) \\ &x(t) \in D \subseteq \Re^n, \quad u(t) \in U \subseteq \Re^m \end{aligned} \tag{2.1}$$

where $D \subseteq \Re^n$ is an open set, $U \subseteq \Re^m$ is a non-empty closed set and the mapping $F : D \times U \to \Re^n$ is locally Lipschitz. The output of system (2.1) is given by

$$y(t) = h(x(t)) \tag{2.2}$$

where the mapping $h : D \to \Re^k$ is continuous. For system (2.1) we adopt the following notion of observability. We assume that for every $x_0 \in D$ and $u \in L^\infty_{loc}(\Re_+; U)$ there exists a unique solution $[0, +\infty) \ni t \to x(t) = x(t, x_0; u) \in D$ for (2.1) satisfying (2.1) for almost every $t \geq 0$ and $x(0) = x(0, x_0; u) = x_0$.

Definition 2.1. Consider system (2.1) with output (2.2). We say that the input $u \in L^\infty([0, r]; U)$ strongly distinguishes the state $x_0 \in D$ in time $r > 0$, if the following condition holds

$$\max_{t\in[0,r]} |h(x(t, x_0; u)) - h(x(t, \xi; u))| > 0, \quad \text{for all } \xi \in D \text{ with } x_0 \neq \xi. \tag{2.3}$$

Remark 2.2. Following the terminology in [26], Definition 2.1 implies that, if the input $u \in L^\infty([0, r]; U)$ strongly distinguishes the state $x_0 \in D$ in time $r > 0$, then for every $\xi \in D$ with $x_0 \neq \xi$ the input $u \in L^\infty([0, r]; U)$ distinguishes between the events $(x_0, 0)$ and $(\xi, 0)$ for system (2.1) with output (2.2) (see [26]).

For system (1.1), we assume that for every $(x_0, y_0) \in O$ and $u \in L^\infty_{loc}(\Re_+; U)$ there exists a unique mapping $[0, +\infty) \ni t \to (x(t), y(t)) = (x(t, x_0, y_0; u), y(t, x_0, y_0; u)) \in O$ satisfying (1.1) for almost every $t \geq 0$ and $(x(0), y(0)) = (x(0, x_0, y_0; u), y(0, x_0, y_0; u)) = (x_0, y_0)$.
Denote by $\Phi(t, x_0, y_0; u)$ the transition matrix of the linear time-varying system $\dot{x}(t) = A(y(t), u(t))x(t)$ when $u \in L^\infty_{loc}(\Re_+; U)$ and $y(t) = y(t, x_0, y_0; u)$ for $t \geq 0$ are considered as the inputs. Then the following fact holds for the solutions of system (1.1). It follows directly from integration of the differential equations (1.1).

Fact I. For every $(x_0, y_0) \in O$ and $u \in L^\infty([0, r]; U)$ the following equations hold for all $t \geq 0$:

$$x(t, x_0, y_0; u) = \Phi(t, x_0, y_0; u)x_0 + \theta(t, x_0, y_0; u) \tag{2.4}$$

$$p(t, x_0, y_0; u) = q'(t, x_0, y_0; u)x_0 \tag{2.5}$$

where $f(y, u) := (f_1(y, u), \ldots, f_k(y, u))'$ and

$$q(t, x_0, y_0; u) := \int_0^t \Phi'(s, x_0, y_0; u)C(s, x_0, y_0; u)ds \tag{2.6}$$

$$\theta(t, x_0, y_0; u) := \int_0^t \Phi(t, x_0, y_0; u) \times \Phi^{-1}(\tau, x_0, y_0; u)b(y(\tau, x_0, y_0; u), u(\tau))d\tau \tag{2.7}$$

$$C'(t, x_0, y_0; u) := \begin{bmatrix} c_{1,1}(y(t, x_0, y_0; u)) & \ldots & c_{1,n}(y(t, x_0, y_0; u)) \\ \vdots & & \vdots \\ c_{k,1}(y(t, x_0, y_0; u)) & \ldots & c_{k,n}(y(t, x_0, y_0; u)) \end{bmatrix} \in \Re^{k\times n} \tag{2.8}$$

$$\begin{aligned} p(t, x_0, y_0; u) := \; & y(t, x_0, y_0; u) - y_0 \\ & - \int_0^t f(y(s, x_0, y_0; u), u(s))ds \\ & - \int_0^t C'(s, x_0, y_0; u)\theta(s, x_0, y_0; u)ds. \end{aligned} \tag{2.9}$$

It is important to note at this point that all expressions involved in (2.4)–(2.9) are evaluated by means of the output trajectory

$y(\tau) = y(\tau, x_0, y_0; u)$ for $\tau \in [0, t]$ and the input $u(\tau)$ for $\tau \in [0, t]$. Particularly, the transition matrix $\Phi(t, x_0, y_0; u)$ can be evaluated by solving the linear matrix differential equation $\frac{d}{d\tau}\Phi(\tau) = A(y(\tau), u(\tau))\Phi(\tau)$ for $\tau \in [0, t]$ with initial condition $\Phi(0) = I$, where I denotes the identity matrix. Similarly, $C(\tau) := C(\tau, x_0, y_0; u)$ is simply $C'(\tau) := \{c_{i,j}(y(\tau)), i = 1, \ldots, k, j = 1, \ldots, n\}$ for $\tau \in [0, t]$ and $\theta(t) = \theta(t, x_0, y_0; u)$ can be computed by solving the linear system of differential equations $\frac{d}{d\tau}\theta(\tau) = A(y(\tau), u(\tau))\theta(\tau) + b(y(\tau), u(\tau))$ for $\tau \in [0, t]$ with initial condition $\theta(0) = 0 \in \Re^n$. Finally, the differential equations $\frac{d}{d\tau}q(\tau) = \Phi'(\tau)C(\tau)$ and $\frac{d}{d\tau}\xi(\tau) = (f_1(y(\tau), u(\tau)), \ldots, f_k(y(\tau), u(\tau)))' + C'(\tau)\theta(\tau)$, for $\tau \in [0, t]$ can be utilized to provide the quantities $q(t) = q(t, x_0, y_0; u)$ and $p(t, x_0, y_0; u) = y(t) - y(0) - \xi(t)$.

The following proposition provides characterizations of the class of inputs $u \in L^\infty([0, r]; U)$ which strongly distinguish the state $(x_0, y_0) \in O$ in time $r > 0$ for system (1.1). The basic idea of Proposition 2.3 is the conversion of the observability property to the minimization of an appropriate L^2 norm. Therefore, our approach is close to the procedure used for optimization-based observers (see [1] and references therein). The proof of the following proposition is postponed to the Appendix.

Proposition 2.3. *Consider system (1.1). The following statements are equivalent:*

(a) *The input $u \in L^\infty([0, r]; U)$ strongly distinguishes the state $(x_0, y_0) \in O$ in time $r > 0$.*
(b) *The problem*

$$\min_{\xi \in B(y_0)} \int_0^r \left|p(t, x_0, y_0; u) - q'(t, x_0, y_0; u)\xi\right|^2 dt \tag{2.10}$$

where $B(y_0) := \{\xi \in \Re^n : (\xi, y_0) \in O\}$, admits the unique solution $\xi = x_0$.
(c) *The symmetric matrix*

$$Q(r, x_0, y_0; u) := \int_0^r q(t, x_0, y_0; u)q'(t, x_0, y_0; u)dt \tag{2.11}$$

is positive definite. Moreover, it holds that

$$x_0 = Q^{-1}(r, x_0, y_0; u) \int_0^r q(t, x_0, y_0; u)p(t, x_0, y_0; u)dt. \tag{2.12}$$

(d) *The following implication holds:*

$$q'(t, x_0, y_0; u)\xi = 0, \quad \forall t \in [0, r] \Rightarrow \xi = 0 \in \Re^n. \tag{2.13}$$

The following corollary utilizes Proposition 2.3 and provides sufficient conditions for an input $u \in L^\infty([0, r]; U)$ to strongly distinguish the state $(x_0, y_0) \in O$ in time $r > 0$ for system (1.1).

Corollary 2.4. *Consider system (1.1) with $k = 1$ and let $(x_0, y_0) \in O$, $u \in L^\infty([0, r]; U)$ for which there exist $t, t_1, \ldots, t_{n-1} \in [0, r]$ such that*

$$\det\left(\begin{bmatrix} C'(t, x_0, y_0; u)\Phi(t, x_0, y_0; u) \\ C'(t_1, x_0, y_0; u)\Phi(t_1, x_0, y_0; u) \\ \vdots \\ C'(t_{n-1}, x_0, y_0; u)\Phi(t_{n-1}, x_0, y_0; u) \end{bmatrix}\right) \neq 0. \tag{2.14}$$

Then the input $u \in L^\infty([0, r]; U)$ strongly distinguishes the state $(x_0, y_0) \in O$ in time $r > 0$. Moreover, the symmetric matrix $Q(r, x_0, y_0; u)$ defined by (2.11) is positive definite and (2.12) holds.

Proof. Suppose that $k = 1$ and that the input $u \in L^\infty([0, r]; U)$ does not strongly distinguish the state $(x_0, y_0) \in O$ in time $r > 0$ for system (1.1). The equivalence (a) $\Leftrightarrow$ (d) shows that there exists $\xi \in \Re^n$, $\xi \neq 0$ such that $q'(t, x_0, y_0; u)\xi = 0$ for all $t \in [0, r]$. It follows that $\frac{d}{dt}q'(t, x_0, y_0; u)\xi = C'(t, x_0, y_0; u)\Phi(t, x_0, y_0; u)\xi = 0$, for all $t \in [0, r]$. Consequently, we obtain:

$$\det\left(\begin{bmatrix} C'(t, x_0, y_0; u)\Phi(t, x_0, y_0; u) \\ C'(t_1, x_0, y_0; u)\Phi(t_1, x_0, y_0; u) \\ \vdots \\ C'(t_{n-1}, x_0, y_0; u)\Phi(t_{n-1}, x_0, y_0; u) \end{bmatrix}\right) = 0,$$
$$\text{for all } t, t_1, \ldots, t_{n-1} \in [0, r]. \tag{2.15}$$

This contradicts (2.14). The proof is complete. □

It is convenient to exploit condition (2.15) in order to construct inputs which do not strongly distinguish the state $(x_0, y_0) \in O$ in time $r > 0$. The following example illustrates how the results of Proposition 2.3 and Corollary 2.4 can allow the study of the observability properties of a nonlinear system.

Example 2.5. Consider the system

$$\begin{aligned} &\dot{x}_1(t) = a_1(y(t))x_1(t) \\ &\dot{x}_2(t) = a_2(y(t))x_2(t) \\ &\dot{y}(t) = u(t) + c_1(y(t))x_1(t) + c_2(y(t))x_2(t) \\ &x(t) = (x_1(t), x_2(t)) \in \Re^2, \quad y(t) \in \Re, \quad u(t) \in \Re \end{aligned} \tag{2.16}$$

where $c_i : \Re \to (0, +\infty)$, $a_i : \Re \to \Re$, $i = 1, 2$ are continuously differentiable functions. For system (2.16) we assume forward completeness for all $u \in L^\infty_{loc}(\Re_+; \Re)$, which can be guaranteed if the functions $a_i : \Re \to \Re$, $i = 1, 2$ are bounded from above and the functions $c_i : \Re \to (0, +\infty)$, $i = 1, 2$ satisfy a linear growth condition, i.e., $|c_i(y)| \le K_1 |y| + K_2$, $i = 1, 2$ for all $y \in \Re$ and for certain constants $K_1, K_2 \ge 0$.

By virtue of Corollary 2.4, if the input $u \in L^\infty([0, r]; \Re)$ does not strongly distinguish the state $(x_0, y_0) \in \Re^3$ in time $r > 0$, then the following condition must hold:

$$\begin{aligned} &c_2(y_0)c_1(y(t, x_0, y_0; u)) \\ &\quad\times \exp\left(\int_0^t (a_1(y(s, x_0, y_0; u)) - a_2(y(s, x_0, y_0; u)))\, ds\right) \\ &\quad= c_1(y_0)c_2(y(t, x_0, y_0; u)), \quad \text{for all } t \in [0, r]. \end{aligned} \tag{2.17}$$

Notice that in this case we have $\Phi(t, x_0, y_0; u) = \mathrm{diag}\big(\exp\big(\int_0^t a_1 (y(s, x_0, y_0; u))ds\big), \exp\big(\int_0^t a_2(y(s, x_0, y_0; u))ds\big)\big)$. Condition (2.17) coincides with condition (2.15) for $n = 2$, $t_1 = 0$. By differentiating (2.17) we obtain:

$$\begin{aligned} &\kappa(y(t, x_0, y_0; u))\dot{y}(t, x_0, y_0; u) + (a_1(y(t, x_0, y_0; u)) \\ &\quad - a_2(y(t, x_0, y_0; u))) = 0, \quad \text{for almost all } t \in [0, r] \end{aligned} \tag{2.18}$$

where

$$\kappa(y) = \frac{d}{dy} \ln\left(\frac{c_1(y)}{c_2(y)}\right). \tag{2.19}$$

Moreover, using (2.16) and (2.17) with $x_0 = (x_{1,0}, x_{2,0})' \in \Re^2$, it holds:

$$\begin{aligned} &\dot{y}(t, x_0, y_0; u) = u(t) + c_2(y(t, x_0, y_0; u)) \\ &\quad\times \exp\left(\int_0^t a_2(y(s, x_0, y_0; u))ds\right)\left(x_{2,0} + x_{1,0}\frac{c_1(y_0)}{c_2(y_0)}\right), \\ &\quad \text{for almost all } t \in [0, r]. \end{aligned} \tag{2.20}$$

If we further assume that $\kappa(y) \neq 0$, for all $y \in \Re$, then we conclude from (2.18) and (2.20) that

"If the input $u \in L^\infty([0,r];\Re)$ does not strongly distinguish the state $(x_0,y_0) \in \Re^2 \times \Re$ in time $r > 0$ with $x_0 = (x_{1,0}, x_{2,0})' \in \Re^2$ for system (2.16), then the input $u \in L^\infty([0,r];\Re)$ satisfies for almost all $t \in [0,r]$:

$$u(t) = \frac{a_2(y(t)) - a_1(y(t))}{\kappa(y(t))} - c_2(y(t)) \times \exp\left(\int_0^t a_2(y(s))\mathrm{d}s\right)\left(x_{2,0} + x_{1,0}\frac{c_1(y_0)}{c_2(y_0)}\right) \quad (2.21)$$

where $y : [0,r] \to \Re$ is the solution of the initial value problem

$$\dot{y}(t) = \frac{a_2(y(t)) - a_1(y(t))}{\kappa(y(t))}, \quad \text{with } y(0) = y_0\text{"}. \quad (2.22)$$

Therefore, condition (2.15) has allowed us to construct inputs which do not strongly distinguish the state $(x_0,y_0) \in \Re^3$ in time $r > 0$. Indeed, without additional hypotheses it is unclear that every input $u \in L^\infty([0,r];\Re)$ will strongly distinguish every state $(x_0,y_0) \in \Re^3$ in time $r > 0$. For example, if there exists $y^* \in \Re$ such that $a_1(y^*) = a_2(y^*) = 0$ then the input $u(t) \equiv u^* = -c_2(y^*)x_{2,0} - c_1(y^*)x_{1,0}$ cannot distinguish between the state $(x_0,y^*) = (x_{1,0},x_{2,0},y^*) \in \Re^2 \times \Re$ and the state $(\xi,y^*) = (\xi_1,\xi_2,y^*) \in \Re^2 \times \Re$ with $\xi_2 = x_{2,0} + \frac{c_1(y^*)}{c_2(y^*)}(x_{1,0} - \xi_1)$. Indeed, both states produce the same output response $y(t) \equiv y^*$ when the constant input $u(t) \equiv u^* = -c_2(y^*)x_{2,0} - c_1(y^*)x_{1,0}$ is applied. □

We next define the notion of strongly observable systems in time $r > 0$.

Definition 2.6. Consider system (2.1) with output (2.2). We say that (2.1) is strongly observable in time $r > 0$ if every input $u \in L^\infty([0,r];U)$ strongly distinguishes every state $x_0 \in D$ in time $r > 0$.

Remark 2.7. Proposition 2.3 guarantees that system (1.1) is strongly observable in time $r > 0$, if the symmetric matrix $Q(r,x_0,y_0;u)$ defined by (2.11) is positive definite for all $(x_0,y_0) \in O$ and $u \in L^\infty([0,r];U)$. It is clear that observability for the linear time-invariant system $\dot{x} = Ax + Bu$, $y = c'x$, where $x \in \Re^n$, $y \in \Re^k$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $c \in \Re^{k\times n}$ is equivalent to strong observability in time $r > 0$ for every $r > 0$. In general, strong observability in time $r > 0$ implies observability in time $r > 0$ in the sense of [26]. However, for nonlinear systems the converse statement does not hold. Notice that system (2.16) of Example 2.5 is not strongly observable in time $r > 0$; however, it is observable in time $r > 0$ in the sense described in [26]: every input $u \in L^\infty([0,r];\Re)$ which does not satisfy (2.21) for almost all $t \in [0,r]$ is an input which distinguishes between the events $(x_0,y_0,0)$ and $(\xi,y_0,0)$ in time $r > 0$, where $(x_0,y_0) = (x_{1,0},x_{2,0},y_0) \in \Re^3$, $(\xi,y_1) = (\xi_1,\xi_2,y_1) \in \Re^3$.

3. Observer design under strong observability

In this section we utilize the results of the previous section under the assumption of strong observability in time $r > 0$ for system (1.1). More specifically, Proposition 2.3 shows that under the following hypothesis for system (1.1):

(H1) *System (1.1) is strongly observable in time $r > 0$.*

We are in a position to define the operator:

$$P : L^\infty([0,r];\Omega) \times L^\infty([0,r];U) \to \Re^n$$

where $\Omega = \{y \in \Re^k : \exists x \text{ such that } (x,y) \in O\}$. For each $(y,u) \in L^\infty([0,r];\Omega) \times L^\infty([0,r];U)$, $P(y,u)$ is defined by

$$P(y,u) = \Phi(r,y;u)Q^{-1}\int_0^r q(\tau)p(\tau)\mathrm{d}\tau + \theta(r) \quad (3.1)$$

where $\Phi(t,y;u)$ is the transition matrix of the linear system $\dot{z}(t) = A(y(t),u(t))z(t)$, $Q = \int_0^r q(\tau)q'(\tau)\mathrm{d}\tau$, $q(\tau) = \int_0^\tau \Phi'(s,y;u)C(s)\mathrm{d}s$, $C'(\tau) := \{c_{i,j}(y(\tau)), i = 1,\dots,k, j = 1,\dots,n\}$, $p(\tau) = y(\tau) - y(0) - \int_0^\tau f(y(s),u(s))\mathrm{d}s - \int_0^\tau C'(s)\theta(s)\mathrm{d}s$, $f(y,u) := (f_1(y,u),\dots,f_k(y,u))'$, $\theta(\tau) := \int_0^\tau \Phi(\tau,y;u)\Phi^{-1}(s,y;u)b(y(s),u(s))\mathrm{d}s$ for all $\tau \in [0,r]$. Proposition 2.3 guarantees that, if hypothesis (H1) holds for system (1.1), then for every $(x_0,y_0) \in O$ and $u \in L^\infty_{loc}(\Re_+;U)$ the following equality holds:

$$x(t,x_0,y_0;u) = P(\delta_{t-r}y, \delta_{t-r}u), \quad \text{for all } t \geq r \quad (3.2)$$

where $(\delta_{t-r}y)(s) = y(t-r+s,x_0,y_0;u)$, $(\delta_{t-r}u)(s) = u(t-r+s)$ for $s \in [0,r]$.

Therefore, if hypothesis (H1) holds for system (1.1), then we are in a position to provide a hybrid, dead-beat observer for system (1.1). Given $t_0 \geq 0$, $(z_0,w_0) \in O$, we calculate $(z(t),w(t)) \in O$ by the following algorithm:

Step i: Calculation of $z(t)$ for $t \in [t_0+ir, t_0+(i+1)r]$.

(1) Calculate $z(t)$ for $t \in [t_0+ir, t_0+(i+1)r)$ as the solution of $\dot{z}(t) = A(w(t),u(t))z(t) + b(w(t),u(t))$, $\dot{w}_i(t) = f_i(w(t),u(t)) + \sum_{j=1}^n c_{i,j}(w(t))z_j(t)$ $(i = 1,\dots,k)$, with $w(t) = (w_1(t),\dots,w_k(t))' \in \Re^k$.

(2) Set $z(t_0+(i+1)r) = P(\delta_{t_0+ir}y, \delta_{t_0+ir}u)$ and $w(t_0+(i+1)r) = y(t_0+(i+1)r)$, where $P : C^0([0,r];\Omega) \times L^\infty([0,r];U) \to \Re^n$ is the operator defined by (3.1).

For $i = 0$ we take $(z(t_0),w(t_0)) = (z_0,w_0)$ (initial condition).

The proposed observer can be represented by the following system of equations:

$$\begin{aligned}
&\dot{z}(t) = A(w(t),u(t))z(t) + b(w(t),u(t)), \quad t \in [\tau_i,\tau_{i+1}) \\
&\dot{w}_i(t) = f_i(w(t),u(t)) + \sum_{j=1}^n c_{i,j}(w(t))x_j(t), \\
&\quad i = 1,\dots,k,\ t \in [\tau_i,\tau_{i+1}) \\
&z(\tau_{i+1}) = P(\delta_{\tau_i}y, \delta_{\tau_i}u), \qquad w(\tau_{i+1}) = y(\tau_{i+1}) \\
&\tau_{i+1} = \tau_i + r \\
&z(t) = (z_1(t),\dots,z_n(t))' \in \Re^n, \\
&w(t) = (w_1(t),\dots,w_k(t))' \in \Re^k \\
&(z(t),w(t)) \in O \subseteq \Re^n \times \Re^k.
\end{aligned} \quad (3.3)$$

Thus, from all the above results, we obtain the following corollary.

Corollary 3.1. *Consider system (1.1) and assume that hypothesis (H1) holds. Consider the unique solution $(x(t),y(t),z(t),w(t)) \in O \times O$ of (1.1), (3.3) with arbitrary initial condition $(x_0,y_0,z_0,w_0) \in O \times O$ corresponding to arbitrary input $u \in L^\infty_{loc}(\Re_+;U)$. Then the solution $(x(t),y(t),z(t),w(t)) \in O \times O$ of (1.1), (3.3) satisfies:*

$$z(t) = x(t) \quad \text{and} \quad w(t) = y(t), \quad \text{for all } t \geq r. \quad (3.4)$$

Proof. The solution $(z(t),w(t)) \in O$ of $\dot{z}(t) = A(w(t),u(t))z(t) + b(w(t),u(t))$ and $\dot{w}_i(t) = f_i(w(t),u(t)) + \sum_{j=1}^n c_{i,j}(w(t))x_j(t)$, $i = 1,\dots,k$ exists for all $t \in [0,r)$ (notice that this is just a copy of system (1.1), for which we have assumed forward completeness) and it is bounded on the interval $[0,r)$. Moreover, it follows from (2.4), (2.12) and definition (3.1) that $x(\tau_1) = x(r) = P(\delta_0 y, \delta_0 u) = P(\delta_{\tau_0}y, \delta_{\tau_0}u)$. Therefore, the switching rules of system (3.3) imply that $z(\tau_1) = x(\tau_1)$ and $w(\tau_1) = y(\tau_1)$. By virtue of uniqueness of solutions of system (1.1) we obtain $z(t) = x(t)$ and $w(t) = y(t)$ for all $t \in [r,2r)$. Applying trivial induction arguments we can guarantee that (3.4) holds. □

Remark 3.2. The proposed observer (3.3) is a hybrid system which uses delays: the history of the output is utilized in order to estimate the state component x of system (1.1). For the overall system (1.1) with (3.3) the classical semigroup property does not hold: however, the weak semigroup property holds (see [27–29]). Also, the overall system (1.1) with (3.3) is autonomous in the sense described in [27–29]. Finally, it should be noted that by virtue of Corollary 2.4 for the case $k = 1$ a sufficient condition for hypothesis (H1) is the following condition:

"*For every* $(x_0, y_0) \in O$, $u \in L^\infty([0, r]; U)$ *there exist* $t, t_1, \dots, t_{n-1} \in [0, r]$ *such that* (2.14) *holds*".

Next assume that the following hypothesis holds in addition to hypothesis (H1).

(H2) *There exist open sets* $D \subseteq \Re^n$ *and* $\Omega \subseteq \Re^k$ *such that* $O = D\times\Omega$. *Moreover, for every* $\xi \in D$ *and for every* $(y, u) \in L^\infty([0, r]; \Omega) \times L^\infty([0, r]; U)$, *the solution* $z(t)$ *of* $\dot z(t) = A(y(t), u(t))z(t) + b(y(t), u(t))$ *with* $z(0) = \xi$ *satisfies* $z(t) \in D$ *for all* $t \in [0, r]$.

If hypothesis (H2) holds, then we can design a reduced order, hybrid, dead-beat observer for system (1.1) of the form:

$$\begin{aligned}
&\dot z(t) = A(y(t), u(t))z(t) + b(y(t), u(t)), \quad t \in [\tau_i, \tau_{i+1})\\
&z(\tau_{i+1}) = P(\delta_{\tau_i} y, \delta_{\tau_i} u)\\
&\tau_{i+1} = \tau_i + r\\
&z(t) = (z_1(t), \dots, z_n(t))' \in D \subseteq \Re^n
\end{aligned} \tag{3.5}$$

where $P : L^\infty([0, r]; \Omega) \times L^\infty([0, r]; U) \to \Re^n$ is the operator defined by (3.1).

Corollary 3.3. *Consider system* (1.1) *and assume that hypotheses* (H1), (H2) *hold. Consider the unique solution* $(x(t), y(t), z(t)) \in D\times\Omega\times D$ *of* (1.1), (3.5) *with arbitrary initial condition* $(x_0, y_0, z_0) \in D \times \Omega \times D$ *corresponding to arbitrary input* $u \in L^\infty_{loc}(\Re_+; U)$. *Then the solution* $(x(t), y(t), z(t)) \in D \times \Omega \times D$ *of* (1.1), (3.5) *satisfies:*

$$z(t) = x(t), \quad \textit{for all } t \ge r. \tag{3.6}$$

Sketch of Proof. The proof is exactly the same as the proof of Corollary 3.1. The only difference is that hypothesis (H2) guarantees that the solution $z(t) \in D$ of $\dot z(t) = A(y(t), u(t))z(t) + b(y(t), u(t))$ exists for all $t \in [0, r)$ and it is bounded on the interval $[0, r)$. □

The following example illustrates how Corollary 3.3 can be applied for the dead-beat hybrid observer design of nonlinear systems.

Example 3.4. Consider the system

$$\begin{aligned}
&\dot x(t) = a(y(t), u(t))x(t)\\
&\dot y(t) = f(y(t), u(t)) + c(y(t))x(t)\\
&x(t) \in D, \quad y(t) \in \Omega, \quad u(t) \in U
\end{aligned} \tag{3.7}$$

where $D = \Re$ or $D = (0, +\infty)$, $\Omega \subseteq \Re$ is an open set, $U \subseteq \Re$ is a closed non-empty set, $a : \Omega \times U \to \Re$, $f : \Omega \times U \to \Re$, $c : \Omega \to \Re$ are continuously differentiable mappings satisfying the following hypothesis:

(H3) $\frac{dc}{dy}(y)f(y, u) \neq 0$, for all $(y, u) \in \Omega \times U$ with $c(y) = 0$.

We also assume that for every $(x_0, y_0) \in D \times \Omega$ and $u \in L^\infty_{loc}(\Re_+; U)$ there exists a unique mapping $[0, +\infty) \ni t \to (x(t), y(t)) = (x(t, x_0, y_0; u), y(t, x_0, y_0; u)) \in D \times \Omega$ satisfying (3.7) for almost every $t \ge 0$. It is clear from (3.7), the fact that $D = \Re$ or $D = (0, +\infty)$ and the integral formula $x(t) = x(0) \exp\left(\int_0^t a(y(s), u(s))ds\right)$ that hypothesis (H2) holds. Moreover, hypothesis (H3) and Corollary 2.4 guarantee that system (3.7) is strongly observable in time $r > 0$, for every $r > 0$. Indeed, notice that $\det\left(\left[C'(t, x_0, y_0; u)\Phi(t, x_0, y_0; u)\right]\right) = c(y(t, x_0, y_0; u)) \exp\left(\int_0^t a\,(y(s, x_0, y_0; u), u(s))\,ds\right)$. Let $r > 0$ be arbitrary. We claim that there exists $t \in [0, r]$ such that $\det\left(\left[C'(t, x_0, y_0; u)\Phi(t, x_0, y_0; u)\right]\right) \neq 0$, i.e., the hypotheses of Corollary 2.4 hold for every $(x_0, y_0) \in D \times \Omega$ and $u \in L^\infty_{loc}(\Re_+; U)$. Notice that if we assume the contrary (i.e., $\det([C'(t, x_0, y_0; u)\Phi(t, x_0, y_0; u)]) = 0$ for all $t \in [0, r]$), then we must have $c(y(t, x_0, y_0; u)) = 0$ for all $t \in [0, r]$. Consequently, we must have $\frac{d}{dt}c(y(t, x_0, y_0; u)) = 0$ for almost all $t \in (0, r)$, which combined with (3.7) gives $\frac{dc}{dy}(y(t, x_0, y_0; u))f(y(t, x_0, y_0; u), u(t)) = 0$ for almost all $t \in (0, r)$. The previous conclusion is in complete contradiction with hypothesis (H3).

Using the formulas $\Phi(t, y, u) = \exp\left(\int_0^t a(y(s), u(s))ds\right)$, $q(\tau) = \int_0^\tau c(y(s)) \exp\left(\int_0^s a(y(w), u(w))dw\right) ds$, $p(\tau) = y(\tau) - y(0) - \int_0^\tau f(y(s), u(s))ds$, $Q = \int_0^r \left(\int_0^\tau c(y(s)) \exp\left(\int_0^s a(y(w), u(w))dw\right)ds\right)^2 d\tau$, we conclude that the system:

$$\begin{aligned}
&\dot z(t) = a(y(t), u(t))z(t), \quad t \in [\tau_i, \tau_{i+1})\\
&z(\tau_{i+1}) = P(\delta_{\tau_i} y, \delta_{\tau_i} u)\\
&\tau_{i+1} = \tau_i + r
\end{aligned} \tag{3.8}$$

where, $P(y, u)$ (see Box I), is a dead-beat reduced order observer for system (3.7). □

We end this section by providing a discussion for the case where $O = \Re^n \times \Re^k$, the matrix $A(y, u)$ is constant, i.e., $A(y, u) \equiv A \in \Re^{n\times n}$, and the matrix $C \in \Re^{n\times k}$ defined by (2.8) is constant. In this case, system (1.1) takes the form

$$\begin{aligned}
&\dot x(t) = Ax(t) + b(y(t), u(t))\\
&\dot y(t) = C'x(t) + f(y(t), u(t))\\
&x(t) \in \Re^n, \quad y(t) \in \Re^k, \quad u(t) \in U \subseteq \Re^m
\end{aligned} \tag{3.10}$$

where $f(y, u) := (f_1(y, u), \dots, f_k(y, u))'$. Strong observability in (arbitrary) time $r > 0$ is guaranteed if the pair of matrices (A, C') is observable. Notice that system (3.10) is a linear time-invariant system with nonlinear output injection terms. In this case, the mapping $P : L^\infty([0, r]; \Omega) \times L^\infty([0, r]; U) \to \Re^n$ defined by (3.1) is expressed by the formula

$$\begin{aligned}
P(y, u) := S \int_0^r \left(\int_0^t \exp(A's)ds\right)
&\times C\left(y(t) - y(0) - \int_0^t \left(f(y(s), u(s))\right.\right.\\
&\left.\left. - \int_s^r C' \exp\,(A(s-\tau))\, b(y(\tau), u(\tau))d\tau\right)ds\right)dt
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
S := \exp\,(Ar) \Bigg(\int_0^r \left(\int_0^t \exp(A's)ds\right)
\times CC' \left(\int_0^t \exp(As)ds\right) dt\Bigg)^{-1}.
\end{aligned} \tag{3.12}$$

A standard Luenberger-type reduced order observer (see [10,26]) for system (3.10) would be a system of the form $\dot z = (A+LC')z(t) + b(y(t), u(t)) + Lf(y(t), u(t))$, $\hat x(t) = z(t) - Ly(t)$, where $L \in \Re^{n\times k}$ is a matrix which guarantees that the matrix $(A + LC') \in \Re^{n\times n}$ is Hurwitz and $\hat x(t)$ is the state estimate. The reduced order observer (3.5) is a completely different observer and is based on a completely different philosophy. A comparison between the two observers can be made:

- Observer (3.5) is a dead-beat observer, which guarantees zero estimation error after r time units. Fast convergence of the estimation error for the Luenberger-type reduced order

$$P(y,u)=\exp\left(\int_0^r a(y(s),u(s))\mathrm{d}s\right)\frac{\int_0^r\left(y(\tau)-y(0)-\int_0^\tau f(y(s),u(s))\mathrm{d}s\right)\left(\int_0^\tau c(y(s))\exp\left(\int_0^s a(y(w),u(w))\mathrm{d}w\right)\mathrm{d}s\right)\mathrm{d}\tau}{\int_0^r\left(\int_0^\tau c(y(s))\exp\left(\int_0^s a(y(w),u(w))\mathrm{d}w\right)\mathrm{d}s\right)^2\mathrm{d}\tau} \tag{3.9}$$

Box I.

observer would mean that the eigenvalues of the matrix $(A+LC')\in\Re^{n\times n}$ have large real parts, which means a "large" gain matrix $L\in\Re^{n\times k}$ and sensitivity with respect to measurement errors. The same sensitivity to measurement errors is noticed for the dead-beat reduced order observer (3.5), if r is made very small (see following section).

- The matrix $S\in\Re^{n\times n}$ defined by (3.12) can be computed off-line. However, for large scale systems the integrations involved in the operator $P:L^\infty([0,r];\Omega)\times L^\infty([0,r];U)\to\Re^n$ defined by (3.11) may have a high computational cost. On the other hand, the implementation of the Luenberger-type reduced order observer is straightforward even for large scale systems.

In general, the implementation of the dead-beat hybrid observer (3.3) requires the on-line solution of $2n^2+2n+nk+2k$ differential equations as well as the solution of a system of n linear algebraic equations. This may demand a high computational effort when n is large (large scale systems). This is a disadvantage of the proposed observer (3.3), which must be taken into account for practical implementation purposes.

4. Applications

This section is devoted to the study of an important application: the robust estimation of the frequency $\omega>0$ of a sinusoidal signal $y(t)=A\sin(\omega t+\varphi)$. The problem can be cast as an observer problem for the following system:

$$\begin{aligned}&\dot{y}(t)=x_1(t),\qquad \dot{x}_1(t)=x_2(t)y(t),\qquad \dot{x}_2(t)=0\\&(x_1(t),x_2(t),y(t))\in O\end{aligned}\tag{4.1}$$

where $O:=\left\{(x_1,x_2,y)\in\Re^3:y^2+x_1^2>0,x_2<0\right\}$ and $x_2(t)=-\omega^2$. It should be noticed that system (4.1) is forward complete and satisfies hypothesis (H1) for every $r>0$. Indeed, only initial states on the manifold $y=x_1=0$ can give identical output responses. The results of the previous section can be applied in order to give the hybrid full order observer:

$$\begin{aligned}&\dot{w}(t)=z_1(t),\qquad \dot{z}_1(t)=z_2(t)w(t),\qquad \dot{z}_2(t)=0\\&(z_1(t),z_2(t),w(t))\in O,\quad \text{for } t\in[\tau_i,\tau_{i+1})\end{aligned}\tag{4.2}$$

$$\begin{aligned}z_1(\tau_{i+1})&=3\Delta^{-1}\left(\int_{\tau_i}^{\tau_{i+1}}\phi^2(t)\mathrm{d}t-\int_{\tau_i}^{\tau_{i+1}}y(\tau)\mathrm{d}\tau\right.\\&\quad\times\left.\int_{\tau_i}^{\tau_{i+1}}(t-\tau_i)\phi(t)\mathrm{d}t\right)\int_{\tau_i}^{\tau_{i+1}}(y(t)-y(\tau_i))(t-\tau_i)\mathrm{d}t\\&\quad+\Delta^{-1}\left(r^3\int_{\tau_i}^{\tau_{i+1}}y(\tau)\mathrm{d}\tau-3\int_{\tau_i}^{\tau_{i+1}}(t-\tau_i)\phi(t)\mathrm{d}t\right)\\&\quad\times\int_{\tau_i}^{\tau_{i+1}}(y(t)-y(\tau_i))\phi(t)\mathrm{d}t\end{aligned}\tag{4.3}$$

$$\begin{aligned}z_2(\tau_{i+1})&=-3\Delta^{-1}\int_{\tau_i}^{\tau_{i+1}}(t-\tau_i)\phi(t)\mathrm{d}t\\&\quad\times\int_{\tau_i}^{\tau_{i+1}}(y(t)-y(\tau_i))(t-\tau_i)\mathrm{d}t+r^3\Delta^{-1}\\&\quad\times\int_{\tau_i}^{\tau_{i+1}}(y(t)-y(\tau_i))\phi(t)\mathrm{d}t\end{aligned}\tag{4.4}$$

$$\begin{aligned}&w(\tau_{i+1})=y(\tau_{i+1})\\&\tau_{i+1}=\tau_i+r\end{aligned}\tag{4.5}$$

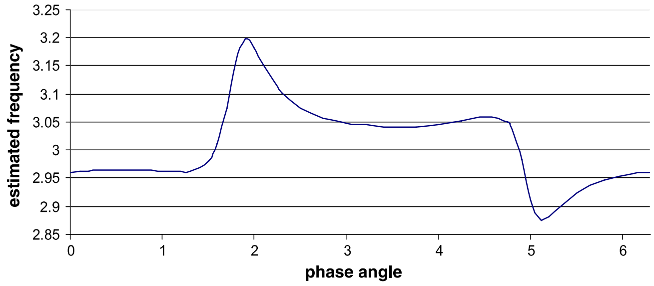

Fig. 1. Estimated frequency from formula (4.4) with $r=1,\tau_i=0,f=10$ as a function of the phase angle φ.

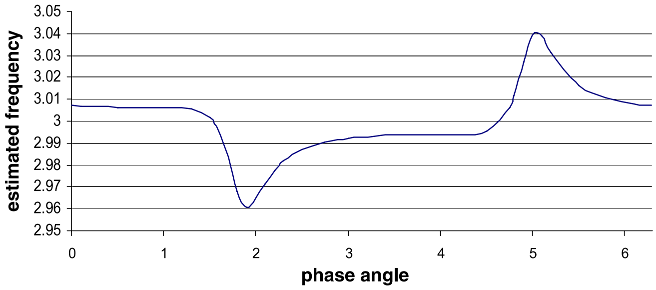

Fig. 2. Estimated frequency from formula (4.4) with $r=1,\tau_i=0,f=100$ as a function of the phase angle φ.

with $\phi(t)=\int_{\tau_i}^t\left(\int_{\tau_i}^s y(l)\mathrm{d}l\right)\mathrm{d}s$ and $\Delta=r^3\int_{\tau_i}^{\tau_{i+1}}\phi^2(t)\mathrm{d}t-3\left(\int_{\tau_i}^{\tau_{i+1}}(t-\tau_i)\phi(t)\mathrm{d}t\right)^2$. It should be noted that hypothesis (H2) does not hold for system (4.1). The frequency ω is estimated by means of the formula $\hat{\omega}=\sqrt{-z_2(t)}$.

We assume next that the measured signal is corrupted by high frequency noise, i.e.,

$$y(t)=A\sin(\omega t+\varphi)+a\sin(ft).\tag{4.6}$$

Exactly the same test of robustness as the one in [30] is performed: the parameters are chosen to be $A=2,a=0.2,\omega=3$. Three cases are considered for the frequency of the noise: $f=10,f=100$ and $f=1000$.

The effectiveness of formula (4.4) with $r=1,\tau_i=0$ is shown in Figs. 1–3 as a function of the phase angle φ. It is shown that the greatest estimation error is 6.6%, 1.3% and 0.083% for the cases $f=10,f=100$ and $f=1000$, respectively. The accuracy of the estimation is similar to the one obtained in [30], where the steady state estimation error was 10%, 1% and 0.1% for the cases $f=10$, $f=100$ and $f=1000$, respectively. It should be noted that the estimated frequency for the hybrid observer is provided only after $r=1$ s, while in [30] at least 5 s are needed in order to obtain an accurate estimate for the frequency.

However, if larger values for $r>0$ are used, then the accuracy of the estimation can be increased significantly. Fig. 4 shows the estimated frequency from formula (4.4) with $\tau_i=0$ as a function of r for the case $f=10$. The phase angle was selected to be $\varphi=1.9$: this is the value of the phase angle that the largest error of the estimation occurs (see also Fig. 1). For $r=3$ the estimation error is 0.066%, i.e., it is 100 times less than the error obtained for $r=1$.

Finally, it should be noted that the full order observer (4.2)–(4.5) can be used for system (4.1) even if the open set O is defined to be $O:=\left\{(x_1,x_2,y)\in\Re^3:y^2+x_1^2>0\right\}$. This is the case studied in [33].

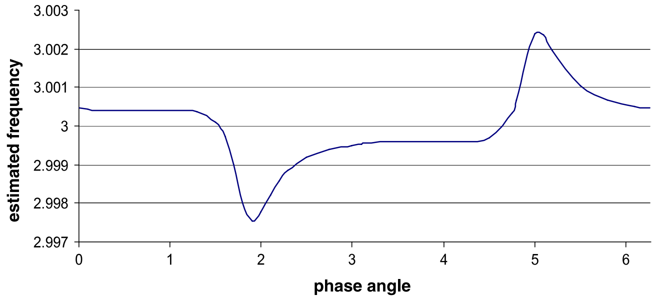

Fig. 3. Estimated frequency from formula (4.4) with $r = 1$, $\tau_i = 0$, $f = 1000$ as a function of the phase angle φ.

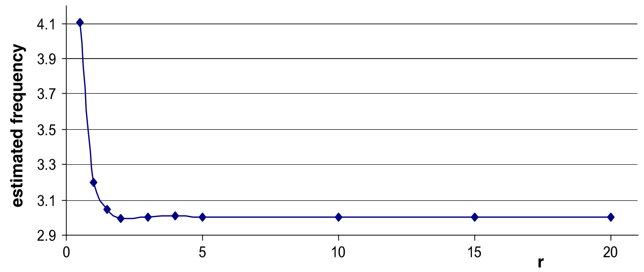

Fig. 4. Estimated frequency from formula (4.4) with $\varphi = 1.9$, $\tau_i = 0$, $f = 10$ as a function of r.

5. Robustness issues

In this section, we discuss briefly the robustness properties of the proposed observer. We focus on the case where both hypotheses (H1), (H2) are satisfied with $D = \Re^n$, $\Omega = \Re^k$ (although the same discussion can be applied to systems satisfying only hypothesis (H1)). It should be noted that a systematic study of the robustness properties of observers for nonlinear systems is rare and the topic is completely "untouched".

The previous application showed (numerically) that the proposed observer may be robust to measurement errors. By measurement error we mean a measurable and locally essentially bounded input $e : \Re_+ \to \Re^k$ which corrupts the output values that are fed to the observer, i.e., the observer is described by the equations:

$$\begin{aligned} &\dot{z}(t) = A(\tilde{y}(t), u(t))z(t) + b(\tilde{y}(t), u(t)), \quad t \in [\tau_i, \tau_{i+1}) \\ &z(\tau_{i+1}) = P(\delta_{\tau_i}\tilde{y}, \delta_{\tau_i}u) \\ &\tau_{i+1} = \tau_i + r \\ &z(t) = (z_1(t), \ldots, z_n(t))' \in \Re^n \end{aligned} \tag{5.1}$$

where $P : L^\infty([0, r]; \Re^k) \times L^\infty([0, r]; U) \to \Re^n$ is the operator defined by (3.1) and

$$\tilde{y}(t) = y(t) + e(t), \quad \forall t \geq 0. \tag{5.2}$$

First of all it should be noted that the dead-beat property cannot be guaranteed under the presence of measurement errors. However, different robustness properties can hold under appropriate assumptions. Taking into account the formula $z(\tau_{i+1}) = P(\delta_{\tau_i}\tilde{y}, \delta_{\tau_i}u) = P(\delta_{\tau_i}y + \delta_{\tau_i}e, \delta_{\tau_i}u)$ for all $i \geq 0$ and Corollary 3.3, which guarantees that $x(\tau_{i+1}) = P(\delta_{\tau_i}y, \delta_{\tau_i}u)$ for all $i \geq 0$, we conclude that the observer error induced by the measurement error at $t = \tau_{i+1}$ ($i \geq 0$) will satisfy:

$$|z(\tau_{i+1}) - x(\tau_{i+1})| = \left|P(\delta_{\tau_i}y + \delta_{\tau_i}e, \delta_{\tau_i}u) - P(\delta_{\tau_i}y, \delta_{\tau_i}u)\right|, \quad \text{for all } i \geq 0. \tag{5.3}$$

At this point, a pair of hypotheses is introduced in order to analyze further the time evolution of the observer error.

(R1) *For every $(x_0, y_0) \in \Re^n \times \Re^k$ and $u \in L^\infty(\Re_+; U)$, the unique solution $(x(t), y(t)) \in \Re^n \times \Re^k$ of (1.1), with initial condition $(x_0, y_0) \in \Re^n \times \Re^k$ corresponding to input $u \in L^\infty(\Re_+; U)$ satisfies $\sup_{t\geq 0}|y(t)| + \sup_{t\geq 0}|x(t)| < +\infty$.*

(R2) *The operator $L^\infty([0, r]; \Re^k) \times L^\infty([0, r]; U)(y, u) \to P(y, u) \in \Re^n$ is completely continuous with respect to $y \in L^\infty([0, r]; \Re^k)$, i.e., for every pair of bounded sets $S \subset L^\infty([0, r]; \Re^k)$, $V \subseteq L^\infty([0, r]; U)$ the image set $P(S \times V) \subset \Re^n$ is bounded and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\left|P(y, u) - P(\hat{y}, u)\right| < \varepsilon$ for every $y, \hat{y} \in S$, $u \in V$ with $\sup_{t\in[0,r]}\left|y(t) - \hat{y}(t)\right| < \delta$.*

Hypothesis (R1) imposes restrictions on the dynamic behavior of system (1.1). On the other hand, hypothesis (R2) is a continuity hypothesis which can be guaranteed easily for certain cases. A case where hypothesis (R2) holds is the case where for every pair of bounded sets $S \subset L^\infty([0, r]; \Re^k)$, $V \subseteq L^\infty([0, r]; U)$ there exists $a > 0$ such that $\det(Q) \geq a$ for all $y \in S$, $u \in V$, where $Q = \int_0^r q(\tau)q'(\tau)d\tau$, $q(\tau) = \int_0^\tau \Phi'(s, y; u)C(s)ds$, $C'(\tau) := \{c_{i,j}(y(\tau)), i = 1, \ldots, k, j = 1, \ldots, n\}$ and $\Phi(t, y; u)$ is the transition matrix of the linear system $\dot{z}(t) = A(y(t), u(t))z(t)$.

Another thing that should be noted here is that the estimation of the state of system (1.1) under hypothesis (R1) cannot be performed in general by means of a high-gain observer (see [7]). Indeed, although the subsystem $\dot{x}(t) = A(y(t), u(t))x(t) + b(y(t), u(t))$ is globally Lipschitz when (R1) holds and a bounded input $u \in L^\infty(\Re_+; U)$ is applied, we are not aware of the Lipschitz constant of the system (since we do not assume knowledge of $\sup_{t\geq 0}|y(t)| < +\infty$), which in general will depend on the initial conditions $(x_0, y_0) \in \Re^n \times \Re^k$ and the applied input $u \in L^\infty(\Re_+; U)$.

Using hypotheses (R1), (R2), we are in a position to show the following robustness result.

Proposition 5.1. *Consider system (1.1) for which hypotheses (H1), (H2) hold with $D = \Re^n$, $\Omega = \Re^k$. Moreover, assume that hypotheses (R1), (R2) hold as well. Then*

(a) *for every $(x_0, y_0, z_0) \in \Re^n \times \Re^k \times \Re^n$, $(u, e) \in L^\infty(\Re_+; U) \times L^\infty(\Re_+; \Re^k)$ the solution $(x(t), y(t), z(t)) \in \Re^n \times \Re^k \times \Re^n$ of (1.1), (5.1), (5.2) with initial condition $(x_0, y_0, z_0) \in \Re^n \times \Re^k \times \Re^n$ corresponding to inputs $(u, e) \in L^\infty(\Re_+; U) \times L^\infty(\Re_+; \Re^k)$ satisfies $\sup_{t\geq 0}|z(t) - x(t)| < +\infty$,*

(b) *for every $(x_0, y_0, z_0) \in \Re^n \times \Re^k \times \Re^n$, $u \in L^\infty(\Re_+; U)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for every $e \in L^\infty(\Re_+; \Re^k)$ with $\sup_{t\geq 0}|e(t)| < \delta$ the solution $(x(t), y(t), z(t)) \in \Re^n \times \Re^k \times \Re^n$ of (1.1), (5.1), (5.2) with initial condition $(x_0, y_0, z_0) \in \Re^n \times \Re^k \times \Re^n$ corresponding to inputs $(u, e) \in L^\infty(\Re_+; U) \times L^\infty(\Re_+; \Re^k)$ satisfies $\sup_{t\geq r}|z(t) - x(t)| < \varepsilon$,*

(c) *for every $(x_0, y_0, z_0) \in \Re^n \times \Re^k \times \Re^n$, $(u, e) \in L^\infty(\Re_+; U) \times L^\infty(\Re_+; \Re^k)$ with $\lim_{t\to+\infty}|e(t)| = 0$, the solution $(x(t), y(t), z(t)) \in \Re^n \times \Re^k \times \Re^n$ of (1.1), (5.1), (5.2) with initial condition $(x_0, y_0, z_0) \in \Re^n \times \Re^k \times \Re^n$ corresponding to inputs $(u, e) \in L^\infty(\Re_+; U) \times L^\infty(\Re_+; \Re^k)$ satisfies $\lim_{t\to+\infty}|z(t) - x(t)| = 0$.*

Proof. Let $(x_0, y_0, z_0) \in \Re^n \times \Re^k \times \Re^n$, $(u, e) \in L^\infty(\Re_+; U) \times L^\infty(\Re_+; \Re^k)$ be given and consider the solution $(x(t), y(t), z(t)) \in \Re^n \times \Re^k \times \Re^n$ of (1.1), (5.1), (5.2) with initial condition $(x_0, y_0, z_0) \in \Re^n \times \Re^k \times \Re^n$ corresponding to inputs $(u, e) \in L^\infty(\Re_+; U) \times L^\infty(\Re_+; \Re^k)$. By virtue of hypothesis (R1) we have $\|y\| = \sup_{t\geq 0}|y(t)| < +\infty$ and $\|x\| = \sup_{t\geq 0}|x(t)| < +\infty$. The proof is based on the following fact, which exploits the fact that $A(y, u) = \{a_{i,j}(y, u), i, j = 1, \ldots, n\}$ and all mappings $a_{i,j} : \Re^k \times U \to \Re$ ($i, j = 1, \ldots, n$), $b : \Re^k \times U \to \Re^n$, are locally Lipschitz. Its proof is standard and is omitted.

Fact. There exist non-decreasing functions $\kappa : \Re_+ \to \Re_+$ and $R : \Re_+ \to \Re_+$ such that

$$|x(t) - z(t)| \le [|x(\tau_i) - z(\tau_i)| + r\,\|e\|\,R(s)] \exp(r\kappa(s)), \quad \forall t \in (\tau_i, \tau_{i+1}) \tag{5.4}$$

where $s := \|y\| + \|e\| + \|u\| + \|x\|$, $\|y\| = \sup_{t\ge 0}|y(t)| < +\infty$, $\|x\| = \sup_{t\ge0}|x(t)| < +\infty$, $\|u\| = \sup_{t\ge0}|u(t)| < +\infty$ and $\|e\| = \sup_{t\ge0}|e(t)| < +\infty$.

By virtue of hypothesis (R2), which implies that the image set $P(S \times V) \subset \Re^n$ is bounded, where $S \subset L^\infty([0,r];\Re^k)$ is the bounded set of measurable and essentially bounded functions $z : [0,r] \to \Re^k$ with $\sup_{t\ge0}|z(t)| \le \|y\| + \|e\|$ and $V \subseteq L^\infty([0,r];U)$ is the bounded set of measurable and essentially bounded functions $v : [0,r] \to U$ with $\sup_{t\ge0}|v(t)| \le \|u\|$, there exists $K \ge 0$ such that $\left|P(\delta_{\tau_i}y + \delta_{\tau_i}e, \delta_{\tau_i}u)\right| \le K$ and $\left|P(\delta_{\tau_i}y, \delta_{\tau_i}u)\right| \le K$, for all $i \ge 0$. Consequently, we obtain from (5.3):

$$|z(\tau_{i+1}) - x(\tau_{i+1})| \le 2K, \quad \text{for all } i \ge 0. \tag{5.5}$$

Combining (5.4) and (5.5) we can conclude that $\sup_{t\ge0}|z(t) - x(t)| < +\infty$.

Inequality (5.4) implies that for every $\varepsilon > 0$ there exists $\delta_1 \in (0,\varepsilon)$ such that $|x(t) - z(t)| < \varepsilon$ for all $t \in (\tau_i, \tau_{i+1})$, provided that $\|e\| < \delta_1$ and $|z(\tau_i) - x(\tau_i)| < \delta_1$. Moreover, hypothesis (R2) and (5.3) implies the existence of $\delta_2 \in (0,\delta_1)$ such that $|z(\tau_{i+1}) - x(\tau_{i+1})| < \delta_1$, for all $i \ge 0$, provided that $\|e\| < \delta_2$. Combining the two previous inequalities, we conclude that $|x(t) - z(t)| < \varepsilon$ for all $t \ge r$, provided that $\|e\| < \delta_2$. On the other hand, if $\lim_{t\to+\infty}|e(t)| = 0$ then there exists $i \ge 0$ sufficiently large such that $\sup_{t\ge\tau_i}|e(t)| < \delta_2$. The previous inequalities imply that $|x(t) - z(t)| < \varepsilon$ for all $t \ge \tau_i + r$ and since $\varepsilon > 0$ is arbitrary we conclude that $\lim_{t\to+\infty}|z(t) - x(t)| = 0$.

The proof is complete. □

Remark 5.2. It should be noted that if $A(y,u) = A(u)$, then there is no need to assume that $\sup_{t\ge0}|x(t)| < +\infty$. Therefore, in this case the conclusions of Proposition 5.1 hold if hypothesis (R1) is replaced by the following weaker hypothesis:

(R3) *For every* $(x_0, y_0) \in \Re^n \times \Re^k$ *and* $u \in L^\infty(\Re_+;U)$, *the unique solution* $(x(t), y(t)) \in \Re^n \times \Re^k$ *of* (1.1), *with initial condition* $(x_0,y_0) \in \Re^n \times \Re^k$ *corresponding to input* $u \in L^\infty(\Re_+;U)$ *satisfies* $\sup_{t\ge0}|y(t)| < +\infty$.

Example 5.3. Consider system (3.7) with $D = \Re$, $\Omega = \Re$, $U = \Re$, which was studied in Example 3.4. Here, we assume that hypothesis (R1) holds for system (3.7) and that $c(y) > 0$ for all $y \in \Re$. Therefore, hypothesis (H3) holds automatically. Moreover, for every pair of bounded sets $S \subset L^\infty([0,r];\Re)$, $V \subseteq L^\infty([0,r];U)$, there exists $M > 0$ such that $\sup_{t\in[0,r]}|y(t)| \le M$ and $\sup_{t\in[0,r]}|u(t)| \le M$ for all $(y,u) \in S \times V$. Using the fact that the functions $a : \Omega \times U \to \Re$, $f : \Omega \times U \to \Re$, $c : \Omega \to \Re$ are locally Lipschitz and that the denominator in (3.9) satisfies

$$\int_0^r \left(\int_0^\tau c(y(s)) \exp\left(\int_0^s a(y(w), u(w))\mathrm{d}w \right) \mathrm{d}s \right)^2 \mathrm{d}\tau \ge \frac{r^3}{3}\mu^2,$$

$$\text{where } \mu := \min_{|y|\le M} c(y) \exp\left(\min_{\substack{|y|\le M, |u|\le M \\ s\in[0,r]}} (s\,a(y,u)) \right) > 0$$

it is straightforward to show that hypothesis (R2) holds for the mapping $P(y,u)$ defined by (3.9).

Therefore, Proposition 5.1 guarantees the BIBO and CICO properties for the output $Y(t) = z(t) - x(t)$ from the input $e \in L^\infty(\Re_+;\Re)$ for system (3.7) and the dead-beat hybrid observer

$$\begin{aligned} &\dot z(t) = a(y(t) + e(t), u(t))z(t), \quad t \in [\tau_i, \tau_{i+1}) \\ &z(\tau_{i+1}) = P(\delta_{\tau_i}y + \delta_{\tau_i}e, \delta_{\tau_i}u) \\ &\tau_{i+1} = \tau_i + r \end{aligned} \tag{5.6}$$

where $P(y,u)$ is defined by (3.9). □

6. Concluding remarks

In this work, a novel hybrid strategy has been developed for solving the dead-beat observer design problem for a class of nonlinear systems with unmeasured states appearing linearly in the differential equations. To this end, the notion of strong observability of a nonlinear control system is introduced and utilized. The proposed methodology is applied for the estimation of the frequency of a sinusoidal signal. The results show that accurate estimates can be provided even if the signal is corrupted by high frequency noise. The results can be applied to processes, which operate only for a finite time (see the example of the batch reactor in [34]).

Future work can shed new light to the problem of dynamic output feedback stabilization, already studied in [18–23]. The dead-beat feature of the proposed observer implies that any static feedback stabilizer for (1.1) can be used in conjunction with the hybrid dead-beat observer (3.3), provided that the inputs produced by the applied feedback can distinguish all states in finite time and that the solution does not blow up during the initial transient period. Another direction for future work is the application of the hybrid, dead-beat observer to systems of mathematical biology: the chemostat model (see [35]) takes the form of system (1.1), when the nutrient concentration is measured. Preliminary results in this research direction can be found in [36].

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Appendix

Proof of Proposition 2.3. We prove the implications (a) ⇔ (b), (b) ⇔ (c) and (c) ⇔ (d).

(a) ⇒ (b) The proof of this implication will be made by contradiction. Suppose that the input $u \in L^\infty([0,r];U)$ strongly distinguishes the state $(x_0,y_0) \in O$ in time $r > 0$. Notice that Fact I and definitions (2.6)–(2.9) imply that problem (2.10) is always solvable and always admits the solution $\xi = x_0$ with

$$\begin{aligned} 0 &= \int_0^r \left| p(t,x_0,y_0;u) - q'(t,x_0,y_0;u)x_0 \right|^2 \mathrm{d}t \\ &= \min_{\xi\in B(y_0)} \int_0^r \left| p(t,x_0,y_0;u) - q'(t,x_0,y_0;u)\xi \right|^2 \mathrm{d}t. \end{aligned} \tag{A.1}$$

Consequently, the negation of (b) is the following statement:

"Problem (2.10) admits the solution $\xi = x_1 \in \Re^n$ with $x_1 \ne x_0$ and $(x_1,y_0) \in O$".

Therefore, we assume that the above statement holds. By virtue of (A.1) we must have $0 = \int_0^r \left|p(t,x_0,y_0;u) - q'(t,x_0,y_0;u)x_1\right|^2 \mathrm{d}t$. Continuity of the mappings $t \to p(t,x_0,y_0;u)$ and $t \to q(t,x_0,y_0;u)$ implies that the following statement holds:

"there exists $x_1 \ne x_0$ with $(x_1,y_0) \in O$ such that $p(t,x_0,y_0;u) = q'(t,x_0,y_0;u)x_1$ for all $t \in [0,r]$".

The above statement in conjunction with definitions (2.6), (2.7), (2.9) shows that (by direct differentiation):

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t,x_0,y_0;u) = f(y(t,x_0,y_0;u),u(t)) + C'(t,x_0,y_0;u) \times (\Phi(t,x_0,y_0;u)x_1 + \theta(t,x_0,y_0;u))$$

where $f(y,u) := (f_1(y,u), \ldots, f_k(y,u))'$ and

$$\frac{d}{dt}((\Phi(t,x_0,y_0;u)x_1 + \theta(t,x_0,y_0;u))) = A(y(t,x_0,y_0;u),u(t))((\Phi(t,x_0,y_0;u)x_1 + \theta(t,x_0,y_0;u))) + b(y(t,x_0,y_0;u),u(t))$$

for almost all $t \in [0,r]$. Consequently, uniqueness of solutions for (1.1) implies that $x(t,x_1,y_0;u) = \Phi(t,x_0,y_0;u)x_1+\theta(t,x_0,y_0;u)$ and $y(t,x_1,y_0;u) = y(t,x_0,y_0;u)$ for all $t \in [0,r]$. Hence, it holds that:

"there exists $x_1 \neq x_0$ with $(x_1,y_0) \in O$ such that $y(t,x_1,y_0;u) = y(t,x_0,y_0;u)$ for all $t \in [0,r]$".

The above statement contradicts the assumption that the input $u \in L^\infty([0,r];U)$ strongly distinguishes the state $(x_0,y_0) \in O$ in time $r > 0$.

(b) $\Rightarrow$ (a) Again the proof of this implication will be made by contradiction. Suppose that problem (2.10) admits the unique solution $\xi = x_0$.

Assume that the input $u \in L^\infty([0,r];U)$ does not strongly distinguish the state $(x_0,y_0) \in O$ in time $r > 0$. This implies that

"there exists $x_1 \in \Re^n$ with $x_1 \neq x_0$ and $(x_1,y_0) \in O$ such that $y(t,x_1,y_0;u) = y(t,x_0,y_0;u)$ for all $t \in [0,r]$".

The reader should notice that the y-components of the different initial states $(x_0,y_0) \in O$ and $(x_1,y_0) \in O$ which produce identical outputs for $t \in [0,r]$, necessarily coincide. By virtue of Fact I and definitions (2.6)–(2.9) it follows that $p(t,x_0,y_0;u) = q'(t,x_0,y_0;u)x_1$ for all $t \in [0,r]$. Hence, we must have:

$$0 = \int_0^r \left|p(t,x_0,y_0;u) - q'(t,x_0,y_0;u)x_1\right|^2 dt.$$

The above equality shows that $x_1 \in \Re^n$ with $x_1 \neq x_0$ and $(x_1,y_0) \in O$ is a solution of problem (2.10), which contradicts the uniqueness of the solution for problem (2.10).

(b) $\Rightarrow$ (c) Again the proof of this implication will be made by contradiction. Suppose that problem (2.10) admits the unique solution $\xi = x_0$. Notice that the objective function for problem (2.10) is the quadratic function:

$$\begin{aligned} R(\xi) &:= \int_0^r \left|p(t,x_0,y_0;u) - q'(t,x_0,y_0;u)\xi\right|^2 dt \\ &= \int_0^r |p(t,x_0,y_0;u)|^2 dt \\ &\quad - 2\int_0^r p'(t,x_0,y_0;u)q'(t,x_0,y_0;u)dt\,\xi \\ &\quad + \xi' Q(r,x_0,y_0;u)\xi \end{aligned} \tag{A.2}$$

where $Q(r,x_0,y_0;u)$ is defined by (2.11) and for which it holds that

$$\begin{aligned} R(\xi) = R(x_0) + 2&\left[-\int_0^r q(t,x_0,y_0;u)p(t,x_0,y_0;u)dt + Q(r,x_0,y_0;u)x_0\right]'(\xi - x_0) + (\xi - x_0)' \\ &\times Q(r,x_0,y_0;u)(\xi - x_0). \end{aligned} \tag{A.3}$$

Since $\xi = x_0$ is a solution of problem (2.10) and since O is open the above equality shows that we must necessarily have

$$\int_0^r q(t,x_0,y_0;u)p(t,x_0,y_0;u)dt = Q(r,x_0,y_0;u)x_0. \tag{A.4}$$

On the other hand assume that statement (c) does not hold, i.e., assume that the symmetric and positive semidefinite matrix $Q(r,x_0,y_0;u) := \int_0^r q(t,x_0,y_0;u)q'(t,x_0,y_0;u)dt$ is not positive definite. Therefore there exists $\zeta \in \Re^n$, $\zeta \neq 0$ such that $0 = \zeta' Q(r,x_0,y_0;u)\zeta$. It follows from (A.3), (A.4) and for sufficiently small $\lambda > 0$ that the vector $\xi = x_0 + \lambda\zeta$ will satisfy $(\xi,y_0) \in O$ (because O is open) and $R(\xi) = R(x_0)$, i.e., the vector $\xi = x_0 + \lambda\zeta$ is an additional solution of problem (2.10) with $\xi \neq x_0$, a contradiction.

Therefore $Q(r,x_0,y_0;u) := \int_0^r q(t,x_0,y_0;u)q'(t,x_0,y_0;u)dt$ is positive definite. Eq. (2.12) is a direct consequence of Eq. (A.4).

(c) $\Rightarrow$ (b) This implication is a direct consequence of (A.2)–(A.4), which show that

$$R(\xi) = (\xi - x_0)' Q(r,x_0,y_0;u)(\xi - x_0), \quad \text{for all } \xi \in \Re^n \text{ with } (\xi,y_0) \in O.$$

Notice that equality (2.5) guarantees that $R(x_0) = 0$.

(c) $\Rightarrow$ (d) This implication follows from the fact that $\xi' Q(r,x_0,y_0;u)\xi := \int_0^r \left|q'(t,x_0,y_0;u)\xi\right|^2 dt$, for all $\xi \in \Re^n$.

(d) $\Rightarrow$ (c) Statement (c) follows from the fact that $\xi' Q(r,x_0,y_0;u)\xi := \int_0^r \left|q'(t,x_0,y_0;u)\xi\right|^2 dt$, for all $\xi \in \Re^n$ and the fact that the mapping $t \to q(t,x_0,y_0;u)$ is continuous. Equality (2.12) is a direct consequence of Fact I and equality (2.5).

The proof is complete. □

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